section section

Asymptotically exact spectral estimates for left triangular matrices

Michael Blank*

Russian Ac. of Sci., Inst. for Information Transmission Problems,
B.Karetnij 19, 101447, Moscow, Russia, blank@iitp.ru

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Abstract

For a family of $n * n$ left triangular matrices with binary entries we derive asymptotically exact (as $n \to \infty$) representation for the complete eigenvalues-eigenvectors problem. In particular we show that the dependence of all eigenvalues on n is asymptotically linear for large n. A similar result is obtained for more general (with specially scaled entries) left triangular matrices as well. As an application we study ergodic properties of a family of chaotic maps.

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1 Binary left triangular matrices

By an $n \times n$ left triangular matrix with binary entries we mean the matrix $A_n = (a_{ij})$ whose entries up to (and including) the secondary diagonal are ones, while all others are zeros, i.e. $a_{ij} = 1$ if and only if $1 \le i \le n - j + 1$. Despite its very classical appearance spectral properties of such matrices were not known, probably due to the fact that for a finite matrix

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size n there is no reasonable representation for the spectrum. Denote by $\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_n^{(n)}$ the eigenvalues of the matrix A_n ordered by their modules (in the decreasing order) and by $\{e_k^{(n)}\}_{k=1}^n$ the corresponding orthogonal system of eigenvectors (which exists due to symmetry of the matrix). As usual by $(e)_i$ we denote the i-th entry of the vector $e \in \mathbf{R}^n$ and by $||e||$ – its $\mathbf{L}^2$-norm.

We start from numerical results. It turns out that numerically with very high accuracy for each fixed $k \in \{1, \dots, n\}$ the dependence of the k-th eigenvalue $\lambda_k^{(n)}$ is linear on n. ¿From this result one might expect that the determinant $\det(A_n)$ grows as n^n for large n, however expanding the matrix with respect to the last line one shows that $\det(A_n) = (-1)^{n+1}$. So there is no surprise that another numerical test shows that for a fixed value n the modules of eigenvalues $|\lambda_k^{(n)}|$ decay hyperbolically on k. The following statement gives an asymptotic representation for the spectrum, which confirms these numerical findings.

Theorem 1.1 *Let A_n be the left triangular matrix. Then for each fixed $k \in \{1, \dots, n\}$*

$$\lambda_k^{(n)} = (-1)^{k+1} \frac{n}{(k-1/2)\pi} + O(1),$$

$$(e_k^{(n)})_i = \cos\left(\frac{(k-1/2)(i-1)}{n}\pi\right) + O(1/n).$$

The **proof** is based on the following statements. First notice that $(A_n v)_i = \sum_{j=1}^{n-i+1} v_j$ for any vector $v \in \mathbf{R}^n$.

Lemma 1.1 *For any C^1 function $f : [0,1] \to [-1,1]$ with the derivative not exceeding 1 by absolute value we have*

$$\sum_{j=1}^{n-i+1} f((j-1)/n) = n \sum_{j=1}^{n-i+1} f((j-1)/n) \cdot \frac{1}{n} = n \int_0^{1-\frac{i-1}{n}} f(s)\, ds + O(1).$$

Proof. Direct computation. ∎

Lemma 1.2 *The operator $L : f(x) \to \int_0^{1-x} f(s)\, ds$ considered as an operator acting in the space $\mathbf{L}^2$ has the complete system of orthogonal eigenfunctions*

$$E_k(x) := \cos((k-1/2)\pi x)$$

with eigenvalues

$$\mu_k := (-1)^{k+1}\frac{1}{(k-1/2)\pi}$$

for $k = 1, 2, \ldots$.

Proof. Indeed,

$$\int_0^{1-x} \cos((k-1/2)\pi s)\ ds = -\frac{1}{(k-1/2)\pi}\sin((k-1/2)\pi s)|_0^{1-x}$$

$$= \frac{1}{(k-1/2)\pi}\sin((k-1/2)\pi(1-x)) = \frac{(-1)^{k+1}}{(k-1/2)\pi}\cos((k-1/2)\pi x).$$

■

Lemma 1.3 *Let the equality* $Av = \mu v + \xi$ *be satisfied for a symmetric matrix* A*, two vectors* $v, \xi \in \mathbf{R}^n$ *with* $||\xi|| \le \varepsilon||v||$ *and a scalar* μ*. Then the closest to* μ *eigenvalue* λ *of the matrix* A *satisfies the inequality* $|\lambda - \mu| \le \varepsilon$ *and if its multiplicity is equal to 1, then the corresponding eigenvector* e *is such that* $||e - v|| \le O(\varepsilon)\cdot||v||$.

Proof. Probably this statement is not quite new (see for example close statements in [3]), however since we were not able to find exact references to needed estimates we give a sketch of the proof. Notice that $(A - \xi v^*/||v||^2)v = \mu v$, i.e. the vector v is the eigenvector of the perturbed matrix $A_\mu := A - \xi v^*/||v||^2$. Indeed,

$$A_\mu v = \mu v + \xi - \xi v^* v/||v||^2 = \mu v + \xi - \xi = \mu v.$$

On the other hand,

$$||\xi v^*/||v||^2|| \le \frac{||\xi||}{||v||} \le \varepsilon.$$

Thus the matrix A_μ can be considered as an ε-perturbation of the symmetric matrix A. Thus the desired estimates can be obtained by the standard perturbation argument.

Moreover, to prove the first inequality about the closest eigenvalue λ one can use a more direct argument [3]. Namely, for $\mu = \lambda$ the inequality becomes trivial, while otherwise

$$||v|| \le ||(A - \mu I)^{-1}|| \cdot ||(A - \mu I)v|| = \frac{1}{|\mu - \lambda|}\cdot||\xi||.$$

Thus

$$|\mu - \lambda| \le \frac{||\xi||}{||v||} \le \varepsilon.$$

■

Proof of Theorem 1.1. It suffices to show that for each $k = 1, 2, \dots, n$ a projection to the space of piecewise constant functions of the k-th eigenvector of the linear operator L introduced in Lemma 1.2 satisfies the equality of type described in Lemma l=apr with $\varepsilon = O(1)$. Let $\mathbf{P}_n : C^1([0,1] \to \mathbf{R}^1, R^n)$ be a projection operator acting from the space of C^1 functions $f : [0,1] \to \mathbf{R}^1$ to $\mathbf{R}^n$ defined as follows:

$$(\mathbf{P}_n f)_i := f((i-1)/n).$$

Then, according to Lemma 1.1, the following equality holds true:

$$A_n \mathbf{P}_n E_k = n\mu_k \mathbf{P}_n E_k + \xi$$

for a vector $\xi \in \mathbf{R}^n$ such that $||\xi|| \le O(1)$. Now the application of Lemma 1.3 finishes the proof. ■

2 Properly scaled left triangular matrices

Results of the previous section might be generalized for a more general class of left triangular matrices. Let $\phi : [0,1] \to (0,1]$ be a C^1 positive function. We define a family of $n \times n$ left triangular matrices generated by the function ϕ as follows: $(A_n^\phi)_{i,j} := \phi((j-i+1)/n)$ for $i \le j$ and zero otherwise. Clearly the binary left triangular matrices satisfy this property for $\phi \equiv 1$.

Again similarly to the argument in the previous section one can prove the asymptotically linear dependence on n of the eigenvalues of the matrix A_n^ϕ using the property that the main contribution to the "shape" of eigenvectors comes from the corresponding eigenfunctions of the integral operator L_ϕ : $f(x) \to \int_0^{1-x} \phi(s) f(s)\ ds$.

Theorem 2.1 *Let μ and f_μ be respectively an eigenvalue and an eigenfunction of the operator L_ϕ (i.e. $L_\phi f_\mu = \mu f_\mu$). Then for each positive integer n there exists an eigenvalue $\lambda = n\mu + O(1)$ and an eigenvector $e \in \mathbf{R}^n$ of the matrix A_n^ϕ such that $(e)_i = f_\mu((i-1)/n) + O(1/n)$ for each $i = 1, 2, \dots, n$.*

Since in this more general case we are not able to give an explicit expression for the asymptotic spectrum, we shall demonstrate also another (more direct) way to derive the asymptotic linearity of the spectrum. Assume that the function ϕ is nonincreasing and denote by S_1^n the set of C^1 monotonically decreasing concave positive functions $f : [0,1] \to [0,1]$ with $f(0) = 1$ and by $\mathbf{S}_1^n \subset \mathbf{R}^n$ – the set of vectors v such that $v_i := f((i-1)/n)$ for some $f \in S_1^n$. We introduce also the following nonlinear operator $B_n : \mathbf{R}^n \to \mathbf{R}^n \cup (\infty)^n$ defined by the relation: $(B_n v)_i := (A_n^\phi v)_i / (A_n^\phi v)_1$.

Lemma 2.1 *The set $\mathbf{S}_1^n$ is invariant with respect to the operator B_n and for any vector $v \in \mathbf{S}_1^n$ the sequence $\{B_n^m v\}_m$ converges as $m \to \infty$ in $\mathbf{L}^2$-norm to the leading eigenvector $e_1^{(n)}$ of the matrix A_n^ϕ.*

Proof. Since the coordinates of the vector v are positive we get that for any i

$$(A_n^\phi v)_{i+1} - (A_n^\phi v)_i = -\phi((n-i+1)/n) v_{n-i+1} < 0.$$

On the other hand, due to the fact that the coordinates of the vector v decrease monotonically we obtain

$$(A_n^\phi v)_{i+1} - 2(A_n^\phi v)_i + (A_n^\phi v)_{i-1} = \phi(\frac{n-i+2}{n}) v_{n-i+2} - \phi(\frac{n-i+1}{n}) v_{n-i+1}$$

$$= \phi(\frac{n-i+1}{n})(v_{n-i+2} - v_{n-i+1}) - \left(\phi(\frac{n-i+1}{n}) - \phi(\frac{n-i+2}{n}) v_{n-i+2}\right) v_{n-i+2}$$

$$< \phi(\frac{n-i+1}{n})(v_{n-i+2} - v_{n-i+1}) < 0.$$

These two inequalities prove the monotonicity and the concavity of $A_n^\phi v$ respectively. The normalization by the positive number $(A_n^\phi v)_1$ finishes the proof of the first statement of Lemma 2.1. Now noticing that the matrix A_n^ϕ has only non negative entries we derive the second statement from the well known Perron theorem. ∎

Thus choosing vectors v from the set $\mathbf{S}_1^n$ providing the smallest and the largest contribution to $(A_n^\phi v)_1$ (namely $(\tilde{v})_i = (n-i+1)/n$ and $(\hat{v})_i \equiv 1$ respectively) we immediately obtain that

$$n/2 \int_0^1 (1-s)\phi(s)\ ds + O(1) \le \lambda_1^{(n)} \le n \int_0^1 \phi(s)\ ds + O(1).$$

In a similar (but somewhat involved) way one can study also other eigenvalues and eigenvectors of this matrix.

3 Application to chaotic dynamics

Now we apply above results for the analysis of ergodic properties of a certain family of chaotic maps. We refer the reader for the necessary definitions and statements for example to [2, 1].

Fix a positive integer n and consider a piecewise linear map map $T_n : [0,1] \to [0,1]$ from the unit interval into itself defined as

$$T_n|_{[(i-1)/n, i/n)}(x) := \frac{i}{n}x$$

for all $i \in \{1, 2, \ldots, n\}$. This map is topologically mixing and its second iterate is piecewise expanding (i.e. the modulus of the derivative of the map is larger than 1 for all points where it is well defined), therefore this map has a unique absolutely continuous invariant measure (see [1]).

Lemma 3.1 *The topological entropy of the map T_n satisfies the equality $h_{\text{top}}(T_n) = \ln\left(\frac{2n}{\pi} + O(1)\right)$.*

Proof. Notice that the map T_n is Markov with respect to the partition to its intervals of monotonicity, i.e. it maps each interval of monotonicity to the union of some other intervals of monotonicity. Therefore one can construct the corresponding topological dynamics [2] as a shift operator σ_n in the state of sequences $\bar{y} = \{y_1, y_2, \ldots\}$ with the alphabet consisting of n symbols (i.e. $y_i \in \{1, 2, \ldots, n\}$ and $(\sigma_n \bar{y})_j := y_{j+1}$), satisfying the property that for each $k \in \{1, 2, \ldots, n\}$ the symbol k may be followed by any symbol from the set $\{1, 2, \ldots, n-k+1\}$, but not the symbol from the complement to this set. Thus the corresponding transition matrix A_n [2] (matrix consisting of zeros and ones, describing possible transitions between symbols of the alphabet under the dynamics) is the left triangular binary $n \times n$ matrix. Now since the topological entropy h_{top} for both the Markov map T_n and the shift operator σ_n is known to be equal to the logarithm of the largest eigenvalue of the transition matrix [2] we derive from Theorem 1.1 the desired relation for the topological entropy. ■

References

[1] M.L. Blank, *Discreteness and continuity in problems of chaotic dynamics*, Monograph, Amer. Math. Soc., 1997.

[2] N.S.G. Martin, G.W. England, *Mathematical Theory of Entropy*, Enc. of Math. and Appl., v.12, 1981, Addison-Wesley Publ. Comp.

[3] B.N. Parlett, *The symmetric eigenvalue problem*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1980.