A Class of Nonlinear Schrödinger Equations with Concentrated Nonlinearity

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Abstract

We consider the nonlinear Schrödinger equation in dimension one with a nonlinearity concentrated in a finite number of points. Detailed results on the local existence of the solution in fractional Sobolev spaces H^ρ are given. We also prove the conservation of the L^2-norm and the energy of the solution and give a global existence result for repulsive and weakly attractive interaction in the space H^1. Finally we prove the existence of blow-up solutions for strongly attractive interaction.

1. Introduction

In this paper we study the nonlinear Schrödinger equation (NLSE) in dimension one with nonlinearity concentrated in a finite number of points.
The NLSE arises in various physical applications and it has been intensively studied, both from the mathematical and from the numerical point of view ([C1],[C2],[GV],[LPSS],[K],[RR]).
Despite of this, there are various interesting problems which are still open. We only mention the problem of the existence and the analysis of the detailed structure of the blow-up solutions which is especially relevant in applications.
In recent years the study of the resonant tunnelling or, in general, the interaction with impurities in semiconductors ([BKB],[J-LPS],[MA],[N]) have motivated the introduction of a new class of NLSE, where the nonlinearity is concentrated in a bounded region of space.
Our aim is to provide a rigorous mathematical treatment in the simple case of a nonlinearity concentrated on a finite number of points (in the case of one point a preliminary analysis is given in [AT]).
We shall give results on local and global existence, on conservation laws and on the blow-up of the solution.
The relatively simple structure of the equation allows explicit computations and rather detailed estimates on the solution. As a consequence our model seems particularly suitable for the understanding of the qualitative behaviour of the solution in various situations.
This is surely true in the case of the blow-up, where one can exhibit a class of explicit solutions ([AT]). We believe that a further analysis could also provide more information on blow-up solutions for the standard NLSE. This is due to the fact that, at least formally, near the blow-up time the standard NLSE behaves like a NLSE with concentrated nonlinearity.
In this paper we shall limit to the one dimensional case. The case of dimension $d > 1$ with nonlinearity on a set of lower dimension, in particular a finite set of points, requires a non trivial extension of the techniques used here and it will be discussed in a further work.

Now we introduce some notation and give the precise formulation of the evolution problem.
Let us fix the set $Y \equiv \{y_j,\ 1 \leq j \leq n,\ y_j \in \mathbf{R}\}$ of n points on the real line.
For a given function $\alpha : \mathbf{R} \longrightarrow \mathbf{R}^n$, $\alpha(t) = (\alpha_1(t), ...\alpha_n(t))$, one can consider the linear non autonomous evolution problem

$$i\frac{\partial}{\partial t}\psi(t) = -\Delta\psi(t) + \sum_{j=1}^{n} \alpha_j(t)\delta_{y_j}\psi(t), \qquad \psi(0) = \psi_0 \tag{1.1}$$

where δ_{y_j} denotes the Dirac measure in y_j.
Problem (1.1) describes an evolution with interactions concentrated in y_1,....y_n of given time-dependent strengths $\alpha_1(t)$,....$\alpha_n(t)$.
A precise formulation of (1.1) is obtained replacing the formal generator in the r.h.s. of (1.1) with the self-adjoint operator $H_{\alpha(t),Y}$ in $L^2(\mathbf{R})$, namely the Schrödinger operator with point interaction of strengths $\alpha(t)$ placed at Y ([AGH-KH]).

The form domain of $H_{\alpha(t),Y}$ coincides with the form domain of the laplacian $H^1(\mathbf{R})$, which means that point interactions in dimension one are a small form perturbation of the laplacian. The operator domain is

$$\mathcal{D}(H_{\alpha(t),Y}) = \left\{\psi \in H^2(\mathbf{R}\backslash Y) \cap H^1(\mathbf{R})\text{s.t. } \psi'(y_j+) - \psi'(y_j-) = \alpha_j(t)\psi(y_j), \;\; 1 \leq j \leq n\right\} \tag{1.2}$$

$$= \{\psi \in L^2(\mathbf{R}) \text{ s.t. } \psi = \phi_\lambda - \sum_{j=1}^n \alpha_j(t)\psi(y_j)G_\lambda(\cdot - y_j), \;\; \phi_\lambda \in H^2(\mathbf{R})\} \tag{1.3}$$

where $\lambda > 0$ and G_λ is the Green function of the laplacian in the one dimensional case

$$G_\lambda(x - x') = \left(-\frac{d^2}{dx^2} + \lambda\right)^{-1}(x - x') = \frac{e^{-\sqrt{\lambda}|x-x'|}}{2\sqrt{\lambda}} \tag{1.4}$$

The action of the operator is described by

$$H_{\alpha(t),Y}\psi = -\phi''_\lambda + \lambda \sum_{j=1}^n \alpha_j(t)\psi(y_j)G_\lambda(\cdot - y_j) \tag{1.5}$$

It is clear from (1.5) that the action of $H_{\alpha(t),Y}$ reduces to the action of the free laplacian on functions vanishing in Y. This means that the interaction is supported on Y and it acts as a boundary condition on Y (see (1.2)).
It is not hard to see that problem (1.1) has a unique solution for α smooth.
In this paper we shall consider the nonlinear evolution problem obtained replacing the given function α with a function of the solution $\psi(t)$ itself. In particular we shall assume

$$\alpha_j(z) = \gamma_j z^{\sigma_j}, \;\; z = |\psi(t, y_j)|^2 \tag{1.6}$$

where $\gamma_j \in \mathbf{R}$, $\sigma_j \in \mathbf{R}^+$. The nonlinear evolution problem (1.1),(1.6) is more conveniently analyzed in its integral form

$$\psi(t, x) = (U(t)\psi_0)(x) - i \sum_{j=1}^n \gamma_j \int_0^t ds\, U(t - s; x - y_j)\, |\psi(s, y_j)|^{2\sigma_j}\, \psi(s, y_j) \tag{1.7}$$

where $U(t)$ is the one-dimensional free unitary group

$$(U(t)\phi)(x) = \int_{\mathbf{R}} dy\, U(t; x - y)\, \phi(y) = \frac{1}{(4\pi i t)^{1/2}} \int_{\mathbf{R}} dy\, e^{i\frac{|x-y|^2}{4t}}\, \phi(y) \tag{1.8}$$

¿From (1.7) it is clear that the solution is entirely determined by its value at Y. Evaluating (1.7) on Y, one obtains the following system of coupled equations

$$\psi(t, y_j) + \sqrt{\frac{i}{4\pi}} \sum_{l=1}^n \gamma_l \int_0^t ds\, \frac{q_l(s)}{\sqrt{t - s}} \varphi_{jl}(t - s) = (U(t)\psi_0)(y_j), \quad 1 \leq j \leq n \tag{1.9}$$

where we have defined

$$q_j(t) = |\psi(t,y_j)|^{2\sigma_j}\,\psi(t,y_j) \tag{1.10}$$

$$\varphi_{jl}(t-s) = \exp\left(i\frac{(y_j-y_l)^2}{4(t-s)}\right) \tag{1.11}$$

Note that (1.9) is a system of nonlinear integral equations involving only the time variable which can be solved in suitable function spaces. The solution of our evolution problem is then obtained by inserting the solution of (1.9) in (1.7).

The rest of the paper is organized as follows.
In Section 2 we prove the local existence theorem in fractional Sobolev spaces for an initial datum in $H^\rho(\mathbf{R})$, $\rho > 1/2$.
In Section 3 we prove the conservation of the L^2-norm and of the energy of the solution. As a consequence we obtain the global existence theorem in $H^1(\mathbf{R})$ for repulsive and weakly attractive interactions.
In Section 4 we give an approximation lemma for the solution which is a technical ingredient in the analysis of the blow-up.
In Section 5 we obtain the existence of blow-up solutions using the uncertainty principle and the explicit expression of the second time derivative of the moment of inertia.

2. Local existence

We shall consider the local (in time) existence problem for an initial datum $\psi_0 \in H^\rho(\mathbf{R})$, $\rho > 1/2$. The assumption $\rho > 1/2$ guarantees the continuity of $(U(t)\psi_0)(x)$ in the space variable which is required in the formulation of equations (1.7), (1.9).
We start with some preliminary results.

Lemma 1 *Let $\psi_0 \in H^\rho(\mathbf{R})$ and $x \in \mathbf{R}$. Then, as a function of the time, $(U(\cdot)\psi_0)(x) \in H^{\rho/2+1/4}_{loc}(\mathbf{R})$.*

Proof
First we prove that $(U(\cdot)\psi_0)(x)$ belongs to $L^\infty(\mathbf{R})$ and then, in particular, to $L^2_{loc}(\mathbf{R})$.

$$\begin{aligned} |(U(t)\psi_0)(x)| &= \frac{1}{\sqrt{2\pi}}\left|\int_{\mathbf{R}} dk\, e^{-ik^2t}e^{-ikx}\hat{\psi}_0(k)\right| \\ &\leq \frac{1}{\sqrt{2\pi}}\|\psi_0\|_{H^\rho(\mathbf{R})}\left(\int_{\mathbf{R}} dk\,(k^2+1)^{-\rho}\right)^{\frac{1}{2}} \quad \forall t\in\mathbf{R} \end{aligned} \tag{2.1}$$

Here and in the sequel we denote by $\hat{}$ the Fourier transform with respect to the space variable and by $\mathcal{F}$ the Fourier transform with respect to the time variable.

Using the expression of the free Schrödinger evolution in the Fourier space, one has

$$(U(t)\psi_0)(x) = \frac{1}{\sqrt{8\pi}} \int_0^{+\infty} d\omega \frac{e^{-i\omega t}}{\sqrt{\omega}} \left(e^{i\sqrt{\omega}x}\, \hat{\psi}_0(-\sqrt{\omega}) + e^{-i\sqrt{\omega}x}\, \hat{\psi}_0(\sqrt{\omega}) \right) \tag{2.2}$$

Then

$$\mathcal{F}\left[(U(\cdot)\psi_0)(x)\right](\omega) = \frac{\left(e^{i\sqrt{\omega}x}\, \hat{\psi}_0(-\sqrt{\omega}) + e^{-i\sqrt{\omega}x}\, \hat{\psi}_0(\sqrt{\omega}) \right)}{2\sqrt{\omega}} \theta(\omega) \tag{2.3}$$

where θ is the Heaviside step function.
Now we can determine the Sobolev degree of $(U(t)\psi_0)(x)$ as a function of t.

$$\begin{aligned} \int_{\mathbf{R}} |\omega|^{2\eta} \left|\mathcal{F}\left[(U(\cdot)\psi_0)(x)\right](\omega)\right|^2 d\omega &= \frac{1}{4}\int_0^{+\infty} d\omega\, \omega^{2\eta-1} \left| e^{i\sqrt{\omega}x}\, \hat{\psi}_0(-\sqrt{\omega}) + e^{-i\sqrt{\omega}x}\, \hat{\psi}_0(\sqrt{\omega}) \right|^2 \\ &\leq \int_{\mathbf{R}} |k|^{4\eta-1} \left|\hat{\psi}_0(k)\right|^2 dk \end{aligned} \tag{2.4}$$

where the last expression is obtained with the changes of variable $\omega = \pm k^2$. Since $\psi_0 \in H^\rho(\mathbf{R})$, the last integral is finite if $\eta = \rho/2 + 1/4$ and this concludes the proof.

□

In the sequel we shall often deal with the following integral operators:

$$(A_a f)(t) \equiv \int_0^t ds\, f(s) \frac{\exp\left(\frac{ia^2}{t-s}\right)}{\sqrt{t-s}}, \qquad a \geq 0 \tag{2.5}$$

Lemma 2 *The integral kernel*

$$A_a(t) \equiv \theta(t)\, \frac{\exp\left(\frac{ia^2}{t}\right)}{\sqrt{t}} \tag{2.6}$$

of the operator A_a is a tempered distribution and

$$(\mathcal{F}A_a)(\omega) = \sqrt{\frac{\pi}{|\omega|}} \left(\frac{\sqrt{i}}{2}\theta(\omega) e^{2ia\sqrt{|\omega|}} + \frac{1}{\sqrt{2}}(\theta(-\omega) + i\theta(\omega))\, e^{-2a\sqrt{|\omega|}} \right) \tag{2.7}$$

Proof
The kernel $A_a(\cdot)$ is locally integrable and bounded at infinity and then it is a tempered distribution. Its Fourier transform is given in [E].

□

In the next lemma we characterize the smoothing properties of A_a.

Lemma 3 *Let $f \in H^\beta(\mathbf{R})$ with support in $[0,T]$, $T < \infty$ and $\beta \geq 0$. Then $F = A_a f$ is an element of $H^{\beta+1/2}_{loc}(\mathbf{R})$.*

Proof
First we observe that A_0 is bounded in $L^2(I)$, where I is any bounded interval on the real line ([GoVe]). Since

$$|(A_a f)(t)| \leq (A_0 |f|)(t) \quad \forall t \in I \tag{2.8}$$

we have $A_a f \in L^2(I)$ which implies $A_a f \in L^2_{loc}(\mathbf{R})$.
Now we show that the inverse Fourier transform of $\omega^{\beta+1/2}[\mathcal{F}(A_a f)](\omega)$ is in $L^2_{loc}(\mathbf{R})$.
¿From distribution theory we know that the convolution of two tempered distributions, one of which is compactly supported, is a well defined tempered distribution. Moreover,

$$(\mathcal{F}F)(\omega) = (\mathcal{F}A_a)(\omega)\,(\mathcal{F}f)(\omega) \tag{2.9}$$

and $\mathcal{F}f \in C^\infty(\mathbf{R})$ since f has compact support. Thus, from (2.7) we have:

$$|(\mathcal{F}F)(\omega)| \leq \frac{\sqrt{\pi}}{2}\left(1+\sqrt{2}\right)\frac{|(\mathcal{F}f)(\omega)|}{\sqrt{|\omega|}} \tag{2.10}$$

Since $\mathcal{F}f \in C^\infty(\mathbf{R}) \cap L^2(\mathbf{R})$, it is clear that $|\omega|^{\beta+1/2}\,|(\mathcal{F}F)(\omega)|$ is in $L^2(\mathbf{R})$ and this concludes the proof of the lemma. □

Definition 4 *Given $T>0$ and $f \in C^0([0,T])$, we define the function $\tilde{f}^{a,b} : \mathbf{R} \to \mathbf{C}$, $0 \leq a < b \leq T$*

$$\tilde{f}^{a,b}(\tau) = f(\tau) \ \text{ if } \tau \in [a,b], \quad \tilde{f}^{a,b}(\tau) = 0 \ \text{ if } \tau \notin [a,b] \tag{2.11}$$

In the next lemma we characterize the Sobolev degree of $\tilde{f}^{a,b}$.

Lemma 5 *Given $\rho \in (0,1/2)$ and $T>0$ consider $f \in H^\rho(0,T) \cap C^0([0,T])$. Then $\tilde{f}^{a,b} \in H^\rho(\mathbf{R})$.*

Proof
It is sufficient to estimate (see e.g. [A])

$$\int_{\mathbf{R}^2} dx\, dy\, \frac{|\tilde{f}^{a,b}(x) - \tilde{f}^{a,b}(y)|^2}{|x-y|^{1+2\rho}} \tag{2.12}$$

It is convenient to distinguish the following nine regions in $\mathbf{R}^2$:

$$\begin{aligned}
E_1 &= \left\{(x_1,x_2) \in \mathbf{R}^2 \text{ s.t. } x_1 \leq a,\ x_2 \geq b\right\} \\
E_2 &= \left\{(x_1,x_2) \in \mathbf{R}^2 \text{ s.t. } a \leq x_1 \leq b,\ x_2 \geq b\right\} \\
E_3 &= \left\{(x_1,x_2) \in \mathbf{R}^2 \text{ s.t. } x_1 \geq b,\ x_2 \geq b\right\} \\
E_4 &= \left\{(x_1,x_2) \in \mathbf{R}^2 \text{ s.t. } x_1 \leq a,\ a \leq x_2 \leq b\right\}
\end{aligned}$$

$$
\begin{aligned}
E_5 &= \Big\{(x_1,x_2)\in \mathbf{R}^2 \text{ s.t. } a\le x_1\le b,\ a\le x_2\le b\Big\}\\
E_6 &= \Big\{(x_1,x_2)\in \mathbf{R}^2 \text{ s.t. } x_1\ge b,\ a\le x_2\le b\Big\}\\
E_7 &= \Big\{(x_1,x_2)\in \mathbf{R}^2 \text{ s.t. } x_1\le a,\ x_2\le a\Big\}\\
E_8 &= \Big\{(x_1,x_2)\in \mathbf{R}^2 \text{ s.t. } a\le x_1\le b,\ x_2\le a\Big\}\\
E_9 &= \Big\{(x_1,x_2)\in \mathbf{R}^2 \text{ s.t. } x_1\ge b,\ x_2\le a\Big\}
\end{aligned}
$$

The integral on E_5 is finite due to the fact that $f\in H^\rho(0,T)$. In $E_1,\ E_3,\ E_7,\ E_9$ the integrand vanishes. It remains to consider $E_2,\ E_4,\ E_6,\ E_8$. We give the details only for E_2, since the remaining ones are similar:

$$
\begin{aligned}
\int_{E_2} dx\,dy\,\frac{|\tilde f^{a,b}(x)-\tilde f^{a,b}(y)|^2}{|x-y|^{1+2\rho}} &= \int_{E_2} dx\,dy\,\frac{|f(x)|^2}{|x-y|^{1+2\rho}}\\
&\le \|f\|^2_{L^\infty(0,T)}\,\frac{(b-a)^{1-2\rho}}{2\rho(1-2\rho)} \qquad (2.13)
\end{aligned}
$$

□

We are now ready to formulate the local existence and uniqueness result in fractional Sobolev spaces.

Theorem 6 *Let $\psi_0\in H^\rho(\mathbf{R})$, $\rho>1/2$. Then $\exists T>0$ such that problem (1.7) has a unique solution belonging to $C([0,T];H^\rho(\mathbf{R}))$, if $\rho<3/2$, and to $C([0,T];H^\mu(\mathbf{R}))$, for any $\mu<3/2$, if $\rho\ge 3/2$.*
In particular, the space of finite energy $H^1(\mathbf{R})$ is preserved by the evolution.

Proof
For $\rho>1/2$ the function $(U(\cdot)\psi_0)(y_j)$ is continuous.
¿From a direct application of the Schauder-Tychonoff fixed point theorem, we know that $\exists T>0$, $\psi(\cdot,Y):[0,T]\longrightarrow \mathbf{C}^n$, such that $\psi(\cdot,Y)$ is the unique solution of (1.9), and this solution is continuous in $[0,T]$ (see [M]). ¿From (1.9), lemma 3 and lemma 5 we conclude that $\psi(t,Y)\in H^\nu(0,T)$, for any $\nu\le\rho/2+1/4$ if $\rho<3/2$, for any $\nu<1$ if $\rho\ge 3/2$.
In what follows $\psi(t,x)$ will denote the solution of (1.7).
In the Fourier space equation (1.7) reads:

$$
\hat\psi(t,k)+\frac{i}{\sqrt{2\pi}}\sum_{j=1}^n\gamma_j\int_0^t ds\,e^{-ik^2(t-s)}\,e^{iky_j}\,q_j(s) \;=\; e^{-ik^2t}\,\hat\psi_0(k) \qquad (2.14)
$$

Observe that

$$
\int_0^t ds\,e^{ik^2s}q_j(s) \;=\; \sqrt{2\pi}\,(\mathcal{F}\tilde q_j^{0,t})(k^2) \qquad (2.15)
$$

Then

$$\int_{\mathbf{R}} |k|^{2\beta} \left| \int_0^t ds\, e^{-ik^2(t-s)}\, q_j(s) \right|^2 dk \;=\; \sqrt{2\pi} \int_0^1 \omega^{\beta-\frac{1}{2}} \left| (\mathcal{F}\tilde{q}_j^{0,t})(\omega) \right|^2 d\omega + \sqrt{2\pi} \int_1^{+\infty} \omega^{\beta-\frac{1}{2}} \left| (\mathcal{F}\tilde{q}_j^{0,t})(\omega) \right|^2 d\omega \tag{2.16}$$

Let us fix $\beta = 0$. The first term in (2.16) is finite due to the fact that $\tilde{q}_j^{0,t}$ has compact support, so its Fourier transform is analytic on the real line, and this makes the factor $\omega^{-1/2}$ irrelevant for the convergence of the integral.
The second term is estimated by $\int_1^{+\infty} \left| (\mathcal{F}\tilde{q}_j^{0,t})(\omega) \right|^2 d\omega$, which is finite because $\tilde{q}_j^{0,t}$ is square integrable. This proves that the l.h.s. of (2.16) is an element of L^2.
Now let us consider $1/2 < \beta < 3/2$; we have

$$|q_j(t_1) - q_j(t_2)| \leq C|\psi(t_1, y_j) - \psi(t_2, y_j)| \tag{2.17}$$

where C depends on $\|\psi(\cdot, y_j)\|_{L^\infty(0,T)}$. Then $q_j \in H^\nu(0,T)$, where ν can be chosen greater than $1/2$, and from lemma 5, $\tilde{q}_j^{0,t} \in H^\zeta(\mathbf{R})$, $\forall \zeta < 1/2$. Since $\beta - 1/2 < 1$, then each integral term in (1.7) is in $H^\mu(\mathbf{R})$, $\forall \mu < 3/2$, while the free evolution term preserves the regularity of the initial datum.
In order to show the continuity of the time evolution in Sobolev spaces, we only observe that the free evolution term in (2.14) is continuous in $H^\rho(\mathbf{R})$, while for each integral term we have, for all $\beta < 3/2$,

$$\begin{aligned} \int_{\mathbf{R}} dk\, (1 + |k|^{2\beta}) \left| \int_{t_1}^{t_2} ds\, e^{ik^2 s} q_j(s) \right|^2 &= \int_{\mathbf{R}} dk \left| \int_{t_1}^{t_2} ds\, e^{ik^2 s} q_j(s) \right|^2 \\ &\quad + \sqrt{2\pi} \int_0^{+\infty} d\omega\, \omega^{\beta - 1/2} \left| (\mathcal{F}\tilde{q}_j^{t_1,t_2})(\omega) \right|^2 \\ &= \left\| \tilde{q}_j^{t_1,t_2} \right\|^2_{H^{\beta/2-1/4}(\mathbf{R})} \end{aligned} \tag{2.18}$$

Now, following the line of the computation in the proof of lemma 5 one yields

$$\begin{aligned} \int_{\mathbf{R}} dk\, (1 + |k|^{2\beta}) \left| \int_{t_1}^{t_2} ds\, e^{ik^2 s} q_j(s) \right|^2 &= \left| \int_{t_1}^{t_2} ds\, e^{ik^2 s} q_j(s) \right|^2 + \int_{[t_1,t_2]^2} dx\, dy\, \frac{|q_j(t_1) - q_j(t_2)|^2}{|t_1 - t_2|^{\beta+1/2}} \\ &\quad + 4 \|q_j\|^2_{L^\infty([t_1,t_2])} \frac{4\,|t_1 - t_2|^{3/2-\beta}}{(2\beta - 1)(3 - 2\beta)} \end{aligned} \tag{2.19}$$

where the r.h.s. obviously vanishes as t_1 tends to t_2. The proof of the theorem is now complete. □

Remark 1
¿From well-known results on nonlinear Volterra integral equations ([M]), it follows that the solution can be continued up to a maximal time T_{max} which depends on the initial datum. Moreover, the following alternative holds: either $T_{max} = \infty$ or $\limsup_{t\to T_{max}} |\psi(t,Y)| = \infty$.

Remark 2
The choice of an initial datum $\psi_0 \in H^\rho(\mathbf{R})$, $\rho > 1/2$, implies that $(U(t)\psi_0)(x)$ is a continuous function of t for any fixed x and then eq. (1.9) is well defined.
We leave untouched the question of a weaker formulation of our evolution problem for an initial datum $\psi_0 \in H^\rho(\mathbf{R})$, $\rho \le 1/2$.

Remark 3
For an initial datum $\psi_0 \in H^\rho(\mathbf{R})$, $\rho \ge 3/2$, the solution does not belong to $C([0,T]; H^\rho(\mathbf{R}))$. In fact, integrating by parts in the integral terms of (2.14), one easily sees that the local behaviour of the solution near y_j coincides with the behaviour of the Green's function $G_\lambda(\cdot - y_j)$ (see (1.4)), which belongs to $H^\mu(\mathbf{R})$ only for $\mu < 3/2$.

3. Conservation laws and global existence

First we prove the conservation of L^2-norm of the solution.

Theorem 7 *Consider $\psi_0 \in H^\rho(\mathbf{R})$, $\rho > 1/2$, and let $\psi(t)$ be the corresponding solution of problem (1.7) in the time interval $[0,T]$. Then $\|\psi(t)\|_{L^2(\mathbf{R})} = \|\psi_0\|_{L^2(\mathbf{R})}$.*

Proof
Using the representation (2.14), we have

$$\begin{aligned}\|\psi(t)\|^2_{L^2(\mathbf{R})} &= \|\psi_0\|^2_{L^2(\mathbf{R})} + \frac{1}{2\pi}\int_{\mathbf{R}} dk \left|\sum_{j=1}^n \gamma_j \int_0^t ds\, e^{iky_j}\, e^{-ik^2(t-s)}\, q_j(s)\right|^2 \\ &\quad -\sqrt{\frac{2}{\pi}}\Re i \sum_{j=1}^n \gamma_j \int_{\mathbf{R}} dk\, e^{ik^2t}\overline{\psi_0(k)} \int_0^t ds\, e^{iky_j}\, e^{-ik^2(t-s)}\, q_j(s)\end{aligned} \tag{3.1}$$

We shall denote the terms in the r.h.s. of (3.1) respectively by (I), (II), (III). Applying Fubini-Tonelli theorem we obtain:

$$(II) = \frac{1}{2\pi}\sum_{j,l=1}^n \gamma_j\gamma_l \int_0^t ds\, \overline{q_j(s)} \int_0^t ds'\; q_l(s') \int_{\mathbf{R}} dk\, e^{ik^2(s'-s)}\, e^{ik(y_l-y_j)} \tag{3.2}$$

We note that

$$i\left(k^2(s'-s)+k(y_l-y_j)\right)=\begin{cases} i\left(\sqrt{s'-s}\,k+\frac{y_l-y_j}{2\sqrt{s'-s}}\right)^2-i\frac{(y_l-y_j)^2}{4(s'-s)} & \text{if } s'\geq s \\ -i\left(\sqrt{s-s'}\,k-\frac{y_l-y_j}{2\sqrt{s-s'}}\right)^2+i\frac{(y_l-y_j)^2}{4(s-s')} & \text{if } s>s' \end{cases} \tag{3.3}$$

Thus

$$\begin{aligned}(II) &= \frac{1}{2\pi}\sum_{j,l=1}^{n}\gamma_j\gamma_l\left\{\int_{0\leq s'\leq s\leq t}\frac{ds\,ds'}{\sqrt{s-s'}}\overline{q_j(s)}q_l(s')\int_{\mathbf{R}}dy\,e^{-iy^2}\,\varphi_{lj}(s-s')\right.\\ &\left.+\int_{0\leq s\leq s'\leq t}\frac{ds\,ds'}{\sqrt{s-s'}}\overline{q_j(s)}q_l(s')\int_{\mathbf{R}}dy\,e^{iy^2}\,\varphi_{lj}(s-s')\right\} &(3.4)\\ &= \frac{1}{2\pi}\sum_{j,l=1}^{n}\gamma_j\gamma_l\left\{\sqrt{-i\pi}\int_{0\leq s'\leq s\leq t}\frac{ds\,ds'}{\sqrt{s-s'}}\,\overline{q_j(s)}q_l(s')\,\varphi_{lj}(s-s')\right.\\ &\left.+\sqrt{i\pi}\int_{0\leq s\leq s'\leq t}\frac{ds\,ds'}{\sqrt{s-s'}}\,\overline{q_j(s)}q_l(s')\,\varphi_{lj}(s-s')\right\} &(3.5)\end{aligned}$$

where we have used

$$\int_{\mathbf{R}}e^{iy^2}dy=\sqrt{\pi i} \tag{3.6}$$

Finally we obtain:

$$\begin{aligned}(II) &= \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{n}\gamma_j^2\int_{0\leq s'\leq s\leq t}\frac{ds\,ds'}{\sqrt{s-s'}}\left[\Re(\overline{q_j(s)}q_j(s'))+\Im(\overline{q_j(s)}q_j(s'))\right]\\ &+\frac{1}{\sqrt{2\pi}}\sum_{j\neq l}\gamma_j\gamma_l\int_{0\leq s'\leq s\leq t}\frac{ds\,ds'}{\sqrt{s-s'}}\left[\Re(\overline{q_j(s)}q_l(s')\varphi_{lj}(s-s'))\right.\\ &\left.+\Im(\overline{q_j(s)}q_l(s')\varphi_{lj}(s-s'))\right] &(3.7)\end{aligned}$$

Now we turn to (III):

$$\begin{aligned}(III) &= \sqrt{\frac{2}{\pi}}\Re\sum_{j=1}^{n}\int_{\mathbf{R}}dk\,(-i\gamma_j)\overline{\hat{\psi}_0(k)}\int_0^t ds\,e^{iky_j}e^{ik^2s}q_j(s)\\ &= -2\Re\sum_{j=1}^{n}i\gamma_j\int_0^t ds\,q_j(s)\overline{(U(s)\psi_0)\,(y_j)} &(3.8)\\ &= 2\Im\sum_{j=1}^{n}i\gamma_j\int_0^t ds\,q_j(s)\left[\overline{\psi(t,y_j)}+\sqrt{\frac{-i}{4\pi}}\gamma_j\int_0^s ds'\frac{\overline{q_j(s')}}{\sqrt{s-s'}}\right.\\ &\left.+\sqrt{\frac{-i}{4\pi}}\sum_{l\neq j}\gamma_l\int_0^s ds'\frac{\overline{q_l(s')}}{\sqrt{s-s'}}\overline{\varphi_{jl}(s-s')}\right] &(3.9)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\Bigg\{\sum_{j=1}^{n}\gamma_j^2\int_{0\le s'\le s\le t}\frac{ds\,ds'}{\sqrt{s-s'}}\left[\Im(q_j(s)\overline{q_j(s')})-\Re(q_j(s)\overline{q_j(s')})\right]\\
&\quad+\sum_{l\neq j}\gamma_j\gamma_l\int_{0\le s'\le s\le t}\frac{ds\,ds'}{\sqrt{s-s'}}\left[\Im(q_j(s)\overline{q_l(s')})\overline{\varphi_{jl}(s-s')}\right.\\
&\quad\left.-\Re(q_j(s)\overline{q_l(s')\varphi_{jl}(s-s')})\right]\Bigg\}
\end{aligned}
\tag{3.10}
$$

Since $(II)+(III)=0$, the L^2-norm is conserved.

$\square$

Now we shall consider the conservation of the energy.
The "potential function" associated to the nonlinear strength (1.6) is a function $\beta_j : \mathbf{R}^+ \to \mathbf{R}$ s.t. $\frac{d\beta_j}{dz}(z)=\alpha_j(z)$, $z=|\psi(t,y_j)|^2$. This motivates the following definition:

Definition 8 *For any $\psi \in H^1(\mathbf{R})$, the energy of ψ is defined by:*

$$
E(\psi)=\|\psi'\|_{L^2(\mathbf{R})}+\sum_{j=1}^{n}\frac{\gamma_j}{\sigma_j+1}|\psi(y_j)|^{2\sigma_j+2}
\tag{3.11}
$$

Here and in the sequel ψ' denotes the derivative of ψ with respect to the space variable.
We note that from lemma 2 it follows that for an initial datum ψ_0 in $\mathcal{S}(\mathbf{R})$ the solution of (1.9) is in $W^{1,1}(0,T;\mathbf{C}^n)$. In fact, from lemma 3 it is easy to prove that this solution belongs to $H^\nu(0,T)$ for any $\nu<1$, then it is also an element of $W^{\nu,1}(0,T)$. Now, exploiting the regularizing properties of the Abel operator ([GoVe]), one can conclude that $\psi(t,Y)$ is in $W^{1,1}(0,T;\mathbf{C}^n)$.
For a smooth initial datum it is then possible to give a direct proof of the energy conservation.

Proposition 9 *Let $\psi_0 \in \mathcal{S}(\mathbf{R})$. Then $E(\psi(t))=E(\psi_0)$.*

Proof
Using the representation (2.14), we have

$$
\begin{aligned}
\|\psi'(t)\|^2_{L^2(\mathbf{R})} &= \int_{\mathbf{R}}dk\,|k\hat{\psi}_0|^2-2\Re i\sum_{j=1}^{n}\gamma_j\int_{\mathbf{R}}dk\,ke^{ik^2t}\overline{\hat{\psi}_0(k)}\frac{k}{\sqrt{2\pi}}\int_0^t ds\,e^{-ik^2(t-s)}q_j(s)e^{iky_j}\\
&\quad+\frac{1}{2\pi}\int_{\mathbf{R}}dk\left|k\sum_{j=1}^{n}\int_0^t ds\,e^{-ik^2(t-s)}\gamma_j q_j(s)e^{iky_j}\right|^2
\end{aligned}
\tag{3.12}
$$

The first term in the r.h.s. of (3.12) coincides with $\|\psi_0'\|^2_{L^2(\mathbf{R})}$.

Denoting by (IV) the second term in (3.12), after an integration by parts in the variable s, we find

$$
\begin{aligned}
(IV) &= -\sqrt{\frac{2}{\pi}}\sum_{j=1}^{n}\gamma_j \Re\Big[\int_{\mathbf{R}} dk\, e^{ik^2 t}\,\overline{\hat{\psi}_0(k)}e^{iky_j}q_j(t) - \int_{\mathbf{R}} dk\overline{\hat{\psi}_0(k)}e^{iky_j}q_j(0) \\
&\quad - \int_{\mathbf{R}} dk\,\overline{\hat{\psi}_0(k)}e^{iky_j}\int_0^t ds\, e^{ik^2 s}\dot{q}_j(s)\Big] \qquad (3.13)\\
&= 2\sum_{j=1}^{n}\gamma_j \Re\left[q_j(0)\overline{\psi_0(y_j)} - q_j(t)\overline{(U(t)\psi_0)\,(y_j)} - \int_0^t ds\,\dot{q}_j(s)\overline{(U(s)\psi_0)\,(y_j)}\right] \qquad (3.14)
\end{aligned}
$$

Using (1.9), the last integral in (3.14) becomes:

$$
\begin{aligned}
&\int_0^t ds\;\dot{q}_j(s)\;\left[\overline{\psi(s,y_j)}\sqrt{\frac{-i}{4\pi}}\gamma_j\int_0^s ds'\,\frac{q_j(s')}{\sqrt{s-s'}} + \sqrt{\frac{-i}{4\pi}}\sum_{l\neq j}\gamma_l\int_0^s ds'\,\frac{\overline{q_l(s')}}{\sqrt{s-s'}}\overline{\varphi_{jl}(s-s')}\right] \\
&= \int_0^t ds\,\overline{\psi(s,y_j)}\,\frac{d}{ds}\left(|\psi(s,y_j)|^{2\sigma_j}\psi(s,y_j)\right) + \sqrt{\frac{-i}{4\pi}}\sum_{l=1}^{n}\gamma_l\int_0^s ds'\,\frac{\dot{q}_j(s)\overline{q_l(s')}}{\sqrt{s-s'}}\overline{\varphi_{jl}(s-s')} \qquad (3.15)
\end{aligned}
$$

Moreover we have

$$
\left(|f|^{2\sigma}f\right)'\overline{f} \;=\; \sigma\left(f^{\sigma}\overline{f}^{\sigma+1}f' + \overline{f}^{\sigma}f^{\sigma+1}f'\right) + f^{\sigma}\overline{f}^{\sigma+1}f' \qquad (3.16)
$$

and

$$
\Re\left[\left(|f|^{2\sigma}\overline{f}\right)'f\right] \;=\; \frac{2\sigma+1}{2\sigma+2}\,|f|^{2\sigma+2} \qquad (3.17)
$$

Then (3.14) can be rewritten as

$$
\begin{aligned}
(IV) &= 2\sum_{j=1}^{n}\gamma_j\Re\Bigg[|\psi_0(y_j)|^{2\sigma_j+2} - |\psi(t,y_j)|^{2\sigma_j+2} - \sum_{l=1}^{n}\sqrt{\frac{-i}{4\pi}}\gamma_l\,q_j(t)\int_0^t ds'\frac{\overline{q_l(s')}}{\sqrt{t-s'}}\overline{\varphi_{jl}(t-s')} \\
&\quad + \frac{2\sigma+1}{2\sigma+2}\left(|\psi(t,y_j)|^{2\sigma_j+2} - |\psi_0(y_j)|^{2\sigma_j+2}\right) + \sqrt{\frac{-i}{4\pi}}\sum_{l=1}^{n}\gamma_l\int_0^s ds'\,\frac{\dot{q}_j(s)\overline{q_l(s')}}{\sqrt{s-s'}}\overline{\varphi_{jl}(s-s')}\Bigg] \\
&= \sum_{j=1}^{n}\frac{\gamma_j}{\sigma_j+1}\left(|\psi_0(y_j)|^{2\sigma_j+2} - |\psi(t,y_j)|^{2\sigma_j+2}\right) \\
&\quad - \sum_{j,l=1}^{n}\frac{1}{\sqrt{\pi}}\gamma_l\gamma_j\int_0^t\frac{ds'}{\sqrt{t-s'}}\,\Re\left[e^{-i\pi/4}\overline{q_l(s')}q_j(t)\right]\overline{\varphi_{jl}(t-s')} \\
&\quad + \sum_{j,l=1}^{n}\frac{1}{\sqrt{\pi}}\gamma_l\gamma_j\int_{0\le s'\le s\le t}\frac{ds\,ds'}{\sqrt{s-s'}}\Re\left[e^{-i\pi/4}\overline{q_l(s')}\dot{q}_j(s)\varphi_{jl}(s-s')\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \gamma_j \frac{|\psi_0(y_j)|^{2\sigma_j+2} - |\psi(t,y_j)|^{2\sigma_j+2}}{\sigma_j+1} \\
&\quad - \sum_{j,l=1}^{n} \frac{1}{\sqrt{2\pi}} \gamma_l \gamma_j \int_0^t \frac{ds'}{\sqrt{t-s'}} \left[\Re\left(\overline{q_l(s')} q_j(t) \overline{\varphi_{jl}(t-s')}\right) + \Im\left(\overline{q_l(s')} q_j(t) \overline{\varphi_{jl}(t-s')}\right)\right] \\
&\quad + \sum_{j,l=1} \frac{1}{\sqrt{2\pi}} \gamma_l \gamma_j \int_{0\le s'\le s\le t} \frac{ds\, ds'}{\sqrt{s-s'}} \left[\Re\left(\overline{q_l(s')} \dot{q}_j(s) \overline{\varphi_{jl}(s-s')}\right) + \Im\left(\overline{q_l(s')} \dot{q}_j(s) \overline{\varphi_{jl}(s-s')}\right)\right]
\end{aligned}
\tag{3.18}
$$

Denoting by (V) the third term in r.h.s. of (3.12), we obtain

$$
\begin{aligned}
(V) &= \frac{1}{2\pi} \int_{\mathbf{R}} dk\, k^2 \left(\int_0^t ds\, e^{ik^2(t-s)} \sum_{j=1}^{n} \gamma_j\, \overline{q_j(s)} e^{-iky_j}\right) \left(\int_0^t ds'\, e^{-ik^2(t-s')} \sum_{l=1}^{n} \gamma_l\, q_l(s') e^{iky_l}\right) \\
&= \frac{1}{2\pi} \lim_{R\to+\infty} \left[\int_{-R}^{R} dk\, k^2 \int_{0\le s'\le s\le t} ds\, ds' e^{ik^2(s'-s)} \sum_{j,l=1}^{n} \gamma_j \gamma_l\, e^{ik(y_l-y_j)} \overline{q_j(s)} q_l(s')\right. \\
&\quad \left. + \int_{-R}^{R} dk\, k^2 \int_{0\le s\le s'\le t} ds\, ds' e^{ik^2(s'-s)} \sum_{j,l=1}^{n} \gamma_j \gamma_l\, e^{ik(y_l-y_j)} \overline{q_j(s)} q_l(s')\right]
\end{aligned}
\tag{3.19}
$$

Exchanging s with s' and l with j in the sum the last expression equals

$$
\lim_{R\to+\infty} \frac{1}{\pi} \Re\left[\int_{-R}^{R} dk\, k^2 \int_{0\le s'\le s\le t} ds\, ds' e^{ik^2(s'-s)} \sum_{j,l=1}^{n} \gamma_j \gamma_l e^{ik(y_l-y_j)} \overline{q_j(s)} q_l(s')\right]
\tag{3.20}
$$

After an integration by parts in the variable s one has:

$$
\begin{aligned}
(V) &= \frac{1}{\pi} \sum_{j,l=1}^{n} \gamma_j \gamma_l \lim_{R\to+\infty} \Re\left[i \int_{-R}^{R} dk\, e^{ik(y_l-y_j)} \int_0^t ds'\, e^{ik^2(s'-t)} q_l(s') \overline{q_j(t)}\right. \\
&\quad -i \int_{-R}^{R} dk\, e^{ik(y_l-y_j)} \int_0^t ds'\, q_l(s') \overline{q_j(s')} \\
&\quad \left. -i \int_{-R}^{R} dk\, e^{ik(y_l-y_j)} \int_0^t ds'\, q_l(s') \int_{s'}^{t} ds\, e^{ik^2(s'-s)} \overline{\dot{q}_j(s)}\right]
\end{aligned}
\tag{3.21}
$$

The second term in (3.21) vanishes in the limit $R\to\infty$ for $l\neq j$, while for $l=j$ it is purely imaginary.
Applying Fubini-Tonelli theorem and using

$$
\int_{\mathbf{R}} dk\, e^{i[k(y_l-y_j)+k^2(s'-t)]} = \sqrt{\frac{-i\pi}{t-s'}}\, e^{i\frac{(y_l-y_j)^2}{4(t-s')}}
\tag{3.22}
$$

we obtain

$$
\begin{aligned}
(V) &= \frac{1}{\pi}\sum_{j,l=1}^{n}\gamma_j\gamma_l\Re\left[\sqrt{i\pi}\int_0^t ds'\,\frac{q_l(s')\overline{q_j(t)}}{\sqrt{t-s'}}e^{i\frac{(y_l-y_j)^2}{4(t-s')}}-\sqrt{i\pi}\int_{0\le s'\le s\le t}ds'\,ds\,\frac{q_l(s')\overline{\dot q_j(s)}}{\sqrt{s-s'}}e^{i\frac{(y_l-y_j)^2}{4(s-s')}}\right] \\
&= \frac{1}{\sqrt{2\pi}}\sum_{j,l=1}^{n}\gamma_j\gamma_l\left\{\int_0^t\frac{ds'}{\sqrt{t-s'}}\left[\Re\left(q_l(s')\overline{q_j(t)}\varphi_{jl}(t-s')\right)-\Im\left(q_l(s')\overline{q_j(t)}\varphi_{jl}(t-s')\right)\right]\right. \\
&\quad\left.-\int_{0\le s'\le s\le t}\frac{ds'\,ds}{\sqrt{s-s'}}\left[\Re\left(q_l(s')\overline{\dot q_j(s)}\varphi_{jl}(s-s')\right)-\Im\left(q_l(s')\overline{\dot q_j(s)}\varphi_{jl}(s-s')\right)\right]\right\} \qquad (3.23)
\end{aligned}
$$

¿From (3.12), (3.18) and (3.23) we have:

$$
\|\psi'(t)\|^2_{L^2(\mathbf{R})}=\|\psi_0'\|^2_{L^2(\mathbf{R})}+\sum_{j=1}^{n}\frac{\gamma_j}{\sigma_j+1}|\psi_0(y_j)|^{2\sigma_j+2}-\sum_{j=1}^{n}\frac{\gamma_j}{\sigma_j+1}|\psi(t,y_j)|^{2\sigma_j+2} \qquad (3.24)
$$

Hence conservation of energy is proven. □

Now we shall extend the conservation of energy to the case of an initial data in $H^1(\mathbf{R})$ using an approximation procedure.
Given $\psi_0\in H^1(\mathbf{R})$ we consider a sequence $\psi_0^{(k)}\in\mathcal{S}(\mathbf{R})$, $k\in\mathbf{N}$, such that $\psi_0^{(k)}\to\psi_0$ in $H^1(\mathbf{R})$ for $k\to\infty$. The corresponding sequence of solutions $\psi^{(k)}(t)$ will be shown to converge to $\psi(t)$ in $H^1(\mathbf{R})$ for any t in the interval of existence of the solutions. We start proving the existence of such an interval. Following [M], we define:

$$
\begin{aligned}
B_k &\equiv \left\{x\in\mathbf{C}^n s.t. \left|x-\left(U(t)\psi_0^{(k)}\right)(Y)\right|\le 1 \text{ for some } t\in[0,1]\right\} &(3.25)\\
M_k &\equiv \sup\{|x|, x\in B_k\} &(3.26)\\
\sigma &\equiv \max\nolimits_{1\le j\le n}\sigma_j &(3.27)
\end{aligned}
$$

Proposition 10 *Consider $\psi_0^{(k)}\in\mathcal{S}(\mathbf{R})$. Then there exists $\bar T>0$, indipendent of k, such that the solution $\psi^{(k)}(t,Y)$ is defined in $[0,\bar T]$.*

Proof
¿From the local existence for Volterra-type integral equations we know that $\psi^{(k)}(t,Y)$ is defined in $[0,T_k]$, where T_k can be chosen equal to $M_k^{-4\sigma-2}/5$ ([M]). Hence, we only need to prove that $M\equiv\sup_{k\in\mathbf{N}}M_k$ is finite. We observe that given $x\in B_k$

$$
|x|\le\left|x-\left(U(t)\psi_0^{(k)}\right)(Y)\right|+\left|\left(U(t)\psi_0^{(k)}\right)(Y)\right| \quad \forall t\in\mathbf{R} \qquad (3.28)
$$

We can choose $t\in[0,1]$ s.t. $\left|x-\left(U(t)\psi_0^{(k)}\right)(Y)\right|\le 1$. Then, from the Sobolev inequalities we obtain

$$
|x|\le 1+\frac{\sqrt{n}\left\|U(t)\psi_0^{(k)}\right\|_{H^1(\mathbf{R})}}{\sqrt{2}}\le 1+\frac{\sqrt{n}\left\|\psi_0^{(k)}\right\|_{H^1(\mathbf{R})}}{\sqrt{2}} \qquad (3.29)
$$

The sequence $\left\|\psi_0^{(k)}\right\|_{H^1(\mathbf{R})}$ is obviously bounded. Defining

$$K \equiv \sup_{k\in\mathbf{N}} \left\|\psi_0^{(k)}\right\|_{H^1(\mathbf{R})} \tag{3.30}$$

one gets

$$|x| \leq 1 + \frac{\sqrt{n}K}{\sqrt{2}} \quad \forall x \in B_k \ \ \forall k \in \mathbf{N} \tag{3.31}$$

so choosing $0 < M \leq 1 + \sqrt{n}K/\sqrt{2}$ and $T \geq M^{-4\sigma-2}/5 > 0$ the proposition is proved. □

The following proposition provides a uniform estimate for the solution.

Proposition 11 *There exists $L > 0$ s.t. $|\psi^{(k)}(t,Y)| \leq L, \quad \forall k \in \mathbf{N}, \ \forall t \in [0,T]$.*

Proof
For any local solution of (1.9) we have ([M])

$$\left|\psi^{(k)}(t,Y) - \left(U(t)\psi_0^{(k)}\right)(Y)\right| \leq 1 \ \text{ if } \ 0 \leq t \leq T_k \tag{3.32}$$

Now the result is obtained following the line of the proof of proposition 10.

□

In the next lemma we give an approximation result for the solutions of (1.9).

Lemma 12 *There exists $0 < T < \bar{T}$ s.t. for $k \to \infty$ we have $\psi^{(k)}(t,Y) \to \psi(t,Y)$ in $H^{3/4}([0,T],\mathbf{C}^n)$.*

Proof
¿From the system (1.9) we have

$$\psi(t,y_j) - \psi^{(k)}(t,y_j) = \left(U(t)(\psi_0 - \psi_0^{(k)})\right)(y_j) + \sqrt{\frac{i}{4\pi}}\sum_{l=1}^{n}\gamma_l \int_0^t ds\, \frac{q_l^{(k)}(s) - q_l(s)}{\sqrt{t-s}}\varphi_{jl}(t-s) \tag{3.33}$$

where $1 \leq j \leq n$ and

$$q_l^{(k)}(t) = \left|\psi^{(k)}(t,y_l)\right|^{2\sigma_l}\psi^{(k)}(t,y_l) \tag{3.34}$$

$$\left|\psi(t,y_j) - \psi^{(k)}(t,y_j)\right| \leq \frac{\left\|\psi_0 - \psi_0^{(k)}\right\|_{H^1(\mathbf{R})}}{\sqrt{2}} + \frac{1}{\sqrt{4\pi}}\sum_{l=1}^{n}|\gamma_l|(A_0|q_l^{(k)} - q_l|)(t) \tag{3.35}$$

where A_0 has been defined in (2.5). For the last term in (3.35) we obtain

$$\begin{aligned} \left\|A_0|q_l^{(k)} - q_l|\right\|_{H^{3/4}(0,T;\mathbf{C})} &\leq C(T)\left\|q_l^{(k)} - q_l\right\|_{H^{1/4}(0,T;\mathbf{C})} && (3.36)\\ &\leq C(T)\left\|\psi^{(k)}(\cdot,y_l) - \psi(\cdot,y_l)\right\|_{H^{1/4}(0,T;\mathbf{C})} && (3.37)\\ &\leq C(T)\left\|\psi^{(k)}(\cdot,y_l) - \psi(\cdot,y_l)\right\|_{H^{3/4}(0,T;\mathbf{C})} && (3.38) \end{aligned}$$

where $C(T) \to 0$ for $T \to 0$. The inequality (3.36) is due to the continuity of Abel operator from $H^{1/4}$ to $H^{3/4}$ ([GoVe]), while (3.37) follows from the uniform boundedness of $\psi^{(k)}(t, y_j)$. Moreover

$$\left\|\psi(\cdot, Y) - \psi^{(k)}(\cdot, Y)\right\|_{H^{3/4}(0,T;\mathbf{C}^n)} \leq n\frac{\left\|\psi_0 - \psi_0^{(k)}\right\|_{H^1(\mathbf{R})}}{\sqrt{2}}\sqrt{T}+C(T)\left\|\psi(\cdot, Y) - \psi^{(k)}(t, Y)\right\|_{H^{3/4}(0,T;\mathbf{C}^n)} \tag{3.39}$$

where we have defined $\|z\|_{H^r(0,T;\mathbf{C}^n)} = \sum_{m=1}^n \|z_m\|_{H^r(0,T;\mathbf{C})}$.
Choosing T such that $|C(T)| < 1$, the proof is complete. □

We are now ready to prove the conservation of the energy for any initial datum in $H^1(\mathbf{R})$.

Theorem 13 *Consider $\psi_0 \in H^1(\mathbf{R})$ and let $\psi(t)$ the corresponding solution at time t. Then we have $E(\psi_0) = E(\psi(t))$.*

Proof
First we observe that the convergence in $H^{3/4}(0, t; \mathbf{C}^n)$ of the charges $q_j^{(k)}$ implies the convergence in $H^{1/4}(\mathbf{R})$ of $\tilde{q}_j^{(k),0,t}$. This follows immediately from (2.13):

$$\left\|\tilde{q}_j^{(k),0,t} - \tilde{q}_j^{0,t}\right\|^2_{H^{1/4}(\mathbf{R})} \leq 2\left\|\tilde{q}_j^{(k),0,t} - \tilde{q}_j^{0,t}\right\|^2_{L^2(0,t)} + 8t^{1/4}\left\|q_j^{(k)} - q_j\right\|^2_{L^\infty(0,t)} \tag{3.40}$$

Now, from (2.18) we have:

$$\left\|\int_0^t ds\, U(t-s; \cdot - y_j)(q_j^{(k)}(s) - q_j(s))\right\|^2_{H^1(\mathbf{R})} \leq C\left\|\tilde{q}_j^{(k),0,t} - \tilde{q}_j^{0,t}\right\|^2_{H^{1/4}(\mathbf{R})} \tag{3.41}$$

We can conclude that $\psi^{(k)}(t) \to \psi(t)$ in $H^1(\mathbf{R})$, and since the energy is continuous in $H^1(\mathbf{R})$, the theorem is proved. □

Remark 4 (Blow-up alternative)
¿From the conservation of the energy and remark 1, it follows that for an initial datum $\psi_0 \in H^1(\mathbf{R})$ we have the following alternative: either the solution is global in time or $\limsup_{t\to T_{max}} \|\psi'(t)\|_{L^2(\mathbf{R})} = \infty$, where T_{max} is the maximal time of existence of the solution of (1.9).

The conservation laws (theorem 7 and theorem 13) are the basic ingredients for the global existence result in $H^1(\mathbf{R})$ in the case of repulsive or weakly attractive interactions.

Theorem 14 *Let $\psi_0 \in H^1(\mathbf{R})$. Furthermore, assume that $\sigma_j < 1$ if $\gamma_j < 0$. Then, the solution of problem (1.7) is global in time.*

Proof
Due to the blow-up alternative it is sufficient to show that the H^1-norm of the solution $\psi(t)$ is bounded.
¿From Sobolev inequalities ([A]) and conservation of the L^2-norm

$$|\psi(t,y_j)|^{2\sigma_j+2} \leq \|\psi(t)\|_{L^2(\mathbf{R})}^{\sigma_j+1} \|\psi'(t)\|_{L^2(\mathbf{R})}^{\sigma_j+1} = \|\psi_0\|_{L^2(\mathbf{R})}^{\sigma_j+1} \|\psi'(t)\|_{L^2(\mathbf{R})}^{\sigma_j+1} \tag{3.42}$$

Then one obtains the following lower bound for the energy:

$$\begin{aligned} E(\psi_0) &\geq \|\psi'(t)\|_{L^2(\mathbf{R})}^2 + \sum_{\gamma_j<0} \frac{\gamma_j}{\sigma_j+1} |\psi(t,y_j)|^{2\sigma_j+2} && (3.43) \\ &\geq \|\psi'(t)\|_{L^2(\mathbf{R})}^2 \left(1 - \sum_{\gamma_j<0} \frac{|\gamma_j|}{\sigma_j+1} \frac{\|\psi_0\|_{L^2(\mathbf{R})}^{\sigma_j+1}}{\|\psi'(t)\|_{L^2(\mathbf{R})}^{1-\sigma_j}}\right) && (3.44) \end{aligned}$$

Thus, if $\|\psi(t)'\|_{L^2(\mathbf{R})}$ is not bounded then $E(\psi_0) = \infty$, which is absurd.

□

Remark 5
¿From (3.44) it is easily seen that the solution is global also in the case $\gamma_j < 0$, $\sigma_j = 1$, provided that $\|\psi_0\|_{L^2(\mathbf{R})}$ is chosen sufficiently small.

4. Approximation results

In the next Section we shall analyze some cases in which the solution cannot be global in time. The basic ingredient to reveal the blow-up of the solution is the moment of inertia ([C2]).
Let us introduce the space $D = \{\psi \in L^2(\mathbf{R}) s.t.\, x\psi \in L^2(\mathbf{R})\}$, and denote $F = D \cap H^1(\mathbf{R})$. Then we have

Definition 15 *Given $\psi \in F$, the moment of inertia $I(\psi)$ is given by*

$$I(\psi) = \|x\psi\|_{L^2(\mathbf{R})}^2 \tag{4.1}$$

Note that the range of the Fourier transform restricted to D is $H^1(\mathbf{R})$, so F is invariant under Fourier transform.
In the rest of this Section we shall concentrate on the construction of a smooth approximation for the elements in F and we analyze the continuity of the time evolution in F.
These technical results will be used in the next Section for the proof of the blow-up.
Given $\psi_0 \in F$, we want to construct a sequence $\psi_{0,m}$, $m \in \mathbf{N}$, in the Schwarz space $\mathcal{S}(\mathbf{R})$ such that, for $m \to \infty$, one has $\psi_{0,m} \to \psi_0$ in $H^1(\mathbf{R})$, and $x\psi_{0,m} \to x\psi_0$ in $L^2(\mathbf{R})$.

Let us define the sequence $\delta_m(x) = \sqrt{\frac{m}{\pi}} e^{-mx^2}$, which converges to a Dirac measure in the sense of tempered distributions. We also define $g_m(k) = \sqrt{2\pi}\hat{\delta}_m(k) = e^{-k^2/4m}$.
It is an easy exercise to verify that $\delta_m * \psi_0 \to \psi_0$ in $H^1(\mathbf{R})$ and, moreover, $g_m\psi_0 \to \psi_0$ in $H^1(\mathbf{R})$.
We define the sequence $\psi_{0,m}$ as follows:

$$\psi_{0,m} = \delta_m * (g_m\psi_0) + g_m(\delta_m * \psi_0) \tag{4.2}$$

In Fourier transform the sequence reads

$$\hat{\psi}_{0,m} = g_m\left(\delta_m * \hat{\psi}_0\right) + \delta_m * (g_m\psi_0) \tag{4.3}$$

So we have the same approximating sequence both in position space and in momentum space. Again it is easy to check that all the functions in the sequence (4.2) belong to the Schwarz space and, moreover, $\psi_{0,m} \to \psi_0$ in $H^1(\mathbf{R})$, and $x\psi_{0,m} \to x\psi_0$ in $L^2(\mathbf{R})$.
We shall now prove that the convergence established for the sequence of initial data holds for the corresponding solutions. Moreover the convergence is uniform in time.

Theorem 16 *Given $\psi_0 \in F$, consider $\psi_{0,m}$ constructed in (4.2), and let $\psi_m(t)$ be the solution corresponding to the initial datum $\psi_{0,m}$.*
Then, for $m \to \infty$ we have

$$\sup_{t\in[0,T]} \|\psi_m(t) - \psi(t)\|_{H^1(\mathbf{R})} \;\to\; 0 \tag{4.4}$$

$$\sup_{t\in[0,T]} \|x\psi_m(t) - x\psi(t)\|_{L^2(\mathbf{R})} \;\to\; 0 \tag{4.5}$$

Proof
First we show that F is preserved by time evolution. The stability of $H^1(\mathbf{R})$ is guaranteed by theorem 6. We show now that

$$\begin{aligned}
\frac{\partial}{\partial k}\hat{\psi}(t,k) \;=\;& e^{-ik^2t}\frac{\partial}{\partial k}\hat{\psi}_0(k) - 2ikte^{-ik^2t}\hat{\psi}_0(k) \\
& -\sqrt{\frac{2}{\pi}}kt\sum_{j=1}^n \gamma_j e^{iky_j}\int_0^t ds\, e^{-ik^2(t-s)}q_j(s) \\
& +\sqrt{\frac{2}{\pi}}k\sum_{j=1}^n \gamma_j e^{iky_j}\int_0^t ds\, e^{-ik^2(t-s)}sq_j(s) \\
& \frac{1}{\sqrt{2\pi}}\sum_{j=1}^n \gamma_j y_j e^{iky_j}\int_0^t ds\, e^{-ik^2(t-s)}q_j(s)
\end{aligned} \tag{4.6}$$

is square integrable.

This is obviously true for the first two terms in (4.6).
¿From the proof of theorem 6 we know that if $\psi_0 \in H^1(\mathbf{R})$ then the integral term in (2.14) is in $H^\mu(\mathbf{R})$, for any $\mu < 3/2$. Moreover, if $\tilde{q}^{0,t} \in H^\lambda(0,T)$, with $\lambda < 1/2$, then $s\tilde{q}^{0,t}(s)$ has the same regularity. So, the integral terms in (4.6) are square integrable.
Now, for any t in a compact interval $[0,T]$ we have

$$\begin{aligned}\|\psi_m(t)-\psi(t)\|^2_{H^1(\mathbf{R})} &\leq \|\psi_{0,m}-\psi_0\|^2_{H^1(\mathbf{R})} \\ &+\frac{n}{2\pi}\sum_{j=1}^n|\gamma_j|^2\int_{\mathbf{R}}dk\,(1+k^2)\left|\int_0^t ds\, e^{-ik^2(t-s)}e^{iky_j}(q_{j,m}-q_j)\right|^2 \quad (4.7)\end{aligned}$$

The first term in the r.h.s. of (4.7) converges uniformly in time. Moreover

$$\begin{aligned}\int_{\mathbf{R}}dk\left|\int_0^t ds\, e^{ik^2s}(q_{j,m}(s)-q_j(s))\right|^2 &\leq 2\sqrt{\pi}\|q_j-q_{j,m}\|^2_{L^\infty(0,T)}\int_0^t ds\int_0^s ds'\frac{1}{\sqrt{s-s'}} \\ &\leq \frac{8}{3}\sqrt{\pi}\,\|q_j-q_{j,m}\|^2_{L^\infty(0,T)}\,T^{3/2} \quad (4.8)\end{aligned}$$

Exploiting the fact that convergence in $H^{3/4}$ implies convergence in L^∞, the r.h.s. of (4.8) goes to zero for $m\to\infty$ uniformly in t.
Now we analyze the last term in (4.7)

$$\begin{aligned}\int_{\mathbf{R}}dk\,k^2\left|\int_0^t ds\, e^{ik^2s}(q_{j,m}(s)-q_j(s))\right|^2 &= 2\pi\int_{\mathbf{R}}dk\,k^2\left|(\mathcal{F}\tilde{q}_j^{0,t}-\mathcal{F}\tilde{q}_{j,m}^{0,t})(k^2)\right|^2 \quad (4.9)\\ &\leq 2\pi\left\|\tilde{q}_j^{0,t}-\tilde{q}_{j,m}^{0,t}\right\|^2_{H^{1/4}(\mathbf{R})} \quad (4.10)\\ &\leq C\,\|q_j-q_{j,m}\|^2_{H^{1/4}(0,T)}+C\,\|q_j-q_{j,m}\|^2_{L^\infty(0,T)}\,T^{1/2} \quad (4.11)\end{aligned}$$

where the last estimate comes from (2.13) and C is a positive constant.
Now we consider the convergence of $x\psi_m(t)$:

$$\begin{aligned}\frac{\partial}{\partial k}\hat{\psi}(t)-\frac{\partial}{\partial k}\hat{\psi}_m(t) &= e^{-ik^2t}\left(\frac{\partial}{\partial k}\hat{\psi}_0-\frac{\partial}{\partial k}\hat{\psi}_{0,m}\right)-2ikte^{-ik^2t}(\hat{\psi}_0-\hat{\psi}_{0,m}) \\ &-\sqrt{\frac{2}{\pi}}kt\sum_{j=1}^n\gamma_je^{iky_j}\int_0^t ds\, e^{-ik^2(t-s)}(q_j(s)-q_{j,m}(s)) \\ &+\sqrt{\frac{2}{\pi}}k\sum_{j=1}^n\gamma_je^{iky_j}\int_0^t dse^{-ik^2(t-s)}s(q_j(s)-q_{j,m}(s))\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{\sqrt{2\pi}}\sum_{j=1}^{n}\gamma_j y_j e^{iky_j}\int_0^t ds\, e^{-ik^2(t-s)}(q_j(s)-q_{j,m}(s)) \\
= & \ (VI)+(VII)+(VIII)+(IX)+(X) \qquad (4.12)
\end{aligned}
$$

A direct computation gives:

$$
\|(VI)\|_{L^2(\mathbf{R})} \leq \left\|\frac{\partial}{\partial k}\hat{\psi}_0 - \frac{\partial}{\partial k}\hat{\psi}_{0,m}\right\|_{L^2(\mathbf{R})} \qquad (4.13)
$$

$$
\|(VII)\|_{L^2(\mathbf{R})} \leq 2T\,\|\psi_0-\psi_{0,m}\|_{H^1(\mathbf{R})} \qquad (4.14)
$$

The estimate of $\|(VIII)\|_{L^2(\mathbf{R})}$ follows as in (4.11) and, after the substitution $q \to sq$, the same is true for $\|(IX)\|_{L^2(\mathbf{R})}$. Finally the last term is estimated like in (4.8) and the proof is complete. □

5. Blow-up solutions

In this Section we shall prove the existence of blow-up solutions. We refer to the standard definition

Definition 17 *Given an initial datum $\psi_0 \in H^1(\mathbf{R})$, the corresponding solution $\psi(t)$ is called a blow-up solution if $\exists\, t_c < \infty$ s.t.*

$$
\limsup_{t\to t_c}\|\psi(t)\|_{H^1(\mathbf{R})} = +\infty \qquad (5.1)
$$

In what follows we shall restrict to the case $n=1$ and, without loss of generality, we fix $y_1 = 0$. The technique that we use is based on the analysis of the moment of inertia and it cannot be directly generalized to the case $n > 1$.
Given $\psi_0 \in F$, we construct the approximating sequence $\{\psi_{0,m}\}$ described in (4.2), and we use it to compute the time derivatives of the moment of inertia of $\psi(t)$, the solution corresponding to the initial datum ψ_0.
We shall make use of a regularized moment of inertia, defined as the ultraviolet cutoff of the moment of inertia computed in the Fourier space.

Definition 18 *For any $\psi \in F$ we define the regularized moment of inertia as follows:*

$$
g_R(t) = \int_{-R}^{R} dk\,\left|\frac{\partial}{\partial k}\hat{\psi}(k)\right|^2 \qquad (5.2)
$$

Proposition 19 *For any given* $\psi_{0,m} \in \mathcal{S}(\mathbf{R})$ *we have* $I_m \in C^1([0,T];\mathbf{R}^+)$ *and*

$$\dot{I}_m(t) = 4\Im \int_{\mathbf{R}} dk\, k\, \hat{\psi}_m(t,k) \frac{\partial}{\partial k} \overline{\hat{\psi}_m(t,k)} \tag{5.3}$$

where $I_m(t) \equiv I(\psi_m(t))$.

Proof
First we calculate $\dot{g}_R(t)$. We note that

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial}{\partial k}\hat{\psi}_m(t) &= -2ike^{-ik^2t}\hat{\psi}_{0,m} - \sqrt{\frac{2}{\pi}}k\gamma \int_0^t ds\, e^{-ik^2(t-s)} q_m(s) \\ &\quad + i\sqrt{\frac{2}{\pi}}k^3 t\gamma \int_0^t ds\, e^{-ik^2(t-s)} q_m(s) \\ &\quad - i\sqrt{\frac{2}{\pi}}k^3 \gamma \int_0^t ds\, e^{-ik^2(t-s)} s q_m(s) \\ &\quad - 2k^3 t e^{-ik^2t}\hat{\psi}_{0,m} - ik^2 e^{-ik^2 t}\frac{\partial}{\partial k}\hat{\psi}_{0,m} \end{aligned} \tag{5.4}$$

is jointly continuous in k and t.
Let us fix t, and consider an arbitrary bounded neighbourhood of t, which we call V_t. The joint continuity in k and t allows to estimate $|\frac{\partial}{\partial t}\frac{\partial}{\partial k}\hat{\psi}(t,k)|$ with a constant in $[-R,R] \times V_t$. So we can apply dominated convergence theorem and differentiate under the integral sign.

$$\begin{aligned} \dot{g}_R(t) &= \int_{-R}^{+R} dk\, 2\Re \left[\frac{\partial}{\partial k}\hat{\psi}_m(t,k) \frac{\partial}{\partial k}\frac{\partial}{\partial t}\overline{\hat{\psi}_m(t,k)} \right] \\ &= 4\Im \int_{-R}^{R} dk\, k\, \hat{\psi}_m(t,k) \frac{\partial}{\partial k}\overline{\hat{\psi}_m(t,k)} \end{aligned} \tag{5.5}$$

Now we take the limit $R \to \infty$ and we find:

$$I_m(t) = I_m(0) + 4 \lim_{R\to\infty} \Im \int_0^t ds \int_{-R}^{R} dk\, k\, \hat{\psi}_m(s,k) \frac{\partial}{\partial k}\overline{\hat{\psi}_m(s,k)} \tag{5.6}$$

Now observe that

$$\left| \int_{-R}^{R} dk\, k\, \hat{\psi}_m(s,k) \frac{\partial}{\partial k}\overline{\hat{\psi}_m(s,k)} \right| \leq \|\psi_m'(s)\|_{L^2(\mathbf{R})} \|x\psi_m(s)\|_{L^2(\mathbf{R})} \tag{5.7}$$

Then, by dominated convergence theorem, the result is proved.

□

In the next proposition we compute the limit $m \to \infty$.

Proposition 20 *Let ψ_0 be an element of F. Then,*

$$\dot{I}(t) = \dot{I}(\psi(t)) = 4\Im \int_{\mathbf{R}} dk\, k\, \hat{\psi}(t,k) \frac{\partial}{\partial k} \overline{\hat{\psi}(t,k)} \tag{5.8}$$

Proof
¿From (5.3), we have

$$\|x\psi_m(t)\|^2_{L^2(\mathbf{R})} = \|x\psi_{0,m}\|^2_{L^2(\mathbf{R})} + 4\Im \int_0^t ds \int_{\mathbf{R}} dk\, k\, \hat{\psi}_m(s,k)\, \frac{\partial}{\partial k} \overline{\hat{\psi}_m(s,k)} \tag{5.9}$$

Since

$$\left| \int_{\mathbf{R}} dk\, k\, \hat{\psi}_m(s,k) \frac{\partial}{\partial k} \overline{\hat{\psi}_m(s,k)} \right| \leq \sup_{s\in[0,T]} \|\psi'_m(s)\|_{L^2(\mathbf{R})} \sup_{s\in[0,T]} \|x\psi_m(s)\|_{L^2(\mathbf{R})} \tag{5.10}$$

by theorem 16 we conclude that the l.h.s. is bounded. Thus by dominated convergence the result is proved.

□

Following the same line we compute the second derivative of the moment of inertia. First, we define the regularized first derivative of I,

$$f_R(t) = 4\Im \int_{-R}^{R} dk\, k\, \hat{\psi}(t,k) \frac{\partial}{\partial k} \overline{\hat{\psi}(t,k)} \tag{5.11}$$

Next, we compute the derivative for an approximating element.

Proposition 21

$$\ddot{I}_m(t) = 8E(\psi_{0,m}) + 4\gamma \frac{\sigma-1}{\sigma+1} |\psi_m(t,0)|^{2\sigma+2} \tag{5.12}$$

Proof
As before, we can differentiate $f_R(t)$ under the integral sign

$$\begin{aligned}
\dot{f}_R(t) &= 4\Im \int_{-R}^{R} k\, dk \left\{ \frac{\partial}{\partial t} \hat{\psi}(t,k) \frac{\partial}{\partial k} \overline{\psi(t,k)} + \hat{\psi}(t,k) \frac{\partial}{\partial k} \frac{\partial}{\partial t} \overline{\hat{\psi}(t,k)} \right\} \\
&= 4\Im \int_{-R}^{R} k\, dk \left\{ \frac{-i\gamma}{\sqrt{2\pi}} q(t) \frac{\partial}{\partial k} \overline{\hat{\psi}(t,k)} + 2ik|\psi(t,k)|^2 \right\} \\
&= 8 \int_{-R}^{R} k^2\, dk\, |\hat{\psi}(t,k)|^2 - 2\sqrt{\frac{2}{\pi}} \gamma \Re \int_{-R}^{R} k\, dk\, q(t)\, \frac{\partial}{\partial k} \overline{\hat{\psi}(t,k)}
\end{aligned} \tag{5.13}$$

Integrating by parts, one obtains

$$\dot{f}_R(t) = 8 \int_{-R}^{R} k^2\, |\hat{\psi}(t,k)|^2 - 2\gamma \sqrt{\frac{2}{\pi}} \Re\, q(t) \left[R(\overline{\hat{\psi}(t,R)} + \overline{\hat{\psi}(t,-R)}) - \int_{-R}^{R} dk\, \overline{\hat{\psi}(t,R)} \right] \tag{5.14}$$

We can now compute $I_m(t)$

$$
\begin{aligned}
I_m(t) &= I_m(0) + \lim_{R\to\infty} \int_0^t ds\, f_R(s) \\
&= I_m(0) + \lim_{R\to\infty} \int_0^t ds\, \left\{ f_R(0) + \int_0^s d\tau\, \dot{f}_R(\tau) \right\} \\
&= I_m(0) + \lim_{R\to\infty} f_R(0) t + \lim_{R\to\infty} \int_0^t ds \int_0^s d\tau \dot{f}_R(\tau) \\
&= I_m(0) + \dot{I}_m(0)\, t + \int_0^t ds \int_0^s d\tau \int_{-R}^{R} 8k^2 |\hat{\psi}_m(\tau,k)|^2 \\
&\quad -2\gamma \int_0^t ds \int_0^s d\tau \sqrt{\frac{2}{\pi}} \Re\, q(\tau) \left[R(\hat{\psi}_m(\tau,R) + \hat{\psi}_m(\tau,-R)) - \int_{-R}^{R} dk\, \overline{\hat{\psi}_m(\tau,k)} \right]
\end{aligned}
\tag{5.15}
$$

We want to take the limit $R \to \infty$.
First we have

$$
\lim_{R\to\infty} 8 \int_{-R}^{R} dk\, k^2 |\hat{\psi}_m(\tau,k)|^2 = 8E(\psi_{0,m}) - 8\frac{\gamma}{\sigma+1} |\psi_m(\tau,0)|^{2\sigma+2} \tag{5.16}
$$

For the first term in the square brackets in (5.15) we have

$$
\begin{aligned}
R(\hat{\psi}_m(\tau,R) + \hat{\psi}_m(\tau,-R)) &= R(e^{-iR^2\tau}\hat{\psi}_{0,m}(R) + e^{iR^2\tau}\hat{\psi}_{0,m}(-R)) \\
&\quad + \sqrt{\frac{2}{\pi}} \gamma R \int_0^\tau ds'\, e^{-iR^2(\tau - s')} q_m(s')
\end{aligned}
\tag{5.17}
$$

After an integration by parts, (5.17) can be estimated as follows:

$$
\begin{aligned}
|R(\hat{\psi}_m(\tau,R) + \hat{\psi}_m(\tau,-R))| &\le R|\hat{\psi}_{0,m}(R)| + R|\hat{\psi}_{0,m}(-R)| + \sqrt{\frac{2}{\pi}} |\gamma| \frac{|e^{iR^2\tau} q_m(\tau) - q_m(0)|}{R} \\
&\quad + \sqrt{\frac{2}{\pi}} \frac{|\gamma|}{R} \left| \int_0^\tau ds'\, e^{iR^2 s'} \dot{q}_m(s') \right| \\
&\le R|\hat{\psi}_{0,m}(R)| + R|\hat{\psi}_{0,m}(-R)| + \frac{2\sqrt{2}}{R\sqrt{\pi}} |\gamma|\, \|q_m\|_{L^\infty(0,T)} && (5.18) \\
&\quad + \frac{\sqrt{2}}{R\sqrt{\pi}} |\gamma|\, \|\dot{q}_m\|_{L^1(0,T)} && (5.19)
\end{aligned}
$$

For the last term in (5.15) we have

$$
\int_{-R}^{R} dk\, |\hat{\psi}(t,k)| \le \sqrt{\pi}\, \|\psi_m(t)\|_{H^1(\mathbf{R})} \tag{5.20}
$$

and taking the limit $R \to \infty$ we obtain

$$\lim_{R \to +\infty} \int_{-R}^{R} dk\, \hat{\psi}_m(\tau, k) = \sqrt{2\pi}\psi_m(\tau, 0) \tag{5.21}$$

We can now compute the limit $R \to \infty$ in (5.15):

$$\begin{aligned} \ddot{I}_m(t) &= 8E(\psi_{0,m}) - 8\frac{\gamma}{\sigma+1}|\psi_m(t,0)|^{2\sigma+2} + 4\gamma|\psi_m(t,0)|^{2\sigma+2} \\ &= 8E(\psi_{0,m}) + 4\gamma\frac{\sigma-1}{\sigma+1}|\psi_m(t,0)|^{2\sigma+2} \end{aligned} \tag{5.22}$$

□

Now we extend the result of the previous proposition to the case of initial datum in F.

Theorem 22 *For any given $\psi_0 \in F$ we have $\ddot{I}(t) = 8E(\psi_0) + 4\gamma\frac{\sigma-1}{\sigma+1}|\psi(t,0)|^{2\sigma+2}$*

Proof
We approximate the inital datum as in (4.2) and we have $\lim_{m\to\infty} I_m(t) = I(t)$, where $I(t) \equiv I(\psi(t))$. Then

$$\begin{aligned} I(t) &= I(0) + \dot{I}(0)t \\ &\quad + \lim_{m\to\infty} \int_0^t ds \int_0^s d\tau \left\{ 8E(\psi_{0,m}) + 4\gamma\frac{\sigma-1}{\sigma+1}|\psi_m(\tau,0)|^{2\sigma+2} \right\} \end{aligned} \tag{5.23}$$

We have $E(\psi_{0,m}) \to E(\psi_0)$ and

$$|\psi_m(\tau,0)| \le \|\psi_m(\tau)\|_{H^1(\mathbf{R})} \le \sup_{\tau\in[0,T]} \|\psi_m(\tau)\|_{H^1(\mathbf{R})} \tag{5.24}$$

By the dominated convergence theorem we finally prove the theorem.

□

In the following theorem we prove the existence of blow-up solutions as a simple consequence of the uncertainty principle, theorem 22 and the blow-up alternative.

Theorem 23 *If $\gamma < 0$, $\sigma \ge 1$, then there exist initial data ψ_0 such that $\psi(t)$ is a blow-up solution.*

Proof
If $\gamma < 0$ there exists $\psi_0 \in F$ with negative energy. In fact for any $\phi_0 \in F$ with $\phi_0(0) \neq 0$ one has

$$E(\lambda\phi_0) = |\lambda|^2 \|\phi_0'\|^2_{L^2(\mathbf{R})} + \frac{\gamma|\lambda|^{2\sigma+2}}{\sigma+1}|\phi_0(0)|^{2\sigma+2} \to -\infty \quad \text{if} \quad \lambda \to \infty \tag{5.25}$$

¿From theorem 22

$$\ddot{I}(t) \leq 8E(\psi_0) < 0 \tag{5.26}$$

and this implies that either $I(t) \downarrow 0$ for $t \to \tilde{T} < \infty$, or $\inf_t \; I(t) > 0$ in the maximal interval of existence of the solution, which is then finite.
In the former case, using the uncertainty principle and the conservation of the L^2-norm

$$\|\psi_0\|_{L^2(\mathbf{R})} = \|\psi(t)\|_{L^2(\mathbf{R})} \leq 2 \, \|x\psi(t)\|_{L^2(\mathbf{R})} \, \|\psi'(t)\|_{L^2(\mathbf{R})} \tag{5.27}$$

we conclude that the solution blows up in finite time.
In the latter case, the blow-up alternative (remark 4) ensure us that the solution blows up.

□

Comparing theorems 14 and 23 we see that $\sigma = 1$ is the critical power in our nonlinear evolution problem.
Such a critical case is of special interest in quantum mechanics since it corresponds to a point interaction with a strength proportional to the probability to find the particle in $x = 0$.
Similarly to the standard NLSE, in the critical case the evolution problem has an additional symmetry (pseudo-conformal invariance) and a corresponding conservation law.
Exploiting this fact in [AT] we have constructed the following class of blow-up solutions

$$\psi(t,x) = \frac{1}{\sqrt{a(t_0-t)}} \exp\left(-\frac{|x|}{2a(t_0-t)} - i\frac{x^2}{4(t_0-t)} + i\frac{1}{4a^2(t_0-t)} \right), \quad a, t_0 > 0 \tag{5.28}$$

An interesting question is to establish whether the behaviour near the blow-up time exhibited by (5.28) is generic and then to extend the analysis to the n-centers case.
We remark that the study of blow-up solutions near the blow-up time is a difficult problem in the standard NLSE and only few rigorous results are known.
On the other hand, in our model the solution is completely determined by the simple integral equation (1.9). This makes the analysis considerably easier and one can hope to get more detailed information on the blow-up rate. We plan to approach the problem in a further work.

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