Bounded Fluctuations and Translation Symmetry Breaking in One-Dimensional Particle Systems*

M. Aizenman [a]   S. Goldstein [b]   J. L. Lebowitz [b]

[(a)] *Departments of Physics and Mathematics, Princeton University, Jadwin Hall, P. O. Box 708, Princeton, NJ 08544.*

[(b)] *Departments of Mathematics and Physics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019.*

July 12, 2000

Abstract

Abstract We present general results for one-dimensional systems of point charges (signed point measures) on the line with a translation invariant distribution μ for which the variance of the total charge in an interval is uniformly bounded (instead of increasing with the interval length). When the charges are restricted to multiples of a common unit, and their average charge density does not vanish, then the boundedness of the variance implies translation-symmetry breaking — in the sense that there exists a function of the charge configuration that is nontrivially periodic under translations — and hence that μ is not "mixing." Analogous results are formulated also for one dimensional lattice systems under some constraints on the values of the charges at the lattice sites and their averages. The general results apply to one-dimensional Coulomb systems, and to certain spin chains, putting on common grounds different instances of symmetry breaking encountered there.

*Dedicated to the memory of J. M. (Quin) Luttinger, a master of one-dimensional systems and much more.

1. Introduction

To fluctuate is normal, and normally fluctuations grow like the square root of the volume. There are however non-trivial exceptions to this rule; among the notable examples are systems of charges with Coulomb interaction and certain quantum spin chains in their ground states. Curiously, the known one-dimensional systems with uniformly bounded fluctuations in the total charge or component of spin in an interval exhibit also other exceptional properties, such as translation symmetry breaking (in jellium and quantum spin chains), or non-uniqueness of the Gibbs state (in the two-component neutral Coulomb system). We prove here that this is not accidental: under some general assumptions, bounded fluctuations in a one-dimensional particle system imply the existence there of a periodic, or in the lattice case a quasi-periodic, structure. In particular, in such systems the correlations of certain local variables do not decay.

We briefly describe the examples mentioned above to which our results apply.

a) Coulomb systems: In general terms, a Coulomb system consists of several (m) species of particles with charges q_α, $\alpha = 1, ..., m$. For a classical system containing N_α particles of species α in a finite domain $V \subset \mathbb{R}^d$, the (configurational) Gibbs canonical distribution is the probability measure on the space of configurations $\{(x_j, \alpha_j)\}_{j=1,\dots,N}$, $x_j \in V$, $\alpha_j = 1, \dots, m$, and $N = \sum_{\alpha=1}^m N_\alpha$, given by the density $\exp[-\beta U(\{(x_j, \alpha_j)\})]/Z$, with

$$U(\{(x_j, \alpha_j)\}) \;=\; \sum_{i \neq j} q_j q_i \; V_C(x_i - x_j) + \sum_j q_j V_{bg}(x_j) + \sum_{i,j} V_{sr}^{(\alpha_i, \alpha_j)}(x_i - x_j) \; , \qquad (1.1)$$

where $q_j \equiv q_{\alpha_j}$, $V_C(x)$ is the Coulomb potential, satisfying $\Delta V_C(x) = -\delta(x)$, $V_{bg}(x)$ is the potential induced by a (uniform) background charge, of *charge* density ρ_{bg}, and $V_{sc}^{(\alpha,\alpha')}(x)$ is a short range interaction between particles of species α and α'. For quantum systems the measure on the space of configurations is given by a more complicated formula; for details, we refer the reader to the recent review article by Brydges and Martin [1].

Under suitable conditions, such measures, with $\{q_j\}$ and ρ_{bg} not all of the same sign, admit translation invariant infinite volume limits. In the limiting states the total charge density, including the background, is zero, i.e., the *particle densities* ν_α satisfy $\sum_\alpha q_\alpha \nu_\alpha + \rho_{bg} = 0$ [1-8]. It is further expected, and proven in some cases, including all classical one-dimensional Coulomb systems [2, 4], that with respect to these limiting measures the variance of the net charge in a region Λ increases with Λ only like its surface area [5, 6]. That is,

$$\mathrm{Var}(Q_\Lambda) := \langle Q_\Lambda^2 \rangle - \langle Q_\Lambda \rangle^2 \ \sim \ |\partial\Lambda| \qquad (\text{as } \Lambda \nearrow \mathbb{R}^d) \ , \tag{1.2}$$

where $Q_\Lambda = \sum_{x_i \in \Lambda} q_{\alpha_i}$. In one dimension this corresponds to the statement that the variance of the charge in an interval I remains bounded as $|I| \to \infty$. Analogous statements apply to the case where the charges are restricted to lattice sites [7]. (Eq. (1.2) follows from the "zero sum rule," which holds whenever the charge correlations have sufficiently rapid decay [1, 5]).

A simple example of a Coulomb system is the so-called *jellium* model, or the *one-component plasma* (OCP), with particles of unit charge dispersed in a negatively charged uniform background. For this system surface growth of charge fluctuations, which now correspond to particle-number fluctuations, has been proven in $d = 1$ [2, 4] and in $d \geq 2$ at high temperatures ($\beta << 1$) and in $d = 2$ also for $\beta = 2$ [1, 9, 10].

For the one-dimensional jellium (with the 1D Coulomb potential $V_C(x) = -\frac{1}{2}|x|$) it is also known that the limits of the Gibbs measures exhibit "translation-symmetry breaking" [8, 4]. The periodic structure found there has been regarded as an example of the 'Wigner lattice.' However, in this case the symmetry breaking was also understood to be related to the boundedness of the charge fluctuations, the two being connected through the properties of the electric field [4] (for which the existence of the limit was previously established in ref. [2]). Here we show that this relation is an example of a more general phenomenon.

b) Spin chains: Another example of a one-dimensional system with reduced fluctuations is provided by the ground state of certain quantum spin systems. These consist of 'chains' (also of interest are arrangements into 'ladders') of quantum spins $\{\underline{\sigma}_n\}_{n\in\mathbb{Z}}$, of a common spin S, such that $2S$ is an integer, with the Hamiltonian

$$H \;=\; -\sum_{n\in\mathbb{Z}} P^{(0)}_{n,n+1} \;, \tag{1.3}$$

where $P^{(0)}_{n,n+1}$ is the projection onto the singlet state, i.e., on the subspace in which $(\underline{\sigma}_n + \underline{\sigma}_{n+1}) = 0$. This class of Hamiltonians was introduced by Affleck [11] as an interesting extension of the spin 1/2 Heisenberg antiferromagnetic spin chain. (It is also related to the classical Potts antiferromagnet with $Q = 2S+1$ [12]).

The 'z-components' of the spins, $\{\sigma^{(3)}_n\}$, form a family of commuting observables. Thus in any quantum state their joint distribution may be described by a 'classical' probability measure on the space of configurations of one-component spin variables with values ranging over $\{-S,\ -S+1, \ldots\ ,\ S\}$. To distinguish these from the full quantum spin variables let us refer to their values as 'charges.' For these systems the ground state ($|0\rangle$) admits a representation in which the charges are organized into <u>neutral clusters</u> [12], i.e., randomly organized clusters of sites for which $\sum \sigma^{(3)} = 0$, with relative spin flip symmetry between different clusters. The clusters can intermingle, so this perspective can be of value only when the distribution of these clusters is such that for any interval I, the number of sites in I belonging to clusters which are not entirely contained in the given interval is seldom large – in a sense uniform in I. A condition implying that this scenario is realized is rapid decay of correlations:

$$\sum_{n>0} n\, |\langle 0|\ \sigma^{(3)}_0 \sigma^{(3)}_n\ |0\rangle| \;<\; \infty\ . \tag{1.4}$$

One can show, using the cluster representation, that when eq. (1.4) holds the two point function satisfies the neutrality condition $\sum_x \langle 0|\ \sigma^{(3)}_0 \sigma^{(3)}_x\ |0\rangle = 0$, which is analogous to the "zero sum rule" of Coulomb systems [1, 5, 6], and the block spins $S_I = \sum_{n\in I} \sigma^{(3)}_n$

have uniformly bounded variance, with

$$\langle 0|\ |S_I|^2\ |0\rangle|\ \le\ 2\sum_{n>0} n\,|\langle 0|\ \sigma_0^{(3)}\sigma_n^{(3)}\ |0\rangle|\ \ . \tag{1.5}$$

The validity of the condition (1.4) depends on S; exact calculations indicate that (1.4) is satisfied for all $S \ge 1$, but not for $S = 1/2$ [11, 13, 14].

In [12] it was shown, using the aforementioned cluster representation, that for "half-integer" spins (with $2S$ an odd integer greater than one) condition (1.4) implies, in addition to the bound on block spin fluctuations, also that the translation invariant infinite volume ground state decomposes into a mixture of two states of period 2. The symmetry breaking which occurs in the above spin systems is akin to *dimerization*, though the neutral clusters need not consist just of pairs of neighboring spins. (A notable fact is that the lack of mixing in the ground state is expressed through other correlators than the two-point correlation function, since the spin-spin correlation does decay to zero.)

The general results presented here place this translation symmetry breaking within a broader context, as we show that it is not coincidental that the same condition (1.4) implies, for the quantum spin chains with *odd* values of $2S$, both bounded variance of the block spins and translation symmetry breaking.

2. Formulation and results

Our main results relate reduced fluctuations to *translation symmetry breaking.* A more precise formulation of the conclusion is that the state, given by a translation invariant measure μ on the space of configurations, has a *cyclic factor.* Let us first introduce the relevant terminology.

2.a Cyclic factors and symmetry breaking

Definition 2.1 *A system described by a measure $\mu(d\omega)$, on the space of configurations ω, which is invariant under either the continuous group of translations (shifts T_x by $x \in \mathbb{R}$) or the group of lattice shifts ($x \in \mathbb{Z}$) is said to have a cyclic factor, of period $0 < \lambda < \infty$, if there is a measurable function $\phi(\omega)$ with values in $[0, 2\pi)$ that evolves under the shifts by*

$$\phi(T_x\omega) \; = \; \phi(\omega) - 2\pi\, x/\lambda \quad (\text{mod } 2\pi), \tag{2.1}$$

with, in the lattice case, $\lambda^{-1} \notin \mathbb{Z}$.

An equivalent formulation is that the measure μ can be decomposed into a mixture:

$$\mu(\cdot) = \int_0^{2\pi} \frac{d\theta}{2\pi} \;\; \mu_\theta(\cdot) \tag{2.2}$$

of mutually singular measures which are cycled under the shifts T_x:

$$T_x \; \mu_\theta \; = \mu_{(\theta - 2\pi x/\lambda)_{(\text{mod } 2\pi)}} \,, \qquad d\theta \text{ a.e..} \tag{2.3}$$

In the lattice case, if λ is rational then $\frac{d\theta}{2\pi}$ in (2.2) should be replaced by, or understood as, a probability measure $\nu(d\theta)$ invariant under $\theta \longmapsto \theta - \frac{2\pi}{\lambda}$ (mod 2π).

The two formulations are related by the observation that in the decomposition (2.2) the cyclic component $\mu_\theta(\cdot)$ is supported by configurations ω with $\phi(\omega) = \theta$.

If the invariance is under the continuous group of translations, the existence of a cyclic factor implies that the measure μ is not ergodic under the smaller group of shifts by λ. For lattice systems, which are invariant only under shifts by multiples of the lattice period, if λ is incommensurate with the period (here 1) then each of the lattice shifts may still act ergodically on μ and the translation symmetry breaking is expressed only through the existence of a quasi-periodic structure; if the two lengths are commensurate then there is loss of ergodicity under shifts by some *multiple* of λ.

Nevertheless, in either the continuum or the lattice cases, the existence of a cyclic factor implies that μ does not have good clustering properties. In particular, the *mixing* condition, which implies that for any bounded function $g(\omega)$ with expectation $\int g\, d\mu = 0$

$$\int g\, T_x g\, d\mu \to 0\ , \quad \text{as } |x| \to \infty\ , \tag{2.4}$$

is not satisfied.

Let us note also that it follows from (2.1) that $\phi(\omega)$ is a tail function (under translations), in fact measurable with respect to the σ-algebra at $+\infty$ (or $-\infty$). Thus, if μ is a Gibbs state for some potential, or weakly Gibbs [16], then its cyclic components should also be Gibbs states, in the appropriate sense, for that potential.

2.b Results

Our first result concerns systems of particles in the continuum. The two cases of primary interest to us are: i) point processes, described by measures on the space of locally finite particle configurations on the line $\mathbb{R}$, and ii) point-charge processes on $\mathbb{R}$. A process of either type may be described in terms of a random atomic measure on $\mathbb{R}$ for which the collection of atoms (the locations of the points or the charges) is (a.s.) locally finite. In the former case the measure, which will be denoted by $N_\omega(I)$, is the number of particles in the interval I; in the latter case the measure is the sum of the point charges in I, $Q_\omega(I) = \sum_{x_i \in I} q_{\alpha_i}$, in the notation of example **1.a**.

Theorem 2.1 *Let $\mu(d\omega)$ be a translation invariant probability measure describing a point process on $\mathbb{R}$. If the variance of the number of points in an interval I (i.e., of the random variable $N_I(\omega) \equiv N_\omega(I)$) is bounded uniformly in the size of the interval*

$$\mathbb{E}\left([N_I - \mathbb{E}(N_I)]^2\right) \ \leq\ C\ (< \infty) \tag{2.5}$$

then μ has a cyclic factor, of period

$$\lambda \ =\ 1/\,\mathbb{E}(N_{[0,1]})\ . \tag{2.6}$$

More generally, let μ describe a translation invariant point-charge process on the line, with the charge values restricted to multiples of a common unit e. Assume that for some $\rho \neq 0$ the family of random variables $\{F_\omega(a,b)\}$, defined by

$$F_\omega(a,b) \;=\; Q_\omega((a,b]) - \rho(b-a) \tag{2.7}$$

where $-\infty < a < b < \infty$ and $Q_\omega(I)$ is the total charge in the interval I, is tight. Then the measure μ has a cyclic factor, of period

$$\lambda = e/\rho\,. \tag{2.8}$$

The tightness of the family of random variables $\{F_\omega(a,b)\}_{a,b}$ means that the probabilities of large fluctuations have asymptotically vanishing bounds that are uniform in $a, b \in \mathbb{R}$:

$$\mathrm{Prob}(\,|F_\omega(a,b)| \geq t\,) \;\leq\; p(t) \tag{2.9}$$

with $p(t) \searrow 0$ as $t \to \infty$. A sufficient condition for this is that the variances of Q_I are uniformly bounded, as in (2.5).

The quantity ρ introduced in Theorem 2.1 is the *asymptotic mean* of the charge density. Under the tightness condition formulated above it is well defined even if $\mathbb{E}(\,|Q_\omega(I)|\,) = \infty$. Note also that ergodicity is not assumed in Theorem 2.1.

Our second result deals with systems of charges $\{q_k\}_{k\in\mathbb{Z}}$ on the one-dimensional lattice $\mathbb{Z}$. Here, for any interval $I \subset \mathbb{Z}$, the charge in I is given by $Q_\omega(I) \;=\; \sum_{k\in I} q_k$, with q_k the charge variable at the lattice site $k \in \mathbb{Z}$.

Theorem 2.2 *Let μ describe a translation invariant system of charges on the lattice $\mathbb{Z}$ with the following properties:*

a. *The charges* (q_k) *are restricted to values of the form* $(\gamma + n\,e)$ *with* γ *and* e *fixed, and* n *taking only integer values.*

b. *For some* $\rho \in \mathbb{R}$ *the family of random variables* $\{F_\omega(a,b) \equiv Q_\omega((a,b]) - \rho(b-a)\}$, *with* $a, b \in \mathbb{Z}$ *and* $a < b$, *is tight.*

c. *Furthermore,*

$$\rho - \gamma \;\neq\; 0 \text{ (mod } e) \ . \tag{2.10}$$

Then the measure μ *has a cyclic factor, of period*

$$\lambda = e/\alpha \ , \tag{2.11}$$

with α *defined by*

$$\alpha \in [0, e) \ , \qquad \alpha \;=\; \rho - \gamma \text{ (mod } e) \ . \tag{2.12}$$

Before we turn to the proof, let us note that the conditions *a.* and *c.* are satisfied in the following examples:

Ex. 1 The charges are integer multiples of a common unit e, with the mean density ρ not an integer multiple of e. (This is the most obvious case, with $\gamma = 0$.)

Ex. 2 Lattice systems of *odd* integer charges with the mean ρ not equal to an odd integer. (To cover this case, choose $e = 2$ and $\gamma = 1$.)

Ex. 3 Lattice systems of charges ± 1 (zero not allowed) which fluctuate, i.e., the q_k are neither identically -1 nor identically 1. (This is a special case of 2., of relevance to the spin models mentioned in the introduction. Note that in this situation there is no need to add a separate restriction on ρ.)

The last example is intended also to show that there is some room for maneuver in applying Theorem 2.2. If one regards the charge values ± 1 as integers, with $\gamma = 0$ and $e = 1$, then condition $c.$ requires that $\rho \neq 0$. However, if one presents the two values as $1 + 2n$ (with $\gamma = 1$ and $e = 2$), then the restriction on ρ is $\rho \neq 1 \pmod 2$, but this is automatically satisfied if the charges fluctuate.

2.c Remarks

It is natural to explore by how much one can weaken the conditions which imply translation symmetry breaking. Following are some remarks concerning such questions (1-4), and other matters.

1) The condition that $\rho \neq 0$, in the general part of Theorem 2.1, and the condition $c.$, eq. (2.10), in Theorem 2.2, are needed there. A relevant example is the one-dimensional two-component Coulomb system with charges $\pm e$, of equal densities. This system, in any of its Θ states, has bounded charge fluctuations but it is also mixing under translations [2, 4, 15] and thus has no cyclic factor (the correlations of all bounded local functions decay there exponentially rapidly). A discrete version of this example is obtained by partitioning this system into lattice cells. The state of this lattice model is mixing, as in the continuum model. The model satisfies conditions $a.$ and $b.$, but not $c.$ — since $\rho = \gamma = 0$ for this model.

2) The discreteness of the matter or the charge configurations is crucial — without this condition it is trivial to construct measures on field configurations $\psi_\omega(x)$ in $d = 1$ with bounded fluctuations and arbitrarily good mixing behavior. To do so, one may start from a random field with strong clustering properties whose configurations $u_\omega(x)$ are differentiable functions, and take $\psi_\omega(x) := du_\omega/dx$.

3) The boundedness of the fluctuations cannot be relaxed much beyond the tightness condition. For the spin chain with $S = 1/2$, i.e., the Heisenberg antiferromagnet, there is no translation symmetry breaking even though the growth of the variance of the block spins is subnormal, increasing only logarithmically with the block length. The same is true for a system of point particles on a line with logarithmic interactions, whose distribution, for $\beta = 2$, is the same as that of the eigenvalues of a random Hermitian matrix sampled from the Gaussian ensemble and suitability scaled (the Dyson distribution) [17].

4) We do not see straightforward extensions of our results to higher dimension. It would be natural to guess that the proper extension of the boundedness of the variance in particle number, in one dimension, might be that the variance is of the order of the surface area. In fact, J. Beck [18] proved that the variance in the particle number in a ball of radius r, averaged over r uniformly distributed in an interval $(0, R)$, must grow at least like R^{d-1}, and such a rate is realized in some cases of the OCP. However, examples of the OCP in $d \geq 2$ dimensions show that surface-rate fluctuations are compatible with good mixing properties. More explicitly, the OCP in $d \geq 2$ dimensions, placed in a cube of side L with periodic boundary conditions, will, for small reciprocal temperatures β and small densities ρ, have a limit as $L \to \infty$ which is translation and rotation invariant with good decay of correlations [10, 1] and with the particle numbers having variance of the order of the surface. (The OCP correlations can in fact be computed explicitly for $d = 2$ at $\beta = 2$ and arbitrary densities ρ: the truncated correlations of n particles are then found to decay like $\exp[-\alpha D^2]$, where D is the distance between two subsets of the particle configuration, maximized over the partitions into two parts [19, 9].)

5) While in the introduction we focused on non-trivial examples of translation invariant particle measures in $d = 1$ with bounded fluctuations, one should note that there are also many trivial ones. Just take the points of the integer lattice and rigidly shift them randomly over a unit interval with uniform weight. Or, after shifting, place each particle with uniform density inside an interval of unit length centered on the shifted lattice points. In the first example the periodic structure is clearly visible: the configurations are periodic. It is less obvious, but still not difficult to see, that also in the second example there is a nontrivial periodic structure, though in this case the particle configurations are not themselves periodic. (We remark, as an aside, that the variance on the left side of (2.5) is minimized, for each I, by the shifted integer lattice.)

6) In view of the interest in the "Gibbs state" condition [20, 16], let us note that it follows from our results that the measures satisfying the conditions of our Theorems 2.1 and 2.2 do not admit representation as Gibbs states with potentials for which the interaction across a boundary is bounded. The reason is that in one dimension this boundedness condition implies uniqueness of the Gibbs measure, which excludes the possibility of cyclic decomposition [21].

3. Derivation

The proof of the results in Section 2 rests on two separate observations. The first is that, under the tightness condition, the configuration of the charges, centered by the subtraction of its asymptotic mean ρ, is the distributional derivative of a process, $\mathcal{E}_\omega(x)$, which is a covariant functional of the charge configuration. In the second step we make essential use of the atomic character of the point-charge process, and in particular of the charge constraints, to conclude the existence of a cyclic factor, with cyclic function $\phi(\omega)$ of Definition 2.1 given by a suitable fractional part of $\mathcal{E}_\omega(0) = \mathcal{E}(\omega)$, namely the function $\phi(\omega)$ with values in $[0, 2\pi)$ such that

$$\phi(\omega) \ = \ 2\pi\mathcal{E}(\omega)/e \quad (\mathrm{mod}\ 2\pi)\ . \tag{3.1}$$

The terms used above are defined as follows. A process $\mathcal{E}_\omega(x)$ is said to be a *covariant* functional of ω (in the sense of behaving covariantly under translations) if it satisfies:

$$\mathcal{E}_\omega(x) \ = \ \mathcal{E}_{T_x\omega}(0) \ , \tag{3.2}$$

(equivalently: $\mathcal{E}_\omega(x+a) \ = \ \mathcal{E}_{T_x\omega}(a)$), in which case it is determined by the function

$$\mathcal{E}(\omega) \ \equiv \ \mathcal{E}_\omega(0) \ . \tag{3.3}$$

This stationary process is the *antiderivative* of the charge distribution, "centered" in the sense described above, if

$$Q_\omega\left((a,b]\right) - \rho(b-a) \ = \ \mathcal{E}_\omega(b) - \mathcal{E}_\omega(a) \ . \tag{3.4}$$

More generally, let ω now be any stationary random locally finite signed measure on $\mathbb{R}$, and write $F_\omega(a,b)$ for $\omega((a,b])$. Let $X_\omega(x)$ be its (possibly generalized) "charge" density field, defined through its (possibly formal) integrals over intervals $\int_a^b X_\omega(x)\,dx \ = \ F_\omega(a,b)$. We shall refer to such an $X_\omega(x)$ as a stationary *locally* (weakly) *integrable* process on $\mathbb{R}$. We say that a covariant functional $\mathcal{E}_\omega(x)$ is the antiderivative of $X_\omega(x)$, and that $X_\omega(x)$ is the derivative of $\mathcal{E}_\omega(x)$, if

$$F_\omega(a,b) \ = \ \mathcal{E}_\omega(b) - \mathcal{E}_\omega(a) \ . \tag{3.5}$$

Note that if a stationary locally integrable process $X_\omega(x)$ is the derivative of a covariant functional $\mathcal{E}_\omega(x)$, then its integrals over intervals $F_\omega(a,b)$, $-\infty < a < b < \infty$, form a tight family of random variables. Of key importance to us is that the converse is also true. This fact — as we learned from Y. Peres after informing him of our derivation — is related to an established result in ergodic theory, expressed there as a relation between tight cocycles and coboundaries (see below). In this part of the argument the discreteness of the charges does not play any role, and the statement which we need can be formulated as follows.

Theorem 3.1 *A stationary locally integrable process $X_\omega(x)$, on $\mathbb{R}$, is the derivative of a stationary process $\mathcal{E}_\omega(x)$ given by a covariant functional if and only if its integrals over intervals*

$$F_\omega(a,b) \;=\; \int_a^b X_\omega(x)\,dx \tag{3.6}$$

with $-\infty < a < b < \infty$, form a tight family of random variables.

Let us note that it follows from the theorem that if a process $X_\omega(x)$ has any stationary antiderivative $Y_{\tilde{\omega}}(x)$ at all—possibly defined only on an extension of the original probability space and thus not determined by X_ω—it must also have a stationary covariant antiderivative $\mathcal{E}_\omega(x)$, since the integrals of X_ω over intervals then obviously form a tight family. It is also worth stressing that the stationary antiderivative of $X_\omega(x)$, whose existence is guaranteed by Theorem 3.1 when the tightness condition is satisfied, is completely determined, in a translation invariant manner, by the X-process alone.

For one-dimensional Coulomb systems, the antiderivative of the charge configuration (centered by the subtraction of the asymptotic mean ρ) is, up to an overall shift by a constant, the electric field. While its meaning is clear for finite systems, its existence as a well-defined, and covariant, function of the charge configuration is not immediately obvious for an infinite system [4].

To be more explicit, let us add that if the variables $\{F_\omega(a,b)\}$ are not only tight but also uniformly integrable, satisfying

$$\mathbb{E}(|F(a,b)|) \leq K \tag{3.7}$$

with $K < \infty$ independent of a and b, then the stationary antiderivative can be chosen as:

$$\mathcal{E}_\omega(x) \;=\; E - \lim_{t\to\infty} \int_0^t (1 - \frac{s}{t}) X_\omega(s+x)\,ds\;, \tag{3.8}$$

where the integral is guaranteed to converge almost surely through an application of the ergodic theorem to the process $\mathcal{E}_\omega(x)$ (see the first remark after the proof Theorem 3.1; see also [4]), and the constant E is the mean value, $\mathbb{E}(\mathcal{E}_\omega(x)) = E$. (If $\mathcal{E}_\omega(x)$ is regarded, as it is in this paper, merely as giving a covariant antiderivative of $X_\omega(x)$, then the value of the constant E in eq. (3.8) is of course arbitrary. However, this term is essential in any situation where $\mathcal{E}_\omega(x)$ has significance in its own right, as it does for Coulumb systems. In particular, it plays an important role in the analysis of the Θ-states mentioned in Section 4.b.)

3.a Existence of stationary anti-derivatives

A natural context for Theorem 3.1 is the theory of cocycles, which has been developed within ergodic theory. For a measure preserving transformation $T : \Omega \to \Omega$, a cocycle is a sequence of functions $F_0(\omega), F_1(\omega), \ldots$ $(\omega \in \Omega)$ of the form

$$F_n(\omega) \ = \ F_0(\omega) \ + \ \sum_{j=0}^{n-1} f(T^j\omega) \ . \tag{3.9}$$

The cocycle is coboundary if $f(\cdot)$ is of the form:

$$f(\omega) \ = \ g(T\omega) - g(\omega) \ . \tag{3.10}$$

In the latter case the cocycle is tight, as a sequence of random variables, since then

$$F_n(\omega) - F_0(\omega) \ = \ g(T^n\omega) - g(\omega) \ . \tag{3.11}$$

The converse, derived by Schmidt [22], is also true:

Theorem 3.2 *([22]) Any* tight *cocycle is a coboundary.*

As mentioned above, we owe the reference to Y. Peres. The L^2 version of the result is a yet older result of Leonov [23]; a recent generalization is found in ref. [24].

Theorem 3.1 is a continuum analog of Theorem 3.2, from which it can be easily derived. However, for the completeness of presentation we shall sketch a direct proof.

Proof of Theorem3.1: Given that the argument is not that different from the derivation of Theorem 3.2, we permit ourselves to present here just a brief summary.

We start by introducing a convenient extension of the space Ω over which the random process $X_\omega(x)$ is defined. Let $\widetilde{\Omega} = \Omega \times \mathbb{R}$ and let $\tilde{\mu}(d\widetilde{\omega})$ be a probability measure on $\widetilde{\Omega}$ having Ω-marginal μ . Writing the points in $\widetilde{\Omega}$ as $\widetilde{\omega} = (\omega, Y)$, we extend the translations T_x, originally defined on Ω, into a flow on $\widetilde{\Omega}$ by:

$$\widetilde{T}_x(\omega, Y) \; = \; (T_x\omega, \, Y + F_\omega(0,x)\,) \; . \tag{3.12}$$

Notice that on the enlarged space the covariant functional defined by

$$Y_{\widetilde{\omega}}(x) \; \equiv \; Y(\widetilde{T}_x\widetilde{\omega}) \; , \tag{3.13}$$

with $Y(\widetilde{\omega})$ the second coordinate of $\widetilde{\omega}$, provides an antiderivative of $X_\omega(x)$, since

$$Y_{\widetilde{\omega}}(x) - Y_{\widetilde{\omega}}(0) \; = \; F_\omega(0,x) \; . \tag{3.14}$$

Moreover, if $\tilde{\mu}(d\widetilde{\omega})$ is stationary under the flow $\widetilde{T}_x$, then the random field $Y_{\widetilde{\omega}}(x)$, on the probability space $\{\widetilde{\Omega}, \tilde{\mu}\}$, is stationary under $\widetilde{T}_x$. This field, however, need not yet provide us with the covariant functional $\mathcal{E}_\omega(x)$ of Theorem3.1, since it may depend not only of ω but also on the entire $\widetilde{\omega}$.

The proof of Theorem 3.1 is in two steps. First, it is established that there indeed exists a probability measure $\tilde{\mu}(d\widetilde{\omega})$ on $\widetilde{\Omega}$ such that: *i.* the Ω marginal of $\tilde{\mu}$ is $\mu(d\omega)$, and *ii.* $\tilde{\mu}$ is stationary under $\widetilde{T}_x$. Then, using the stationarity — which permits the application of the ergodic theorem — we construct a modified field on $\widetilde{\Omega}$ which depends only on the first coordinate, ω. It is this field which yields the desired $\mathcal{E}_\omega(x)$.

To prove the first claim, we let $\nu_L(d\widetilde{\omega})$ be the probability measure for which ω has the distribution $\mu(d\omega)$ and the conditional distribution of Y, given ω, is that of the random variable $Y = F_\omega(-u,0)$ with u sampled over $[0,L]$ with the uniform probability distribution (du/L). It is easy to see that $\nu_L(\cdot) = \frac{1}{L}\int_0^L \widetilde{T}_u \nu_0 \; du$, where $\nu_0 = \mu \times \delta_0$, is close to being invariant under $\widetilde{T}_x$, with the variational distance between $\widetilde{T}_x \nu_L(\cdot)$ and $\widetilde{\nu}_L(\cdot)$ bounded by $2x/L$.

It is also easy to see that the distribution of Y under the measures $\nu_L(\cdot)$ is tight:

$$\begin{aligned} \mathrm{Prob}_{\nu_L}(\,|Y| \geq t\,) \quad &= \quad \frac{1}{L}\int_0^L \mathrm{Prob}_\mu(\,|F_\omega(-u,0)| \geq t\,)\; du \\ &\leq \quad \sup_{u>0} \mathrm{Prob}_\mu(\,|F_\omega(-u,0)| \geq t\,)\;. \end{aligned} \tag{3.15}$$

Under the tightness assumption of Theorem 3.1, the right hand side vanishes as $t \to \infty$.

The above considerations suggest that compactness and continuity arguments can be invoked to prove that the sequence of probability measures $\nu_L(\cdot)$ has a convergent subsequence, as $L \to \infty$, and that the limiting measure is strictly stationary under the flow $\widetilde{T}$. The existence of a suitable limiting measure can indeed be proven using a topology which is natural for ω in the present set-up (it is at this point that we leave the argument at the level of a sketch), for which eq. (3.15) implies that the sequence of measures $\nu_L(\cdot)$ is tight, and thus has a convergent subsequence, with limit $\tilde{\mu}$.

Clearly the $\Omega-$marginal of $\tilde{\mu}$ is μ. That $\tilde{\mu}$ is also stationary would be immediate were $\widetilde{T}_x$ continuous on $\widetilde{\Omega}$. However, there is here a slight complication in that the second component on the right-hand-side of eq. (3.12) is not continuous in ω. Nevertheless, the set of configurations ω for which $\widetilde{T}_x\widetilde{\omega}$ is discontinuous—those configurations having atoms at 0 or at x—has μ-measure 0, (an easy consequence of the stationarity of μ). Using this fact, the conclusion of stationarity follows.

The above implies that in the larger space $\widetilde{\Omega}$, the process $X_\omega(x)$ has a stationary antiderivative $Y_{\widetilde{\omega}}(x)$. While the antiderivative thus obtained need not be determined by ω alone, its stationarity implies that it has the necessary asymptotic statistical regularity to yield an antiderivative of the field X_ω which is a covariant function of ω alone. This can be done by selecting from among the one-parameter family of possible antiderivatives (differing only by an overall shift) the one whose median value, averaged along the positive x-axis, is set at $Y = 0$.

The median level for $Y(\cdot)$ is defined by

$$M[Y] \;= \inf_{M} \left\{ M \;:\; \lim_{L\to\infty} \frac{1}{L} \int_0^L I[Y(x) \geq M]\, dx \;\; \leq 1/2 \right\} \;, \tag{3.16}$$

provided the limit exists. For $Y_{\widetilde{\omega}}(\cdot)$, which is sampled with a stationary distribution, the existence of the limit (simultaneously for a countable collection of M) follows by the ergodic theorem. Using the median, we define

$$\mathcal{E}_{\widetilde{\omega}}(x) \;\; = \;\; Y_{\widetilde{\omega}}(x) - M[Y_{\widetilde{\omega}}] \,. \tag{3.17}$$

Because $M[Y + \text{Const.}] = M[Y] + \text{Const.}$, $\mathcal{E}_{\widetilde{\omega}}$ depends only on ω (and not on the other coordinate of $\widetilde{\omega}$), and thus it defines a suitable covariant antiderivative of $X_\omega(x)$. ■

Remarks:

1. Under the additional assumption that eq. (3.7) holds, the stationary antiderivative $Y_{\widetilde{\omega}}$ constructed in the first step of the argument above satisfies $\mathbb{E}[Y_{\widetilde{\omega}}(0)] < \infty$, as may easily be seen from the construction of $\tilde{\mu}$. One may then alternatively define an antiderivative functional $\mathcal{E}$ using the average $\bar{Y}$ of Y, $\bar{Y} = \lim_{L\to\infty} \frac{1}{L}\int_0^L Y(x)\, dx$, in place of the median $M[Y]$. Moreover, for this antiderivative we may obtain an explicit formula: Averaging the expression $\mathcal{E}_\omega(0) - \mathcal{E}_\omega(u) = -F_\omega(0,u)$ with respect to du/L over the interval $(0, L]$ and taking the limit $L \to \infty$, using the fact that, by construction, $\lim_{L\to\infty} \frac{1}{L}\int_0^L \mathcal{E}_\omega(u)\, du = 0$, one arrives, in fact, at the formula given by eq. (3.8) (see ref. [4] for further discussion of eq. (3.8)).

2. Suppose $X_\omega(x)$ is ergodic. Then it is easy to see—either by using the covariant antiderivative of Theorem 3.1 or directly—that a stationary antiderivative $Y_{\widetilde{\omega}}(x)$ of $X_\omega(x)$

is a covariant functional of ω alone if and only if $Y_{\tilde{\omega}}(x)$ is ergodic. Moreover, two ergodic antiderivatives of $X_\omega(x)$ differ by an absolute constant. Thus if $X_\omega(x)$ is ergodic, instead of proceeding to step 2 as described above, we could obtain a covariant antiderivative $\mathcal{E}$ by simply decomposing $\tilde{\mu}$, corresponding to the antiderivative $Y_{\tilde{\omega}}(x)$ from step 1, into its ergodic components and choosing any one of these. Moreover, using the ergodic theorem, it follows from the manner of construction of ν_L that the process $Y_{\tilde{\omega}}(x)$ from step 1 is a mixture of the antiderivatives $\mathcal{E}(x) - y$ with weights $\mu_{\mathcal{E}(0)}(dy)$, i.e., with y random, with distribution that of $\mathcal{E}(0)$. In particular, the distribution of $Y_{\tilde{\omega}}(0)$ is that of the difference of two independent copies of $\mathcal{E}(0)$. Note also that it follows that all limits of ν_L, $L \to \infty$, must agree, independent of which convergent subsequence is chosen, so that in fact $\lim_{L\to\infty} \mu_L$ itself exists, without passage to a subsequence.

3.b Proof of the main results

Proof of Theorem 2.1: It suffices to consider the point-charge processes, since any point process may be regarded as a special case with unit charges. Under the assumption of tight fluctuations, Theorem 3.1 says that there is a measurable function $\mathcal{E}(\omega)$ of the charge configuration such that

$$\mathcal{E}(T_x\omega) \ = \mathcal{E}(\omega) + Q_\omega((0,x]) - \rho x \ . \tag{3.18}$$

Since the charge $Q_\omega((0,x])$ assumes only values which are integer multiples of e, we may conclude that $\phi(\omega) \equiv \frac{2\pi}{e}\mathcal{E}(\omega) \pmod{2\pi}$ defines a cyclic factor as described in Theorem 2.1 and Definition 2.1. ■

Proof of Theorem 2.2: The argument in the discrete case is similar to the above, with the only notable difference being that the charge $Q_\omega(\,(0,x]\,)$ now includes γx in addition to an integer multiple of e. Thus in this case the "antiderivative functional"

$\mathcal{E}(\omega) = g(T\omega)$ provided by Theorem 3.2 satisfies

$$\mathcal{E}(T_x\omega) \ = \mathcal{E}(\omega) + (\gamma - \rho)x \pmod{e} \tag{3.19}$$

and hence $\phi(\omega) \equiv \frac{2\pi}{e}\mathcal{E}(\omega) \pmod{2\pi}$ defines a cyclic factor as described in Theorem 2.2 and Definition 2.1. ∎

4. Extensions

4.a Jellium tubes

Our discussion of one-dimensional systems applies also to elongated tubes which are locally of higher dimension, but are of finite cross section and infinite in one direction. An example of such a system is the three dimensional OCP with the flux lines of the three dimensional field restricted to stay in the tube. (this is an idealized situation, possibly mimicking some high contrast dielectric materials, or a Kalutza-Klein model with only one unconfined dimension.) If the variances of the total charge in tubes of length L stays bounded, as may be expected under the Coulomb interactions, then the theory presented here implies translation symmetry breaking, along the unconfined direction. In such a situation the projections of the positions of the point charges along the free direction yield a one dimensional system of the type discussed in this paper. For $d = 2$ this system is considered in [19] for the explicitly solvable case $\beta = 2$.

4.b Θ-states of one-dimensional Coulomb gas

Unlike the OCP the one-dimensional Coulomb gas consisting of a continuum system of point charges of value $\pm e$, with respective densities $\rho_+ = \rho_-$, does not break translation symmetry, see Remark 1) of Section 2.c. This system does however exhibit another form of non-uniqueness of the Gibbs state: it admits a one-parameter family of infinite-volume translation invariant Gibbs states, indexed by the fractional part (Θ) of the boundary charge "imposed at infinity" [4, 15]. While at first glance this "anomaly" appears to be of a different kind than what is discussed here, let us point out that it can also be viewed as a remnant of the translation symmetry breaking in asymmetric charge systems.

The charge-symmetric model may be arrived at as a limit of Coulomb systems with a uniform background of small charge density $\rho_{bg} = -\epsilon$, and $\rho_+ = \rho_- + \epsilon$. In this situation our results do apply, and prove that the system exhibits translation symmetry breaking, the corresponding infinite volume Gibbs states having a cyclic factor of period $\lambda = 1/\epsilon$. For each ϵ the resulting cyclic components $\mu_\theta^{(\epsilon)}$ (of equation (2.2)) are periodic under translations, with period $1/\epsilon$. Moreover, the dependence of $\mu_\theta^{(\epsilon)}$ on θ can be so chosen that when the asymmetry parameter ϵ is taken down to zero, these converge (locally) to a one-parameter family of translation invariant states (i.e., probability measures on the space of configurations) which form the Θ-states of the symmetric Coulomb gas. The proof can be obtained using the methods of ref. [4].

Acknowledgments

We thank Yuval Peres for calling our attention to the relation of our work with the theory of tight cocycles [22-24]. MA gratefully acknowledges the hospitality accorded him at Rutgers University. The work was supported in part by the NSF grants PHY-9971149, DMR-9813268 and AFOSR Grant F49620–98–1–0207.

References

[1] D. C. Brydges and P. A. Martin, "Coulomb systems at low density: A review," *J. Stat. Phys.*, **96**, 1163 (1999). See also Ph. Martin, "Sum Rules in Charged Fluids", *Rev. Mod. Phys.*, **60**, 1075–1127 (1988).

[2] A. Lenard, "Exact statistical mechanics of a one-dimensional system with Coulomb forces. III. Statistics of the electric field," *J. Math. Phys.*, **4**, 533 (1963).

[3] J. L. Lebowitz and E. H. Lieb, "The existence of thermodynamics for real matter with Coulomb forces," *Phys. Rev. Lett.*, **22**, 631 (1969). E. H. Lieb and J. L. Lebowitz, "The constitution of matter: Existence of thermodynamics for systems composed of electrons and nuclei," *Advances in Math.*, **9**, 316, (1972).

[4] M. Aizenman and P. A. Martin, "Structure of Gibbs states of one-dimensional Coulomb systems," *Comm. Math. Phys.*, **78**, 99. (1980).

[5] H. van Beijeren and B.U. Felderhof, *Mol. Phys.*, **38**, 1179 (1979); Ch. Gruber, Ch. Lugrin, and Ph. A. Martin, *J. Stat. Phys.*, **22**, 193 (1980); Ph.A. Martin and T. Yalcin, *J. Stat. Phys.*, **22**,435 (1980).

[6] J.L. Lebowitz, *Phys. Rev. A* **72**, 773 (1983); J.L. Lebowitz in *Strongly Coupled Coulomb Systems*, G.J. Kalman, J.M. Rommel and K. Blagoev, eds., Plenum Press, New York and London, 1998; D. Levesque, J.-J. Weis and J. L. Lebowitz, "Charge Fluctuations in the Two-Dimensional One-Component Plasma", *J. Stat. Phys.*, **100**, 209-222, 2000.

[7] M. Gaudin, *J. Phys.*, **34**, 511 (1973).

[8] H. J. Brascamp and E. H. Lieb, "Some inequalities for Gaussian measures and the long-range order of the one-dimensional plasma," in Functional Integration and its Applications, A. M. Arthurs (ed.), (Oxford Univ. Press, 1975).

[9] B. Jancovici, *Phys. Rev. Lett.*, **46**, 366, (1981).

[10] J. Imbrie, *Commun. Math. Phys.*, **87**, 515, (1983).

[11] I. Affleck, "Exact results on the dimerization transition in SU(n) antiferromagnetic chains," *J. Phys. C: Cond. Matt.*, 405, (1990).

[12] M. Aizenman and B. Nachtergaele, "Geometric aspects of quantum spin states," *Comm. Math. Phys.*, **164**, no. 1, 17, (1994).

[13] M. T. Batchelor and M. N. Barber, "Spin-s quantum chains and Temperley-Lieb algebras," *J. Phys. A*, **23**, no. 1, L15–L21, (1990).

[14] A. Klümper, "The spectra of q-state vertex models and related antiferromagnetic quantum spin chains," *J. Phys. A*, **23**, no. 5, 809, (1990).

[15] M. Aizenman and J. Fröhlich, *J. Stat. Phys.*, "States of one-dimensional Coulomb systems as simple examples of Θ-vacua and confinement", **26**, 247 (1981).

[16] Ch. Maes, A. van Moffaert, and F. Redig, "Almost Gibbsian versus Weakly Gibbsian Measures", *Stoch. Proc. and Appl.*, **79**, 1–15, (1999).

[17] M. L. Mehta, *Random Matrices.* Academic Press, New York, 1990, 2nd ed.

[18] J. Beck, *Acta Math*, 1, **159**, (1987).

[19] P. Choquard, P.J. Forrester and E.R. Smith, "The Two Dimensional One Component Plasma at $T = 2$; The Semiperiodic Strip", *J. Stat. Phys.*, **33**, 13 (1983); B. Jancovici, private communication.

[20] A. C. D. van Enter, R. Fernández, and A. D. Sokal, "Regularity properties and pathologies of position-space renormalization-group transformations: scope and limitations of Gibbsian theory," *J. Stat. Phys.*, **72**, no. 5-6, 879–1167, (1993).

[21] H.-O. Georgii, "Gibbs Measures and Phase Transitions", W. De Gruyter, Berlin (1988).

[22] K. Schmidt, *Cocycles of Ergodic Transformation Groups.* Lect. Notes in Math. Vol. 1, MacMillan Co. of India, 1977.

[23] V. Leonov, "On the dispersion of time averages of a stationary random process," *Theory Probab. Appl.*, **6**, 93, (1961).

[24] J. Aaronson and B. Weiss, "Remark on tightness of cocycles," 1999 preprint.