On the validity of two-face equivalent source approximations

Preliminary Report

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Abstract

A family of fast surface scattering solvers developed recently [1, 2] is based on approximation of the surface field by the field of so-called equivalent sources. Use of certain spatially sparse distributions of equivalent sources is advantageous for reducing the operation count of the algorithms. It turns out, however, that certain types of sparse distributions cannot produce the required approximations. In the present work we prove that use of the two-face source distributions of equivalent sources we introduced in [1] is indeed sufficient to produce approximations of required accuracy.

Introduction

Numerical evaluation of wave scattering from acoustically large bodies remains one of the most challenging computational problems. One of the main difficulties here is related to the (large) size of the discretization grid necessary to provide a satisfactory accuracy of the solution. Typically, some kind of acceleration scheme has to be used to make possible such large scale computations. A family of acceleration schemes developed recently [1, 2] utilizes approximation of the surface field by the field of so-called equivalent sources. These sources, being placed at the nodes of 3D Cartesian grid, allow for the acceleration of computations through the use the 3D Fast Fourier Transform.

Use of certain spatially sparse distributions of equivalent sources is advantageous for reducing operation count of the algorithms. It turns out, however, that certain types of sparse distributions cannot produce the required approximations, as shown by a counterexample we present in this paper. In present work we prove that the use of so-called two-face source distributions of equivalent sources introduced in [1] is indeed sufficient to produce approximations of required accuracy.

According to the prescriptions of the algorithm [1], one subdivides the volume occupied by the scatterer into a number of cubic cells. Then, equivalent sources are placed on the faces of each non-empty cell, i.e. each cell containing some part of the scattering surface. The intensities of these sources are chosen to approximate the field scattered by the part of the surface contained inside the cell. In order to reduce the number of operations required by the algorithm, sources are placed only on two parallel faces of each cell. Both types of singular sources: monopoles (as would arise from a discretized single layer distribution) and dipoles (which would result from discretization of a double layer distribution), are utilized in the algorithm.

The main purpose of the present work is to provide a mathematical proof of the fact that the two-face approximations introduced in [1] can indeed produce accurate approximations to the field generated by the original surface sources contained inside the cell. Further, we show that use of monopoles or dipoles alone does not give rise to convergent approximations.

Convergence of various two-face approximations

We show first that it is not possible, in general, to approximate the field of sources located inside of a cubic cell by the field of monopoles located on two parallel disks containing two faces of the cell. We prove this statement by means of the following counterexample.

Let us assume that it is possible to approximate a field of a unit singular source located in the origin by a field of a single layer potential distributed over the surface of two disks D_h and D_{-h} of radius R placed parallel to the plane Oxy with the centers in the points $\mathbf{r}_h = (0,0,h)$ and $\mathbf{r}_{-h} = (0,0,-h)$. Then, due to the symmetries of the problem, the required density distributions will be necessarily radially symmetric and equal on both disks; thus, we denote both the corresponding density by $\varphi(\rho)$, where ρ is the distance from the center of a disk. The far-field $u_\infty(\hat{\mathbf{x}})$ in the direction $\hat{\mathbf{x}}$ created by charges of density $\phi(\mathbf{y})$ distributed over a region Ω is defined by the formula

$$u_\infty(\hat{\mathbf{x}}) \equiv \lim_{t\to\infty} te^{-ikt}\int_\Omega \Phi(t\hat{\mathbf{x}},\mathbf{y})\phi(\mathbf{y})d\mathbf{y} = \frac{1}{4\pi}\int_\Omega e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}}\phi(\mathbf{y})d\mathbf{y}, \quad \hat{\mathbf{x}}\in\mathbb{S}^1.$$

Thus, the far field $u_{\infty,D_0}(\hat{\mathbf{x}})$ of a disk D_0 of radius R centered at the origin with the single layer density $\varphi(\rho)$ is given by the following expression:

$$u_{\infty,D_0}(\hat{\mathbf{x}}) = \frac{1}{4\pi}\int_{D_0} e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}}\varphi(|\mathbf{y}|)d\mathbf{y}.$$

Now the combined far field $u_{\infty,D_h\cup D_{-h}}(\hat{\mathbf{x}})$ of the disks D_h and D_{-h} can be expressed through $u_{\infty,D_0}(\hat{\mathbf{x}})$ as follows:

$$u_{\infty,D_h\cup D_{-h}}(\hat{\mathbf{x}}) = \left(e^{-ik\hat{\mathbf{x}}\cdot\mathbf{r}_h} + e^{-ik\hat{\mathbf{x}}\cdot\mathbf{r}_{-h}}\right)u_{\infty,D_0}(\hat{\mathbf{x}}) = 2u_{\infty,D_0}(\hat{\mathbf{x}})\cos(kh\cos\varsigma),$$

where ς is the angle between $\hat{\mathbf{x}}$ and the z-axis. Obviously, if $kh \geq \pi/2$ (i.e., if the distance between the disks D_h and D_{-h} is greater than a half of wavelength λ) then there exist at least one solution ς_0 to the equation $kh\cos\varsigma = \pi/2$. In other words, the far field $u_{\infty,D_h\cup D_{-h}}(\hat{\mathbf{x}})$ vanishes for all directions $\hat{\mathbf{x}}$ such that the angle between $\hat{\mathbf{x}}$ and the z-axis is equal to ς_0. However, the far-field created by a unit singular source located in the origin is equal, simply, to unity. Therefore, the field of monopoles distributed over two parallel disks located more than half-wavelength apart cannot be, in general, used to approximate the field of sources located in between the disks. (Remarkably, the radius R of the disks does not play any role in this statement.) We, therefore, have proved the following

Proposition 1 *If the side of a cubic cell C is larger than $\lambda/2$, then outside the cell the field created by sources cointained in C cannot be, in general, approximated by a single layer potential with density supported on two parallel faces of C.*

Let us now try to use for the approximation the field of monopoles supported on L equally spaced parallel disks (where L is even) placed symmetrically with respect to the origin. A consideration similar to the one presented above shows that if the distance between the disks is greater than $\lambda/2$, the approximation of a general field is, again, not possible. The same is true, of course, for the odd number of disks L, since, if the field cannot be approximated using L disks, it obviously cannot be approximated using $L-1$ disks. This immediately implies the following

Proposition 2 *The field created by singular sources (monopoles) located at the nodes of the 3D Cartesian grid inside a cubic cell C, cannot, in general, approximate the field of arbitrary sources contained in the cell C, if the spacing of the Cartesian grid is larger than $\lambda/2$.*

The above Proposition shows that it is impossible, in general, to speed up FFT acceleration schemes (behind some limit) by coarsening the grid.on which equivalent sources are placed. According to Proposition prop1, the approximation fails if the grid spacing is larger than $\lambda/2$. (Actual grid spacings used in [2] range from $\lambda/10$ to $\lambda/6$.)

As the above analysis shows, construction of spatially sparse equivalent source approximations is not a trivial task. In what follows we will show that it is possible to approximate the field created by sources contained inside a sphere by a field of both double- and single layer potentials supported on a concentric disk located inside that sphere. Such an approximation, valid outside of a larger concentric sphere, can be of any given finite accuracy. In order to prove this statement we will first construct an arbitrarily accurate approximation to a single term of spherical wave expansion.

Theorem 3 *A spherical wave function $W_n^m(\mathbf{r}) \equiv Y_n^m(\mathbf{r}/|\mathbf{r}|)h_n^{(1)}(k|\mathbf{r}|)$ (with arbitrary $m,n \in \mathbb{Z}, n \geq 0, |m| \leq n$) can be approximated outside a sphere S_b of radius b within arbitrary accuracy $\varepsilon > 0$ by a field of a layer potential with densities supported on an open disk $D = \{(x,y,z)| \quad z=0, \quad x^2+y^2 < a^2\}, .$with $a < b$.*

Proof. For fixed m,n, let us define the following function

$$H(\mathbf{r},\mathbf{r}') = W_n^m(\mathbf{r}-\mathbf{r}').$$

Introduce vectors $\mathbf{r}_0(\delta), \mathbf{r}_1(\delta)$, and $\mathbf{r}_2(\delta)$ defined as

$$\begin{aligned}\mathbf{r}_0 &= (0,0,0),\\ \mathbf{r}_1 &= (0,0,-\delta),\\ \mathbf{r}_2 &= (0,0,\delta),\end{aligned}$$

see Figure 1. Obviously, $H(\mathbf{r},\mathbf{r}_0(\delta))$ coincides with $W_n^m(\mathbf{r})$, whereas $H(\mathbf{r},\mathbf{r}_1(\delta))$ and

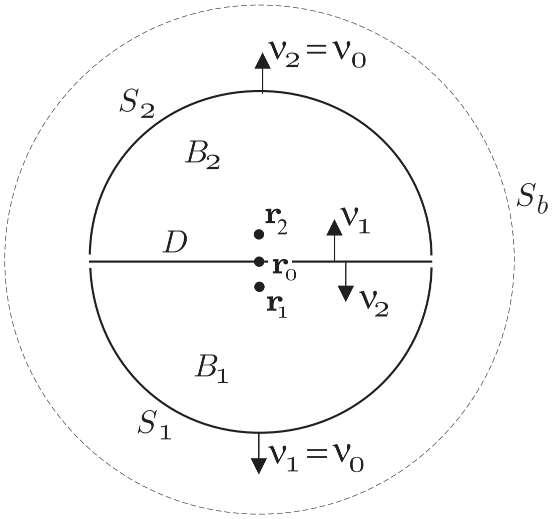

Figure 1: Illustration to the Theorem 3

$H(\mathbf{r}, \mathbf{r}_2(\delta))$ are obtained by shifting function $W_n^m(\mathbf{r})$ along z-axis by $-\delta$ and δ respectively. Let us approximate $W_n^m(\mathbf{r})$ by a combination

$$\breve{H}_\delta(\mathbf{r}) = H(\mathbf{r}, \mathbf{r}_1(\delta)) + H(\mathbf{r}, \mathbf{r}_2(\delta)) - H(\mathbf{r}, \mathbf{r_0}(\delta)). \tag{1}$$

Since $W_n^m(\mathbf{r})$ is uniformly continuous outside of any fixed sphere centered in the origin, for any $\varepsilon > 0$ there exist such δ_0 that

$$\left| W_n^m(\mathbf{r}) - \breve{H}_\delta(\mathbf{r}) \right| < \varepsilon/2, \tag{2}$$

for any $\delta < \delta_0$ and for any $\mathbf{r}$ outside S_b.

Consider a closed ball B_0 of radius a centered at the origin. Let us denote the lower and upper halves of the ball B_0 by B_1 and B_2 respectively. Introduce sphere S_0 of radius a and its open lower and upper halves S_1 and S_2. Obviously, $\partial B_0 = S_0 = S_1 \cup S_2 \cup \partial D$, $\partial B_1 = S_1 \cup D \cup \partial D$, and $\partial B_2 = S_2 \cup D \cup \partial D$, where ∂D denotes the boundary of disk D (i.e. intersection of S_0 and the plane Oxy).

Let us now define ν_i to be exterior normals to the domains B_i; normals ν_i are well defined at any point of S_i and D. By construction, the following equalities hold

$$\begin{aligned} \nu_1(\mathbf{r}) &= \nu_0(\mathbf{r}), \quad \forall \mathbf{r} \in S_1, \\ \nu_2(\mathbf{r}) &= \nu_0(\mathbf{r}), \quad \forall \mathbf{r} \in S_2, \\ \nu_1(\mathbf{r}) &= -\nu_2(\mathbf{r}), \quad \forall \mathbf{r} \in D. \end{aligned} \tag{3}$$

Since each one of the functions $H(\mathbf{r}, \mathbf{r}_i)$, $i = 1, 2, 3$ is a radiating solution to the Helmholtz equation outside of the domain B_i, one can represent these functions (using Green's identity) as a sum of double- and single- layer potentials with densities supported on the boundaries of the domains B_i, i.e.

$$H(\mathbf{r}, \mathbf{r}_i(\delta)) = \int\limits_{\partial B_i} \left(H(\mathbf{r}', \mathbf{r}_i(\delta)) \frac{\partial \Phi(\mathbf{r}, \mathbf{r}')}{\partial \nu_i(\mathbf{r}')} - \frac{\partial H(\mathbf{r}', \mathbf{r}_i(\delta))}{\partial \nu_i(\mathbf{r}')} \Phi(\mathbf{r}, \mathbf{r}') \right) ds(\mathbf{r}'), \quad i = 1, 2, 3, \tag{4}$$

where $\Phi(\mathbf{r},\mathbf{r}') = \exp(ik|\mathbf{r}-\mathbf{r}'|)/|\mathbf{r}-\mathbf{r}'|$ is the fundamental solution to the Helmholtz equation. All representations (4) are valid, in particular, outside of sphere S_b.

Let us substitute identities (4) into Equation (1). Then, using Equations (3) to combine integrals over surfaces S_1, S_2, and D we arrive at the following expression for $\breve{H}_\delta(\mathbf{r})$

$$\breve{H}_\delta(\mathbf{r}) = I_1(\mathbf{r},\delta) + I_2(\mathbf{r},\delta) + I_3(\mathbf{r},\delta) + I_4(\mathbf{r},\delta),$$

where

$$\begin{aligned}
I_1(\mathbf{r},\delta) &= \int\limits_{\mathcal{S}_1} \left(H(\mathbf{r}',\mathbf{r}_1(\delta)) - H(\mathbf{r}',\mathbf{r}_0(\delta))\right) \frac{\partial\Phi(\mathbf{r},\mathbf{r}')}{\partial\nu_1(\mathbf{r}')} ds(\mathbf{r}') \\
&\quad - \int\limits_{\mathcal{S}_1} \Phi(\mathbf{r},\mathbf{r}') \frac{\partial\left(H(\mathbf{r}',\mathbf{r}_1(\delta)) - H(\mathbf{r}',\mathbf{r}_0(\delta))\right)}{\partial\nu_1(\mathbf{r}')} ds(\mathbf{r}'), \\
I_2(\mathbf{r},\delta) &= \int\limits_{\mathcal{S}_2} \left(H(\mathbf{r}',\mathbf{r}_2(\delta)) - H(\mathbf{r}',\mathbf{r}_0(\delta))\right) \frac{\partial\Phi(\mathbf{r},\mathbf{r}')}{\partial\nu_2(\mathbf{r}')} ds(\mathbf{r}') - \\
&\quad - \int\limits_{\mathcal{S}_2} \Phi(\mathbf{r},\mathbf{r}') \frac{\partial\left(H(\mathbf{r}',\mathbf{r}_2(\delta)) - H(\mathbf{r}',\mathbf{r}_0(\delta))\right)}{\partial\nu_2(\mathbf{r}')} ds(\mathbf{r}'), \\
I_3(\mathbf{r},\delta) &= \int\limits_{\mathcal{D}} \left(H(\mathbf{r}',\mathbf{r}_1(\delta)) - H(\mathbf{r}',\mathbf{r}_2(\delta))\right) \frac{\partial\Phi(\mathbf{r},\mathbf{r}')}{\partial\nu_1(\mathbf{r}')} ds(\mathbf{r}') \\
I_4(\mathbf{r},\delta) &= \int\limits_{\mathcal{D}} \frac{\partial(H(\mathbf{r}',\mathbf{r}_2(\delta)) - H(\mathbf{r}',\mathbf{r}_1(\delta)))}{\partial\nu_1(\mathbf{r}')} \Phi(\mathbf{r},\mathbf{r}') ds(\mathbf{r}')
\end{aligned}$$

Since functions $H(\mathbf{r},\mathbf{r}_i(\delta))$ and their gradients are uniformly continuous for $|\mathbf{r}-\mathbf{r}_i(\delta)| > a/2$, and since $\Phi(\mathbf{r},\mathbf{r}')$ and $\nabla\Phi(\mathbf{r},\mathbf{r}')$ are bounded when $|\mathbf{r}-\mathbf{r}'| > a/2$, integrals $I_1(\mathbf{r},\delta)$ and $I_2(\mathbf{r},\delta)$ vanish in the limit $\delta \to 0$. Therefore, there exist such δ_1 that

$$|I_1(\mathbf{r},\delta) + I_2(\mathbf{r},\delta)| < \varepsilon/2, \quad \forall\delta < \delta_1. \tag{5}$$

Choosing δ_2 equal to $\min(\delta_0,\delta_1)$ and combining inequalities (2) and (5), one shows that for any non-zero $\delta < \delta_2$ and $|\mathbf{r}| > a$

$$|W_n^m(\mathbf{r}) - (I_3(\mathbf{r},\delta) + I_4(\mathbf{r},\delta))| < \varepsilon. \tag{6}$$

Integral $I_3(\mathbf{r},\delta)$ can be viewed as a double layer potential with a density supporter on the disk D, whereas $I_4(\mathbf{r},\delta)$ can be understood as a single layer potential. Thus, Equation (6) shows that function $W_n^m(\mathbf{r})$ is approximated by a combination of a potential of single- and double- layer densities continuously distributed over disk D. Moreover, notice that $W_n^m(\mathbf{r})$ is an even function of z if $m+n$ is even, and an odd function otherwise. Therefore if $m+n$ is even, $I_3(\mathbf{r},\delta)$ vanishes and, thus, required approximation is given by a single-layer potential $I_4(\mathbf{r},\delta)$. Otherwise, if $m+n$ is odd, function $W_n^m(\mathbf{r})$ is approximated by a double layer potential $I_3(\mathbf{r},\delta)$. ■

Remark 4 *Notice that we use potentials $I_3(\mathbf{r},\delta)$ and $I_3(\mathbf{r},\delta)$ which correspond to finite values of δ, thus, avoiding hypersingular integrations which would arise in the case $\delta = 0$.*

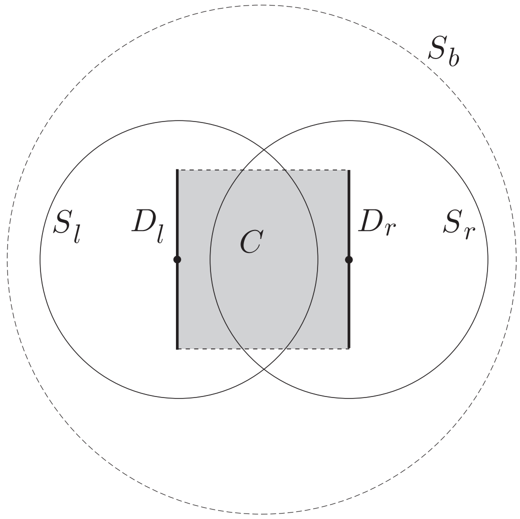

Figure 2: Illustration to the Theorem 5

Theorem 5 *The field $u_{S_0}(\mathbf{r})$ created by sources contained in a sphere S_0 can be approximated outside a larger concentric sphere S_b within arbitrary finite accuracy ε by a sum of a single- and a double- layer potentials with densities supported on a planar unit disk $D = \{(x, y, z)| \quad z = 0, \quad x^2 + y^2 < a^2\}$.*

Proof. According to the Theorem 2.14 [3] the field $u_{S_0}(\mathbf{r})$ can be represented outside S_b by a spherical wave expansion which in our notation has the following form

$$u_{S_0}(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m W_n^m(\mathbf{r}).$$

This expansion converges absolutely and uniformly on an arbitrary compact subsets of $\mathbb{R}^3 \setminus S_b$. Therefore, there exists such a value of a truncation parameter $n_t(\varepsilon)$ that the truncated spherical wave expansion

$$\sum_{n=0}^{n_t} \sum_{m=-n}^{n} a_n^m W_n^m(\mathbf{r}).$$

approximates $u_{S_0}(\mathbf{r})$ with a given finite accuracy $\varepsilon/2$. According to the previous Theorem, one can, in turn, approximate each one of $W_n^m(\mathbf{r})$ with an arbitrary accuracy by a single- and double- layer potentials supported on a disk D, in such a way that the total accuracy of approximation of $u_{S_0}(\mathbf{r})$ by the layer potentials is less than ε ■

Let us now establish the possibility of the two-face approximations. Consider a square cell C of side $2h$ and spheres S_1 and S_2 of radius $\sqrt{3}h$ centered at the centers of the left and right faces of the cell C, as shown in Figure 2. Now, all the sources contained in the left half of the cell C are also contained inside the sphere S_l. Therefore, according to the statement of Theorem 5, the field of these sources can be approximated by a double- and single- layer potentials with densities supported on a disk D_l contained in the left face of the cell C. This approximation will be valid, in particular, outside the sphere S_b of radius $3h$ centered in the origin. The error of this approximation can be made less than $\varepsilon/2$ for arbitrary positive ε. Similarly, the field of sources contained in the right half of cell C can be approximated within accuracy $\varepsilon/2$ by double- and single- layer potentials distributed over the right face of the cell. Hence, the combined potentials of the densities supported on the left and right faces of the cell C approximate within accuracy ε the field created by all sources contained inside C, and this approximation is valid everywhere outside sphere S_b. We have, therefore, proven the following

Theorem 6 *The field created by sources contained inside a cubic cell of side $2h$ can be approximated with arbitrary finite accuracy by single- and double- layer potentials with densities distributed over any two parallel faces of the cell; such approximation is valid, in particular, outside of the sphere of radius $3h$ concentric with the cell.*

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