THE DISCRETE SPECTRUM OF THE SCHRÖDINGER OPERATOR IN THE LARGE COUPLING CONSTANT LIMIT

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0. Introduction

In a simple quantum mechanical model for impurities in a crystal, one studies the spectrum of a periodic Schrödinger operator $A = -\Delta + p(x)$, perturbed by a decaying potential $-\alpha V_0$, which models the impurity ($\alpha > 0$ is a coupling constant). The basic assumptions are that H has a spectral gap Λ in its essential spectrum; eigenvalues of $A(\alpha) = A - \alpha V_0$ inside Λ correspond to the eigenstates of the electrons observed in solid state physics. We study the number of eigenvalues of $A(\alpha)$ inside an interval $(\lambda_1, \lambda_2) \subset \Lambda$. The main purpose of our study is to obtain an asymptotics of this number, as $\alpha \to \infty$. We point out that the problem has been solved only for one-dimensional Schrödinger operators in [3]. The results of the present paper deal with multidimensional Schrödinger operators. However we consider only the case of spherically symmetric potentials V_0.

1. Formulation of the main result. Preliminary information

1. Basic notation. a. Below M is a densely defined linear operator in a Hilbert space $\mathfrak{H}$. We denote by $D(M)$, M^*, $\sigma(M)$ the domain, the adjoint operator and the spectrum of the operator respectively. In the case when $M = M^*$, the symbol $E_M(\cdot)$ denotes the spectrum measure for M. Let the spectrum of the operator $M = M^*$ in $\mathfrak{H}$ be bounded from below and be discrete to the left from the point $\lambda_0 \in \mathbb{R}$. Then the operator M is semibounded from below and is generated by a certain quadratic form $m[\cdot,\cdot]$ in $\mathfrak{H}$. We define the spectral distribution function of M:

$$N(\lambda, M) = \mathrm{rank} E_M(-\infty, \lambda), \quad \lambda < \lambda_0.$$

Furthermore, let $d[m]$ be the domain of the form m. Then for the quantity $N(\lambda, M)$ the following relation holds:

$$N(\lambda, M) = \max(\mathrm{dim} F), \quad F \subset d[m], m[u,u] < \lambda(u,u), \forall u \in F, u \neq 0.$$

b. Below $d \geq 2, \mathbb{Q} = [0,1)^d$, $S = \{x \in \mathbb{R}^d : |x| = 1\}$ and $B_r = \{x \in \mathbb{R}^d : |x| < r\}, r > 0$. Notation $H^s(\Omega)$, $\Omega \subset \mathbb{R}^d$, is taken for the Sobolev classes of oder $s \in \mathbb{N}$; $H_0^s(\Omega)$ is the closure of the class $C_0^\infty(\Omega)$ in the metric of $H^s(\Omega)$. For Laplace operators on domains $\Omega \subset \mathbb{R}^d$ and on the sphere S we take the notations Δ, Δ_θ respectively. The gradient operator is denoted by ∇. Different estimate constants are denoted by C, c (sometimes with indices).

2. Let a function $p \in L_\infty(\mathbb{R}^d)$ take real values and

$$p(x+n) = p(x), \quad x \in \mathbb{R}^d, n \in \mathbb{Z}^d.$$

Consider the selfadjoint operator

$$Au = -\Delta u + pu, \quad u \in D(A) = H^2(\mathbb{R}^d),$$

in $L_2(\mathbb{R}^d)$. Obviously A is an elliptic $\mathbb{Z}^d$-periodic operator. The spectrum $\sigma(A)$ can have gaps besides the semi-infinite one. Let $\Lambda = (\lambda_-, \lambda_+)$ be a fixed gap in the spectrum $\sigma(A)$ (if $\lambda_+ = \inf\sigma(A)$, then $\lambda_- = -\infty$). Below we perturb the operator A so, that the spectrum of the perturbed operator in Λ is discrete.

Let $V : [1, +\infty) \to \mathbb{R}_+$ be a smooth function and

$$\lim_{r\to\infty} r^s V(r) = v_a, \quad 2 < s < 2 + 2/(d-1), v_a > 0. \tag{1}$$

We denote by V_0 the operator of multiplication by the function $V(|x|)$ extended by zero for $|x| < 1$. We put $A(\alpha) := A - \alpha V_0, \alpha > 0$. It is easy to see that the spectrum $\sigma(A(\alpha))$ is discrete in Λ. Let $\lambda_- < \lambda_1 < \lambda_2 < \lambda_+$. The main object of our investigation is the asymptotic behavior of the value $\operatorname{rank} E_{A(\alpha)}(\lambda_1, \lambda_2)$ for $\alpha \to \infty$.

Below we systematically use the following notations for the operators on a domain. Let $\Omega \subset \mathbb{R}^d$ be an open domain. We consider the following operator in $L_2(\Omega)$

$$A^{(\varepsilon)}(\alpha, \Omega) = -\Delta + p - \alpha V_0, \quad \alpha > 0;$$

the index ε takes the values $\mathcal{D}$ or $\mathcal{N}$ and characterizes the boundary (Dirichlet or Neumann) conditions on $\partial\Omega$. In oder to formulate the main result we need to introduce the density of states for the operator A. Let $x_1, x_2 \in \mathbb{R}^d$ and $l > 0$. Let also $\Omega(l) := \Omega(l; x_1, x_2) = l(\mathbb{Q} + x_1) + x_2$. Then for an arbitrary $\lambda \in \mathbb{R}$ the following limit exists

$$\rho(\lambda) = \lim_{l\to\infty} l^{-d} N(\lambda, A^{(\varepsilon)}(0, \Omega(l))).$$

It does not depend on x_1, x_2 and on the values of the index ε. This limit is called the integrated density of states for the operator A at the point λ.

Let $\lambda_1,\ \lambda_2$ be fixed, $\lambda_- < \lambda_1 < \lambda_2 < \lambda_+$. We put

$$b(\varepsilon) \ = \ \begin{cases} \text{v-sup}_{x\in\mathbb{R}^d}(p(x)-\lambda_1), & \varepsilon = \mathcal{D}; \\ \text{v-inf}_{x\in\mathbb{R}^d}(p(x)-\lambda_2), & \varepsilon = \mathcal{N}. \end{cases}$$

We impose an additional condition on the function V:

Condition 1.1. There exists $\eta_* > 0$, such that for all $0 < \eta < \eta_*$ and $\alpha > \eta^{-s}$ the number of the roots of the equation $2b(\varepsilon)r = \alpha(r^2V(r))'$, $\varepsilon = \mathcal{D},\mathcal{N}$ in the interval $r \in [1, \eta\alpha^{1/s}]$ does not exceed a fixed number $l \in \mathbb{N}$.

We note that Condition 1.1 is fulfilled when the function $r^{-1}(r^2V(r))'$ decreases monotonously for $r > 1$. In particular this condition is fulfilled for $V(r) = v_a r^{-s},\ r > 1$.

The main result of the present work gives an asymptotics of the entire spectrum multiplicity of the operator $A(\alpha)$ in the interval (λ_1, λ_2) as $\alpha \to \infty$.

Theorem 1.1. *Let $d \geq 2$ and $\lambda_- < \lambda_1 < \lambda_2 < \lambda_+$. Assume that* (1) *and Condition 1.1 are fulfilled. Then*

$$\begin{gathered}\lim_{\alpha\to\infty} \alpha^{-d/s}\text{rank}E_{A(\alpha)}(\lambda_1,\lambda_2) = \\ \int_{\mathbb{R}^d} (\rho(\lambda_2 + v_a|x|^{-s}) - \rho(\lambda_1 + v_a|x|^{-s}))dx.\end{gathered} \tag{2}$$

Now we discuss the condition $s < 2 + 2/(d-1)$. The quasiclassical exponent for the Laplace operator on the sphere is $æ = (d-1)/2$. The "Dirichlet-Neumann bracketing" method can be applied only to the asymptotics of the oder α^q, $q > æ$. In Theorem 1.1 $q = d/s$. This leads to the discussed estimate for s.

2. Auxiliary result

1. Let $\mathcal{Y} = \{x \in \mathbb{R}^d :\ 1 < |x| < \eta\alpha^{1/s}\}$, $d \geq 2$. We write $A^{(\varepsilon)}(\alpha) = A^{(\varepsilon)}(\alpha, \mathcal{Y})$ for brevity. From the inequality $A^{(\mathcal{N})}(\alpha) < A^{(\mathcal{D})}(\alpha)$ we obtain that $N(\lambda, A^{(\mathcal{D})}(\alpha)) \leq N(\lambda, A^{(\mathcal{N})}(\alpha))$ for an arbitrary $\lambda \in \mathbb{R}$. The following statement gives an estimate of the difference of the distribution functions of the spectrum of the operators $A^{(\mathcal{D})}(\alpha)$ and $A^{(\mathcal{N})}(\alpha)$.

Proposition 2.1. *Let $p \in L_\infty(\mathbb{R}^d)$, $2 < s < 2 + s/(d-1)$ and $\lambda_1 < \lambda_2$. Assume that V satisfies the conditions of Theorem 1.1. Then there is $\eta_0 > 0$ such that for all $0 < \eta < \eta_0$*

$$\mu(\eta) := \limsup_{\alpha\to\infty} \alpha^{-d/s}(N(\lambda_2, A^{(\mathcal{N})}(\alpha)) - N(\lambda_1, A^{(\mathcal{D})}(\alpha))) < \infty.$$

Also one has

$$\lim_{\eta\to 0} \mu(\eta) = 0.$$

The proof of Proposition 2.1 is based on the elementary separation of the variables. We put $\lambda_{\mathcal{D}} = \lambda_1$ and $\lambda_{\mathcal{N}} = \lambda_2$. The value $N(\lambda_\varepsilon, A^{(\varepsilon)}(\alpha))$ is equal to the maximal dimension of the subspaces F on which

$$\int_{\mathcal{Y}} (|\nabla u|^2 + (p(x) - \lambda_\varepsilon)|u|^2) dx \;<\; \alpha \int_{\mathcal{Y}} V(|x|)|u|^2 dx. \tag{3}$$

It is sufficient to consider subspaces $F \subset C_0^\infty(\mathcal{Y})$ (the case $\varepsilon = \mathcal{D}$) and $F \subset C^\infty(\mathcal{Y})$ (the case $\varepsilon = \mathcal{N}$). Let $b(\varepsilon)$ be the same as in §1. We replace the function p in the definition of the operator $A^{(\varepsilon)}(\alpha)$ by $b(\varepsilon) + \lambda_\varepsilon$. Since $b(\mathcal{D}) + \lambda_{\mathcal{D}} \geq p(x)$ and $b(\mathcal{N}) + \lambda_{\mathcal{N}} \leq p(x)$ the difference

$$N(\lambda_{\mathcal{N}}, A^{(\mathcal{N})}(\alpha)) \;-\; N(\lambda_{\mathcal{D}}, A^{(\mathcal{D})}(\alpha)) \tag{4}$$

can only increase. After the passage to the spherical coordinates and the substitutions $r = e^t$, $u = v \exp(-\frac{(d-2)}{2}t)$ and $p \mapsto b(\varepsilon) + \lambda_\varepsilon$ the inequality (3) turns into

$$\begin{gathered}\int_0^T \int_S (|\nabla_\theta v|^2 + |\frac{\partial v}{\partial t} + \frac{(2-d)}{2} v|^2) d\theta dt + \\ \int_0^T \int_S b(\varepsilon) e^{2t} |v|^2 d\theta dt < \\ \alpha \int_0^T \int_S e^{2t} V(e^t)|u|^2 d\theta dt,\end{gathered} \tag{5}$$

where $T = 1/s \log(\eta^s \alpha)$.

Let $\mathcal{F}(\mathcal{D}) = C_0^\infty(0,T)$, $\mathcal{F}(\mathcal{N}) = C^\infty[0,T]$ and let W_j, $j = 0, 1, ...$ be the space of the spherical functions of oder j on the sphere S. The subspaces, obtained by closing $\mathcal{F}(\varepsilon)W_j$, dioganalize the quadratic forms in both parts of (5). Hence the operator $A^{(\varepsilon)}(\alpha)$ is decomposed into the orthogonal sum of the operators $A_j^{(\varepsilon)}(\alpha)$ operating in the corresponding "cells". In the cell with the number j the inequality (5) is transformed into $k_j = \dim W_j$ copies of the inequalities for scalar functions

$$\begin{gathered}\int_0^T (|y' - \frac{(d-2)}{2} y|^2 + (\Lambda_j + b(\varepsilon)e^{(2t)})|y|^2) dt \;< \\ \alpha \int_0^T e^{2t} V(e^t)|y|^2 dt, \quad y \in F \subset \mathcal{F}(\varepsilon),\end{gathered} \tag{6}$$

where Λ_j are the eigenvalues of the operator Δ_θ on the sphere S. We denote by $\Gamma = \Gamma(\alpha)$ the family of the numbers j for which (6) is fulfilled for $\varepsilon = \mathcal{N}$ at least for one $y \in \mathcal{F}(\mathcal{N})$. Obviously the set $\{t \in [0,T] : \;\Lambda_j + b(\mathcal{N})e^{2t} \;\leq\; \alpha e^{2t} V(e^t)\}$ is not empty for $j \in \Gamma$. Therefore for an arbitrary $j \in \Gamma$

$$\Lambda_j \;\leq\; \alpha \sup_{t \in [0,+\infty)} (e^{2t} V(e^t)) + |b(\mathcal{N})| \eta^2 \alpha^{2/s}, \quad j \in \Gamma. \tag{7}$$

By the Weyl law for Λ_j and k_j we have the estimate

$$\sum_{j\in\Gamma} k_j \ \le\ C(\alpha^{\frac{d-1}{2}}+1), \quad 0<\eta<1. \tag{8}$$

Under the substitution of $\mathcal{F}(\mathcal{N})$ by $C_0^\infty(0,T)$ the maximal dimension of subspaces F, on which (6) is fulfilled, decreases at most by two. According to (8) under this procedure the difference (4) would be changed by a quantity of order $O(\alpha^{\frac{d-1}{2}})$ at $\alpha\to\infty$. For $\mathcal{F}(\varepsilon)=C_0^\infty(0,T)$ the inequality (6) turns into:

$$\begin{gathered}\int_0^T (|y'|^2+(\gamma_j+b(\varepsilon)e^{2t})|y|^2)dt < \\ \alpha\int_0^T e^{2t}V(e^t)|y|^2dt, \\ y\in F\subset C_0^\infty(0,T),\ \gamma_j=\Lambda_j+(\frac{d-2}{2})^2.\end{gathered} \tag{9}$$

2. We denote by $Z(\varepsilon,\alpha)$ the set of numbers j, for which (9) is fulfilled at least for one $y\in C_0^\infty(0,T)$. For $j\in Z(\varepsilon,\alpha)$ we consider the set

$$\Delta=\Delta(\varepsilon,j,\alpha):=\{t\in[0,T]:\ \gamma_j+b(\varepsilon)e^{2t}\le\alpha e^{2t}V(e^t)\}.$$

Obviously $\Delta\ne\emptyset$ for every $j\in Z(\varepsilon,\alpha)$. Let us clarify the structure of the set Δ. If a point $t\in(0,T)$ lies on the boundary of the set Δ, then

$$\gamma_j+b(\varepsilon)e^{2t}=\alpha e^{2t}V(e^t). \tag{10}$$

For $e^t=r$ we get

$$\gamma_j+b(\varepsilon)r^2=\alpha r^2V(r). \tag{11}$$

According to the Roll theorem the root of this equation lies between two solutions of the equation

$$2b(\varepsilon)r=\alpha\frac{d}{dr}(r^2V(r)). \tag{12}$$

According to the condition (1.1) the number of the roots of the equation (12) does not exceed a fixed number $l\in\mathbb{N}$. Therefore Δ is a union of at most $l+2$ disjoint (probably degenerated to a point) intervals. Consequently, under the substitution of $\mathcal{F}(\varepsilon)=C_0^\infty(0,T)$ by $C_0^\infty(\Delta)$ the maximal dimension $N(\varepsilon,j,\alpha)$ of the subspaces F, where (9) is fulfilled, will be decreased at most by $2l+4$ (the doubled number of the intervals forming Δ). Since $Z(\varepsilon,\alpha)\subset\Gamma$ and according to (8) the value (4) will be changed only by $O(\alpha^{\frac{d-1}{2}})$ as $\alpha\to\infty$. Let $\Delta=\cup_{k=1}^{l+2}\Delta_k$, where $\Delta_k=\Delta_k(\varepsilon,j,\alpha)$ are disjoint (probably empty or degenerated to a point) intervals. For $F \subset C_0^\infty(\Delta) = C_0^\infty(\Delta_1) + \cdots + C_0^\infty(\Delta_{l+2})$ the inequality (9) is decomposed into $l+2$ inequalities for the intervals Δ_k:

$$\int_{\Delta_k} (|y'|^2 + (\gamma_j + b(\varepsilon)e^{2t})|y|^2)dt < \alpha \int_{\Delta_k} e^{2t}V(e^t)|y|^2 dt, \qquad y \in F \subset C_0^\infty(\Delta_k), \quad k = \overline{1, l+2} \tag{13}$$

The maximal dimension $N_k = N_k(\varepsilon, j, \alpha)$ of the subspaces F from (13) coincides with the entire multiplicity of the negative spectrum of the operator

$$-y'' + (\gamma_j + b(\varepsilon)e^{2t})y - \alpha e^{2t}V(e^t)y, \quad y \in H^2(\Delta_k) \cap H_0^1(\Delta_k), \quad k = \overline{1, l+2}$$

3. In the investigation of the spectrum of this operator we use the technique suggested in [3]. Let us put

$$f = f(\varepsilon, t, \alpha) = (\alpha e^{2t}V(e^t) - \gamma_j - b(\varepsilon)e^{2t} + 1)^{1/2}, t \in \Delta_k.$$

Proposition 2.2. *For the quantity N_k the following inequality holds*

$$|N_k(\varepsilon, j, \alpha) - \int_{\Delta_k(\varepsilon,j,\alpha)} f(\varepsilon, t, \alpha)| \leq C(1 + \log(1 + \alpha)), \tag{14}$$

where $C = C(\eta)$ does not depend on j and α.

Proof. Let $\Delta_k = [R_1^{(k)}, R_2^{(k)}]$, $k = \overline{1, l+2}$. Let us employ the relation between the quantity N_k and the properties of the solution of the problem

$$\begin{cases} -\phi'' + (\gamma_j + b(\varepsilon)e^{2t})\phi - \alpha e^{2t}V(e^t)\phi = 0, \\ \phi(R_1^{(k)}) = 0, \ \phi'(R_1^{(k)}) = 1. \end{cases} \tag{15}$$

Namely the quantity N_k coincides (see [3]) with the number of zeros $\mu_k = \mu_k(\varepsilon, j, \alpha)$ of the function ϕ lying strictly inside Δ_k. In order to estimate μ_k we represent $\phi(t)$ in the polar form. We put

$$\phi(t) = \beta(t)\sin\xi(t), \quad \phi'(t) = f(t)\beta(t)\cos\xi(t).$$

Then it follows from (15) (see [3]) that

$$\xi' = f + f'(2f)^{-1}\sin(2\xi) - f^{-1}\sin^2\xi, \quad \xi(R_1^{(k)}) = 0. \tag{16}$$

We note also that $\beta(t) > 0$ for $t \in \Delta_k$. Consequently the quantity μ_k coincides with the number of zeros of the function $\sin\xi(t)$. In other words μ_k coincides with the number of roots of the equation $\xi(t) = 0 \pmod{\pi}$. In view of (16) $\xi' = f > 0$, if $\xi = \pi n$, $n \in \mathbb{Z}$. Therefore for a fixed $n \in \mathbb{N}$ the function $\xi(t)$ can take the values πn only once. Thus

$$\pi^{-1}\xi(R_2^{(k)}) - 1 \leq \mu_k \leq \pi^{-1}\xi(R_2^{(k)}).$$

Since $N_k = \mu_k$ according to (16) we obtain

$$|N_k - \pi^{-1}\int_{\Delta_k} f\,dt| \le (2\pi)^{-1}\int_{\Delta_k} |f'/f|dt + \pi^{-1}|\Delta_k| + 1, \quad k = \overline{1, l+2}.$$

It is easy to see that the function f' changes its sign on the interval Δ_k not more than l times. Therefore $\int_{\Delta_k} |f'/f|dt \le 2(l+1) \times \max_{t\in\Delta_k} |\log f|$. It remains to take into account that $1 \le f \le (\alpha \sup_{r>1}(r^2V(r)) + |b(\mathcal{N})|\eta^2\alpha^{2/s} + 1)^{1/2}$ for $t \in \Delta_k$. □

4. Let us set

$$\Phi_j(\alpha) := \int_{\Delta(\mathcal{N},j,\alpha)} f(\mathcal{N}, t)dt - \int_{\Delta(\mathcal{D},j,\alpha)} f(\mathcal{D}, t)dt,$$

and $C_1 := (b(\mathcal{D}) - b(\mathcal{N}))^{1/2}$. The following statement gives the estimate of the quantity $\Phi_j(\alpha)$.

Proposition 2.3. *For $\Phi_j(\alpha)$ the following inequality holds*

$$\Phi_j(\alpha) \le C_1 \int_{\Delta(\mathcal{N},j,\alpha)} e^t dt. \tag{17}$$

Proof. It follows from the definition of the value $f(\varepsilon, t)$ that $f(\mathcal{N}, t)^2 = C_1^2 e^{2t} + f(\mathcal{D}, t)^2$ for $t \in \Delta(\mathcal{D}, j, \alpha)$. Consequently

$$f(\mathcal{N}, t) - f(\mathcal{D}, t) =$$

$$C_1^2 e^{2t}/(f(\mathcal{N}, t) + f(\mathcal{D}, t)) \le$$

$$C_1^2 e^{2t}/f(\mathcal{N}, t) \le C_1 e^t, \quad t \in \Delta(\mathcal{D}, j, \alpha).$$

Furthermore, the inequality $f(\mathcal{N}, t) \le C_1 e^t$ is fulfilled for $t \in \Delta(\mathcal{N}, j, \alpha) \setminus \Delta(\mathcal{D}, j, \alpha)$. And $\Delta(\mathcal{D}, j, \alpha) \subset \Delta(\mathcal{N}, j, \alpha)$. Thus,

$$\begin{aligned} \Phi_j(\alpha) = \int_{\Delta(\mathcal{D},j,\alpha)} (f(\mathcal{N}, t) - f(\mathcal{D}, t))dt + \\ \int_{\Delta(\mathcal{N},j,\alpha)\setminus\Delta(\mathcal{D},j,\alpha)} f(\mathcal{N}, t)dt \le \\ C_1 \int_{\Delta(\mathcal{N},j,\alpha)} e^t dt. \end{aligned} \tag{18}$$

□

Above (see Subsections 1, 2) we made transformations that either increase difference (4) or change it by a value of order $o(\alpha^{d/s})$ as $\alpha \to \infty$. Finally this difference equals

$$\sum_{j\in Z(\mathcal{N},\alpha)} k_j \sum_{k=1}^{l+2} (N_k(\mathcal{N}, j, \alpha) - N_k(\mathcal{D}, j, \alpha)). \tag{19}$$

According to (8) and (14) the value (19) coincides with the sum $\sum_{j\in Z(\mathcal{N},\alpha)} k_j\Phi_j(\alpha)$ with accuracy of order $o(\alpha^{d/s})$ when $\alpha\to\infty$. Therefore

$$N(\lambda_{\mathcal{N}},A^{(\mathcal{N})}(\alpha))-N(\lambda_{\mathcal{D}},A^{(\mathcal{D})}(\alpha))\leq \sum_{j\in Z(\mathcal{N},\alpha)} k_j\Phi_j(\alpha)+o(\alpha^{d/s}), \qquad \alpha\to\infty. \tag{20}$$

At the next step of the proof we decompose $Z(\mathcal{N},\alpha)$ into subsets $X_n=X_n(\alpha)$. For the each of these subsets we estimate the value $\sum_{j\in X_n} k_j\Phi_j(\alpha)$ separately. Then we sum up these estimates over n.

For $n\in\mathbb{N}\cup\{0\}$ and $\alpha>0$ we define a set $X_n=X_n(\alpha)$ of numbers j for which

$$n\alpha^{2/s}\leq\Lambda_j<(n+1)\alpha^{2/s}. \tag{21}$$

Our nearest goal is to estimate the sums of the quantities $k_j=\mathrm{dim}W_j$, the numbers j of which are in $X_n(\alpha)$. For that we use the precise values

$$\Lambda_j=j(j+d-2),$$

$$k_j=\mathrm{dim}W_j=C^{d-2}_{d+j-2}+C^{d-2}_{d+j-3}.$$

From the last equation it follows that

$$k_j\leq Cj^{d-2},\quad j\geq 1. \tag{22}$$

Proposition 2.4. *There exists a number $n_0\in\mathbb{N}$ such that for any $n>n_0$ and $\alpha>1$*

$$\sum_{j\in X_n} k_j\leq C(n^{(d-2)/2}\alpha^{(d-2)/s}+n^{(d-3)/2}\alpha^{(d-1)/s}), \tag{23}$$

where $C>0$ does not depend on n and α.

Proof. Let $n\in\mathbb{N}$ and $\alpha>1$. Than for $j\in X_n(\alpha)$

$$n\alpha^{2/s}\leq\Lambda_j=j(j+d-2)=j^2(1+d-2/j)\leq j^2(d-1).$$

Comparing the utmost parts we get

$$j^2\geq(d-1)^{-1}n\alpha^{2/s},\quad \forall j\in X_n(\alpha).$$

Therefore for $j\in X_n(\alpha)$

$$n\alpha^{2/s}\leq\Lambda_j=j^2(1+d-2/j)\leq j^2\Big(1+\frac{(d-2)(d-1)^{1/2}}{n^{1/2}\alpha^{1/s}}\Big).$$

Taking square root of the utmost parts of this inequality we come to the estimate

$$j\geq(1+(d-2)(d-1)^{1/2}/n^{1/2}\alpha^{1/s})^{-1/2}n^{1/2}\alpha^{1/s}=:\xi,\quad \forall j\in X_n(\alpha).$$

Now let us derive an upper estimate for $j \in X_n(\alpha)$. Since for $j \in X_n(\alpha)$

$$j^2 \leq j(j+d-2) = \Lambda_j \leq (n+1)\alpha^{2/s},$$

we have

$$j \leq (n+1)^{1/2}\alpha^{1/s} =: \zeta, \quad \forall j \in X_n(\alpha).$$

According to (22) we get

$$\begin{aligned} \sum_{j\in X_n} k_j \leq C\sum_{\xi}^{\zeta} j^{d-2} \leq C\int_{\xi}^{\zeta+1} x^{d-2}dx = \\ C(((n+1)^{1/2}\alpha^{1/s}+1)^{d-1} - \\ n^{d-1/2}\alpha^{d-1/s}(1+(d-2)(d-1)^{1/2}/n^{1/2}\alpha^{1/s})^{-(d-1)/2}). \end{aligned} \tag{24}$$

It leads to (23) for large $n > n_0$. $\square$

5. Let η be the same as in the statement of the Proposition 2.1,

$$v_+ := \sup_{r>1}(r^s V(r)), \quad C_2 := |b(\mathcal{N})|.$$

In order to estimate a sum of the kind $\sum_{j\in X_n} k_j\Phi_j(\alpha)$ it is enough to estimate the integral in the right side of (17) and to use (23). The investigation of the inequality (17) leads to the following result.

Proposition 2.5. *Let be $\eta > 0$ and*

$$n > \eta^2 C_2 + \eta^{-(s-2)}. \tag{25}$$

Then for $j \in X_n$

$$\Phi_j(\alpha) \leq C_1 v_+^{1/(s-2)}\alpha^{1/s}(n - C_2\eta^2)^{-1/(s-2)}. \tag{26}$$

Proof. For $t \in \Delta(\mathcal{N}, j, \alpha)$ we have

$$\alpha v_+ e^{-(s-2)t} - \Lambda_j + C_2\eta^2\alpha^{2/s} \geq \alpha e^{2t}V(e^t) - \Lambda_j - (\frac{d-2}{2})^2 - b(\mathcal{N})e^{2t} \geq 0.$$

Let be $j \in X_n$. Then $\Lambda_j \geq n\alpha^{2/s}$. Therefore

$$\alpha v_+ e^{-(s-2)t} \geq n\alpha^{2/s} - C_2\eta^2\alpha 2/s, \ \forall t \in \Delta(\mathcal{N}, j, \alpha), j \in X_n.$$

Thus, if $\Delta(\mathcal{N}, j, \alpha) \neq \emptyset$, then the number $\tilde{T} := \frac{1}{s}\log\alpha - \frac{1}{s-2}\log((n - C_2\eta^2)/v_+)$ is positive. Moreover $\Delta(\mathcal{N}, j, \alpha) \subset [0, \tilde{T}]$. Therefore substituting $\Delta(\mathcal{N}, j, \alpha)$ by $[0, \tilde{T}]$ we come to (26). In the case when $\Delta(\mathcal{N}, j, \alpha) = \emptyset$ the estimate (26) is fulfilled by the definition of the value $\Phi_j(\alpha)$. $\square$

Now let us sum up the quantities $\sum_{j\in X_n} k_j\Phi_j(\alpha)$ over n. Namely we consider the set Y_0 - the union of all X_n, for which $X_n \cap Z(\mathcal{N}, \alpha) \neq \emptyset$ and (25) is fulfilled. The following statement gives an estimate of the sum $\sum_{j\in Y_0} k_j\Phi_j(\alpha)$.

Proposition 2.6. *Under the conditions of Proposition 2.1 there exists a function* $\mu_1 : (0,1) \to \mathbb{R}_+$, *such that*

$$\lim_{t\to 0} \mu_1(t) = 0,$$

$$\sum_{j\in Y_0} k_j \Phi_j(\alpha) \le \mu_1(\eta)\alpha^{d/s} + o(\alpha^{d/s}), \quad \alpha \to \infty, 0 < \eta < 1.$$

Proof. If $X_n \subset Y_0$, then according to (25), $n \ge n_1 := [\eta^2 C_2 + \eta^{-(s-2)}]+1$, and according to (7) and (21), $n \le n_2 := [v_+\alpha^{1-2/s}+C_2\eta^2]$. Furthermore, we multiply both parts of (26) by k_j and sum up over $j \in X_n$. Taking into account (23) we get

$$\sum_{j\in X_n} k_j \Phi_j(\alpha) \le C\frac{n^{\frac{d-2}{2}}\alpha^{\frac{d-1}{s}} + n^{\frac{d-3}{2}}\alpha^{\frac{d}{s}}}{(n - C_2\eta^2)^{\frac{1}{s-2}}}.$$

Therefore

$$\begin{aligned} \sum_{j\in Y_0} k_j \Phi_j(\alpha) \le C \sum_{n=n_1}^{n_2} \frac{n^{\frac{d-2}{2}}\alpha^{\frac{d-1}{s}} + n^{\frac{d-3}{2}}\alpha^{\frac{d}{s}}}{(n - C_2\eta^2)^{\frac{1}{s-2}}} \le \\ C\alpha^{d/s} \sum_{n=n_1}^{\infty} \frac{n^{\frac{d-3}{2}}}{(n - C_2\eta^2)^{\frac{1}{(s-2)}}} + O(\alpha^{d/2-2/s}) \end{aligned} \tag{27}$$

as $\alpha \to \infty$. Since in the conditions of Proposition 2.1 $d/2 - 2/s < d/s$, it remains to note that $n_1 \to \infty$ when $\eta \to 0$. □

6. Let us consider the set $Y := Z(\mathcal{N}, \alpha) \setminus Y_0$. An analog of Proposition 2.6 for Y is

Proposition 2.7. *Under the conditions of Proposition 2.1 there exists a function* $\mu_2 : (0,1) \to \mathbb{R}_+$ *such that*

$$\lim_{t\to 0} \mu_2(t) = 0,$$

$$\sum_{j\in Y} k_j \Phi_j(\alpha) \le \mu_2(\eta)\alpha^{d/s} + o(\alpha^{d/s}), \quad \alpha \to \infty, 0 < \eta < 1$$

Proof. If $X_n \cap Y \ne \emptyset$, then the inequality (25) is not fulfilled. Therefore according to (21)

$$\Lambda_j < (1 + \eta^2 C_2 + \eta^{-(s-2)})\alpha^{2/s}$$

for all $j \in Y$. In accordance with the Weyl law for Λ_j and k_j we get

$$\sum_{j\in Y} k_j \le C(1 + \alpha^{\frac{d-1}{s}}(1 + \eta^2 C_2 + \eta^{-(s-2)})^{\frac{d-1}{2}}). \tag{28}$$

Furthermore, since $\Delta(\mathcal{N}, j, \alpha) \subset [0, T]$, due to (17) we have: $\Phi_j(\alpha) \leq C\eta\alpha^{1/s}$. Let us multiply both parts of this inequality by k_j and sum up over $j \in Y$. Now taking into account (28) we get

$$\sum_{j\in Y} k_j\Phi_j(\alpha) \leq C\eta\alpha^{1/s}(1 + \alpha^{\frac{d-1}{s}}(1 + \eta^2 C_2 + \eta^{-(s-2)})^{\frac{d-1}{2}}).$$

It is exactly what we need. □

Let us now complete the proof of Proposition 2.1. Since $Z(\mathcal{N}, \alpha) = Y \cup Y_0$, then from Propositions 2.6 and 2.7 it follows that

$$\limsup_{\alpha\to\infty} \alpha^{-d/s} \sum_{j\in Z(\mathcal{N},\alpha)} k_j\Phi_j(\alpha) = \mu(\eta), \tag{29}$$

where $0 \leq \mu(\eta) \leq \mu_1(\eta) + \mu_2(\eta)$. Thus $\lim_{\eta\to 0} \mu(\eta) = 0$. Combining (20) with (29) we come to the statement of Proposition 2.1. □

3. Proof of Theorem 1.1

1. In this section we prove the relation (3). Necessary auxiliary statements for this are given in Subsections 1 and 2. In Subsection 3 the proof of Theorem 1.1 is completed. Here we follow the way of the article [2] but additionally we use Proposition 2.1.

For $\lambda \in \Lambda$ and $\alpha > 0$ we define a quantity $N_0(\lambda, A(\alpha))$ as a number of eigenvalues of the operator $A(t)$ which cross the point λ with t growing from zero to α (excluding α). It is clear that for $\lambda_1, \lambda_2 \in \Lambda, \lambda_1 < \lambda_2$

$$\mathrm{rank} E_{A(\alpha)}[\lambda_1, \lambda_2) = N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha)).$$

Therefore it is enough to obtain the asymptotics of the right-hand side, as $\alpha \to \infty$. The following statement plays an important role in the proof of Theorem 1.1.

Theorem 3.1. *Let $\lambda \in \Lambda$ and $\varepsilon > 0$. Assume that $\lambda - \varepsilon, \lambda + \varepsilon \in \Lambda$. Then there exist positive numbers r_0, α_0 and c such that for $r > r_0, \alpha > \alpha_0$ the following inequalities are fulfilled:*

$$\begin{aligned} N_0(\lambda, A(\alpha)) \leq N((\lambda + \varepsilon) + 0, A^{(\mathcal{N})}(\alpha, \alpha^{1/s}B_r)) - \\ N(\lambda + \varepsilon, A^{(\mathcal{N})}(0, \alpha^{1/s}B_r)) + cr^{d-1}\alpha^{(d-1)/s}, \end{aligned} \tag{30}$$

$$\begin{aligned} N_0(\lambda, A(\alpha)) \geq N(\lambda - \varepsilon, A^{(\mathcal{D})}(\alpha, \alpha^{1/s}B_r)) - \\ N(\lambda - \varepsilon, A^{(\mathcal{D})}(0, \alpha^{1/s}B_r)) - cr^{d-1}\alpha^{(d-1)/s}, \end{aligned} \tag{31}$$

where c does not depend on r and α.

We postpone the proof of Theorem 3.1 until Subsection 4. Actually all the ideas of the proof are given in [2]. In §4 we reproduce them in the exact form. Below we use the following notations $||p||_\infty = \sup_{x\in\mathbb{R}^d}|p(x)|$, $v_+ = \sup_{r>1} r^s V(r)$. For $\Omega \subset \mathbb{R}^d$ we put $L(\Omega) = \inf_{x\in\Omega}|x|$. For $L(\Omega) > 0, \lambda \in \mathbb{R}$ we introduce the quantity $g = g(\lambda,\Omega) = ||p||_\infty + v_+ L(\Omega)^{-s} + |\lambda|$. The following proposition gives an asymptotical estimate for the quantity $N(\lambda, A^{(\varepsilon)}(\alpha, \alpha^{1/s}\Omega))$, as $\alpha \to \infty$.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a domain with a piecewise smooth boundary. Assume that $L(\Omega) > 0$. Then for any $\lambda \in \mathbb{R}$*

$$\limsup_{\alpha\to\infty} \ \alpha^{-d/s} N(\lambda, A^{(\varepsilon)}(\alpha, \alpha^{1/s}\Omega)) \le (2\pi)^{-d}\omega_d g^{d/2}\mathrm{mes}(\Omega). \tag{32}$$

Proof. For $x \in \alpha^{1/s}\Omega$ we have the estimate $\alpha V(|x|) \le v_+ L(\Omega)^{-s}$. Thus $\alpha V_0(x) + \lambda - p(x) \le g,\ x \in \alpha^{1/s}\Omega$. Then it follows from the variational principle that $N(\lambda, A^{(\varepsilon)}(\alpha, \alpha^{1/s}\Omega))$ does not exceed the amount of the negative spectrum of the operator $-\Delta - gI$ in $L_2(\alpha^{1/s}\Omega)$. Let us substitute $x = \alpha^{1/s}y, y \in \Omega$. As a result we obtain that $N(\lambda, A^{(\varepsilon)}(\alpha,))$ does not exceed the entire multiplicity of the negative spectrum of the operator $-\Delta - \alpha^{2/s} gI$ in $L_2(\Omega)$ (with appropriate boundary conditions). Now the inequality (32) follows from the Weyl law for the eigenvalues of the Laplace operator on a domain with a piecewise smooth boundary.□

We fix $n \in \mathbb{Z}^d$ and $\nu > 0$. Let us consider the operator $A^{(\varepsilon)}(\alpha, \alpha^{1/s}\nu(\mathbb{Q}_n))$. Let $x_n^{(\mathcal{N})}$ be the nearest and $x_n^{(\mathcal{D})}$ be the most remote (with respect to the origin) point of the cube $\mathbb{Q}_n$.

Proposition 3.2. *Let $\nu > 0, n \in \mathbb{Z}^d$ and $\lambda \in \mathbb{R}$. Assume that $|x_n^{(\mathcal{N})}| > 0$. Then*

$$\limsup_{\alpha\to\infty} \alpha^{-d/s} N(\lambda, A^{(\mathcal{N})}(\alpha, \alpha^{1/s}\nu\mathbb{Q}_n)) \le \nu^d \rho(\lambda + v_a(\nu|x_n^{(\mathcal{N})}|)^{-s}), \tag{33}$$

$$\liminf_{\alpha\to\infty} \alpha^{-d/s} N(\lambda, A^{(\mathcal{D})}(\alpha, \alpha^{1/s}\nu\mathbb{Q}_n)) \ge \nu^d \rho(\lambda + v_a(\nu|x_n^{(\mathcal{D})}|)^{-s}). \tag{34}$$

Proof. First of all, let us assume that $V(r) = v_a r^{-s}, r > 1$. Then for $x \in \alpha^{1/s}\nu\mathbb{Q}_n, |x| > 1$ we have

$$v_a(\nu|x_n^{(\mathcal{D})}|)^{-s} \le \alpha V(|x|) \le v_a(\nu|x_n^{(\mathcal{N})}|)^{-s}.$$

After the substitution $V_0(x) = V(|x|)$ by $v_a(\nu|x_n^{(\mathcal{N})}|)^{-s}$ (by $v_a(\nu|x_n^{(\mathcal{D})}|)^{-s}$), the value $N(\lambda, A^{(\varepsilon)}(\alpha, \alpha^{1/s}\nu\mathbb{Q}_n))$ can only increase (decrease). Thus the quantity $N(\lambda, A^{(\varepsilon)}(\alpha, \alpha^{1/s}\nu\mathbb{Q}_n))$ is estimated from above (the case $\varepsilon = \mathcal{N}$) or from below (the case $\varepsilon = \mathcal{D}$) by $M(\varepsilon,\alpha)$, which equals the amount of the negative spectrum of the operator $-\Delta + p - (\lambda + v_a(\nu|x_n^{(\varepsilon)}|)^{-s})I$ in $L_2(\alpha^{1/s}\nu\mathbb{Q}_n)$ with ε-conditions on $\alpha^{1/s}\nu\partial\mathbb{Q}_n$. According to the definition of the quantity $\rho(\tilde{\lambda}), \tilde{\lambda} \in \mathbb{R}$,

$$\lim_{\alpha\to\infty} \alpha^{-d/s} M(\varepsilon, \alpha) = \nu^d \rho(\lambda + v_a(\nu|x_n^{(\varepsilon)}|)^{-s}).$$

This proves (33), (34) for $V(r) = v_a r^{-s}$. In the general case for any $\mu > 0$ there exists $\alpha_0 > 0$, such that for $\alpha > \alpha_0$

$$(v_a - \mu)|x|^{-s} \le V(|x|) \le (v_a + \mu)|x|^{-s}, \ \forall x \in \alpha^{1/s}\nu\mathbb{Q}.$$

For the potentials in the utmost parts of this inequality the estimates of the kind (33), (34) are fulfilled with the substitution of v_a by $v_a + \mu$ in the right-hand sides. It remains to let $\mu \to 0$. □

The following statement gives an asymptotics of the spectral distribution function of the operator on the increasing with the growth of α spherical layer.

Proposition 3.3. *Let* $0 < \delta_1 < \delta_2 < \infty$, $\Omega = \{x \in \mathbb{R}^d : \delta_1 < |x| < \delta_2\}$ *and* $\lambda \in \mathbb{R}$. *Then*

$$\lim_{\alpha\to\infty} \alpha^{-d/s} N(\lambda, A^{(\varepsilon)}(\alpha, \alpha^{1/s}\Omega)) = \int_{\delta_1<|x|<\delta_2} \rho(\lambda_2 + v_a|x|^{-s})dx.$$

Proof. We fix $\nu > 0$. Let $\mathcal{J}$ be a set of all $n \in \mathbb{Z}^d$ for which $\nu\mathbb{Q}_n \subset \Omega$ and $\Omega_1 = \Omega \setminus \cup_{n\in\mathcal{J}}(\nu\mathbb{Q}_n)$. Then

$$\begin{aligned} N(\lambda, A^{(\varepsilon)}(\alpha, \alpha^{1/s}\Omega)) \le N(\lambda, A^{(\mathcal{N})}(\alpha, \alpha^{1/s}\Omega_1)) +& \\ \sum_{n\in\mathcal{J}} N(\lambda, A^{(\mathcal{N})}(\alpha, \alpha^{1/s}\nu\mathbb{Q}_n)).& \end{aligned} \tag{35}$$

Using Proposition 3.2 and 3.3 we obtain the following estimate

$$\begin{aligned} \limsup_{\alpha\to\infty}\alpha^{-d/s} N(\lambda, A^{(\varepsilon)}(\alpha, \alpha^{1/s}\Omega)) \le Cmes(\Omega_1) +& \\ \sum_{n\in\mathcal{J}} \nu^d \rho(\lambda + v_a(\nu|x_n^{(\mathcal{N})}|)^{-s}).& \end{aligned} \tag{36}$$

Let us pass to the limit as $\nu \to 0$. Obviously, $\mathrm{mes}(\Omega_1) \to 0$, $\sum_{n\in\mathcal{J}} \nu^d\rho(\lambda + v_a(\nu|x_n^{(\mathcal{N})}|)^{-s}) \to \int_{\delta_1<|x|<\delta_2} \rho(\lambda + v_a|x|^{-s})dx$ as $\nu \to 0$. Therefore,

$$\limsup_{\alpha\to\infty}\alpha^{-d/s} N(\lambda, A^{(\varepsilon)}(\alpha, \alpha^{1/s}\Omega)) \le \int_{\delta_1<|x|<\delta_2} \rho(\lambda + v_a|x|^{-s})dx.$$

In the same way we obtain the lower estimate:

$$\liminf_{\alpha\to\infty}\alpha^{-d/s} N(\lambda, A^{(\varepsilon)}(\alpha, \alpha^{1/s}\Omega)) \ge \int_{\delta_1<|x|<\delta_2} \rho(\lambda + v_a|x|^{-s})dx. \ \square$$

Let $\Omega \subset \mathbb{R}^d$ be a domain with a piecewise smooth boundary and $\lambda_j \in \mathbb{R}, j = 1, 2$. We put

$$n(\lambda_1, \lambda_2, \Omega, \alpha) = N(\lambda_2, A^{(\mathcal{N})}(\alpha, \alpha^{1/s}\Omega)) - N(\lambda_1, A^{(\mathcal{D})}(\alpha, \alpha^{1/s}\Omega)),\ \alpha > 0.$$

Obviously if $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$, then the following inequalities are fulfilled

$$n(\lambda_1, \lambda_2, \Omega, \alpha) \geq -n(\lambda_2, \lambda_1, \Omega_j, \alpha), \quad j = 1, 2, \tag{37}$$

$$n(\lambda_1, \lambda_2, \Omega, \alpha) \leq n(\lambda_1, \lambda_2, \Omega_1, \alpha) + n(\lambda_1, \lambda_2, \Omega_2, \alpha). \tag{38}$$

Proposition 3.4. *Let $r > 0$ and $\lambda_j \in \mathbb{R}, j = 1, 2$. Then*

$$\lim_{\alpha\to\infty} \alpha^{-d/s} n(\lambda_1, \lambda_2, B_r, \alpha) = \int_{|x|<r} (\rho(\lambda_2 + v_a|x|^{-s}) - \rho(\lambda_1 + v_a|x|^{-s}))dx. \tag{39}$$

Proof. Let us take $\Omega = B_r, \Omega_1 = B_\eta,\ 0 < \eta < r$ and $\Omega_2 = \Omega \setminus \overline{\Omega_1}$ in (38). Now we multiply the obtained inequality by $\alpha^{-d/s}$ and pass to the upper limit as $\alpha \to \infty$. According to Proposition 2.1 and 3.3 we get

$$\limsup_{\alpha\to\infty} \alpha^{-d/s} n(\lambda_1, \lambda_2, \Omega, \alpha) \leq \mu(\eta) + \int_{\eta<|x|<r} (\rho(\lambda_2 + v_a|x|^{-s}) - \rho(\lambda_1 + v_a|x|^{-s}))dx. \tag{40}$$

Taking $\eta \to 0$ we come to the following estimate

$$\limsup_{\alpha\to\infty} \alpha^{-d/s} n(\lambda_1, \lambda_2, \Omega, \alpha) \leq \int_{|x|<r} (\rho(\lambda_2 + v_a|x|^{-s}) - \rho(\lambda_1 + v_a|x|^{-s}))dx. \tag{41}$$

In the same way we obtain the lower estimate from the equation (37)

$$\liminf_{\alpha\to\infty} \alpha^{-d/s} n(\lambda_1, \lambda_2, \Omega, \alpha) \geq \int_{|x|<r} (\rho(\lambda_2 + v_a|x|^{-s}) - \rho(\lambda_1 + v_a|x|^{-s}))dx. \tag{42}$$

□

Let us combine Proposition 3.1 and 3.5. We get the following result.

Proposition 3.5. *Let $\lambda_j \in \Lambda, j = 1, 2$, and $\varepsilon > 0$. Assume that $\lambda_j + \varepsilon \in \Lambda, \lambda_j - \varepsilon \in \Lambda, j = 1, 2$. Then there exists $r > 0$ such that*

$$\limsup_{\alpha\to\infty} \alpha^{-d/s}(N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))) \leq \int_{|x|<r} (\rho(\lambda_2 + \varepsilon + v_a|x|^{-s}) - \rho(\lambda_1 - \varepsilon + v_a|x|^{-s}))dx, \tag{43}$$

$$\liminf_{\alpha\to\infty}\alpha^{-d/s}(N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))) \geq \int_{|x|<r}(\rho(\lambda_2 - \varepsilon + v_a|x|^{-s}) - \rho(\lambda_1 + \varepsilon + v_a|x|^{-s}))dx \tag{44}$$

Proof. The function $\rho(\lambda)$ is constant for $\lambda \in \Lambda$. Therefore if we put $v_a = 0$ in (39), then we get

$$\lim_{\alpha\to\infty} \alpha^{-d/s}(N(\lambda_2\pm\varepsilon, A(0, \alpha^{1/s}B_r)) - N(\lambda_1\pm\varepsilon, A(0, \alpha^{1/s}B_r))) = \operatorname{mes}(B_r)(\rho(\lambda_2\pm\varepsilon) - \rho(\lambda_1\pm\varepsilon)) = 0. \tag{45}$$

We subtract the inequality (31) with $\lambda = \lambda_1$ from (30) with $\lambda = \lambda_2$. We multiply the result by $\alpha^{-d/s}$ and pass to the upper limit at $\alpha \to \infty$. According to (39) and (45) we obtain

$$\begin{aligned}\limsup_{\alpha\to\infty}\alpha^{-d/s}(N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))) &\leq \\ \limsup_{\alpha\to\infty}\alpha^{-d/s} n(\lambda_1 - \varepsilon, (\lambda_2 + \varepsilon) + 0, B_r, \alpha) &= \\ \int_{|x|<r}(\rho(\lambda_2 + \varepsilon + v_a|x|^{-s}) - \rho(\lambda_1 - \varepsilon + v_a|x|^{-s}))dx.&\end{aligned} \tag{46}$$

This coincides with (43). The inequality (44) can be obtained in the same way. We just have to subtract (30) with $\lambda = \lambda_1$ from (31) with $\lambda = \lambda_2$. $\square$

Let us complete the proof of Theorem 1.1. In (43) and (44) we pass to the limit as $r \to \infty$. Then we take $\varepsilon \to 0$. The passage to the limit as $\varepsilon \to 0$ under the integral is valid since the integrated functions in (43), (44) have a integrable majorant for $\varepsilon < \varepsilon_0$. As a result we get

$$\limsup_{\alpha\to\infty}\alpha^{-d/s}(N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))) \leq \int(\rho(\lambda_2 + v_a|x|^{-s}) - \rho(\lambda_1 + v_a|x|^{-s}))dx, \tag{47}$$

$$\liminf_{\alpha\to\infty}\alpha^{-d/s}(N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))) \geq \int(\rho(\lambda_2 + v_a|x|^{-s}) - \rho(\lambda_1 + v_a|x|^{-s}))dx. \tag{48}$$

It remains to note that $\operatorname{rank} E_{A(\alpha)}(\lambda_1, \lambda_2) = N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))$ for $\lambda_1 \in \rho(A(\alpha))$. $\square$

4. Proof of Theorem 3.1

1. We fix an interval $[a, b]$ satisfying the following conditions

$$\lambda - \varepsilon, \lambda + \varepsilon \in (a, b)$$

$$[a, b] \subset \Lambda. \tag{49}$$

Obviously there exist $a', b' \in \mathbb{R}$ such that

$$a' < a,\ b' < b,\ [a', b'] \subset \Lambda. \tag{50}$$

Let the operators $A^{(\varepsilon)}(\alpha, \Omega)$ be the same as in the section 1. We put $\Pi_r = E_{A^{(\mathcal{D})}(0,B_r)}(a', b'), r > 0$. Also let the function $\psi \in C^\infty(\mathbb{R}^d)$ satisfy the following conditions

$$0 \le \psi(x) \le 1, \quad \psi(x) = \begin{cases} 0 & \text{for } |x| < 1/2, \\ 1 & \text{for } |x| > 3/4. \end{cases}$$

We denote by Ψ_r the multiplication operator by the function $\psi_r(x) = \psi(x/r), r > 0$, in $L_2(\mathbb{R}^d)$ and we put

$$\tilde{A}_r = A^{(\mathcal{D})}(0, B_r) + (b' - a')\Psi_r \Pi_r \Psi_r,\ r > 0. \tag{51}$$

In [1] the following statement concerning $\tilde{A}_r$ is proved.

Proposition 4.1. *Let the conditions* (49), (50) *be fulfilled and let $\tilde{A}_r$ be defined as in* (51). *Then there exists $\tilde{r_0} > 0$ such that*

$$[a, b] \subset \rho(\tilde{A}_r), \quad \forall r > \tilde{r_0}.$$

We put

$$\tilde{A}_r(\alpha) = \tilde{A}_r - \alpha V_0. \tag{52}$$

The next statement allows to approximate the quantity $N_0(\lambda, A(\alpha))$ by similar quantities for operators on a ball $B_r, r > 0$. We refer to [2] for its proof.

Proposition 4.2. *Let $\lambda \in (a, b)$ and the conditions* (49), (50) *are fulfilled. Assume that the operator $\tilde{A}_r$ in* (52) *is defined in* (51). *Then*

$$N_0(\lambda, A(\alpha)) \ge \limsup_{r\to\infty}(N(\lambda, \tilde{A}_r(\alpha')) - N(\lambda, \tilde{A}_r)), \qquad 0 < \alpha' < \alpha; \tag{53}$$

$$N_0(\lambda, A(\alpha)) \le \liminf_{r\to\infty}(N(\lambda, \tilde{A}_r(\alpha')) - N(\lambda, \tilde{A}_r)), \qquad 0 < \alpha < \alpha'. \tag{54}$$

We introduce the selfadjoint operators $A_{R,r}^{\varepsilon_1,\varepsilon_2} = -\Delta + p$ in $L_2(B_r \setminus B_R)$, $r > R > 0$, where the indexes ε_1 and ε_2, which take the values $\mathcal{D}$ or $\mathcal{N}$, characterize boundary conditions (Dirichlet or Neumann) on ∂B_R and ∂B_r correspondingly. We put also $\tilde{A}_{R,r}^{\varepsilon_1,\varepsilon_2} = A_{R,r}^{\varepsilon_1,\varepsilon_2} + (b' - a')\Psi_r\Pi_r\Psi_r,\ r > R > 0$. The next statement is contained in Lemma 3.2 and Lemma 4.2 of [2].

Proposition 4.3. *Let $\tilde{r}_0$ be as in Proposition 4.1 and $\lambda \in (a,b)$. Then for $r > \tilde{r}_0, 1 \le R \le r/2$ the following inequalities hold*

$$N(\lambda, A^{(\mathcal{D})}(0,B_R)) + N(\lambda, \tilde{A}^{\mathcal{D},\mathcal{D}}_{R,r}) \ge N(\lambda, \tilde{A}_r) - C_1 R^{d-1}, \qquad C_1 = C_1(\lambda) \tag{55}$$

$$N(\lambda, A^{(\mathcal{N})}(0,B_R)) + N(\lambda, \tilde{A}^{\mathcal{N},\mathcal{D}}_{R,r}) \le N(\lambda, \tilde{A}_r) + C_2 R^{d-1}, \qquad C_2 = C_2(\lambda), \tag{56}$$

where C_1 and C_2 do not depend on r and R.

2. Now let us complete the proof of Theorem 3.1. First of all we establish the relation (30). Let $\alpha' > \alpha$ and $r_0 = (v_+/\varepsilon)^{1/s}$. We put $R = m\alpha^{1/s}, m > r_0$. We note that $\alpha V_0(x) < \varepsilon$ for $|x| > R$. Therefore from the variational principle it follows that

$$N(\lambda, \tilde{A}_r(\alpha')) \le N(\lambda+\varepsilon, A(\alpha', B_R)) + N(\lambda+\varepsilon, \tilde{A}^{\mathcal{N},\mathcal{D}}_{R,r}), \quad R < r \tag{57}$$

Since $\lambda + \varepsilon \in (a,b)$, we obtain from (56) that

$$N(\lambda, \tilde{A}_r) = N(\lambda+\varepsilon, \tilde{A}_r) \ge N(\lambda+\varepsilon, A^{(\mathcal{N})}(0,B_R)) + N(\lambda+\varepsilon, \tilde{A}^{\mathcal{N},\mathcal{D}}_{R,r}) - C_2 R^{d-1}, \quad C_2 = C_2(\lambda+\varepsilon), r > \tilde{r}_0, 1 \le R \le r/2. \tag{58}$$

Estimating the right-hand side of (54) with the help of (57), (58) we get

$$N_0(\lambda, A(\alpha)) \le N(\lambda+\varepsilon, A^{(\mathcal{N})}(\alpha', B_R)) - N(\lambda+\varepsilon, A^{(\mathcal{N})}(0,B_R)) + C_2 R^{d-1}, \; R = m\alpha^{1/s} \ge 1. \tag{59}$$

Taking $\alpha' \to \alpha$ we come to the upper estimate

$$N_0(\lambda, A(E\alpha)) \le N((\lambda+\varepsilon)+0, A^{(\mathcal{N})}(\alpha, B_R)) - N(\lambda+\varepsilon, A^{(\mathcal{N})}(0,B_R)) + C_2 R^{d-1}, \; R = m\alpha^{1/s}, \tag{60}$$

which coincides with (30) for $r = m$ and $c = C_2$. Moreover (60) holds for all $m > r_0, \alpha > \alpha_0 = r_0^{-s}$. The relation (31) can be obtained in the same way from (53) and (55). □

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