Geometric phase related to point-interaction transport on a magnetic Lobachevsky plane

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Abstract

We consider a charged quantum particle living in the Lobachevsky plane and interacting with a homogeneous magnetic field perpendicular to the plane and a point interaction which is transported adiabatically along a closed loop $\mathcal{C}$ in the plane. We show that the bound-state eigenfunction acquires at that the Berry phase equal to 2π times the number of the flux quanta through the area encircled by $\mathcal{C}$.

1 Introduction

The phenomena arising from a geometric phase called Berry phase [Ber] have been put in evidence in many quantum mechanical systems. Recently such a "Berry phase effect" has been observed in some magnetic systems [LSG, MHK]. Moreover, it was shown that the geometric phase can emerge even in a time-independent homogeneous magnetic field when a potential well trapping a two-dimensional particle is transported along a closed loop. An example in which the potential is of zero range is worked out in [EG], where a formula was proved showing that in the absence of an additional confining potential the acquired phase is proportional to the number of flux quanta through the area encircled by the loop.

This result raises the natural question whether it can be extended to systems with a nontrivial configuration-space geometry. In this letter we address this problem in the framework of a solvable model in which the Euclidean plane is replaced by a Riemannian manifold of a constant negative curvature; we shall show that the analogue of the mentioned "planar" formula is valid here. Among other things, our result illustrates an important difference between two kinds of geometric phases, namely those of Berry and of Aharonov–Anandan. Specifically, the Aharonov–Anandan connection is closely related to the metric connection of the parameter spaces $\mathbf{C}P^N$ [Mos], whereas the Berry connection is completely independent of the Levi–Civita connection in the parameter space.

2 The free Hamiltonian

The configuration space of our model is the Lobachevsky plane, i.e. a complete two-dimensional simply connected Riemannian manifold of constant negative curvature R, $R < 0$. We shall employ the Poincaré realization in which the Lobachevsky plane is identified with the upper complex halfplane

$$\mathbb{H}_a^2 = \{z \in \mathbb{C} : \Im z > 0\}$$

endowed with the metric

$$ds^2 = \frac{a^2}{y^2}\,(dx^2 + dy^2)\,,$$

where $x = \Re z$, $y = \Im z$, and $a > 0$ is the parameter related to the curvature R by $R = -2/a^2$. Then the geodesic distance on $\mathbb{H}_a^2$ is given by the formula

$$d_a(z, z') = a \text{ Arcosh} \left[1 + \frac{|z - z'|^2}{2yy'}\right]$$

and the area element $d\mu_a$ has the form

$$d\mu_a = \frac{a^2}{y^2}\, dx \wedge dy\,. \tag{2.1}$$

A constant magnetic field on $\mathbb{H}_a^2$ is given by a 2-form $\mathbb{B}$ defined as

$$\mathbb{B} = \frac{Ba^2}{y^2}\, dx \wedge dy\,,$$

where B is the field intensity. The form $\mathbb{B}$ is obviously exact and any 1-form $\mathbb{A}$ such that $\mathbb{B} = d\mathbb{A}$ is called a vector potential related to the field $\mathbb{B}$. For our purpose it is convenient to choose $\mathbb{A}$ in the Landau gauge

$$\mathbb{A} = \frac{Ba^2}{y}\, dx\,.$$

The Schrödinger operator describing a particle of charge e and mass m_* which lives on the Lobachevsky plane $\mathbb{H}_a^2$ and interacting with a magnetic field is according to [Com] given by

$$H^0 = -\frac{\hbar^2}{2m_* a^2}\left\{ y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2iby\frac{\partial}{\partial x} - b^2 \right\} - \frac{\hbar^2\nu}{8m_* a^2}\,, \tag{2.2}$$

where we have introduced the dimensionless quantity

$$b = \frac{eBa^2}{\hbar c}$$

which has a simple meaning: if $\Phi_e = \frac{2\pi\hbar c}{e}$ is the magnetic flux quantum relative to the charge e, then b is the doubled number of flux quanta through the degenerate triangle (the area of which is πa^2).

The presence of the last term on the r.h.s. of (2.2) (this term is absent in [Com]) can be justified in different ways, e.g. as a "van Vleck correction" [Gut], in which case $\nu = 1$. On the other hand, in one embeds locally the Lobachevsky plane into $\mathbb{R}^3$ and derive the Hamiltonian through a squeezing limit of a saddle-shaped layer [Tol] one obtain the last term with $\nu = 4$. We shall not discuss this difference, however, because it is of no importance for the result we are going to derive in this paper. For the sake of simplicity we put in the following $e = c = \hbar = 2m_* = 1$.

The spectrum of H^0 consists of two parts [Com], the second one being absent for weak fields, $2|b| \le 1$:

(i) an absolutely continuous spectrum in the interval $[b^2/a^2, \infty)$,

(ii) a point spectrum consisting of a finite number of infinitely degenerate eigenvalues E_n^0, $0 \le n < |b| - \frac{1}{2}$. These Landau levels are given explicitly:

$$E_n^0 = \frac{1}{a^2}\left(b^2 - \left(|b| - n - \frac{1}{2}\right)^2\right) = \frac{1}{a^2}\left(|b|(2n+1) - \left(n + \frac{1}{2}\right)^2\right). \tag{2.3}$$

We will need also an explicit expression for the Green's function $G^0(z, z'; \zeta)$ of H^0. Let us introduce the quantity

$$\sigma(z, z') = \cosh^2\left(\frac{d_a(z, z')}{2a}\right). \tag{2.4}$$

It is easy to see that it is independent of a being equal to

$$\sigma(z, z') = \frac{|x - x'|^2 + |y + y'|^2}{4yy'}$$

Given $\zeta \in \mathbb{C} \setminus [b^2/a^2, \infty)$ we put

$$t(\zeta) = \frac{1}{2} + \sqrt{b^2 - a^2\zeta},$$

where the square root $\sqrt{z}$ is defined in the cut plane $\mathbb{C} \setminus (-\infty, 0)$ by the requirement $\Re\sqrt{z} \ge 0$. With this notation the integral kernel of $(H^0 - \zeta)^{-1}$ is of the form [Els]

$$G^0(z, z'; \zeta) = \frac{1}{4\pi}\left(-\frac{z - \bar{z}'}{\bar{z} - z'}\right)^b \frac{\Gamma(t+b)\Gamma(t-b)}{\Gamma(2t)}\, \sigma^{-t}\, F(t+b, t-b; 2t; \sigma^{-1}),$$

where $F(a, b; c; x)$ is the hypergeometric function.

3 Krein's formula

Now we shall consider a point perturbation – introduced in the usual way [BF, AGHH] – of the operator H^0 supported by a point $w \in \mathbb{H}_a^2$, $w = u + iv$. By Krein's formula [AGHH, App. A] the Green's function of the perturbed operator has the form

$$G(z, z'; \zeta) = G^0(z, z'; \zeta) - \frac{G^0(z, w; \zeta)G^0(w, z'; \zeta)}{Q(\zeta) - \alpha}, \tag{3.1}$$

where the parameter α is related to the scattering length λ of the point "potential" by the formula

$$\alpha = \frac{m_*}{\pi\hbar^2}\,\ln\lambda$$

(or $2\pi\alpha = \ln\lambda$ in the rational units). The Krein's function $Q(\zeta)$, defined as the regularized trace of the free-resolvent kernel, was evaluated in [BG] to be

$$Q(\zeta) = \frac{1}{4\pi}\,\left[\psi(t+b) + \psi(t-b) + 2\gamma - 2\ln 2a\right]\,,$$

where $\psi(z)$ is Euler's digamma function and $\gamma = \psi(-1) = 0.577...$ is the Euler number. The perturbed Hamiltonian with the Green's function (3.1) will be denoted as $H_{w,\alpha}$.

Using the well-known behaviour of the digamma function [AS, BE] we find that the Krein's function has in our case the following properties:

(a) $Q(\zeta)$ is a meromorphic function in the cut plane $\mathbb{C}\setminus[b^2/a^2,\infty)$ with the poles at the points E_n^0 of the discrete spectrum of H^0 given by (2.3).

(b) $\lim_{\mathbb{R}\ni\zeta\to-\infty} Q(\zeta) = -\infty$.

(c) At the continuum threshold we have

$$\lim_{\mathbb{R}\ni\zeta\to b^2/a^2} Q(\zeta) = \begin{cases} +\infty & \dots \quad |b| \text{ half}-\text{integer} \\ q_{a,b} & \dots \quad \text{otherwise} \end{cases}$$

where $q_{a,b} = \frac{1}{4\pi}\,\left[\psi(\frac{1}{2}+b) + \psi(\frac{1}{2}-b) + 2\gamma - 2\ln 2a\right]$.

(d) $\frac{\partial Q}{\partial\zeta} > 0$ holds at each point of $\mathbb{R}\setminus\sigma(H^0)$.

As usual we employ the symbol $[x]$ for the integer part of a number x. We set $n_0 = \lim_{\varepsilon\to 0}\left[|b| - \frac{1}{2} - \varepsilon\right]$ and consider the following family of intervals

$$(-\infty, E_0^0),\ (E_0^0, E_1^0),\ \dots, (E_{n_0-1}^0, E_{n_0}^0),\ (E_{n_0}^0, b^2/a^2)\,.$$

We also set $E_{-1}^0 = -\infty$ so for $n_0 = -1$ the family consists of a single interval $(-\infty, b^2/a^2)$. The last interval having b^2/a^2 as the right endpoint will be called special, while all the other intervals are dubbed regular.

The listed properties of the function $Q(\zeta)$ allow us to make the following conclusions. At each regular interval the equation

$$Q(\zeta) = \alpha \tag{3.2}$$

has one and only one solution. We denote it by $E_k(\alpha)$ where the index refers to the right endpoint of the interval in question. If $|b|$ is half-integer the equation (3.2) has at the special interval a solution for any $\alpha \in \mathbb{R}$, otherwise a solution exists there if and only if the inequality

$$4\pi\alpha < \psi(\frac{1}{2}+b) + \psi(\frac{1}{2}-b) + 2\gamma - 2\ln 2a$$

is valid; if the solution at the special interval exists, it is unique and we shall denote it by $E_{n_0+1}(\alpha)$.

It follows from (3.1) that the discrete spectrum of $H_{w,\alpha}$ consists exactly of all solutions of the equation (3.2) in the interval $(-\infty, b^2/a^2)$. Any such solution $E_k(\alpha)$ is at that a simple eigenvalue; the corresponding normalized eigenfunction $\Psi_k(z;w,\alpha)$ is given by

$$\Psi_k(z;w,\alpha) = c_k\, G^0(z;w,E_k(\alpha))$$

with the normalization factor

$$c_k = \left[\left. \frac{\partial Q}{\partial \zeta} \right|_{\zeta=E_k(\alpha)} \right]^{-1/2} .$$

For our future purpose it is important that $E_k(\alpha)$ is independent of w, and therefore it remains to be a simple isolated eigenvalue as the position of the point perturbation is changed.

4 The Berry phase

We shall now realize our aim of finding the Berry phase for an adiabatic evolution of our system in the parameter space $\mathbb{H}^2_a \ni w$. Let us compute the corresponding Berry potential. Since α and the level index k are kept fixed in the following we drop them from the notations. We first remark that the eigenfunction Ψ can be written in the form

$$\Psi(z;w) = \left(-\frac{z-\bar{z}'}{\bar{z}-z'} \right)^b \phi(\sigma(z,w))\,, \tag{4.1}$$

where the function ϕ is real-valued. The derivatives of the first factor with respect to $u = \Re w$ and $v = \Im w$ are

$$\frac{\partial}{\partial u} \left(-\frac{z-\bar{z}'}{\bar{z}-z'} \right)^b \;=\; -b \left(-\frac{z-\bar{z}'}{\bar{z}-z'} \right)^{b-1} \frac{z-\bar{z}+w-\bar{w}}{(\bar{z}-w)^2}\,, \tag{4.2}$$

$$\frac{\partial}{\partial v}\left(-\frac{z-\bar{z}'}{\bar{z}-z'}\right)^{b} = -bi\left(-\frac{z-\bar{z}'}{\bar{z}-z'}\right)^{b-1}\frac{z+\bar{z}-(w+\bar{w})}{(\bar{z}-w)^2}\,. \tag{4.3}$$

The last relations in combination with (4.1) gives

$$\begin{aligned}\langle\Psi|\frac{\partial}{\partial v}\Psi\rangle &= -2bi\int_{\mathbb{H}_a^2}\frac{x-u}{(x-u)^2+(y+v)^2}\,[\phi(\sigma(z,w))]^2 d\mu_a(z)\\ &\quad+\int_{\mathbb{H}_a^2}\phi(\sigma(z,w))\,\frac{\partial}{\partial v}\,\phi(\sigma(z,w))d\mu_a(z)\,.\end{aligned} \tag{4.4}$$

Since ϕ is real-valued and $\int_{\mathbb{H}_a^2}[\phi(\sigma(z,w))]^2 d\mu_a(z) = 1$ holds for all $w\in\mathbb{H}_a^2$, the second integral in (4.4) vanishes. Using the substitution $z-u\to z$ in the first one, we get the expression

$$\langle\Psi|\frac{\partial}{\partial v}\Psi\rangle = -2bi\int_{\mathbb{H}_a^2}\frac{x}{x^2+(y+v)^2}\,[\phi(\sigma(z,iv))]^2 d\mu_a(z)\,. \tag{4.5}$$

By (2.4) the function $\sigma(x,iv)$ is even with respect to x, hence the integrated function in (4.5) is odd and

$$\langle\Psi|\frac{\partial}{\partial v}\Psi\rangle = 0\,.$$

Let us turn to the u-component of the Berry potential. Since ϕ is real-valued, we infer from (4.1) and (4.2)

$$\begin{aligned}\langle\Psi|\frac{\partial}{\partial u}\Psi\rangle &= -2bi\int_{\mathbb{H}_a^2}\frac{y+v}{(x-u)^2+(y+v)^2}\,[\phi(\sigma(z,w))]^2 d\mu_a(z)\\ &= -2bi\int_{\mathbb{H}_a^2}\frac{y+v}{x^2+(y+v)^2}\,[\phi(\sigma(z,iv))]^2 d\mu_a(z)\,.\end{aligned}$$

Another substitution, $z\to vz$, yields

$$\langle\Psi|\frac{\partial}{\partial u}\Psi\rangle = -\frac{2bi}{v}\int_{\mathbb{H}_a^2}\frac{y+1}{x^2+(y+1)^2}\,[\phi(\sigma(z,i))]^2 d\mu_a(z)\,,$$

and since $\sigma(z,i) = [x^2+(y+1)^2]/4y$, we have

$$\langle\Psi|\frac{\partial}{\partial u}\Psi\rangle = -\frac{bi}{2v}\int_{\mathbb{H}_a^2}\frac{y+1}{y\sigma(z,i)}\,[\phi(\sigma(z,i))]^2 d\mu_a(z)\,.$$

We have remarked already that σ is independent of a; then it follows from (2.1) that

$$\langle\Psi|\frac{\partial}{\partial u}\Psi\rangle = -\frac{bia^2}{2v}\int_{\mathbb{H}_a^2}\frac{y+1}{y\sigma(z,i)}\,[\phi(\sigma(z,i))]^2 d\mu_1(z)\,. \tag{4.6}$$

To evaluate the last integral we pass to the polar coordinate system centered at the point i putting $r = d_1(z,i)$. Then

$$y^{-1} = \cosh r + \sinh r\,\cos 2\varphi\,, \qquad d\mu_1(z) = \sinh r\,dr\,d\varphi\,,$$

where φ is the polar angle [Ter]. Since $\sigma(z,i) = \cosh^2\frac{r}{2}$ we can rewrite the r.h.s. of (4.6) as

$$-\frac{bia^2}{2v}\int_0^\infty dr\int_0^{2\pi} d\varphi\,(1+\cosh r + \sinh r\,\cos 2\varphi)\,\frac{\sinh r}{\cosh^2\frac{r}{2}}\,\left[\phi\left(\cosh^2\frac{r}{2}\right)\right]^2.$$

Using $1+\cosh r = 2\,\cosh^2\frac{r}{2}$ and integrating over φ we find

$$\langle\Psi|\frac{\partial}{\partial u}\Psi\rangle = -\frac{2\pi bia^2}{v}\int_0^\infty\left[\phi\left(\cosh^2\frac{r}{2}\right)\right]^2\sinh r\,dr\,.$$

On the other hand,

$$\begin{aligned}2\pi a^2\int_0^\infty\left[\phi\left(\cosh^2\frac{r}{2}\right)\right]^2\sinh r\,dr &= a^2\int_0^\infty dr\int_0^{2\pi} d\varphi\,[\phi\,(\sigma(z,i))]^2\sinh r\\ = a^2\int_{\mathbb{H}_1^2}[\phi(\sigma(z,i))]^2 d\mu_1(z) &= \int_{\mathbb{H}_a^2}[\phi(\sigma(z,i))]^2 d\mu_a(z) = \|\Psi(\cdot\,;i)\|^2 = 1\,,\end{aligned}$$

so finally we arrive at the expression

$$\langle\Psi|\frac{\partial}{\partial u}\Psi\rangle = -\frac{ib}{v}\,.$$

Hence the Berry potential,

$$\mathbb{V}(w) = i\,\langle\Psi(\cdot\,;w)|\nabla_w\Psi(\cdot\,;w)\rangle$$

of our system is of the form

$$\mathbb{V}(w) = \left(\frac{b}{v},0\right) = \left(\frac{Ba^2}{v}\,,0\right)$$

i.e. similarly to the case of the Euclidean plane it coincides with the vector potential $\mathbb{A}$ if we express $\mathbb{V}$ as a 1-form $\frac{Ba^2}{v}\,du$.

Let now $\mathcal{C}$ be a smooth closed contour in the Lobachevsky plane $\mathbb{H}^2_a$, then the Stokes formula yields the sought expression for the Berry phase $\gamma(\mathcal{C})$:

$$\gamma(\mathcal{C}) = \int_{\mathcal{C}} \frac{Ba^2}{v}\,du = \int\!\!\int_S \frac{Ba^2}{v}\,du \wedge dv = BS = \frac{2\pi\Phi_{\mathcal{C}}}{\Phi_e}\,, \tag{4.7}$$

where $\Phi_{\mathcal{C}}$ is the total flux of the field $\mathbb{B}$ through the area S encircled by the loop $\mathcal{C}$. The relation (4.7) is the main result of this letter.

Notice that in distinction of the spectrum of the Hamiltonian $H_{w,\alpha}$ the Berry phase depends neither on the curvature R nor on the coupling parameter α. In particular, $\gamma(\mathcal{C})$ is independent of the energy of the considered particle in the zero-range potential well which confines it.

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