Multidimensional Inverse Scattering for the Nonlinear Klein-Gordon Equation with a Potential*†

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Abstract

In this paper we solve the multidimensional inverse scattering problem for the nonlinear Klein-Gordon equation on $\mathbf{R}^n, n \geq 2$:

$$\frac{\partial^2}{\partial t^2}u(x,t) - \Delta u(x,t) + u(x,t) + V_0(x)u(x,t) + \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u(x,t) = 0.$$

We prove that the small-amplitude limit of the scattering operator determines uniquely all the $V_j, j = 0, 1, \cdots$. Our proof gives, as well, a method for the reconstruction of the $V_j, j = 0, 1, \cdots$.

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1 Introduction

Let us consider the following nonlinear Klein-Gordon equation with a potential

$$\frac{\partial^2}{\partial t^2}u(x,t) - \Delta u(x,t) + u(x,t) + V_0 u(x,t) + F(x,u) = 0,\ u(x,0) = f_1,\ \frac{\partial}{\partial t}u(x,0) = f_2, \tag{1.1}$$

where $u(x,t)$ is a complex-valued funcion of $x \in \mathbf{R}^n, t \in \mathbf{R}$, and $F(x,u)$ is a complex-valued function. Moreover, $\Delta := \sum_{j=1}^n D_j^2$. We use the standard notation, $D_j := \frac{\partial}{\partial x_j}$ and for $\alpha := (\alpha_1, \cdots, \alpha_n)$, $D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, with $|\alpha| := \sum_{j=1}^n \alpha_j$. We introduce some assumptions and definitions that we need to construct the scattering operator for the nonlinear Klein-Gordon equation (1.1).

Assumption A

Let p satisfy: $\rho < p \leq \frac{n+3}{n-1}$, where ρ is the positive root of $\frac{n}{2}\frac{\rho-1}{\rho+1} = \frac{1}{\rho}$. Let k be an integer such that $k > \frac{n}{p+1}$. Let $F = F_1 + iF_2$ with F_1, F_2 real-valued, and $u = r_1 + ir_2, r_1, r_2 \in \mathbf{R}$. We suppose that $F(0) = 0$ and that for all integers β with $1 \leq \beta \leq k+1$ and all α with $\beta + |\alpha| \leq k+1$, we have that,

$$\sum_{j=1}^{2} \left| \frac{\partial^\beta}{\partial r_1^{\beta_1} \partial r_2^{\beta 2}} D^\alpha F_j(x,u) \right| \leq C|u|^{\max[0,p-\beta]},\ \text{for } |u| \leq \gamma, \tag{1.2}$$

for some $\gamma > 0$, and for all nonnegative integers, β_1, β_2, with $\beta = \beta_1 + \beta_2$.

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We denote by H_0 the self-adjoint realization of $-\Delta$ in $L^2(\mathbf{R}^n)$ with domain the Sobolev space $W_{2,2}$. For the definition of the Sobolev spaces $W_{j,p}, j = 1, 2, \cdots, 1 \leq p \leq \infty$ see [1].

Assumption B

We assume that V_0 is real valued and that,

(a)If $n \geq 3$ we suppose that for some $\delta > (3n/2) + 1$,

$$\sup_{x \in \mathbf{R}^n} (1+|x|)^\delta \left(\int_{|x-y| \leq 1} |D^\alpha V_0(y)|^{p_0}\, dy \right)^{1/p_0} < \infty, \tag{1.3}$$

for all $|\alpha| \leq k+2+k_0$, with k as in Assumption A . If $n=3$, $p_0=2$ and $k_0=0$, and if $n \geq 4$, $p_0 > n/2$ and $k_0 := [(n-1)/2]$, where $[\sigma]$ denotes the integral part of σ.

(b)if $n=2$ we suppose that

$$|D^\alpha V_0(x)| \leq C(1+|x|)^{-\delta}, \tag{1.4}$$

for some $\delta > 6$ and for all $|\alpha| \leq k+2$ with k as in Assumption A. Furthermore, we suppose that $\int V_0(x)\, dx \neq 0$.

Moreover, we assume that zero is neither an eigenvalue nor half-bound state (a resonance) of $H := H_0 + V_0$, for $n \geq 2$.

■

For $n \geq 3$, zero is a half-bound state (a resonance) of H if the equation $H\phi = 0$ has a solution $\phi \notin L^2(\mathbf{R}^n)$, such that $(1+|x|)^{-1-\epsilon}\phi \in L^2(\mathbf{R}^n)$ for all $\epsilon > 0$ [23], [24]. In the case $n=2$, zero is a half-bound state (a resonance), [25] if $H\phi = 0$ has a nontrivial solution that is locally in $W_{2,2}$ and that has the following asymptotic behaviour at infinity,

$$\frac{\partial^\alpha}{\partial x^\alpha}\left(\phi - a - \frac{b_1 x_1 + b_2 x_2}{|x|^2}\right) = O\left(|x|^{-1-|\alpha|-\epsilon}\right),\ |\alpha| \leq 1, \tag{1.5}$$

for some $\epsilon > 0$, and where at least one of the constants a, b_1, b_2, is different from zero.

Under Assumption B H is self-adjoint with domain $W_{2,2}$ and it has no singular-continuous spectrum and no positive eigenvalues [6]. Moreover, the wave operators

$$W_\pm := s - \lim_{t \to \pm\infty} e^{itH}\, e^{-itH_0}, \tag{1.6}$$

exist and Range$W_\pm = \mathcal{H}_c$, the subspace of continuity of H. In fact, these results are true under more general conditions. The crucial issue is that Yajima has proven that under Assumption B the wave operators and the adjoints, $W_\pm^*$, are bounded operators on $W_{l,p}, l = 0, 1, \cdots, k+2$, where if $n \geq 3$, $1 \leq p \leq \infty$ and if $n = 2, 1 < p < \infty$. Moreover, there is a constant C such that for all Borel functions f

$$\frac{1}{C}\|f(H_0)\|_{\mathcal{B}\left(W_{l,p}, W_{\acute{l},\acute{p}}\right)} \leq \|f(H)P_c\|_{\mathcal{B}\left(W_{l,p}, W_{\acute{l},\acute{p}}\right)} \leq C\,\|f(H_0)\|_{\mathcal{B}\left(W_{l,p}, W_{\acute{l},\acute{p}}\right)}, \tag{1.7}$$

for the same $p, l, \acute{l}$, as above and where P_c is the projector onto the subspace of continuity of H. For this result with $n \geq 3$ see Theorem 1.2 of [24] and also [23] and for the case $n = 2$ see [25]. For similar results on the case $n = 1$ see [18]. Equation (1.7) allows us [23] to obtain $L^p - L^{\acute{p}}$ estimates for the linear Klein-Gordon equation with a potential from the corresponding results when the potential is zero. More precisely, let us consider the Klein-Gordon equation,

$$\frac{\partial^2}{\partial t^2} u(x,t) - \Delta u(x,t) + u(x,t) + V_0 u(x,t) = 0,\ u(x,0) = 0,\ \frac{\partial}{\partial t} u(x,0) = f. \tag{1.8}$$

We denote by T_t the operator $f \hookrightarrow u(\cdot, t)$. For any pair of Banach spaces X, Y, $\mathcal{B}(X,Y)$ denotes the Banach space of all bounded operators from X into Y. Then by (1.7) and Theorems B and C of [8],

$$\|T_t P_c\|_{\mathcal{B}(W_{l,p}, W_{l,q})} \leq C|t|^a,\ l = 0, 1, \cdots, k+2,\ |t| \geq 1, \tag{1.9}$$

where, $\frac{1}{p} + \frac{1}{q} = 1$ and for $\frac{2(n+1)}{n+3} \leq p \leq \frac{2(n+2)}{n+4}$, $a := -\frac{2}{q}$. For $\frac{2(n+2)}{n+4} \leq p \leq 2$, $a := \frac{n}{2} - \frac{n}{p}$. Moreover,

$$\|T_t P_c\|_{\mathcal{B}(W_{l,p}, W_{l,q})} \leq C|t|^b, \text{ where } b := n + 1 - \frac{2n}{p},\ l = 0, 1, \cdots, k+2, \tag{1.10}$$

$\frac{2(n+1)}{n+3} \leq p \leq 2$.

Estimates (1.9) (1.10 allows us to extend to the case of $n \geq 2$ the method for the construction of the scattering operator for (1.1) and for the solution of the inverse scattering problem, that we gave in [20] in the case of $n = 1$.

We write (1.1) as a first order system in the usual way:

$$\frac{\partial}{\partial t} U = AU + P(U), \text{ where } U := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix};\ A := \begin{bmatrix} 0 & 1 \\ \Delta - V_0 - 1 & 0 \end{bmatrix};\ P(U) := \begin{bmatrix} 0 \\ -F(x, u_1) \end{bmatrix},$$

$$U(x,0) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{1.11}$$

The equation (1.1) is equivalent to (1.11) if we take $u = u_1$, and $\frac{\partial}{\partial t} u = u_2$.

Let us consider the following spaces:

$$\mathcal{H} := W_{k,2} \oplus W_{k-1,2},\ X := W_{k,p+1} \oplus W_{k-2,p+1}, p \geq 1, \tag{1.12}$$

with k and p as in Assumption A and

$$M := \left\{ U \in C(\mathbf{R}, X) : \sup_{t \in \mathbf{R}} (1 + |t|)^d \|U\|_X < \infty \right\},\ \text{ with norm: } \|U\|_M := \sup_{t \in \mathbf{R}} (1 + |t|)^d \|U\|_X, \tag{1.13}$$

where if $1 \leq p \leq 1 + \frac{4}{n}$, $d := \frac{n}{2}\frac{p-1}{p+1}$ and if $1 + \frac{4}{n} \leq p \leq \frac{n+3}{n-1}$, $d := \frac{2}{p+1}$. Finally, if $n \geq 3$, or $n = 2$ and $p \leq 3$,we define

$$N := W_{k+2,1+\frac{1}{p}} \oplus W_{k,1+\frac{1}{p}}, \tag{1.14}$$

and if $n = 2$ and $p > 3$,

$$N := W_{k+2,1+\frac{1}{p}} \cap W_{k+2,2} \oplus W_{k,1+\frac{1}{p}} \cap W_{k,2}. \tag{1.15}$$

Note that by Sobolev's imbedding theorem [1], $N \subset \mathcal{H}$ with a continuous imbedding. For functions $U(x,t)$ defined on $\mathbf{R}^n \times \mathbf{R}$ we write $U(t)$ for $U(\cdot, t)$. Let us denote by $\mathcal{V}(t)$ the operator that gives the solution to the linear initial value problem (1.11) with $P = 0$, i.e., $U(t) = \mathcal{V}(t)U(0)$. The operators $\mathcal{V}(t), t \in \mathbf{R}$, form a strongly-continuous group on $\mathcal{H}$ that is uniformly bounded in t (see (2.9) in Section 2). Let us denote [14],

$$N_\delta(V_0) := \sup_{x \in \mathbf{R}^n} \left[\int_{|x-y|<\delta} |V(y)|^{p_0} dy \right]^{1/p_0}. \tag{1.16}$$

In the following theorem we construct the scattering operator.

THEOREM 1.1. *Suppose that Assumptions A, and B are satisfied,that H has no eigenvalues and that,* $\lim_{\delta \to 0} N_\delta(D^\alpha V_0) = 0$, *where* $|\alpha| \leq k$, *with* k *and* p_0 *as in Assumption B. Then, there is a* $\delta > 0$ *such that for all* $\phi_- \in N$ *with* $\|\phi_-\|_N \leq \delta$ *there is a unique solution,* U, *to (1.11) such that* $U \in C(\mathbf{R}, \mathcal{H}) \cap M$ *and*

$$\lim_{t \to -\infty} \|U(t) - \mathcal{V}(t)\phi_-\|_{\mathcal{H}} = 0. \tag{1.17}$$

Moreover, there is a unique $\phi_+ \in \mathcal{H}$ *such that*

$$\lim_{t \to \infty} \|U(t) - \mathcal{V}(t)\phi_+\|_{\mathcal{H}} = 0. \tag{1.18}$$

Furthermore, $\mathcal{V}(t)\phi_\pm \in M$ *and*

$$\|U(t) - \mathcal{V}(t)\phi_\pm\|_M \leq C \, \|\mathcal{V}(t)\phi_\pm\|_M^p, \tag{1.19}$$

and

$$\|\phi_+ - \phi_-\|_{\mathcal{H}} \leq C \, \|\phi_-\|_N^p. \tag{1.20}$$

The scattering operator, $S_{V_0} : \phi_- \hookrightarrow \phi_+$ *is injective on* N.

Note that for $n = 2$, $\rho \approx 2.41$, for $n = 3$, $\rho = 2$, and that $\lim_{n \to \infty} \rho = 1$. Moreover, observe that in Theorem 1.1 we do restric F in such a way that energy is conserved.

Let us denote $B_1 := \sqrt{H+1}$ and $B_0 := \sqrt{H_0+1}$. Under the conditions of Theorem 1.1 the wave operators

$$W_{1,\pm} := s - \lim_{t \to \pm\infty} e^{itB_1} \, e^{-itB_0} \tag{1.21}$$

exist and are unitary operators on L^2. The limits in the right-hand side of (1.21) are taken in the strong topology in L^2. Also, the invariance principle holds, and

$$W_{1,\pm} = W_{\pm}. \tag{1.22}$$

For these results see [13]. We denote by $\mathcal{V}_0(t)$ the solution operator for the initial value problem (1.11) with $F = 0$ and $V = 0$. It is given by (2.9) in Section 2 with H_0 instead of H. By J we denote the identification operator from $\mathcal{H}$ onto $\mathcal{H}_{V_0}$ (see (2.8)): $JU := U$. The wave operators for the linear Klein-Gordon equation are given by,

$$\Omega_{\pm} := s - \lim_{t \to \pm\infty} \mathcal{V}(-t) \, J \, \mathcal{V}_0(t). \tag{1.23}$$

The limits in the right-hand side of (1.28) are taken in the strong topology in $\mathcal{H}_{V_0}$. It follows from Theorems XI.75 and XI.76 in [13] that $\Omega_{\pm}$ exist and that they are unitary from $\mathcal{H}$ onto $\mathcal{H}_{V_0}$. Moreover, by (1.22), the proof of Theorem XI.75 and Theorem XI.76 of [13]:

$$\Omega_{\pm} = T^{-1} \begin{bmatrix} W_{\pm} & 0 \\ 0 & W_{\mp} \end{bmatrix}; \; T := \frac{1}{\sqrt{2}} \begin{bmatrix} B_1 & i \\ B_1 & -i \end{bmatrix}. \tag{1.24}$$

The scattering operator for the linear Klein-Gordon equation is defined as:

$$S_L := \Omega_+^{-1} \, \Omega_-. \tag{1.25}$$

By (1.25),

$$S_L = \begin{bmatrix} \mathcal{S}_L & 0 \\ 0 & \mathcal{S}_L^* \end{bmatrix}, \tag{1.26}$$

where $\mathcal{S}_L$ is the scattering operator for the linear Schrödinger equation on the line:

$$\mathcal{S}_L := W_+^* W_-. \tag{1.27}$$

By (1.26) and (1.27) S_L is bounded on X and on $\mathcal{H}$. In order to reconstruct the potential V_0 we introduce below the scattering operator that relates asymptotic states that are solutions to the linear Klein-Gordon equation with potential zero:

$$S := \Omega_+^{-1} \, S_{V_0} \, \Omega_- . \tag{1.28}$$

We now reconstruct S_L from S.

THEOREM 1.2. *Suppose that the assumptions of Theorem 1.1 are satisfied. Then for every $\phi \in N$,*

$$\frac{d}{d\epsilon} S(\epsilon\phi)\Big|_{\epsilon=0} = S_L \phi, \tag{1.29}$$

where the derivative in the left-hand side of (1.29) exists in the strong convergence in $\mathcal{H}$.

COROLLARY 1.3. *Under the conditions of Theorem 1.1 the scattering operator, S, determines uniquely the potential V_0.*

Proof: By Theorem 1.2 S determines uniquely S_L. Then, from (1.26) we obtain $\mathcal{S}_L$. From $\mathcal{S}_L$ we uniquely reconstruct V_0 using any of the standard methods (see for example [5]).

Let us now consider the case where $F(x,u) = \sum_{j=1}^{\infty} V_j(x) |u|^{2(j_0+j)} u$. As we prove below we can also reconstruct the $V_j, j = 1, 2, \cdots$.

For $\lambda \geq 1$ and $\acute{x} \in \mathbf{R}^n$ we denote by H_λ the following self-adjoint operator in L^2:

$$H_\lambda := H_0 + V_\lambda(x), \text{ where } V_\lambda(x) = \frac{1}{\lambda^2} V_0\left(\frac{x}{\lambda} + \acute{x}\right). \tag{1.30}$$

As we assume that H has no eigenvalues, we have that H_λ has no eigenvalues, i.e., $H_\lambda > 0$. Let $\Phi \in C_0^\infty(\mathbf{R} \setminus 0)$.

LEMMA 1.4. *Suppose that the conditions of Theorem 1.1 are satisfied with $p := \frac{n+3}{n-1}$, and moreover, that $F(x,u) = \sum_{j=1}^{\infty} V_j(x)|u|^{2(j_0+j)}u$, for $|u| \leq \gamma$, for some $\gamma > 0$, where j_0 is an integer such that $j_0 \geq (p-3)/2$, and where $V_j \in W_{k,\infty}$ with $\|V_j\|_{W_{k,\infty}} \leq C^j, j = 1, 2, \cdots$, for some constant C. Then, for any ϕ such that if $n = 2, \phi \in W_{k,1+1/p} \cap W_{\max(k,2),2}$ and if $n \geq 3, \phi \in W_{k,1+1/p} \cap W_{k+1,2}$, there is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$:*

$$((S_{V_0} - I)(\epsilon U_0), V_0)_{\mathcal{H}_V} = \sum_{j=1}^{\infty} \epsilon^{2j+1} \left[\int_{-\infty}^{\infty} dt \, dx \, V_j(x) \; |(\mathcal{V}(t) U_0)_1|^{2(j+j_0+1)} + Q_j \right], \text{ with} \tag{1.31}$$

$$U_0 := \begin{bmatrix} 0 \\ \phi \end{bmatrix}, V_0 := \begin{bmatrix} (H+I)^{-1}\phi \\ 0 \end{bmatrix}, \tag{1.32}$$

where $Q_{j_0} = 0$, *and* $Q_j, j > j_0$, *depends only on* ϕ *,and on* V_k *with* $k < j$. *Moreover, for any* $\acute{x} \in \mathbf{R}$, *and any* $\lambda \geq 1$, *we denote,* $\phi_\lambda(x) := \lambda\left(\Phi(H_\lambda)\phi\right)(\lambda(x-\acute{x}))$. *Then, if* $\Phi(H_0)\phi \neq 0$*:*

$$V_j(\acute{x}) = \frac{\lim_{\lambda\to\infty} \lambda^{n+1} \int\int \, dt\, dx\, V_j(x)\, \left|(\mathcal{V}(t)U_\lambda)_1\right|^{2(j_0+j+1)}}{\int\int \, dt\, dx\, \left|(\mathcal{V}(t)U_\infty)_1\right|^{2(j_0+j+1)}}, \tag{1.33}$$

where,

$$U_\lambda := \begin{bmatrix} 0 \\ \phi_\lambda \end{bmatrix}, U_\infty := \begin{bmatrix} 0 \\ \Phi(H_0)\phi \end{bmatrix}. \tag{1.34}$$

COROLLARY 1.5. *Under the conditions of Lemma 1.4 the scattering operator,* S*, determines uniquely the potentials,* $V_j, j = 0, 1, \cdots$.

Proof : By Corollary 1.3 S determines uniquely V_0. Then, the $\Omega_\pm$ are uniquely determined, and by (1.28) S_{V_0} is uniquely determined by S. Finally, by (1.31), (1.33) each one of the V_j is uniquely determined.

■

Remark that our method is constructive and that it allows us to reconstruct, in a recursive way, each one of the $V_j, j = 0, 1, \cdots$. The results above extend to the case of $\mathbf{R}^n, n \geq 2$, our previous results, in the case of $n = 1$, that we obtained in [20]. A result as in Corollary 1.5 in three dimensions and with $V_0 = 0$ was proven by Morawetz and Strauss in [10] (see also Morawetz [9]) and by Bachelot in [2] and [3]. They reconstruct the V_j from a skew form. Morawetz and Strauss consider the case where $V_j\, j = 1, \cdots$ are constant. For references in the large literature in direct scattering for the nonlinear Klein-Gordon equation in the case when $V_0 = 0$ see [20].

2 Scattering

In this section we always assume that that H has no eigenvalues. It follows that $H > 0$. By Theorem 3 in page 135 of [15],

$$\|F^{-1}(1+q^2)^{k/2}(Ff)(q)\|_{L^p}, \tag{2.1}$$

is a norm that is equivalent to the norm of $W_{k,p}, 1 < p < \infty$. Then, by (1.7) there are constants C_1, C_2, such that,

$$C_1 \|\phi\|_{W_{l,p}} \leq \left\|(I+H)^{l/2}\phi\right\|_{L^p} \leq C_2 \|\phi\|_{W_{l,p}}, \tag{2.2}$$

for $l = 1, \cdots, k+2, 1 < p < \infty$. Below we will use this equivalences without further comments.

To prove Theorem 1.1 we need a $L^p - L^q$ estimate for $\frac{\partial}{\partial t}u(x,t)$, where $u(x,t)$ is the solution to the initial value problem (1.8). This estimate follows easily from (1.9) and (1.10).Denote, $\acute{T}_t := \frac{\partial}{\partial t}T_t$. Suppose that $f \in C_0^\infty$. By Taylor's expansion,

$$\begin{aligned} u(x,t_0) &= u(x,t) + (t_0 - t)\frac{\partial}{\partial t}u(x,t) + R(t_0,t), \text{ with} \\ R(t_0,t) &= \frac{(t_0-t)^2}{2}\frac{\partial^2}{\partial t^2}u(x, t+\theta(t_0-t)), \, 0 < \theta < 1. \end{aligned} \tag{2.3}$$

For $|t| \geq 1$ we take $t_0 - t = 1/2$. Then, by (1.8) and (2.3)

$$|\frac{\partial}{\partial t}u(x,t)| \leq 2\left[|u(x,t_0)| + |u(x,t)| + \frac{1}{8}|(H+1)u(x,t+\theta(t_0-t))|\right]. \tag{2.4}$$

Since C_0^∞ is dense in L^p, it follows from (1.9) and (2.4) that

$$\left\|\acute{T}_t\right\|_{\mathcal{B}(W_{l,p},W_{l-2,q})} \leq C|t|^a, \; |t| \geq 1, \; l = 0,1,2,\cdots,k+2, \tag{2.5}$$

where, $\frac{1}{p} + \frac{1}{q} = 1$ and for $\frac{2(n+1)}{n+3} \leq p \leq \frac{2(n+2)}{n+4}$, $a := -\frac{2}{q}$. For $\frac{2(n+2)}{n+4} \leq p \leq 2$, $a := \frac{n}{2} - \frac{n}{p}$ .For $|t| \leq 1$ we proceed as follows,

$$\frac{\partial}{\partial t}u(x,t) = f(x) + \int_0^t \frac{\partial^2}{\partial s^2}u(x,s)\,ds. \tag{2.6}$$

By (1.8), (1.10) and (2.6), and since by Sobolev's imbedding theorem [1] $L^q \subset W_{2,p}, p \geq \frac{2n}{n+2}$,

$$\left\|\acute{T}_t\right\|_{\mathcal{B}(W_{l,p},W_{l-2,q})} \leq C, \;\; |t| \leq 1, \; l = 0,1,2,\cdots,k+2, \tag{2.7}$$

where, $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{2(n+1)}{n+3} \leq p \leq 2$. Note that (2.5) and (2.6) hold for $\acute{T}_t P_c$ in the case when H has eigenvalues. This follows from the result in the case $V_0 = 0$ and (1.7).

We denote by $\mathcal{H}_{V_0}$ the space $\mathcal{H}$ endowed with the scalar product

$$(U,D)_{\mathcal{H}_{V_0}} := ((I+H)^{k/2}U_1,(I+H)^{k/2}D_1)_{L^2} + ((I+H)^{(k-1)/2}U_2,(I+H)^{(k-1)/2}D_2)_{L^2}. \tag{2.8}$$

The norm defined by (2.8) is equivalent to the norm of $\mathcal{H}$. The solution operator, $\mathcal{V}(t)$, to the initial value problem (1.11) with $F=0$ is given by

$$\mathcal{V}(t)\left[\begin{array}{c} f_1 \\ f_2 \end{array}\right] = \left[\begin{array}{c} \cos(t\sqrt{I+H})f_1 + \frac{\sin(t\sqrt{I+H})}{\sqrt{I+H}}f_2 \\ -\sin(t\sqrt{I+H})\,\sqrt{I+H}f_1 + \cos(t\sqrt{I+H})f_2 \end{array}\right]. \tag{2.9}$$

The $\mathcal{V}(t)$ are a strongly-continuous unitary group of operators on $\mathcal{H}_{V_0}$. It follows that they are uniformly bounded on $\mathcal{H}$ for $t \in \mathbf{R}$.

Proof of Theorem 1.1: By Sobolev's theorem L^∞ is continuously imbedded in $W_{k,p+1}$, and it follows from standard arguments (see [7])that $U \in C(\mathbf{R},\mathcal{H}) \cap M$ is a solution to (1.11) with $\lim_{t\to-\infty} \|U(t) - \mathcal{V}(t)\phi\|_{\mathcal{H}} = 0$, for some $\phi \in \mathcal{H}$, if and only if U is a solution to the following integral equation

$$U(t) = \mathcal{V}(t)\phi + \int_{-\infty}^t \mathcal{V}(t-\tau)P(U(\tau))\,d\tau. \tag{2.10}$$

As we will see below the integral in the right-hand side of (2.10) converges absolutely in $\mathcal{H}$ and in M. For $U \in M$ we denote,

$$\mathcal{Q}U(t) := \int_{-\infty}^t \mathcal{V}(t-\tau)P(U(\tau))\,d\tau. \tag{2.11}$$

We designate, $M_R := \{U \in M : \|U\| \leq R\}$. It follows from Assumption A, (1.9), (1.10), (2.5), (2.7) and since L^∞ is continuously imbedded in $W_{k,p+1}$ that there is an $R_0 > 0$ such that if, $U, D \in M_{R_0}$, then,

$$\|\mathcal{Q}U(t) - \mathcal{Q}D(t)\|_X \leq C\,(1+|t|)^{-d}(\|U\|_M + \|D\|_M)^{p-1}\|U-D\|_M, \tag{2.12}$$

where we used that $p\,d > 1$. Hence, by (2.12) with $D=0$ and since $\mathcal{V}(t)$ is unitary on $\mathcal{H}_{V_0}$:

$$\|\mathcal{Q}U(t)\|^2_{\mathcal{H}_{V_0}} = 2\Re\int_{-\infty}^t d\tau\;(P(U(\tau)),\mathcal{Q}U(\tau))_{\mathcal{H}_V} \leq C\int_{-\infty}^t d\tau\,\|P(U)(\tau)\|_{W_{k-2,1+\frac{1}{p}}}\,(1+|\tau|)^{-d}\,\|U\|^p_M$$

$$\leq C\int_{-\infty}^{t} d\tau\, \|u_1\|^p_{W_{k-2,1+p}} (1+|\tau|)^{-d}\|U\|^p_M \leq C\int_{-\infty}^{t} d\tau\,(1+|\tau|)^{-d(p+1)}\,\|U\|^{2p}_M$$

$$\leq C(1+\max[0,-t])^{-(d+dp-1)}\,\|U\|^{2p}_M. \tag{2.13}$$

By (2.12) with $D=0$, the integral in the right-hand side of (2.10) converges in M and by (2.13) it converges in $\mathcal{H}$. Let us prove the uniqueness. Let U and D be two solutions to (2.10) that satisfy (1.17). Then,

$$U(t)-D(t)=\mathcal{Q}U(t)-\mathcal{Q}D(t). \tag{2.14}$$

Let us denote $U_T := \chi_{(-\infty,T)}(t)\,U(t)$, where $\chi_{(-\infty,T)}(t)$ is the characteristic function of $(-\infty,T), T\in$ **R**. Arguing as in (2.12) it follows from (2.14) that

$$\|U_T(t)-D_T(t)\|_M < 1/2\|U_T(t)-D_T(t)\|_M, \text{ for some } T \text{ negative enough}. \tag{2.15}$$

It follows that $U(t)=D(t)$ for $t\leq T$. Hence, the standard uniqueness result implies that $U=D$. The uniqueness of ϕ_+ is obvious by the unitarity of $\mathcal{V}(t)$ in $\mathcal{H}_{V_0}$. By (1.9) and (2.5),

$$\|\mathcal{V}(t)\phi_-\|_X \leq C\,\|\phi_-\|_N\,,\ |t|\geq 1. \tag{2.16}$$

If $n\geq 3$, By Sobolev's theorem and (2.9),

$$\|\mathcal{V}(t)\phi_-\|_X \leq C\left[\|(\mathcal{V}(t)\phi_-)_1\|_{W_{k+1,2}} + \|(\mathcal{V}(t)\phi_-)_2\|_{W_{k-1,2}}\right] \leq C\|\phi_-\|_N. \tag{2.17}$$

If $n=2$ and $p\leq 3$, it follows from (1.10) and (2.7) that

$$\|\mathcal{V}(t)\phi_-\|_X \leq C\|\phi_-\|_N,\ |t|\leq 1. \tag{2.18}$$

Moreover, if $n=2$ and $p>3$, using again (2.9) and Sobolev's theorem we obtain that,

$$\|\mathcal{V}(t)\phi_-\|_X \leq C\left[\|(\mathcal{V}(t)\phi_-)_1\|_{W_{k+2,2}} + \|(\mathcal{V}(t)\phi_-)_2\|_{W_{k,2}}\right] \leq C\|\phi_-\|_N. \tag{2.19}$$

By (2.16), (2.17), (2.18) and (2.19),

$$\|\mathcal{V}(t)\phi_-\|_M \leq C\|\phi_-\|_N. \tag{2.20}$$

Let us take $R \leq R_0$ so small that $C(2R)^{p-1} \leq 1/2$, with C as in (2.12), and $\delta > 0$ such that $C\delta \leq R/4$, with C as in (2.20). Then, by (2.12) and (2.20) the map $U \hookrightarrow \mathcal{V}(t)\phi_- + \mathcal{Q}P(U)$ is a contraction from M_R into M_R for all ϕ_- with $\|\phi_-\|_N \leq \delta$. By the contraction mapping theorem (see Theorem V.18 in page 151 of [11]) this map has a unique fixed point that is a solution to (2.10) in M_R, and moreover,

$$\|U\|_M \leq C\,\|\mathcal{V}(t)\phi_-\|_M + \frac{1}{2}\,\|U\|_M \,. \tag{2.21}$$

Then,

$$\|U\|_M \leq C\,\|\mathcal{V}(t)\phi_-\|_M \,. \tag{2.22}$$

Equation(1.19) for ϕ_- follows from (2.10), (2.12) with $D = 0$ and (2.22). By (2.10) and (2.13) $U \in C(\mathbf{R}, \mathcal{H})$ and (1.17) holds.

We define

$$\phi_+ = \phi_- + \int_{-\infty}^{\infty} \mathcal{V}(-\tau)P(U(\tau))\,d\tau. \tag{2.23}$$

By (1.9), (1.10), (2.5)and (2.7)

$$\left\|\int_{-\infty}^{\infty} \mathcal{V}(-\tau)P(U(\tau))\,d\tau\right\|_X \leq C\int_{-\infty}^{\infty} (1+|\tau|)^{-d(p+1)}\|U\|_M^p\,d\tau \leq C\|U\|_M^p. \tag{2.24}$$

Hence, $\phi_+ \in X$. Estimating as in (2.13) we prove that $\phi_+ \in \mathcal{H}$ and that

$$\|\phi_+ - \phi_-\|_{\mathcal{H}} \leq C\,\|U\|_M^p \,. \tag{2.25}$$

Equation (1.20) follows from (2.22) and (2.25). By (2.23) and (2.10):

$$U(t) = \mathcal{V}(t)\phi_+ - \int_t^{\infty} \mathcal{V}(t-\tau)P(U(\tau))\,d\tau, \tag{2.26}$$

and we prove (1.18) estimating as in (2.13). Moreover, estimating as in (2.12) we obtain that

$$\left\|\int_t^{\infty} \mathcal{V}(t-\tau)P(U(\tau))\,d\tau\right\|_M \leq C\,\|U\|_M^p \,. \tag{2.27}$$

By (2.26) and (2.27), $\mathcal{V}(t)\phi_+ \in M$. Using again (2.26) and arguing as in the proof of (2.22) we obtain the following estimate:

$$\|U\|_M \leq C\,\|\mathcal{V}(t)\phi_+\|_M \,. \tag{2.28}$$

Equation (1.19) for ϕ_+ follows from (2.26), (2.27) and (2.28).

If $\phi_+ = S_{V_0}\phi_- = 0$, it follows from (2.26) that

$$U(t) = -\int_t^\infty \mathcal{V}(t-\tau)P(U(\tau))\,d\tau. \tag{2.29}$$

Let us denote: $U_T(t) := \chi_{(T,\infty)}(t)U(t)$. Arguing as in (2.15) it follows from (2.29) that

$$\|U_T(t)\|_M \leq \frac{1}{2}\|U_T(t)\|_M, \text{ for some } T \text{ positive enough.} \tag{2.30}$$

Hence, (2.30) implies that $U(t) = 0$ for $t \geq T$, and by standard uniqueness, $U(t) = 0$. Then, by (1.17) $\phi_- = 0$, proving that S_{V_0} is injective.

Proof of Theorem 1.2: As $S(0) = 0$ it is enough to to prove that

$$s - \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\left(S_{V_0}(\epsilon\phi) - \epsilon\phi\right) = 0. \tag{2.31}$$

By (2.20) and (2.22) with ϕ_- replaced by $\epsilon\phi$:

$$\|U\|_M \leq C\epsilon\|\phi\|_N. \tag{2.32}$$

Hence, by (2.23) and (2.24)

$$\left\|\frac{1}{\epsilon}\left(S_V(\epsilon\phi) - \epsilon\phi\right)\right\|_X \leq \frac{C}{\epsilon}\|U\|_M^p \leq C\epsilon^{p-1}\|\phi\|_N^p, \tag{2.33}$$

and (2.31) follows in X. We prove (2.31) in $\mathcal{H}$ estimating the integral in the right-hand side of (2.23) as in (2.13) and using (2.32).

Proof of Lemma 1.4 By the contraction mapping theorem, [11]

$$U(t) = \mathcal{V}(t)\epsilon U_0 + \sum_{n=1}^\infty \mathcal{Q}^n\mathcal{V}(t)\epsilon U_0. \tag{2.34}$$

Then, (1.31) follows from (2.23) and 2.34).

As $N_\delta(D^\alpha V_\lambda) \leq N_\delta(D^\alpha V_0)$ for $\lambda \geq 1$, equations (2.2)with $p = 2$ hold with H_λ instead of H with the same C_1, C_2 for all $\lambda \geq 1$ (see [22] for a proof of this fact). Let us consider the following initial value problem

$$\frac{\partial}{\partial t}U = A_\lambda U, \text{ where } U := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix};\ A_\lambda := \begin{bmatrix} 0 & 1 \\ \Delta - V_\lambda - \frac{1}{\lambda} & 0 \end{bmatrix},$$

$$U(x,0) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{2.35}$$

Let us denote by $\mathcal{V}_\lambda(t)$ the operator that gives the solution to the linear initial value problem, (2.35), i.e., $U(t) = \mathcal{V}_\lambda(t)U(0)$. Note that if $f_1 = 0$ the first component of the solution to (2.35) is given by,

$$u_1(x,t) = \frac{\sin\left(t(H_\lambda + \frac{1}{\lambda})^{1/2}\right)}{\left(H_\lambda + \frac{1}{\lambda}\right)^{1/2}} f_2. \tag{2.36}$$

We designate, $\tilde{t} := \lambda t$ and $\tilde{x} := \lambda(x - \acute{x})$. Then,

$$\left(\mathcal{V}_\lambda(\tilde{t})\, V_\lambda\right)(\tilde{x},\tilde{t}) = \left(\mathcal{V}(t)\, U_\lambda\right)(x,t), \text{ where } V_\lambda := \begin{bmatrix} 0 \\ \Phi(H_\lambda)\phi \end{bmatrix}, \tag{2.37}$$

where U_λ is given by (1.34). By (2.37),

$$I_j := \lambda^{n+1} \int\int dt\, dx\, V_j(x)\, \left|\left(\mathcal{V}(t)U_\lambda\right)_1\right|^{2(j_0+j+1)}(x) = \int\int d\tilde{t}\, d\tilde{x}\, V_j(\frac{\tilde{x}}{\lambda}+\acute{x}))\, \left|\left(\mathcal{V}_\lambda(\tilde{t})V_\lambda\right)_1\right|^{2(j_0+j+1)}(\tilde{x}). \tag{2.38}$$

By Theorem VIII.20 in page 286 of [11] and (2.36)

$$s - \lim_{\lambda\to\infty} \left(\mathcal{V}_\lambda(\tilde{t})V_\lambda\right)_1 = \left(\mathcal{V}_\infty(\tilde{t})U_\infty\right)_1, \tag{2.39}$$

where if $n = 2$ the limit exists in the strong topology on $W_{\max(k,2),2}$ and if $n \geq 3$ the limit exists in the strong topology on $W_{k+1,2}$. $\mathcal{V}_\infty$ is the solution operator for the wave equation, i.e., the solution operator for the initial value problem (2.35) with $-V_\lambda - \frac{1}{\lambda}$ replaced by zero. Sobolev's imbedding theorem implies that the limit exists also on the strong topology on $L^q, 2 \leq q \leq \infty$. Furthermore,

$$\left\|\left(\mathcal{V}_\lambda(\tilde{t})V_\lambda\right)_1\right\|_{L^\infty} \leq C\,\|\phi\|_{W_{r,2}}, \tag{2.40}$$

where if $n = 2, r = 2$ and if $n \geq 3, r = k+1$. By (1.9) and (2.37),

$$\left\|\left(\mathcal{V}_\lambda(\tilde{t})V_\lambda\right)_1\right\|_{L^{p+1}}^{p+1} = \lambda^n\, \|(\mathcal{V}(t)U_\lambda)_1\|_{L^{p+1}}^{p+1} \leq C\frac{1}{t^2}\lambda^n\, \|\phi_\lambda\|_{L^{1+1/p}}^{p+1} = C\frac{1}{\tilde{t}^2}\, \|\phi\|_{L^{1+1/p}}^{p+1}. \tag{2.41}$$

Equation (1.33) follows from (2.38) -(2.41),and the dominated convergence theorem. Note that $2(j_0 + j + 1) \geq p + 1$ and that V_j is continuous.

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