The Luttinger model: its role in the RG–theory of one dimensional many body Fermi systems

Giovanni Gallavotti
Fisica, Università di Roma
P.le A. Moro 2, 00185 Roma, Italia
e-mail: `gallavotti@roma1.infn.it`

Abstract: *The Luttinger model was introduced to illustrate the theory of Tomonaga via an exactly soluble model. It became soon the subject of great interest also on the part of Mathematical Physics and a key to the investigations of the mathematical properties of Condensed Matter Physics. This paper reviews aspects of the above developments relevant for renormalization group methods.*

§1. The Luttinger model.

The model describes many body systems of two kinds of fermions on a line whose fields are $\tilde\psi^\alpha_{\omega,x}$, with $\alpha=\pm$ specifying the creation and annihilation operators for fermions located at a point $x\in[-L/2,L/2]\subset R^1$ and distinguished by the label $\omega=+$ or $\omega=-$. The Hamiltonian, "(kinetic energy)+ (chemical potential)+(potential energy)", is written as

$$\begin{aligned}H=&\int_{-L/2}^{L/2}dx\,\sum_{\omega=\pm}\tilde\psi^+_{\omega,x}v_F(i\omega\partial_x-p_F)\tilde\psi^-_{\omega,x}+\\&+\lambda\int_{-L/2}^{L/2}dxdy\,\tilde\psi^+_{+,x}\tilde\psi^-_{+,x}\tilde\psi^+_{-,y}\tilde\psi^-_{-,y}\,v(x-y)\end{aligned}\tag{1.1}$$

where λ is a coupling constant and $v(x-y)$ is a smooth short range pair potential ($e.g.$ $v(x-y)\equiv0$ if $|x-y|\ge p_0^{-1}$ for some "range" p_0^{-1}) . The parameter v_F is the "velocity at the Fermi surface", $v_F=p_F/m$ if m is the mass of the particles: we have set $\hbar=m=1$.

This model was introduced in [Lu63] and it illustrated Tomonaga's theory of spin 0 fermions in one dimension, [To50], which shows the remarkable phenomenon of the *anomaly* of the ground state: *i.e.* a ground state with a density of states which does not have a discontinuity at the Fermi momentum $k=p_F$ but its graph becomes vertical with a vertical tangency exponent $a(\lambda)=O(\lambda^2)$ called the *anomaly*" of the Fermi surface.

The model supposes *a priori* that the "physically significant" part of the Hamiltonian is described by *quasi particles*. This means realizing that the Schwinger functions of the ground state of a spin 0 Fermi gas can be identified, to leading non trivial order in the coupling constant λ, by thinking that the system in fact consists of two particles with energy close to the Fermi energy $p_F^2/2$, one with momentum close to $+p_F$ and the other with momentum close to $-p_F$ whose fields, in a suitable superposition, yield the field $\tilde\psi^\pm_x$ of the observable particles.

Writing a momentum close to $\pm p_F$ as $\pm p_F+k$ the free field $\tilde\psi^\pm_x$ is expressed as

$$\tilde\psi^\pm_x\simeq\sum_{\omega=\pm}\psi^\pm_{\omega,x}e^{\pm i\,p_F\,\omega x}\tag{1.2}$$

where calling $a^\pm_{k,\omega}\stackrel{def}{=}\alpha^\pm_{k+p_F\omega,\omega}$, if $\alpha^\pm_p$ are the creation and annihilation operators for the Fermi particles, the *quasi particles fields* are

$$\psi^\pm_{\omega,x}=\frac{1}{\sqrt L}\sum_k e^{\pm ikx}a^\pm_{\omega,k}\tag{1.3}$$

Hence if the operator T_0

$$T_0 \stackrel{def}{=} \sum_\omega \sum_{k>0} v_F\, k\, (a^+_{\omega k,\omega} a^-_{\omega k,\omega} + a^-_{-\omega k,\omega} a^+_{-\omega k,\omega}) \tag{1.4}$$

approximates the free fermionic kinetic energy plus the chemical potential energy for $k \simeq 0$, then the quasi particle fields at time t will be $\psi^\pm_{\omega,x,it} = e^{\pm iT_0 t} \psi^\pm_{\omega,x} e^{\mp iT_0 t}$ $= e^{\pm ip_F t} \psi^\pm_{\omega,x,it}$ with

$$\psi^\pm_{\omega,x,t} \equiv e^{tT_0} \psi^\pm_{x,\omega} e^{-tT_0} = \frac{1}{\sqrt{L}} \sum_k e^{\pm(i\,k\,x + t\,\omega\,k)} a^\pm_{k,\omega} \tag{1.5}$$

I do not repeat here the heuristic analysis showing that taking the two kinds of particles Hamiltonian (1.1) and approximating it with the free kinetic energy (1.4) and expressing the fermionic field $\tilde\psi^\pm$ via (1.2) means
(1) thinking the particles with positive momentum as distinct from the ones with negative momentum and, *furthermore*,
(2) replacing the dispersion relation $\varepsilon(p) = p^2/2$ with $p = \pm p_F + k$ with $\varepsilon(\pm, k) = p_F^2/2 \pm kp_F$ and, also,
(3) allowing k to take all values rather than $k > -p_F$ or $k < p_F$.

Luttinger realized that Tomonaga's theory amounted, to leading order in λ, to
(i) replacing the real fermionic particle fields with the $\tilde\psi^\pm$ fields defined by th r.h.s. of (1.2) and (1.3),
(ii) replacing the "real" fermionic particle kinetic energy defined as

$$T'_0 = \int_{-L/2}^{L/2} dx\, \tilde\psi^+_x \frac{1}{2}(\partial_x^2 - p_F^2) \tilde\psi^-_x \equiv \sum_p \frac{1}{2}(p^2 - p_F^2)\, \alpha^+_p \alpha^-_p \tag{1.6}$$

where $p = 2\pi n/L$, $n = 0, \pm1, \pm2, \ldots$, by the operator T_0 above, which can be written

$$T_0 = \int_{-L/2}^{L/2} dx \sum_\omega \psi^+_{\omega,x} (-iv_F \omega \partial_x) \psi^-_{\omega,x} \tag{1.7}$$

and
(iii) replacing the potential energy by the expression in (1.1), *rather than* considering the usual pair potential which would be written as

$$\lambda \int_{-L/2}^{L/2} dx\, dy\, \tilde\psi^+_x \tilde\psi^-_x\; \tilde\psi^+_y \tilde\psi^-_y\; v(x-y) \tag{1.8}$$

Note that comparing (1.8) with the potential energy in (1.1) and with (1.2) several "cross terms" involving fields with different quasi particles labels are absent.

The Luttinger approximation was stated to be exactly soluble in the remarkable paper [Lu63] and the solution did yield the expected anomaly of the Fermi surface mentioned above, thereby providing a simple explanation of the phenomenon.
The exact solution was, however, not really correct because of an error on the Fock space canonical commutation relations which, in infinitely many degrees of freedom systems, do not have a unique representation: a matter that is now well understood but that was not so clear at the time. Nevertheless the attempt in [Lu63] contained the important ideas indicating strongly that the model could probably be really solved exactly. The exact solution was discovered a little later by Mattis and Lieb, [ML65]. The real value of the anomaly and Luttinger's agree to leading order in λ (*i.e.* to second order) but they differ in the higher orders

(which vanish identically in Luttinger's expression while in [ML65] the anomaly is analytic in λ near $\lambda = 0$ but *it is not* a polynomial of second order); see also [BGM92].

The work [ML65] should be seen as a part of the series of exact solutions of simple models which, starting with the 2–dimensional Ising model led to the Bethe ansatz (which is not necessary in the case of the Luttinger model) and to exactly solvable 6 and 8 vertex models, [ML66], [Ba82].

The Luttinger model is closely related to the Thirring model of quantum field theory, [Lu63]: for an analysis of the relation between the two models see [Ma94].

§2. Schwinger functions and ground state.

More recently there has been renewed interest, particularly after the discovery of high temperature superconductivity, in understanding the properties of condensed matter from a fundamental point of view, *i.e.* without relying on heuristic arguments, whenever possible. One of the directions in which research has been stimulated is the application of multiscale analysis, *i.e.* of the renormalization group methods, to investigate the large distance properties of the correlations in the ground states of the simplest systems, *i.e.* Bose or Fermi gases with weak short range interactions: [BG90], [FT90], [BGPS94], [Sh92], [BG95]. Other methods have also been applied leading to complementary results valid also at strong coupling, for instance [KL73] and [KLY88], [LY00], or for exact solutions [LL63], [LW68].

A mathematically rigorous theory of the ground state of the 1–dimensional Fermi gas turned out surprisingly hard even at small λ, probably because the basic formalism, even today, is still waiting to acquire an established shape. Furthermore the case of spinning electrons with attractive rotationally symmetric pair interaction still defies research. The situation is worse in higher dimension where only formal developments seem to be available (see [BG90], [FT90], [Sh92]): therefore higher dimension will be left out of the present discussion, but for sporadic comments.

In dimension 1 the Hamiltonian of a system of spin 0, mass $m = 1$, fermionic particles in a periodic box $[-L/2, L/2]$ (and in the grand canonical ensemble) is

$$H = \sum_{s=1}^{N} \left(-\frac{1}{2}\Delta_{x_i} - \mu \right) + 2\lambda \sum_{i<j} v(x_i - x_j) \tag{2.1}$$

with N=number of particles or equivalently, in second quantized form,

$$\begin{aligned} H = & \int_{-L/2}^{L/2} dx\, \tilde\psi^+_x (-\frac{1}{2}\Delta_x - \mu)\tilde\psi^-_x + \\ & + \lambda \int_{-L/2}^{L/2} dx\, dy\, v(x-y)\, \tilde\psi^+_x \tilde\psi^-_x\, \tilde\psi^+_y \tilde\psi^-_y \end{aligned} \tag{2.2}$$

where $\tilde\psi$ is the usual Fermi field (rather than its approximation in (1.2), (1.3)).

If ρ is the density of the gas the simplest question that one can ask is about the behavior near $p = \pm\pi\rho \stackrel{def}{=} \pm\, p_F = \pm\sqrt{2\mu}$ of the Fourier transform of the one particle reduced density matrix. Note that if $\lambda = 0$ it is known that the one particle reduced density matrix at inverse temperature $\beta = +\infty$ is

$$\langle \tilde\psi^+_x \tilde\psi^-_0 \rangle \stackrel{def}{=} \lim_{\beta\to\infty} \frac{\mathrm{Tr}\, e^{-\beta H} \tilde\psi^+_x \tilde\psi^-_0}{\mathrm{Tr}\, e^{-\beta H}} \tag{2.3}$$

with Fourier transform $\chi_{p_F}(p) = 1$ if $|p| < p_F$ and zero otherwise.

In the following it will be more convenient to study the more general Schwinger functions

$$S_{\sigma_1\cdots\sigma_n}(t_1x_1,\ldots,t_nx_n)=(-1)^\pi\lim_{\beta\to\infty}\lim_{\Lambda\to\infty}\cdot \tag{2.4}$$
$$\cdot\frac{\mathrm{Tr}\;\left(e^{-(\beta-t_{\pi(1)})H}\psi^{\sigma_{\pi(1)}}_{x_{\pi(1)}}e^{-(t_{\pi(1)}-t_{\pi(2)})H}\psi^{\sigma_{\pi(2)}}_{x_{\pi(2)}}\cdots\psi^{\sigma_{\pi(n)}}_{x_{\pi(n)}}e^{-t_{\pi(n)}H}\right)}{\mathrm{Tr}\;e^{-\beta H}}\;,$$

where $\sigma_i=\pm1$, π is the permutation of $(1,\ldots,n)$, such that $t_{\pi(1)}>t_{\pi(2)}>\ldots>t_{\pi(n)}$ and $(-1)^\pi$ is the permutation parity; n is even. In this way $\chi_{p_F}(p)$ is the Fourier transform of $S_{-+}((0^-,x),(0,0))$.

§3. Functional integration for 1–dim spinless fermions. Luttinger–Kohn–Ward determination of the chemical potential.

Denoting $\xi=(x_0,x),\eta=(y_0,y)\in R^2,\kappa=(k_0,k)\in R^2$ the Schwinger functions introduced in §2 can be usefully expressed as functional integrals

$$S_{\sigma_1\cdots\sigma_n}(\xi_1,\ldots,\xi_n)=\lim_{\beta\to\infty}\lim_{\Lambda\to\infty}\cdot$$
$$\cdot\frac{\int P(d\psi)\;e^{-\lambda\int_{-L/2}^{L/2}\int_0^\beta\psi^+_\xi\psi^-_\xi\,\psi^+_\eta\psi^-_\eta\,\delta(x_0-y_0)v(x-y)d\xi d\eta}\;\psi^{\sigma_1}_{\xi_1}\cdots\psi^{\sigma_n}_{\xi_n}}{\int P(d\psi)e^{-\lambda\int_{-L/2}^{L/2}\int_0^\beta\cdots\cdots}} \tag{3.1}$$

where the $\psi^\pm_\xi$ are "*Grassmanian variables*", see for instance sec. 3.1 in [BG95]. The "integration" with respect to $P(d\psi)$ is defined on monomials $\psi^+_{\xi_1}\ldots\psi^-_{\xi_{2n}}$ by assigning the value of $\langle\psi^+_{\xi_1}\ldots\psi^-_{\xi_{2n}}\rangle\stackrel{def}{=}\int P(d\psi)\,\psi^+_{\xi_1}\ldots\psi^-_{\xi_{2n}}$ via Wick's rule with propagators $\langle\psi^-_\xi\psi^-_\eta\rangle=\langle\psi^+_\xi\psi^+_\eta\rangle=0$ and

$$\begin{aligned}&\langle\psi^-_\xi\psi^+_{\xi'}\rangle=g(\xi-\xi')\\ &g(\xi)=\sum_{n\in Z^1,\,n_0\in Z^1}(-1)^{n_0}\overline{g}(x_0+n_0,x+nL,\beta)\;,\\ \overline{g}(\xi)=&\frac{1}{(2\pi)^{d+1}}\int\frac{e^{-ip_0x_0-ipx}}{-ip_0+(p^2-p_F^2)/2}dp_0dp\end{aligned} \tag{3.2}$$

so that $g(\xi)$ is periodic in x with period L and antiperiodic in x_0 with period β.

The integration is extended linearly to even polynomials and to functions of the Grassmanian fields that admit an entire even power series expansion, like the exponential in (3.1).

In this way the integrals in the numerator and in the denominator of (3.1) are defined as formal power series in λ. The series can be quite easily shown to be convergent for $|\lambda|<O(L^{-1})$ and one of the goals is to find conditions under which they can be analytically continued to values of λ which are, possibly, small but independent of the size L of the system.

If we insist in fixing *a priori* the chemical potential the first effect of the interaction will be that the singularity of the Fourier transform of $\langle\tilde\psi^+_x\tilde\psi^-_0\rangle$, see (1.3), will no longer be where it is located when $\lambda=0$, namely at $k=\pm p_F=\pm\sqrt{2\mu}$, but (reasonably) it will be shifted by $O(\lambda)$.

It is, therefore, more convenient to *keep the location of the singularity fixed* at a prefixed value p_F. This can be achieved by taking a λ–dependent chemical potential $\mu=\frac12p_F^2+\nu$, see (2.1), where ν has to be conveniently chosen as a function of λ.

This means that instead of (3.1) one studies the same expression with the argument of the exponential modified into

$$-\nu\int_{-L/2}^{L/2}\int_0^\beta \psi^+_\xi\psi^-_\xi\,d\xi-\lambda\int_{-L/2}^{L/2}\int_0^\beta \psi^+_\xi\psi^-_\xi\psi^+_\eta\psi^-_\eta\,\delta(x_0-y_0)v(x-y)d\xi d\eta \tag{3.3}$$

In a series of basic papers, [Lu60], [KL60], [LW60], Luttinger, Kohn and Ward point out that this determines ν as a power series in λ and it has the important effect of generating a power series for the Schwinger functions which is finite to all orders, uniformly in the size of the system: had we fixed the chemical potential μ rather than the Fermi momentum p_F we would have obtained a power series in λ with coefficients diverging to all non trivial orders as $L\to\infty$.

Of course the latter divergence does not mean that the theory with a fixed chemical potential cannot be defined: it simply means that such a theory will have Schwinger functions with a singularity at a Fermi momentum which is different from $\sqrt{2\mu}$ so that the expansion with reference to a free field with singularity at $\sqrt{2\mu}$ contains diverging expressions. It makes also clear that it is likely to be more convenient to develop a perturbation theory of the ground state at fixed Fermi momentum rather than at fixed chemical potential. Should one wish to study the problem at fixed chemical potential μ, after developing the theory at fixed p_F and obtaining the chemical potential correction ν in terms of λ, p_F, one can imagine, that $\mu = p_F+\nu(p_F,\lambda)$ and solve this relation for p_F, as a function of μ,λ

A related important result due to Luttinger (*"Luttinger's theorem"*) states that *fixing the Fermi momentum is equivalent to fixing the density which is $\rho=\pi p_F$* whether $\lambda=0$ or not: this was shown by Luttinger, [Lu60], to hold to all orders of perturbation theory. A mathematically rigorous proof would be desirable.

§4. The ultraviolet problem.

If p_0^{-1} is the range of the interaction potential and $\varepsilon(p)=(p^2-p_F^2)/2$ the propagator (3.2) can be written as sum of two terms

$$\begin{aligned}&\overline{g}(\xi)=g^{(>0)}(\xi)+g^{(\le 0)}(\xi)\\ &g^{(>0)}(\xi)=\frac{1}{(2\pi)^2}\int\frac{(1-e^{-(q_0^2+\varepsilon(q)^2)/p_0^4})\,e^{-i\kappa\cdot\xi}}{-iq_0+\varepsilon(q)}d\kappa\\ &g^{(\le 0)}(\xi)=\frac{1}{(2\pi)^{d+1}}\int\frac{e^{-(q_0^2+\varepsilon(q)^2)/p_0^4}\,e^{-i\kappa\cdot\xi}}{-iq_0+\varepsilon(q)}d\kappa\end{aligned}\tag{4.1}$$

where $\xi=(x_0,x)$, $\kappa=(q_0,q)$. The term $g^{(>0)}$ is the "ultraviolet component of the propagator" and $g^{(\le0)}$ is the "infrared component".

The decomposition can be used to introduce two auxiliary "independent" Grassmanian fields $\psi^{(>0)}_\xi$ and $\psi^{(\le0)}_\xi$ so that $\psi_\xi=\psi^{(>0)}_\xi+\psi^{(\le0)}_\xi$. This means that in evaluating the integrals in (3.1) we can replace ψ_x with $\psi^{(>0)}_\xi+\psi^{(\le0)}_\xi$ and perform the Grassmanian integration following the Wick's rule with propagators

$$\langle\psi^{(\alpha)-}_\xi\psi^{(\alpha)+}_\eta\rangle=g^{(\alpha)}(\xi-\eta)\qquad \alpha=(>0),(\le0)\tag{4.2}$$

while all the other propagators vanish. Calling $V(\psi)$ the expression (3.3) this is also written as the identity ("Fubini's theorem" for Grassmanian integrals)

$$\frac{\int P(d\psi)\,e^{V(\psi)}\psi^{\sigma_1}_{\xi_1}\cdots}{\int P(d\psi)\,e^{V(\psi)}}\equiv$$
$$\equiv\frac{\int P(d\psi^{(\le0)})P(d\psi^{(>0)})\,e^{V(\psi^{(>0)}+\psi^{(\le0)})}(\psi^{(>0)\sigma_1}_{\xi_1}+\psi^{(\le0)\sigma_1}_{\xi_1})\cdots}{\int P(d\psi)\,e^{V(\psi^{(>0)}+\psi^{(\le0)})}} \tag{4.3}$$

Remark: attention should be paid to the fact that (4.1) deals with the "infinite volume" ($\beta\to\infty, L\to\infty$) limit $\overline{g}(\xi)$ rather than with the $g(\xi)$ of (3.2). Hence one should really deal with g and decompose the latter into the sum $g^{(>0)}(\xi)+g^{(\le0)}(\xi)$. This generates a great variety of "small" problems both in the ultraviolet and, later, in the infrared analysis. There is no few words way out of this (well known) difficulty. Here we choose to ignore it except for a few necessary comments when needed, because it is discussed widely in the literature, see [BG95], [BM00a].

The idea is to perform first the integral over the "high frequency part" $\psi^{(>0)}$ of the field both in the numerator and in the denominator of (4.3). Fixing attention on the denominator (simpler than the numerator which, however, can be treated in the same way) the result will be written

$$\int P(d\psi)\,e^{V(\psi)}=\int P(d\psi^{(\le0)})\,e^{\tilde V^{(0)}(\psi^{(\le0)})} \tag{4.4}$$

which in fact is a definition of $\tilde V^{(0)}$, the "effective potential on scale p_0".

Technically there is a lot of work behind the latter relation: one has to show that the result of the integration of the field $\psi^{(>0)}$ can be written in the form of an exponential of an effective potential $\tilde V^{(0)}$: this means showing that the result can be expressed as an exponential of

$$\tilde V^{(0)}(\psi^{(\le0)})=\sum_{m\ge0,p\le m}^{\infty}\int\frac{d\xi_1\ldots d\xi_m}{m!}\tilde V^{(0)}_{m,p}(\xi_1\ldots\xi_m)\cdot$$
$$\cdot\,\psi^{(\le0)+}_{\xi_1}\ldots\psi^{(\le0)-}_{\xi_p}\partial_{\xi_{p+1}}\psi^{(\le0)+}_{\xi_p}\ldots\partial_{\xi_m}\psi^{(\le0)-}_{\xi_m} \tag{4.5}$$

and the kernels $\tilde V^{(0)}_{m,p}(\xi_1\ldots\xi_m)$ are

(i) *analytic in* λ,ν.

(ii) *decay exponentially* on scale p_0: which means that they are bounded by $C_n\exp -cp_0^{-1}d(\xi_1,\ldots\xi_{2n})$ where $d(\xi_1,\ldots\xi_{2n})$ is the length pf the shortest path connecting the points $\xi_1,\ldots\xi_{2n}$ in R^{2n} and $c>0, C_n$ are suitable constants.

Note that already the $V(\psi)$, *cf.* (3.3) can be cast in the form (4.5). Furthermore the Grassmanian integral $P(d\psi)$ can be written as the "Lebesgue Grassmanian integral" $d\psi^+d\psi^-$ times an exponential

$$P(d\psi)=const\,e^{-\int_{-L/2}^{L/2}\int_0^\beta dx\,dx_0\,\psi^+_\xi(\partial_{x_0}-\frac12(\partial_x-p_F^2))\psi^-_\xi}\;d\psi^+d\psi^- \tag{4.6}$$

where $d\psi^+d\psi^-$ is the Grassmanian integration with "trivial propagators', *i.e.* with the only non vanishing propagator given by $\langle\psi^-_\xi\psi^+_\eta\rangle=\delta(\xi-\eta)$. Therefore the argument of the exponent can also be written in the form (4.5) (involving only derivatives of first order at most: as the second derivative can be integrated by parts): a property that turns out to be quite important.

Remark: the difficulty mentioned in the remark following (4.3) shows up very clearly here. The integrals in the exponential in (4.5) should be over the rectangle $[-L/2,L/2]\times[0,\beta]$ with periodic–antiperiodic boundary conditions. However they are extended to the whole R^2: this means that we have implicitly taken the limits in (3.1). However strictly speaking this does not make sense unless we explain what it means to integrate over the whole space a monomial in Grassmanian variables. The correct interpretation is the following:

(a) keeping β, L finite and using the propagator $g(\xi)$ in (3.2) one performs the integration over the ultraviolet components $\psi^{(>0)}$ and one obtains (4.4) with a $\tilde V^{(0)}$ similar to (4.5) but with the coefficient kernels which now depend on L, β.

(b) the kernels converge to limits as $\beta, L \to \infty$ and the limits $\tilde V^{(0)}_{m,p}(\xi_1 \ldots \xi_m)$ verify the properties stated after (4.5) (uniformly in β, L).

(c) the identity (4.4) means that if one wishes to compute

$$\frac{\int P(d\psi^{(>0)})P(d\psi^{(\le 0)})e^{V(\psi)}\psi^{(\le 0)\sigma_1}_{\xi_1}\psi^{(\le 0)\sigma_2}_{\xi_2}\ldots}{\int P(d\psi^{(>0)})P(d\psi^{(\le 0)})e^{V(\psi)}} \tag{4.7}$$

then one can "simply" compute

$$\frac{\int P(d\psi^{(\le 0)})e^{\tilde V^{(0)}(\psi^{(\le 0)})}\psi^{(\le 0)\sigma_1}_{\xi_1}\psi^{(\le 0)\sigma_2}_{\xi_2}\ldots}{\int P(d\psi^{(\le 0)})e^{\tilde V^{(0)}(\psi^{(\le 0)})}} \tag{4.8}$$

by developing the $e^{\tilde V^{(0)}}$ in powers of the fields and then apply Wick's rule with propagator $g^{(\le 0)}$ obtaining in this way a combination of integrals of products of the kernels in (4.5). And the series converges.

(d) all divergences due to the infinite extension of the domains of integration over the ξ variables disappear when one considers the ratio in (4.3) because of the fast decay of the kernels. As one may suspect there will also be exchange of limits queries, which are solved again by using the uniformity of the estimates. See [BGPS94], for instance, and [BM00a].

Performing the "remaining integral" is therefore similar to the original problem except that now the expression $\tilde V^{(0)}$ is more involved but the propagator is simpler (being "just" $g^{(\le 0)}$, *i.e.* a propagator with an ultraviolet cut off at scale p_0^{-1}).

In fact the decay of the coefficients $\tilde V^{(0)}_m$ make the $\tilde V^{(0)}$ "essentially local" and the "remaining integral" is not really harder than the one obtained by replacing $\tilde V^{(0)}$ by

$$\begin{aligned} V^{(0)}(\psi^{(\le 0)}) &= \nu_0 \int_{-L/2}^{L/2}\int_0^\beta \psi^{(\le 0)+}_\xi \psi^{(\le 0)-}_\xi \, d\xi + \\ &+ \lambda_0 \int_{-L/2}^{L/2}\int_0^\beta \psi^{(\le 0)+}_\xi \psi^{(\le 0)-}_\xi \psi^{(\le 0)+}_\eta \psi^{(\le 0)-}_\eta \, w(\xi-\eta)\, d\xi\, d\eta \end{aligned} \tag{4.9}$$

where w is a smooth potential with range p_0^{-1}.

In fact in most treatments the above analysis is considered "trivial" and one just poses the problem of studying the ratio of integrals

$$\frac{\int P(d\psi^{(\le 0)})\, e^{-V^{(0)}(\psi^{(\le 0)})}\,\psi^{(\le 0)\sigma_1}_{\xi_1}\ldots}{\int P(d\psi^{(\le 0)})\, e^{-V^{(0)}(\psi^{(\le 0)})}} \tag{4.10}$$

with $V^{(0)}$ given by (4.9).

It should be noted that the ultraviolet problem does not even arise when the fermions are supposed to be located on a lattice because in that case there is a minimum length scale, hence a maximum momentum.

For this reason I concentrate here on the infrared problem, modeled by (4.10). However things are not so simple in the case of the Luttinger model: in that model the inverse propagator diverges at large momenta linearly rather than quadratically. This generates a non trivial ultraviolet problem: it can be treated in a way analogous to the above, but one has to exhibit various cancellations because the expressions for the kernels of the effective potential $V^{(0)}$ are given by apparently non convergent integrals. The analysis for the (harder) Luttinger model case is carried out in detail in [GS93].

§5. The infrared problem and quasi particles.

The infrared problem (*i.e.* understanding the large distance properties of the Schwinger functions, or equivalently the singularities at finite momentum of their Fourier transforms) is more interesting and rich in structure. A naive application of perturbation theory leads to facing the fact that the propagators oscillate on scale p_F^{-1} and decay slowly at infinity.

Quasi particles arise when one attempts to disentangle the oscillations and the decay at ∞. Technically one remarks that the propagator for the functional integrals with respect to $P(d\psi)$ can be written, setting $\xi=(t,\underline{x})$ and $\kappa=(k_0,\underline{k})$ in general dimension $d\ge 1$, as

$$\begin{aligned} g(\xi)=&\int \frac{d^{d+1}q}{(2\pi)^{d+1}}\frac{e^{-i(q_0t+\underline{q}\cdot\underline{x})}}{-iq_0+(\underline{q}^2-p_F^2)/2m}=\int d\underline{\omega}\, e^{-ip_F\,\underline{\omega}\,\underline{x}}g(\underline{x},t,\underline{\omega})=\\ =&\int e^{-ip_F\,\underline{\omega}\,\underline{x}}\,d\underline{\omega}\int\frac{e^{-ik_0t+\underline{k}\cdot\underline{x}}}{\varepsilon(\underline{k},\underline{\omega})}\frac{d^{d+1}\kappa}{(2\pi)^{d+1}}\end{aligned}\tag{5.1}$$

where $\underline{\omega}$ is a unit vector and $d\underline{\omega}$ is the integration over the unit sphere in R^d normalized to 1 (if the dimension $d=1$ integrating over $\underline{\omega}$ means $2^{-1}\sum_{\omega=\pm}\cdot$). An elementary calculation shows that (at least if d is odd) $g(\underline{x},t,\underline{\omega})$, *which is not unique*, can be defined so that $\varepsilon(k,\underline{\omega})=-ik_0+v_F\,\underline{\omega}\cdot\underline{k}+O(\underline{k}^2)$ (see §5 and appendix A in [BG90]).

This means that the free fermion system in a ground state with Fermi momentum at p_F can be considered as a system of *"quasi particles"* in the vacuum carrying an "*intrinsic*" linear momentum equal to a Fermi sphere momentum $p_F\,\underline{\omega}$ in addition to the "*external*" momentum $\underline{k}$. The *dispersion relation* is *almost* linear in the sense that the system on large scales, *i.e.* κ small, will show a dispersion relation essentially identical to $\varepsilon(\underline{k},\underline{\omega})=\underline{\omega}\cdot\underline{k}\,v_F$: this property seems to remain valid even in presence of interaction so that the intuition can be led by the idea that the *quasi particles* must be taken seriously, see sec. 5 in [BG90]. Formally (5.1) can be regarded, *cf.* also (1.2), as the propagator of a composite field defined as

$$\psi^{\pm}_{\underline{x},t}=\int d\underline{\omega}\,e^{\pm ip_F\,\underline{\omega}\,\underline{x}}\psi^{\pm}_{\underline{x},t,\underline{\omega}}\tag{5.2}$$

with the fields $\psi^{\pm}_{\underline{x},t,\underline{\omega}}$ being Grassmanian fields with propagators $g(\xi,\underline{\omega})$.

Remark: It is important to stress that the above fields are just Grassmanian fields rather than the usual fermionic fields. If we wanted, instead, to think of the quasi particles fields as Grassmanian fields which correspond to *physical particles it would be necessary that the propagators $g(\xi,\underline{\omega})$ have the "reflection positivity" property*, [Si74]: and this would also be a criterion to fix the arbitrariness in the choice of the representation (5.1) mentioned above. However it is not known whether such a choice is possible. The success of the quasi particles–based view of the theory of the ground states of fermions leads us to feel that this might be possible.

It is difficult to give a meaning to the simple and captivating *approximation*

$$g(\xi,\underline{\omega})=\int\frac{d^{d+1}\kappa}{(2\pi)^{d+1}}\frac{e^{-i(k_0t+\underline{k}\,\underline{x})}}{-ik_0+v_F\,\underline{\omega}\cdot\underline{k}}\tag{5.3}$$

if $d>1$ because it corresponds to a system of fermions with a linear dispersion relation which *therefore* will give rise to ultraviolet instabilities: possibly when the potential is repulsive and certainly when it is attractive. Note, however, that it makes sense if $d=1$ because in this case the system is stable *even* with a linear kinetic energy: in fact it becomes almost precisely the Luttinger model (it differs from it because of the presence of "extra" cross terms in the interaction, *cf.* comment following (1.8)).

However (5.3) gives a good representation of the propagators *for small* κ. Therefore the approximation can be expected to be reasonable in the sense that its version with an ultraviolet cut–off, *i.e.*

$$g^{(\le 0)}(\xi,\underline{\omega})=\int\frac{d^{d+1}\kappa}{(2\pi)^{d+1}}e^{-i(k_0t+\underline{k}\,\underline{x})}\frac{e^{-p_0^{-4}(k_0^2+v_F^2\,\underline{k}^{\,2})}}{-ik_0+v_F\,\underline{\omega}\cdot\underline{k}},\tag{5.4}$$

can be used to study the infrared problems presented by the (4.9), (4.10).

The papers [BG90] and [BGPS94] consider the integrals in (4.10) with a propagator (5.4) and an interaction $V^{(0)}$ even simpler than the one in (4.9), namely

$$\begin{aligned}V^{(0)}(\psi^{(\le 0)})=&\ \nu_0\int_{-L/2}^{L/2}\int_0^\beta\sum_{\omega=\pm}\psi^{(\le 0)+}_{\omega,\xi}\psi^{(\le 0)-}_{\omega,\xi}\,d\xi+\\&+\lambda_0\int_{-L/2}^{L/2}\int_0^\beta\psi^{(\le 0)+}_{+,\xi}\psi^{(\le 0)-}_{+,\xi}\psi^{(\le 0)+}_{-,\xi}\psi^{(\le 0)-}_{-,\xi}\,d\xi\end{aligned}\tag{5.5}$$

where several crossed terms have been eliminated and the interaction has been made strictly local (which, in the spinless case, is possible only if we think of the system as made with quasi particles because the exclusion principle makes local interactions which are quartic in spinless fermion fields vanish identically).

In the quoted papers it is shown in detail that if $d=1$ the understanding of the integrals in (4.10) with a propagator (5.4) and an interaction $V^{(0)}$ given by (5.5) suffice to solve in a mathematically complete way the problem of the ground state of (1.1) and to obtain Tomonaga's main result that the anomaly at the Fermi surface is not 0 and in fact it is an analytic function of the coupling λ. Of course this is no surprise because of the works [Lu63], [ML65].

Why to redo Tomonaga's work in mathematically precise way? the point is that the notion of "understanding" and of "proof" of a physical result evolve. And many of the results that were considered established at the time ($\sim$ 1950) have come under scrutiny and have been cast into a more rigorous form. This is necessary mostly because the attempts to extend them to other cases (namely higher dimension or even one dimension with spin, or just with more structure, see question (2) in §8) have failed and therefore it becomes necessary to have a clear idea of what really fails and what can still be usefully taken over to attack harder problems. This necessity is similar to the need that became evident in the 1960's to have a more rigorous foundation of the theory of ensembles in equilibrium statistical mechanics. It should be clear that revisiting Tomonaga's theory in no way implies that there are faults in the original work: it simply did not deal with questions that at the time were, rightly, not considered important.

§6. Renormalization group and the infrared problem.

We consider the problem of computing the "partition function", *i.e.* the problem of studying the integral in the denominator of (4.9) with a propagator given by (5.4) and an interaction given by (5.5). We define

$$g^{(h)}(\xi,\omega) \stackrel{def}{=} \int \frac{d\kappa}{(2\pi)^2} \frac{\big(e^{-2^{-2h}\kappa^2} - e^{-2^{-2(h-1)}\kappa^2}\big)}{-ik_0 + v_F\,\omega\,k} e^{i(k_0 t + \omega\, kx)} = 2^h g^{(0)}(2^h\xi,\omega) \tag{6.1}$$

$h=0,-1,-2,\ldots$, $\omega=\pm$, $\kappa=(k_0,k)$, $\kappa^2=(k_0^2+v_F^2k^2)/p_0^4$. Therefore if $Z_0 \stackrel{def}{=} 1$

$$\frac{1}{Z_0} g^{(\le 0)}(\xi,\omega) = \frac{1}{Z_0}\sum_{h=0}^{-\infty} 2^h\, g^{(h)}(2^h\xi,\omega) \stackrel{def}{=} \frac{1}{Z_0} g^{(0)}(\xi,\omega) + \frac{1}{Z_0} g^{(\le -1)}(\xi,\omega) \tag{6.2}$$

and the integration (4.10) can be thought of as an integration over quasi particle fields $\psi^{(\le 0)}_{\omega,\xi}$ which are decomposable as the sum of two fields $\psi^{(0)}_{\omega,\xi}$ and $\psi^{(\le -1)}_{\omega,\xi}$ with propagators $\frac{1}{Z_0} g^{(0)}(\xi,\omega)$ and $\frac{1}{Z_0} g^{(\le -1)}(\xi,\omega)$:

$$\int P(d\psi^{(\le 0)}) e^{V^{(0)}(\sqrt{Z_0}\psi)} \equiv \int P_{Z_0}(d\psi^{(0)}) P_{Z_0}(d\psi^{(\le -1)}) e^{V^{(0)}(\sqrt{Z_0}(\psi^{(0)}+\psi^{(\le -1)}))} \tag{6.3}$$

This is convenient because "we know" how to compute the integral over $\psi^{(0)}$ by perturbation theory. This means that the techniques to study the integral although not trivial are, nevertheless, well established mainly via the results in [Le87] which provide us with a technique designed to take advamtage of the fermionic nature of the fields. The result is the expected one: one can basically compute to second order of perturbation in λ and neglect the rest. More precisely one can prove that the result of the integration (in the sense discussed in the remark following (4.5)) is

$$\begin{aligned} &\int P_{Z_0}(d\psi^{(\le -1)}) e^{\overline{V}^{(0)}(\sqrt{Z_0}\psi^{(\le -1)})}, \qquad \text{with} \qquad \overline{V}^{(0)}(\sqrt{Z_0}\psi^{(\le -1)}) = \\ &= \sum_{n\ge 0,\alpha} \int \sqrt{Z_0}^{\,n} V_\alpha(\xi_1,\ldots,\xi_n)\Phi_{\xi_1}\cdots\Phi_{\xi_n}\, e^{ip_F\sum_i \sigma_i \underline{\omega}_i x_i} \frac{d\xi}{n!} \end{aligned} \tag{6.4}$$

where Φ_{ξ_i} is either $\psi^{\sigma_i}_{\xi_i}$, $\sigma_i=\pm$, or a derivative of this field; and α denotes the labels σ_i as well as the labels necessary to identify which of the fields Φ are differentiated and which are not; we also suppose that the ψ^+ fields are to the left of the ψ^- (recall, however, that the fields are Grassmanian so that they anticommute). The number n must be, obviously, even.

The kernels $V_\alpha(\xi_1,\ldots,\xi_n)$ are analytic functions of λ_0,ν_0 convergent with a convergence radius which *is independent of the sizes L,β of the system* and decay exponentially fast on scale p_0^{-1}, in the sense following (4.5): the technique for this proof is in [Le87] and the analysis can be found in [BGPS94].

It would not be wise to simply iterate the procedure calling $V^{(-1)}$ the "effective interaction" $\overline{V}^{(0)}$ and (with the procedure already used in the previous splitting, *cf.* (6.2)) splitting the field $\psi^{(\le -1)}$ into $\psi^{(-1)}+\psi^{(\le -2)}$: because the result, under further iteration, would be *a progressive worsening* of the bounds on the constants and a decrease in the convergence radius of the expansions in powers of $]l_0,\nu_0$. This is basically due to the fact that ν_0 *cannot be arbitrary* under the only condition that it is small: *it has to be tuned* so that the Schwinger functions have singularities at momentum $\pm p_F$.

The correct procedure will be to adjust recursively the value of ν_0 as a function of λ_0 *keeping the singularity at fixed momentum.* For this purpose one distinguishes the *relevant* part of the interaction $\overline{V}^{(0)}$ from the *irrelevant* part. The latter is *not* small (in a sense it is the most important part!): the relevant part is just a part of $\overline{V}^{(0)}$ whose absence would allow us to perform a naive recursion without loss in the size of the constants or of the convergence radii.

The identification of the relevant terms is a matter of dimensional analysis, at least in the simplest cases, see [Ga85], [BG95]. What we call here "relevant" is a set of terms (see below) which in the usual nomenclature of the renormalization group approaches is further divided into relevant and marginal terms. In the present case the relevant part is the "*local part*" of the terms which are quadratic and quartic in the fields. If $\delta_{a,b}$ denotes a Kronecker delta such local part is defined by

$$\begin{aligned}
\mathcal{L}\psi^+_{\xi_1\omega_1}\psi^+_{\xi_2\omega_2}\psi^-_{\xi_3\omega_3}\psi^-_{\xi_4\omega_4} &= \delta_{\omega_1+\omega_2+\omega_3+\omega_4,0}\,\frac{1}{2}\sum_{j=1}^{2}\psi^+_{\xi_j\omega_1}\psi^+_{\xi_j\omega_2}\psi^-_{\xi_j\omega_3}\psi^-_{\xi_j\omega_4}\\
\mathcal{L}\psi^+_{\xi_1\omega_1}\psi^-_{\xi_2\omega_2} &= \delta_{\omega_1,\omega_2}(\psi^+_{\xi_1\omega_1}\psi^-_{\xi_2\omega_2}+\psi^+_{\xi_1\omega_1}(\xi_2-\xi_1)\cdot D_\xi\,\psi^-_{\xi_2\omega_2})
\end{aligned}\tag{6.5}$$

where $D_\xi=(\partial_t,\,\partial_x)$.

Applying the operator $\mathcal{L}$ to the expression in (6.4) one realizes that the result can be espressed as a linear combination of the following Grassmanian monomials

$$\begin{aligned}
F_1 &= -\int \psi^+_{+,\xi}\psi^-_{+,\xi}\psi^+_{-,\xi}\psi^-_{-,\xi}d\xi\\
F_2 &= -\int\sum_{\omega=\pm}\psi^+_{\omega,\xi}\psi^-_{\omega,\xi}\,d\xi\ ,\\
F_3 &= -\int\sum_{\omega=\pm}\psi^+_{\omega,\xi}(-i\,v_F\,\omega\partial_x)\psi^-_{\omega,\xi}\,d\xi\ ,\\
F_4 &= -\int\sum_{\omega=\pm}\psi^+_{\omega,x}\partial_t\psi^-_{x\omega,x}\,d\xi
\end{aligned}\tag{6.6}$$

Note that there is only one possible non zero local term of fourth order in the field because of the Fermi statistics and the facts that $\omega=\pm1$ and that our fermions are spinless. Note also that the above localization operation would be completely different (*and useless*) if we had not introduced the quasi particles: for instance F_1 would simply vanish because of the Fermi statistics as it would involve the square of Fermi fields.

Applying the operator $\mathcal{L}$ to $\overline{V}^{(0)}(\sqrt{Z_0}\psi^{(\le-1)})$ one obtains an expression

$$\begin{aligned}
&\overline{\lambda}^{(0)}\,Z_0^2F_1(\psi^{(\le-1)})+\overline{\nu}^{(0)}Z_0F_2(\psi^{(\le-1)})+\\
&+\overline{\alpha}^{(0)}Z_0F_3(\psi^{(\le-1)})+\overline{\zeta}^{(0)}Z_0F_4(\psi^{(\le-1)})+(1-\mathcal{L})\overline{V}^{(0)}
\end{aligned}\tag{6.7}$$

where $\overline{\lambda}^{(0)},\overline{\nu}^{(0)},\overline{\alpha}^{(0)},\overline{\zeta}^{(0)}$ are simple combinations of integrals of the kernels, see (6.4), for the monomials quadratic or quartic in the fields and are therefore analytic in λ_0,ν_0.

The simplest way to proceed is to consider a representation of $P_{Z_0}(d\psi)$ analogous to (4.6) for the integrations over the quasi particle fields, with the quadratic form $\psi_\xi^{(\le -1)+}(\partial_t+(-\frac12\partial_x^2-\frac12 p_F^2))\psi_\xi^{(\le -1)-}$ replaced by $Z_0\sum_{\omega=\pm}\psi_{\omega,\xi}^{(\le -1)+}(\partial_t - iv_F\omega\partial_x)\Gamma_{-1}(\underline{\partial})\psi_{\omega,\xi}^{(\le -1)-}$ where $\Gamma_h(\underline{\partial})$ is the operator which multiplies the Fourier transforms by $\Gamma_h(\kappa)=e^{+(2^h p_0)^{-2}(k_0^2+v_F^2k^2)}$: indeed this choice attributes to the field $\psi_{\omega,\xi}^{(\le -1)-}$ the correct propagator $g^{(\le -1)}(\xi)$, *cf.* (6.2). Then the integral to be performed looks like

$$\begin{aligned}
&const\int e^{-Z_0\sum_{\omega=\pm}\int\psi_{\omega,\xi}^{(\le-1)+}(\partial_t-iv_F\omega\partial)\Gamma_{-1}(\underline{\partial})\psi_{\omega,\xi}^{(\le-1)-}d\xi}\cdot\\
&\cdot e^{\overline{\nu}_0 Z_0F_2(\psi^{(\le-1)})+\overline{\lambda}^{(0)}Z_0^2F_1(\psi^{(\le-1)})+(1-\mathcal{L})\overline{V}^{(0)}}\cdot\\
&\cdot e^{\overline{\alpha}^{(0)}Z_0F_3(\psi^{(\le-1)})+\overline{\zeta}^{(0)}Z_0F_4(\psi^{(\le-1)})}\,d\psi^+d\psi^-
\end{aligned}\tag{6.8}$$

and it is natural to collect the quadratic parts defining

$$\begin{aligned}
&Z_{-1}\stackrel{def}{=}Z_0+\overline{\alpha}_0Z_0, &&\zeta_{-1}Z_{-1}\stackrel{def}{=}(\overline{\zeta}^{(0)}-\overline{\alpha}_0)Z_0\\
&\lambda_{-1}Z_{-1}^2=\overline{\lambda}^{(0)}Z_0^2, &&\nu_{-1}Z_{-1}=2\,\overline{\nu}_0Z_0
\end{aligned}\tag{6.9}$$

Here the factor 2 in front of the "relevant coupling" $\overline{\nu}_0$is natural: if the theory is developed only to first order (a deed called "power counting" in the jargon of the renormalization group, see eq. (6.18) in [BG95]) the result is that $Z_{-1}=Z_0$ and $\nu_{-1}=2\nu_0$. This transforms the integral into

$$\begin{aligned}
\int P_{Z_{-1}}(d\psi^{(\le-1)})\,e^{Z_{-1}\zeta_{-1}F_4(\psi^{(\le-1)})+Z_{-1}\nu_{-1}F_2(\psi^{(\le-1)})}\cdot\\
\cdot e^{Z_{-1}^2\lambda_{-1}F_1(\psi^{(\le-1)})+W_h(\sqrt{Z_{-1}}\psi^{(\le-1)})}
\end{aligned}\tag{6.10}$$

The procedure can now be iterated and the integrals can be performed in the same way defining recursively λ_h,ν_h,ζ_h called "*running couplings*", Z_h called "*wave function renormalization*" and W_h called "*irrelevant operators*" for $h=0,-1,-2\ldots$. The ratio Z_h/Z_{h+1} will be called the *"wave function renormalization rate"*.

Remarks: (i) Note that

$$\begin{aligned}
&W_h(\sqrt{Z_{-1}}(\psi^{(\le-1)}))=(1-\mathcal{L})\overline{V}^{(0)}(\sqrt{Z_0}\psi^{(\le-1)})+\\
&+\overline{\alpha}_0Z_0\int\psi_\xi^{(\le-1)+}(\partial_t-iv_F\omega\partial_x)(1-\Gamma_{-1}(\underline{\partial}))\psi_\xi^{(\le-1)-}\,d\xi
\end{aligned}\tag{6.11}$$

(ii) In [BG90], [BGPS94], [BG95] the definitions of W_h and/or of the localization operator $\mathcal{L}$ are slightly different: more involved but, possibly, more convenient for performing estimates. The choices differ, however, by "irrelevant terms" and are (therefore) equivalent.

The basic bound is that the running couplings on scale h and the kernels that are coefficients in the definition of W_h are analytic in $\underline{v}_{h'}=(\lambda_{h'},\nu_{h'},\zeta_{h'})$ for $h'>h$ under the condition that all the latter quantities and $|Z_{h'}/Z_{h'1}-1|$ for $h'>h$ are small enough independently on the size L of the system and also on the scale h.

The relation between the running couplings can be written, therefore, as

$$\begin{aligned}
\underline{v}_{h-1}&=M_h(\underline{v}_h+\overline{B}_h(\underline{v}_h,\underline{v}_{h+1},\ldots,\underline{v}_0;\frac{Z_h}{Z_{h+1}},\ldots)\ ,\\
1&=\frac{Z_h}{Z_{h-1}}\left(1+\overline{A}_h(\underline{v}_h,\underline{v}_{h+1},\ldots,\underline{v}_0;\frac{Z_h}{Z_{h+1}},\ldots)\right)\ ,
\end{aligned}\tag{6.12}$$

where M_h is a diagonal matrix with matrix elements $(Z_h/Z_{h-1})^2$, $2\,Z_h/Z_{h-1}$, Z_h/Z_{h-1} and the functions $\overline{B}_h, \overline{A}_h$ are analytic under the conditions that the running couplings and the renormalization rate are small enough independently of $h\le 0$.

Furthermore one can prove, [BG90], [BGPS94], that after eliminating the wave function renormalization rates in the first of (6.12) by using recursively the second equation the relation (6.12) can be written

$$\begin{aligned}&\frac{Z_h}{Z_{h+1}}=1+B'_h(\underline{v}_{h+1},\underline{v}_h,\ldots,\underline{v}_0)\\&\underline{v}_h=\Lambda\,\underline{v}_{h+1}+\underline{B}_h(\underline{v}_{h+1},\underline{v}_h,\ldots,\underline{v}_0)\end{aligned}\tag{6.13}$$

with B'_h, $\underline{B}_h$ analytic in their arguments $\underline{v}_{h'}$, under the conditions that the running couplings are small enough independently of $h\le 0$, and Λ is the diagonal matrix with diagonal $(1,1,2)$.

At this point the strategy is clear: the (6.13), called the *beta functional*, defines a "flow", called the *renormalization flow*, in the space of the running couplings $\underline{v}=(\lambda,\nu,\zeta)$ which is well defined as long as the running couplings themselves stay small enough in the above sense so that perturbation theory can be applied to pass from one scale to the next (*i.e.* as long as $|\underline{v}_h|$ is so small to be inside the convergence domain of the series expressing the functions B'_h, $\underline{B}_h$). The initial values are λ_0,ν_0,ζ_0: λ_0 will have to be taken small (so that the perturbation theory can be applied at least to perform the first integration, *i.e.* the integration over $\psi^{(0)}$), ν_0 also will have to be taken small and ζ_0 *must* be taken $\zeta_0=0$ (because we start the analysis from (5.5) which contains no term like F_4, see (6.6.).

Note that if we studied the problem without the approximation introduced after (5.5) (starting from the Hamiltonian (3.3)) we would end up after the ultraviolet integration with a $V^{(0)}$ which is much more involved than the expression (5.5) with which we began the infrared analysis. Therefore the localization operator acting on $V^{(0)}$ may (as indeed it does) produce also a term of the type F_4: which implies that ζ_0 will have a non zero value. This however does not help to provide us with more freedom because this ζ_0 is *not arbitrary* being a function of the couplings λ,ν in the original Hamiltonian (3.3): *it is intrinsic to the model that there are only two adjustable constants in the Hamiltonian.*

§7. The vanishing of the beta function: the role of the Luttinger model.

The only freedom that we have to "make things work" is the choice λ_0 (or of λ in the theory without approximations) small "enough" and the selection of ν_0 (or of ν in the theory without approximations).

What has been sketched until now is technically involved, but it is not really difficult because the techniques for a rigorous multiscale analysis has been established since a long time. In many problems, among which the d–dimensional Fermi systems, see [BG90], or the scalar field theories, see [Ga85], [BG95], *it is, in a sense, a pure matter of technical routine work to prove that the successive integration of the components of the fields on various scales can be reduced to the study of a renormalization flow like (6.12), under the assumption that the running constants stay small enough.*

In itself such a work *does not* really mean much unless one is able to exhibit a trajectory of the flow that keeps the running constants small enough to allow us to apply perturbation theory to compute the effective potentials from one scale to the next via convergent series for the kernels coefficients.

Remarks: (1) It is indeed very easy to work out alternative, substantially different, integration algorithms which work as beautifully as the one described above *but* *which are completely useless*: a nice and relevant example is obtained by keeping $Z_h \equiv 1$ and, correspondingly, not eliminating the quadratic terms with coefficients $\overline{\alpha}_h$ that are generated at each integration of a field component $\psi^{(h)}$. Everything works fine: *however one can prove that no initial data* λ_0, ν_0 *exist which keep the constants* $\lambda_h, \nu_h, \alpha_h, \zeta_h$ *small for all* h *if* $\alpha_0 = 0, \zeta_0 = 0$ *(or, in the theory without approximations, if* λ *and* ν *are the only free parameters)!*

(2) The only interest of the latter remark (1) lies in the fact that if it had been possible to find ν_0 as a function of λ_0 so that $\lambda_h, \nu_h, \alpha_h, z_h$ stayed small for all $h \le 0$ then it would follow (with some extra work, see [BGPS94]) that the singularity of the one particle Schwinger function at the Fermi surface would be a discontinuity, *as in the non interacting case*. This failure shows that the latter property cannot be derived as a consequence of a perturbation analysis, if true (it is not! as we know since Tomonaga's work).

(3) Of course the impossibility of a normal Fermi surface in generic 1–dimensional many fermions systems has been known since the early days of many body theory as a consequence of the divergence of the second order corrections to the Schwinger functions (divergence of the "self–energy" which occurs in dimension 1).

Therefore *after* the above (long and tiring) set up of the renormalization flow the real work starts.

It was well known that there were interesting conjectures about a number of cancellations that occur in the theory of one dimensional spinless fermions. In our language they can be summarized by saying that "the beta function effectively vanishes", see [So79].

It is very difficult however to find a proof by computing the beta functional to all orders. Introducing the function, called the "beta function",

$$\underline{\beta}(\underline{v}) \stackrel{def}{=} \lim_{h \to -\infty} \underline{B}_h(\underline{v}, \underline{v}, \ldots, \underline{v}), \qquad \underline{v} = (\lambda, \nu, \zeta) \in R^3 \tag{7.1}$$

a major simplification occurs by realizing, see [BG90], [BGPS94], that *there is a solution to the flow generated by the recursion (6.13) with* $|\underline{v}_h|$ *so small that all running couplings stay within the convergence domain of the functions* $\underline{B}_h$ *provided the components (relative to the contants* λ *and* ζ*)* β_1, β_2 *of the beta function* $\underline{\beta} = (\beta_1, \beta_2, \beta_3)$ *vanish.*

A proof of the latter property looks like an easier problem. However the first proof came from an indirect argument that we describe below.

(1) one first repeats the above analysis for the Luttinger model: this was assumed possible in [BG90], [BGM92], and later proved in [GS93]: the difficulty being essentially in the treatment of the ultraviolet problem.

(2) once the beta functional has been defined for the Luttinger model it has been remarked in [BG90] that although the beta functionals of the Luttinger model and the corresponding one of the "realistic" model in (1.1) are different the beta functions of the two models *coincide*, [BG90].

(3) one checks that if β_1 or β_2 did not vanish this would contradict the exact solution of the Luttinger model, [BG90], [BGM92], [BGPF94].

This allowed to prove the existence of a trajectory of the renormalization flow which stays close to the origin and within the radius of convergence of all the expansions in the running couplings considered (including that of the beta functional) thus completing the theory of the 1–dimensional spinless Fermni gas at small coupling yielding also the analyticity of the anomaly exponent in λ.

The third component of the beta function does not vanish: this is however not necessary. In fact the constant ν_h tends to diverge and very fast ($O(2^{-h})$): but this can be easily counterbalanced because the initial value of ν_0 is free and we can tune it so that ν_h does not diverge. In fact there can be only one value of ν_0 which has this property (essentially because, near a hyperbolic equilibrium attractive or marginal along the λ–axis and repulsive along the ν–axis, given λ small enough one can find only one value of ν small such that the evolution of the datum λ,ν stays close to the origin forever).

§8. Outlook.

One can wonder whether a simpler symmetry argument can be given to prove that the beta function β_1,β_2 components vanish: after [BG90] this has been attempted in [MD93] who however neglect the (necessary) presence of the decreasing cut–off $2^h p_0$, for $h\to-\infty$, arising in the successive integrations of the infrared scales.
Nevertheless such a proof must be possible and recently new attempts at finding it are being considered, [BM00b]: this would be a major achievement which, however, would not diminish the importance of the idea of Luttinger to enucleate out of Tomonaga's model a much simpler model that really catches all its main features. The existence of the indirect proof of the vanishing of the beta function based in an essential way on the [ML65] solution of the Luttinger model, is an important factor in the search of the proof based on symmetry considerations.

Mainly through the work of Mastropietro and collaborators the Luttinger model (together with its extension by Mattis, [Ma64]) has received several other applications in the frame of the renormalization group approach to the ground state of Fermi systems; for a complete review see [GM00] where, among other results, the interesting phenomenon of the λ dependence of the anomalies in the higher Schwinger functions is summarized as well as the results of the theory of one dimensional fermions in a periodic or *quasi periodic* potential (with Fermi momentum p_F in the bands or at the top of a band).

Another somewhat unexpected development has been the determination of the asymptotic behavior of the correlation functions in eight vertex models or XYZ models in a magnetic field for values of the parameters that do not correspond to exactly soluble points, [Ma99b], [BM00a]. A corollary of the latter papers should be the determination of the critical exponents of Ising type models even with non nearest neighbour interactions, see [Ma99b]. Applying the remark in [Sp99] concerning the translation of the next neighbor 2–dimensional Ising model into a fermionic spinless 1–dimensional problem it could lead to a strong extension of the theory of the critical point in such model covering a wide range of other models which (unlike the case considered in [Sp99]) show non universal critical exponents, see also remarks to sec. 1.6 of [BM00a], [Ma00] and the review [GM00]. Furthermore the technique used in [Ma99b], [BM00a] is quite different from that in [Sp99].

I conclude by mentioning a few open problems for small coupling:

(1) spinning fermions on a line with attractive interaction. See [BM95] for the repulsive case
(2) spinless fermions on two parallel lines with attractive interaction, see [Ma99a] for the repulsive case.
(3) two and three dimensional Fermi surface properties(!).
(4) a mathematical proof of the identity $\rho=\pi p_F$ between the density ρ and the Fermi momentum p_F, see [Lu60] and §3 above, at small coupling λ.
and it is, perhaps, surprising to find the first two in a list of open problems, while (3) is well known to require new ideas to be understoon and (4) seems understandable with the techniques that we know.

We can say that the work of Luttinger [Lu63] proposing the exactly soluble model bearing his name and providing some of the ideas that contributed to its exact solution in [ML65] has been, and remains, a landmark in Condensed Matter Physics (with its clarifying function of Tomonaga's work) and in Mathematical Physics (as an example of how to build a model that catches all the relevant features of a realistic model and, yet, it is more tractable) and it has shed new light on the difficult subject of the theory of low temperature quantum systems.

References

[BG90] *Benfatto, G., Gallavotti, G.: Perturbation theory of the Fermi surface in a quantum liquid. A general quasi particle formalism and one dimensional systems*, J. Statistical Physics, **59**, 541-660, 1990. See also [BGPS94].

[BG95] *Benfatto, G., Gallavotti, G.: Renormalization group*, p. 1–143, Princeton University Press, 1995.

[BGM92] *Benfatto, G., Gallavotti, G., Mastropietro, V.: Renormalization group and the Fermi surface in the Luttinger model*, Physical Review **B 45**, 5468- 5480, 1992.

[BGPS94] *Benfatto, G., Gallavotti, G., Procacci, A., Scoppola, B.: Beta function and Schwinger functions for many fermion systems in one dimension. Anomaly of the Fermi surface*, Communications in Mathematical Physics **160**, 93–172, 1994.

[BM00a] *Benfatto, G., Mastropietro, V.: Renormalization group, hidden symmetries and approximate Ward identities in the XYZ model,* I, II, in print on Communications in Mathematical Physics, preprint 2000, in `http://ipparco. roma1.infn.it`.

[BM00b] *Benfatto, G., Mastropietro, V.:* private communication.

[BM95] *Bonetto, F., Mastropietro, V. Beta function and anomaly of the Fermi surface for a* $d = 1$ *system of interacting fermions in a periodic potential*, Communications in Mathematical Physics, **172**, 57–93, 1995.

[FT90] *Feldman, J., Trubowitz, E.: Perturbation theory for many fermion systems*, Helvetica Physica Acta **63**, 157–260, 1990. And: *The flow of an electron-positron system to the superconducting state*, Helvetica Physica Acta **64**, 213–357, 1991.

[Ga85] *Gallavotti, G.: Renormalization Theory and Ultraviolet Stability for Scalar Fields via Renormalization Group Methods*, Reviews of Modern Physics, **57**, 471–562, 1985.

[GM00] *Gentile, G., Mastropietro, V.: Renormalization Group for One-Dimensional Fermions: a Review on Mathematical Results*, preprint, `http://ipparco. roma1.infn.it`, in print in Physics reports.

[GS93]: *Gentile, G., Scoppola, B.: Renormalization group and the ultraviolet problem in the Luttinger model*, Communications in Mathematical Physics, **154**, 135–179, 1993. And *Gentile, G.*, Thesis, 1991, unpublished.

[KL60] *Kohn, W., Luttinger, J.M.: Ground state energy of a many fermion system*, Physical Review **118**, 41–45, 1960.

[KL73] *Kac, M., Luttinger, J.M.: Bose–Einstein condensation in the presence of impurities I* and *II*, Journal of athematical Physics, **14**, 1626–1630, 1973, and **15**, 183–186, 1974.

[KLY88] *Kennedy, T., Lieb, E., Shastry, ?.: The XY model has long range order for all spins and all dimensions greater than one*, Physical Review Letters, **61**, 2582–2584, 1988.

[Le87] *Lesniewski, A.: Effective actions for the Yukawa*$_2$ *quantum field theory*, Communications in Mathematical Physics, **108**, 437- 467, 1987.

[LL63] *Lieb E., Lininger, W.: Exact analysis of an interacting Bose gas. I. The general solution and the ground state*, Physical Review, **130**, 1606–1615, 1963, reprinted in [ML66]. See also *Lieb E.: Exact analysis of an interacting Bose gas. I. The excitation spectrum*, Physical Review, **130**, 1616–1624, 1963.

[Lu60] *Luttinger, J.M.: Fermi surface and some simple equilibrium properties of a system of interacting fermions*, Physical Review, **119**, 1153–1163, 1960.

[Lu63] *Luttinger, J.M.: An exactly soluble model of a many fermion system*, Journal of Mathematical Physics, **4**, 1154-1162, 1963.

[LW60] *Luttinger, J.M., Ward, J.: Ground state energy of a many fermion system. II*, Physical Review **118**, 1417–1427, 1960.

[LW68] *Lieb, E., Wu, F.Y.: Absence of Mott transition in an exact solution of the short-range, one-band model in one-dimension*, Physical Review Letters, **20**, 1445–1448, 1968.

[LY00] *Lieb, E., Yngvason, J.: The Ground State Energy of a Dilute Two-dimensional Bose Gas*, preprint 00-63 in `mp_arc@ma.utexas.ed`, 2000.

[Ma00] *Mastropietro, V.:* private communication.

[Ma64] *Mattis, D.: Band theory of magnetism in metals in context of exactly soluble model*, Physics, **1**, 183-193, 1964.

[Ma91] *Mastropietro, V.: Interacting soluble Fermi systems in one dimension*, Nuovo Cimento **B 1**, 304–312, 1994.

[Ma99a] *Mastropietro, V.: Anomalous BCS equation for a Luttinger superconductor*, Modern Physics Letters, **13**, 585–597, 1999. And *Anomalous superconductivity for coupled Luttinger liquids*, in print on Reviews in Mathematical Physics, preprint 1999, in `http://ipparco.roma1.infn.it`.

[Ma99b] *Mastropietro, V.: A renormalization group computation of the XYZ correlation functions*, Letters in Mathematical Physics, **47**, 339–352, 1999.

[Ma00] *Mastropietro, V.:* private communication.

[MD93] *Metzner, W., Di Castro, C.: Conservation laws and correlation functions in the Luttinger liquids*, Physical Review **B 47**, 16107–16123, 1993.

[ML65] *Mattis, D., Lieb, E.: Exact solution of a many fermion system and its associated boson field*, Journal of Mathematical Physics, **6**, 304–310?, 1965 (reprinted in: [ML66]).

[ML66] *Mattis, D., Lieb, E.: Mathematical Physics in one dimension*, Academic Press, New York, 1966.

[Ba82] *Baxter, R.: Exactly solved models*, Academic Press, London, 1982.

[Sh92] *Shankar, R.: Renormalization group approach to interacting fermions*, Review of Modern Physics, **66**, 129–192, 1994.

[Si74] *Simon, B.: The $P(\varphi)_2$ euclidean (quantum) field theory*, Princeton University Press, 1974.

[So79] *Sólyom, J.: The Fermi gas model of one-dimensional conductors*, Advances in Physics **28**, 201–303, 1979.

[Sp99] *Spencer, T.: A mathematical approach to universality in two dimensions*, preprint, 1999.

[To50] *Tomonaga, S.: Remarks on Bloch's method of sound waves applied to many fermion problems*, Progress of Theoretical Physics, **5**, 349–374, 1950, (reprinted in [ML66]).

Preprints at: `http://ipparco.roma1.infn.it`
e-mail: `gallavotti@roma1.infn.it`