Constructive Fractional-Moment Criteria for Localization in Random Operators

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Abstract

We present a family of finite-volume criteria which cover the regime of exponential decay for the fractional moments of Green functions of operators with random potentials. Such decay is a technically convenient characterization of localization for it is known to imply spectral localization, absence of level repulsion, dynamical localization and a related condition which plays a significant role in the quantization of the Hall conductance in two-dimensional Fermi gases. The constructive criteria also preclude fast power-law decay of the Green functions at mobility edges.

Key words: Random operators, localization, constructive criteria, mobility edge.

1 Introduction

In the study of Anderson localization the analogy with the statistical mechanics of spin systems has often served as a source of insight [1,2]. In this note we report on some recent results [3] which were motivated in part by this analogy, and in part by the desire to develop elementary methods for the study of different aspects of the localization phenomena. Our goal is to present only an outline – the detailed statements and proofs are given in ref. [3].

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The subject of our discussion are random operators acting on the Hilbert space $\ell^2(\mathbb{Z}^d)$ of square summable functions defined over the regular d-dimensional lattice $\mathbb{Z}^d$. A prototypical example is the discrete Schrödinger operator:

$$H_\omega \;=\; -\Delta \;+\; V_\omega(x) \;, \tag{1}$$

where $V_\omega(x)$ is a random potential, and Δ is the nearest neighbor difference operator ("discrete Laplacian"). The operator may be augmented by the addition of a magnetic field, and/or random off-diagonal hopping terms. However, for our results we assume translation invariance at least in the stochastic sense and up to gauge transformations (i.e., magnetic shifts). (The statements may also be adapted to periodic and quasi-periodic structures.) It is now well understood that such operators have energy regimes in which the spectrum consists of an infinite collection of eigenvalues associated to exponentially localized eigenfunctions. It is also expected, although the theory is sorely lacking, that in certain situations such operators also possess energy regions associated to extended states.

A very useful tool is provided by the Green function of H_ω:

$$G_\omega(x,y;E+\mathrm{i}\eta) \;:=\, < x|\frac{1}{H_\omega - E - \mathrm{i}\eta}|y> \;\; . \tag{2}$$

To illustrate the relation with spin systems one may note the analogy between G_ω and the spin-spin correlation functions suggested by the functional integral expression for the former as a Gaussian integral:

$$G_\omega(x,y;E+\mathrm{i}\eta) \;=\; (-\mathrm{i})\frac{\int[\mathcal{D}\Phi]\ \mathrm{e}^{-\mathrm{i}\langle\Phi,(H_\omega - E - \mathrm{i}\eta)\Phi\rangle}\ \Phi(x)\Phi(y)}{\int[\mathcal{D}\Phi]\ \mathrm{e}^{-\mathrm{i}\langle\Phi,(H_\omega - E - \mathrm{i}\eta)\Phi\rangle}} \;. \tag{3}$$

The relevant rigorous methods familiar from the study of spin systems do not apply here since the integral involves a complex action. Nevertheless, exponential bounds for G_ω in the limit $\eta \to 0$ are a "signature" of the localized regime, much as exponential decay of spin-spin correlations indicates the high-temperature regime.

Since the Green function depends on the disorder, it is tempting to consider the averaged function $\mathbb{E}\left(|G_\omega(x,y;E+\mathrm{i}\eta)|\right)$, where $\mathbb{E}(\cdot)$ indicates the expectation with respect to the potential. However, for E in the spectrum of H_ω, this quantity may diverge as $\eta \to 0$. As was realized in ref. [4], we can avoid this problem by considering a "fractional moment" of the Green function: $\mathbb{E}\left(|G_\omega(x,y;E+\mathrm{i}\eta)|^s\right)$ for $s < 1$.

Thus, a technically convenient "signature" of localization is the exponential decay of such fractional moments, at some suitable $s \in (0,1)$,

$$\mathbb{E}(| < x| \frac{1}{H_\omega - E - \mathrm{i}\eta} |y > |^s) \ \leq \ A(s) \ \mathrm{e}^{-\mu(s)|x-y|} \ , \tag{4}$$

where the bound is satisfied uniformly in $\eta \in \mathbb{R}$ for all energies in some range $E \in (a,b)$.

Before we turn to the new results, which offer constructive criteria for the validity of the fractional moment condition (4), let us list several known implications of this condition, each of which also presumes some mild regularity conditions on the distribution of the random potential:

i. *Spectral localization ([4] - using [5]):* The spectrum of H_ω within the interval (a,b) is almost-surely of the pure-point type, and the corresponding eigenfunctions are exponentially localized.

ii. *Dynamical localization ([6]):* Wave packets with energies in the specified range do not spread (and in particular the *SULE* condition of [7] is met):

$$\mathbb{E}\left(\sup_{t \in \mathbb{R}} | < x| \ \mathrm{e}^{-\mathrm{i}tH} \, P_{H \in (a,b)} \ |y > | \right) \ \leq \widetilde{A} \, \mathrm{e}^{-\tilde{\mu}|x-y|} \ . \tag{5}$$

iii. *Absence of level repulsion ([8]).* Minami has shown that (4) implies that in the range (a,b) the energy gaps have Poisson-type statistics.

iv. *Exponential decay of the projection kernel ([9]):*

$$\mathbb{E}(\, | < x| \ P_{H \leq E} \ |y > | \,) \ \leq \widehat{A} \, \mathrm{e}^{-\hat{\mu}|x-y|} \ . \tag{6}$$

This condition plays an important role in the quantization of Hall conductance in the ground state of the two–dimensional electron gas with Fermi level $E_F \in (a,b)$ [10,11,9].

The fractional moment condition (4) has already been established for certain regimes: large disorder and extreme energies [4], and also at weak disorder but for energies far from the spectrum of $-\Delta$ [6]. Our new results extend the reach of the fractional moment method by showing that the entire region in which (4) holds can, in principle, be determined by a sequence of finite calculations.

In their general appearance, these results may remind one of some of the constructive criteria for the high temperature phases in certain models of statistical mechanics, which are mentioned below. As in that case, the results also yield some conclusions about the critical behavior, which in the present context refers to the behaviour in the vicinity of the mobility edge – wherever such an edge occurs.

Before the introduction of the fractional-moment method, localization regimes have been established using the multiscale analysis of Fröhlich and Spencer

[12] which yields exponential bounds of the form:

$$|G_\omega(x,y;E)| \;\le\; A(\omega,x)\,\mathrm{e}^{-\mu(\omega)|x-y|}\;, \tag{7}$$

with $\mu(\omega) > 0$ and $A(\omega, x) < \infty$ for almost every ω and all $x \in \mathbb{Z}^d$. The bounds which the multiscale analysis provides for the probability of the exceptional cases decay faster than any power of $|x - y|$ but not exponentially fast. It is difficult to use such results to demonstrate exponential bounds on *expectation values* such as those seen in equations (6) and (5). Nevertheless we note that dynamical localization was recently established also by arguments starting from the bounds provided by the multiscale analysis [13], using methods not related to this work. We shall return to the relation between the fractional-moment method and the multiscale analysis in the final section of this note.

2 The finite-volume criteria

The results presented herein describe certain conditions which when satisfied by the operator $H_{\Lambda;\omega}$ obtained by restricting H_ω to some finite volume Λ are sufficient to deduce the fractional moment condition (4) for the full operator H_ω. For simplicity we state these results only in the case of random Schrödinger operators. The reader is directed to [3] for versions which apply to more general operators.

To guarantee that the fractional moments are finite, we require certain regularity of the joint probability distribution of the site potentials $V(x)$. An additional technical assumption related to the "decoupling lemmas" used in [4,6,9] is also required. In their mildest form the conditions required are somewhat technical to state, so for the present note we shall call a probabilty distribution of the potential *regular* if the site values $V(x)$ are independent identically distributed random variables whose distribution has a bounded density with compact support. The interested reader may find the more general assumptions in [3].

In order to state our results, we must introduce some notation. Given a finite region $\Lambda \subset \mathbb{Z}^d$, we denote by $\Gamma(\Lambda)$ the set of lattice bonds (nearest neighbor pairs) connecting sites in Λ with sites in $\mathbb{Z}^d \setminus \Lambda$, and by Λ^+ the region obtained from Λ by adding to it all of its nearest neighbors. The number of elements of a set W is denoted $|W|$.

As mentioned above, $\mathbb{E}(\cdot)$ indicates the expectation with respect to the random potential. We also let

$$\mathbb{E}_{+\mathrm{i}0\,(-\mathrm{i}0)}(\,G(E)\,) \;:=\; \lim_{\substack{\eta\searrow 0\\(\eta\nearrow 0)}} \mathbb{E}(\,G(E+\mathrm{i}\eta)\,)\;. \tag{8}$$

Following is the first of our results.

Theorem 1 *Let H_ω be a random Schrödinger operator with a regular distribution of the random potential. Then for each $s < 1$ there exists $C_s < \infty$ such that if for some $E \in \mathbb{R}$ (in fact also $E \in \mathbb{C}$) and some finite region $\Lambda \subset \mathbb{Z}^d$ which contains the origin O:*

$$\left(1 + \frac{C_s}{\lambda^s} \, |\Gamma(\Lambda)|\right)^2 \sum_{<u,u'>\in\Gamma(\Lambda)} \mathbb{E}\left(|<O|\frac{1}{H_{\Lambda;\omega} - E}|u>|^s \right) \; < \; 1 \; , \tag{9}$$

then H_ω satisfies the fractional-moment condition (4), and there exist $\mu(s) > 0, A(s) < \infty$, which depend on E only through the value of the LHS in eq. (9), so that for any region $\Omega \subset \mathbb{Z}^d$,

$$\mathbb{E}_{\pm i0}\left(|<x|\frac{1}{H_{\Omega;\omega} - E}|y>|^s \right) \; \leq \; A(s) \; \mathrm{e}^{-\mu(s) \, \mathrm{dist}_\Omega(x,y)} \; , \tag{10}$$

with

$$\mathrm{dist}_\Omega(x,y) = \min\{|x-y|, [\mathrm{dist}(x,\partial\Omega) + \mathrm{dist}(y,\partial\Omega)]\} \; . \tag{11}$$

The modified metric, $\mathrm{dist}_\Omega(x,y)$, is a distance function relative to which the entire boundary of Ω is regarded as one point. It permits us to state that there is exponential decay in the “bulk” without ruling out the possible existence of extended *boundary states* in some geometry.

One may also formulate finite-volume criteria which rule out extended boundary states, that is which permit us to conclude exponential decay of the fractional moments in *any* region Ω. The trade-off is that the finite volume test, presented in the next result, is a bit more involved.

Theorem 2 *Let H_ω be a random Schrödinger operator with a regular distribution of the random potential. Then for each $s < 1$ there exists $\widetilde{C}_s > 0$ such that if for some $E \in \mathbb{R}$ (alternatively, complex E) and some finite region $O \in \Lambda \subset \mathbb{Z}^d$:*

$$\max_{W\subset\Lambda} \left\{ |\Gamma(\Lambda^+)| \, \frac{\widetilde{C}_s}{\lambda^s} \sum_{<u,u'>\in\Gamma(\Lambda)} \mathbb{E}\left(|<O|\frac{1}{H_{W;\omega} - E}|u>|^s \right) \right\} \; < \; 1 \; , \tag{12}$$

then there are $\mu(s) > 0$ and $A(s) < \infty$ — which depend on the energy E only through the value of the LHS of eq. (12) — such that for any region $\Omega \subset \mathbb{Z}^d$

$$\mathbb{E}_{\pm i0}\left(|<x|\frac{1}{H_{\Omega;\omega} - z}|y>|^s \right) \; \leq \; A(s) \; \mathrm{e}^{-\mu(s) \, |x-y|} \; . \tag{13}$$

It is rather obvious that the collection of finite-volume criteria provided in Theorem 2 covers the entire regime in which the conclusion, eq. (13), holds. The corresponding statement for Theorem 1 is a bit less immediate, but it is also true:

Theorem 3 *Let H_ω be a random Schrödinger operator with a regular distribution of the random potential, and fix $s < 1$. If at some energy E (or $E \in \mathbb{C}$) the localization condition (4) is satisfied, with some $A < \infty$ and $\mu > 0$, then for all large enough (but finite) L the condition (9) is met for $\Lambda = [-L, L]^d$.*

3 Some Implications

We shall now mention a number of implications of the finite-volume criteria for fractional moment localization.

First, of course, are explicit bounds, and we already obtain such bounds with a single site estimate corresponding to $\Lambda = \{0\}$. The test provided by Theorem 2 is met for all λ and E such that:

$$\frac{2d^2(2d+1)\, C_s}{\lambda^s} \; \mathbb{E}\Big(\frac{1}{|\lambda V - E|^s} \Big) \;<\; 1. \tag{14}$$

This implies localization for strong disorder, and at extremal energies, in the manner of ref. [4].

The above explicit criterion may now be systematically improved. However, since the calculations quickly become quite laborious, perhaps the main benefit are certain qualitative statements. Those bear some resemblance to results derived using the multiscale approach; however the conclusions drawn here go beyond the latter by yielding results on the exponential decay of the *mean values*.

3.1 Fast power decay $\Rightarrow$ exponential decay

An interesting and useful implication (as seen below) is that fast enough power law implies exponential decay. In this sense, random Schrödinger operators join other statistical mechanical models in which such principles have been previously recognized. The list includes the general Dobrushin-Shlosman results [14] and the more specific two-point function bounds for: percolation [15,16], Ising ferromagnets [17,18], certain $O(N)$ models [19], and time-evolution models [20,21].

Theorem 4 *Let H_ω be a random Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ with a regular potential. Then there are $L_o, B_1, B_2 < \infty$ such that if for some $E \in \mathbb{R}$ and some finite $L \geq L_o$, either*

$$\sup_{L/2 \leq \|y\| \leq L} \mathbb{E}\left(| < O | \frac{1}{H_{\Lambda_L,\omega} - E} | y > |^s \right) \ \leq \ B_1 / L^{3(d-1)} \ , \tag{15}$$

or

$$\sup_{L/2 \leq \|y\| \leq L} \mathbb{E}\left(| < O | \frac{1}{H_\omega - E} | y > |^s \right) \ \leq \ B_2 / L^{4(d-1)} \ , \tag{16}$$

where $\Lambda_L = [-L, L]^d$ and $\|y\| \equiv \sum_j |y_j|$, then the exponential *localization (4) holds for all energies in some open interval (a, b) containing E.*

3.2 Lower bounds for $G_\omega(x, y; E_{\text{edge}} + i0)$ at mobility edges

Boundary points of the continuous spectrum are referred to as *mobility edges*. The random Schrödinger operators considered here are ergodic, hence the location of such points does not depend on the realization [22]. However, except for the Bethe lattice [23], the proof of the occurrence of continuous spectrum is still an open problem. Nevertheless, it is interesting to note that Theorem 4 yields the following pair of lower bounds on the decay rate of the Green function at mobility edges, E_{edge}, for a random Schrödinger operator with regular potential:

$$\sup_{L/2 \leq \|y\| \leq L} \mathbb{E}\left(| < O | \frac{1}{H_{[-L,L]^d;\omega} - E_{\text{edge}}} | y > |^s \right) \ \geq \ B_1 \ L^{-3(d-1)} \ , \tag{17}$$

and

$$\sup_{L/2 \leq \|y\| \leq L} \mathbb{E}\left(| < O | \frac{1}{H_\omega - E_{\text{edge}}} | y > |^s \right) \ \geq \ B_2 \ L^{-4(d-1)} \ . \tag{18}$$

We do not expect these bounds to be optimal.

Vaguely similar bounds are known for the critical two-point functions in the statistical mechanical models mentioned above.

3.3 Extending off the real axis

The following statement is of somewhat technical interest, but it has interesting implications, such as the decay of the projection kernel, for which it is useful to have bounds on the resolvent at $E + i\eta$ which are uniform in η. Such bounds permit integrating the resolvent estimates along contours which cut the real axis, as in the derivation of (eq. (6)) in ref. [9].

Theorem 5 *Let H_ω be a random Schrödinger operator with a regular potential. Suppose that for some $E \in \mathbb{R}$, and $\Delta E > 0$, the following bound holds uniformly for $\xi \in [E - \Delta E, E + \Delta E]$:*

$$\mathbb{E}\left(|<x| \frac{1}{H_\omega - \xi - i0} |y>|^s \right) \;\leq\; A\, e^{-\mu|x-y|} \; . \tag{19}$$

Then for all $\eta \in \mathbb{R}$:

$$\mathbb{E}\left(|<x| \frac{1}{H_\omega - E - i\eta} |y>|^s \right) \;\leq\; \widetilde{A} \;\; e^{-\tilde{\mu}|x-y|} \; , \tag{20}$$

with some $\widetilde{A} < \infty$ and $\tilde{\mu} > 0$ – which depend on ΔE and the bound (19).

3.4 Localization in spectral tails.

The finite volume criteria presented above allow us to conclude exponential localization from suitable bounds on the density of states of the operators in regions $\Lambda_L = [-L, L]^d$. The following statement will be useful for such a purpose.

Theorem 6 *Let H_ω be a random Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ with a regular potential. For each $L > 0$ there exist $\delta_L > 0$ and $P_L > 0$ such that if*

$$\mathrm{Prob}\left[\mathrm{dist}\left(\sigma(H_{\Lambda_L;\omega}), E\right) \leq \delta_L\right] \;<\; P_L \; , \tag{21}$$

then the exponential localization condition (4) holds in some open interval containing E. Furthermore, given $\beta \in (0,1)$ and $\xi > 3(d-1)$, it is possible to choose δ_L and p_L such that $\underline{\lim}\, L^\beta \delta_L > 0$ and $\underline{\lim}\, L^\xi P_L > 0$.

Remarks: 1. It is of interest to combine the criterion presented above with Lifschitz tail estimates on the density of states at the bottom of the spectrum and at band edges. As an example, consider the bottom of the spectrum of H_ω: $E_0 = -\lambda V_0$ where V_0 is the minimum value in the support of V. Using Lifschitz tail estimates, it is possible to show that [24]:

$$\mathrm{Prob}\left[\inf \sigma(H_{\Lambda_L;\omega}) \leq E_0 + \Delta E\right] \;\leq\; \mathrm{Const.}\; L^d e^{-\Delta E^{-d/2}} \; . \tag{22}$$

By choosing $\Delta E \propto L^{-\beta}$ with $\beta \in (0,1)$ for large enough L, we infer fractional moment localization from this bound via Theorem 6. Previous results in this vein may be found in [25–29].

2. The input conditions (21) are similar to the input used in the multiscale analysis. In fact, there it is not important that $\xi > 3(d-1)$, and it suffices to assume the condition is met for some $\xi > 0$. However, one may note that wherever the multiscale analysis applies, its conclusion allows to deduce the condition as stated here. Thus, the exponential localization in the stronger sense discussed in our work may be concluded also for the regime for which localization may be established through the multiscale analysis.

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