SCATTERING THEORY FOR A TWO-BODY QUANTUM SYSTEM IN A CONSTANT MAGNETIC FIELD

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Abstract. We study the scattering theory for a two-body quantum system in a constant magnetic field. The system consists of one neutral particle and one charged particle. We shall show the asymptotic completeness for this system.

1991 Mathematics Subject Classification Primary 81U10; Secondary 81V70

§1. Introduction

In this paper, we study the scattering theory for a two-body quantum system in a constant magnetic field. We consider the case that the system consists of one neutral particle and one charged particle.

The scattering theory for N-body systems in a constant magnetic field has been studied by Gérard-Łaba [GŁ1,2,3]. What we should emphasize is that they have assumed that all particles in the systems are charged, that is, there is no neutral particle in the systems under consideration, even if the systems consist of only two particles (see also [Ł1,2]). Under this assumption, if there is no neutral proper subsystem, one has only to observe the behavior of all subsystems parallel to the magnetic field. Skibsted [Sk1,2] studied the scattering theory for N-body systems in combined constant electric and magnetic fields, but he needed the same assumption implicitly. Considering such circumstances, we shall study the scattering theory for the system which consists of one neutral and one charged particles in a constant magnetic field. This paper seems to be the first which deals with such a situation.

We consider a system of two particles moving in a given constant magnetic field $\boldsymbol{B} = (0, 0, B) \in \boldsymbol{R}^3$, $B > 0$. Let m_j and q_j $(j = 1,\, 2)$ be the mass and charge of the j-th particle, respectively, and let $x \in \boldsymbol{R}^3$ and $y \in \boldsymbol{R}^3$ be the position vectors of the first and second particles, respectively. Throughout this paper, we assume that the first particle is neutral and the second particle is charged, i.e. $q_1 = 0$ and $q_2 = q \neq 0$.

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We suppose that these particles interact with one another through the pair potential $V(x-y)$. Then the total Hamiltonian for the system is given by

$$\tilde{H} = \frac{1}{2m_1} D_x^2 + \frac{1}{2m_2} \left(D_y - q\boldsymbol{A}(y) \right)^2 + V(x-y) \tag{1.1}$$

acting on $L^2(\boldsymbol{R}^{3\times 2})$, where $D_x = -i\nabla_x$ and $D_y = -i\nabla_y$ are the momentum operators of the first and second particles, respectively, and $\boldsymbol{A}(y)$ is the vector potential. Using the Coulomb gauge, the vector potential $\boldsymbol{A}(y)$ is given by

$$\boldsymbol{A}(y) = \frac{B}{2}(-y_2, y_1, 0), \quad y = (y_1, y_2, y_3). \tag{1.2}$$

As is well-known, the separation of the center of mass motion in the direction parallel to the field $\boldsymbol{B}$ may be done easily (see e.g. [AHS2]):

$$\tilde{H} = \frac{1}{2M} D_Z^2 \otimes I + I \otimes H \tag{1.3}$$

on $L^2(\boldsymbol{R}^6) = L^2(\boldsymbol{R}_Z) \otimes L^2(\boldsymbol{R}^5_{(x_\perp, y_\perp, z)})$, where $M = m_1 + m_2$ is the total mass,

$$Z = \frac{m_1 x_3 + m_2 y_3}{M} \tag{1.4}$$

is the position of the center of mass in the direction parallel to the field $\boldsymbol{B}$, $D_Z = -i\partial_Z$,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{1.5}$$

is the reduced mass,

$$z = x_3 - y_3 \tag{1.6}$$

is the relative position in the direction parallel to the field $\boldsymbol{B}$, $D_z = -i\partial_z$, $x_\perp = (x_1, x_2)$, $y_\perp = (y_1, y_2)$, $p_x = (-i\partial_{x_1}, -i\partial_{x_2})$, $p_y = (-i\partial_{y_1}, -i\partial_{y_2})$, and

$$H = \frac{1}{2\mu} D_z^2 + \frac{1}{2m_1} p_x^2 + \frac{1}{2m_2} \left(p_y - q\boldsymbol{A}(y) \right)^2 + V(x-y) \tag{1.7}$$

acting on $L^2(\boldsymbol{R}^5_{(x_\perp, y_\perp, z)})$. Here we identified the vector potential $\boldsymbol{A}(y)$ with the vector $(B/2)(-y_2, y_1) \in \boldsymbol{R}^2$. In this paper, we study the scattering theory for this Hamiltonian H.

Now we shall state the assumptions on the potential V. Under the assumptions, the Hamiltonian H is self-adjoint. We write $r_\perp = (r_1, r_2)$ for $r = (r_1, r_2, r_3) \in \boldsymbol{R}^3$. The dot $\cdot$ means the usual Euclidean metric. We shall use the following convention for smooth cut-off functions F with $0 \le F \le 1$, which is often used throughout this paper. For sufficiently small $\eta > 0$, we define

$$\begin{aligned} F(s \le d) &= 1 \quad \text{for} \quad s \le d - \eta, \quad = 0 \quad \text{for} \quad s \ge d, \\ F(s \ge d) &= 1 \quad \text{for} \quad s \ge d + \eta, \quad = 0 \quad \text{for} \quad s \le d, \end{aligned}$$

and $F(d_1 \le s \le d_2) = F(s \ge d_1)\, F(s \le d_2)$. The choice of $\eta > 0$ does not matter to the argument below.

$(V1)$ $V = V(r) \in L^2(\boldsymbol{R}^3) + L^\infty_\epsilon(\boldsymbol{R}^3)$ is a real-valued function.
$(V2)$ $\partial^\alpha_{r_\perp} V$ $(|\alpha| = 1,2)$, $(r\cdot\nabla_r)^l V$ $(l = 1,2)$, $\partial^\beta_{r_\perp}(r\cdot\nabla_r V)$ $(|\beta| = 1)$ and $r_\perp\cdot\nabla_{r_\perp}(r\cdot\nabla_r V)$ are all $-\Delta_r$-bounded. Moreover, for some $\mu_1 > 0$, as $R \to \infty$,

$$\left\| F\left(\frac{|r|}{R} \ge 1\right) \nabla_{r_\perp} V(-\Delta_r + 1)^{-1} \right\| = O(R^{-\mu_1}),$$
$$\left\| F\left(\frac{|r|}{R} \ge 1\right) r\cdot\nabla_r V(-\Delta_r + 1)^{-1} \right\| = O(R^{-\mu_1})$$

are satisfied.
(SR) V satisfies

$$\left\| F\left(\frac{|r|}{R} \ge 1\right) V(-\Delta_r + 1)^{-1} \right\| = O(R^{-\mu_S})$$

as $R \to \infty$, with $\mu_S > 1$.
(LR) V is decomposed as $V = V_S + V_L$, where a real-valued $V_L \in C^\infty(\boldsymbol{R}^3)$ such that $|\partial^\alpha_r V(r)| \le C_\alpha \langle r\rangle^{-|\alpha|-\mu_L}$ with $\mu_L > 1/2$, and V_S satisfies

$$\left\| F\left(\frac{|r|}{R} \ge 1\right) V_S(-\Delta_r + 1)^{-1} \right\| = O(R^{-\mu_S})$$

as $R \to \infty$, with $\mu_S > 1$.

Remark 1.1. As is seen from the above assumptions, we may not allow certain 'satisfactory' potentials, that is, for instance Coulomb potentials. This is caused by that we need some relative boundedness of the derivatives of the potential on the coordinates perpendicular to the field $\boldsymbol{B}$ with respect to $-\Delta_r$. In excuse of not allowing Coulomb potentials, we do not have to consider them physically because one particle is neutral. Anyway, though we do not know if our assumptions seem to be natural or not, it seems natural that we will have to consider the problem with distinguishing the direction parallel to the field $\boldsymbol{B}$ from the directions perpendicular to it.

To formulate the results precisely, we shall introduce the usual and the modified wave operators. We denote the free Hamiltonian by

$$H_0 = \frac{1}{2\mu} D_z^2 + \frac{1}{2m_1} p_x^2 + \frac{1}{2m_2} \left(p_y - q\boldsymbol{A}(y)\right)^2. \tag{1.8}$$

The usual wave operators $W^\pm$ are defined by

$$W^\pm = \mathrm{s}\text{-}\lim_{t\to\pm\infty} e^{itH} e^{-itH_0}. \tag{1.9}$$

Writing $V(r) = V(r_\perp, z)$, we define the modified wave operators $W_D^\pm$ by

$$W_D^\pm = \mathrm{s}\text{-}\lim_{t\to\pm\infty} e^{itH} e^{-itH_0 - i\int_0^t V(sp_x/m_1, sD_z/\mu)\, ds}. \tag{1.10}$$

The main results of this paper are the following two theorems:

Theorem 1.1. *Assume that* $(V1)$, $(V2)$ *and* (SR) *are fulfilled. Then the usual wave operators* $W^{\pm}$ *exist and are asymptotically complete*

$$L_c^2(H) = Ran\, W^{\pm}.$$

Here $L_c^2(H)$ *is the continuous spectral subspace of the Hamiltonian* H.

Theorem 1.2. *Assume that* $(V1)$, $(V2)$ *and* (LR) *are fulfilled. Then the modified wave operators* $W_D^{\pm}$ *exist and are asymptotically complete*

$$L_c^2(H) = Ran\, W_D^{\pm}.$$

Remark 1.2. In Theorem 1.2, we assumed that $\mu_L > 1/2$. This is caused by our choice of the modifier. We use the so-called Dollard modifier. If one takes other modifiers by constructing solutions of some Hamilton-Jacobi equations, the condition may be relaxed.

Remark 1.3. We consider the problem in the space $\boldsymbol{R}^3$. But our analysis may be applied also in studying the same problem in the space $\boldsymbol{R}^2$.

The plan of this paper is as follows: In §2, we study the spectral properties of the reduced Hamiltonian $\hat{H}$ defined below by using the Mourre theory. In §3, we show some propagation estimates which are useful for proving the asymptotic completeness. In §4 and §5, we prove the main results of this paper. We shall give the proofs in the case $t \to \infty$ only. The case $t \to -\infty$ may be proved in the same way.

Acknowledgement

We would like to express our deep gratitude to Professor Christian Gérard for his valuable discussions as well as his hospitality, because the contents of this paper were developed during our staying at the École Polytechnique. This work was supported by the Ministry of Education, Science, Sports and Culture, the Government of Japan.

§2. The Mourre estimate

In this section, we study the spectral properties of the Hamiltonian H, by using the Mourre theory.

First we have to 'remove' the center of mass motion in the directions perpendicular to the field $\boldsymbol{B}$. To this end, we shall introduce the total pseudomomentum k perpendicular to the field $\boldsymbol{B}$ (see e.g. [AHS2]):

$$k = p_x + p_y + q\boldsymbol{A}(y), \tag{2.1}$$

where we identified $\boldsymbol{A}(y) \in \boldsymbol{R}^3$ with $(B/2)(-y_2, y_1) \in \boldsymbol{R}^2$. We note that this operator k commutes with H. Since in this case the total charge of the system is non-zero, the two components of the total pseudomomentum k cannot commute with each other, but satisfy the Heisenberg commutation relation. Then we may 'remove' the center of mass motion perpendicular to the field $\boldsymbol{B}$ by following the arguments of [AHS2], [GL1,2,3] and [Sk1,2]. Before doing that, we introduce the relative coordinates $v = (v_1, v_2)$ and the center of mass coordinates $w = (w_1, w_2)$ perpendicular to the field $\boldsymbol{B}$:

$$v = x_\perp - y_\perp, \quad w = \frac{m_1 x_\perp + m_2 y_\perp}{M}. \tag{2.2}$$

The associated momenta are denoted by $p_v = (p_{v_1}, p_{v_2}) = (-i\partial_{v_1}, -i\partial_{v_2})$ and $p_w = (p_{w_1}, p_{w_2}) = (-i\partial_{w_1}, -i\partial_{w_2})$. We introduce the unitary operator U on $L^2(\boldsymbol{R}^5)$ as follows:

$$U = e^{-iw\cdot q\boldsymbol{A}(y)} e^{iqBw_1w_2/2} e^{ip_{w_1}p_{w_2}/(qB)}. \tag{2.3}$$

We should note that $w\cdot q\boldsymbol{A}(y) = -(m_1 q/M) w\cdot \boldsymbol{A}(v)$, where we identified $v = (v_1, v_2) \in \boldsymbol{R}^2$ with $(v_1, v_2, 0) \in \boldsymbol{R}^3$ and $\boldsymbol{A}(v)$ with a vector in $\boldsymbol{R}^2$. Then the total pseudomomentum k may be transformed by the unitary operator U as follows:

$$U^* k_1 U = p_{w_1}, \quad U^* k_2 U = qBw_1. \tag{2.4}$$

And the Hamiltonian H may be transformed by the unitary operator U as follows:

$$\begin{aligned} U^*HU =& \frac{1}{2\mu} D_z^2 \\ &+ \frac{1}{2m_1}\left\{\left(p_{v_1} + \frac{m_1 qB}{M} w_2 - \frac{m_1^2 qB}{2M^2} v_2\right)^2 + \left(p_{v_2} + \frac{m_1}{M} p_{w_2} + \frac{m_1^2 qB}{2M^2} v_1\right)^2\right\} \\ &+ \frac{1}{2m_2}\left\{\left(-p_{v_1} + \frac{m_2 qB}{M} w_2 - \frac{m_1(m_1+2m_2)qB}{2M^2} v_2\right)^2\right. \\ &\qquad \left. + \left(-p_{v_2} + \frac{m_2}{M} p_{w_2} + \frac{m_1(m_1+2m_2)qB}{2M^2} v_1\right)^2\right\} + V. \end{aligned} \tag{2.5}$$

This Hamiltonian U^*HU does not depend on the variable w_1. Then we may identify this operator U^*HU with an operator acting on $L^2(\boldsymbol{R}^4_{(v_1,v_2,w_2,z)})$. We denote this reduced Hamiltonian by $\hat{H}$.

From now on we shall analyze the spectral properties of the Hamiltonian $\hat{H}$. To this end, we want to introduce a conjugate operator $\hat{A}$ for $\hat{H}$. Now we consider the following operator on $L^2(\boldsymbol{R}^5)$ for H (not $\hat{H}$):

$$\tilde{A} = \frac{1}{2}\{(zD_z + D_z z) + (x_\perp \cdot p_x + p_x \cdot x_\perp)\}. \tag{2.6}$$

Putting $H_0 = H - V$, we may obtain the following commutation relation by a straightforward computation:

$$i[H_0, \tilde{A}] = \frac{1}{\mu} D_z^2 + \frac{1}{m_1} p_x^2 = 2\left(H_0 - \frac{1}{2m_2}(p_y - q\boldsymbol{A}(y))^2\right). \tag{2.7}$$

As is well-known, the spectrum of the last term consists of the Landau levels

$$\tau = \left\{\frac{|q|B}{m_2}\left(n + \frac{1}{2}\right) \,\middle|\, n = 0, 1, 2, \dots\right\}. \tag{2.8}$$

We want to use this commutation property (2.7) in studying the spectral theory for the Hamiltonian $\hat{H}$, that is, we want to use the operator $U^*\tilde{A}U$ as a conjugate operator for the Hamiltonian $\hat{H}$. However, this operator $U^*\tilde{A}U$ cannot commute with the total pseudomomentum (p_{w_1}, qBw_1). Noting that the right-hand side of (2.7) and H_0 commute with the total pseudomomentum k, we may expect the existence of some alternative conjugate operator which satisfies the commutation relation (2.7) and commutes with the total pseudomomentum k. In order to find such an operator, we shall remove the dependence on the total pseudomomentum (p_{w_1}, qBw_1) from the operator $U^*\tilde{A}U$. For doing that, we have only to consider the part $x_\perp \cdot p_x$. Using the relative coordinates v and the center of mass coordinates w, we have

$$x_\perp \cdot p_x = w \cdot p_v + \frac{m_2}{M} v \cdot p_v + \frac{m_1}{M} w \cdot p_w + \frac{m_1 m_2}{M^2} v \cdot p_w.$$

By a straightforward computation, we have the following equality:

$$\begin{aligned} U^*(x_\perp \cdot p_x)U =& w_2 p_{v_2} - \frac{1}{qB} p_{w_2} p_{v_1} + \frac{m_2}{M} v \cdot p_v + \frac{m_1(m_1+2m_2)}{2M^2} v_2 p_{w_2} \\ &+ \frac{m_1(m_1+2m_2)qB}{2M^2} v_1 w_2 + R, \end{aligned}$$

where R has the dependence on the total pseudomomentum (p_{w_1}, qBw_1). Thus we define the self-adjoint operator $\hat{A}$ on $L^2(\boldsymbol{R}^4)$ as follows:

$$\begin{aligned} \hat{A} =& \frac{1}{2}\left\{(zD_z + D_z z) + \frac{m_2}{M}(v \cdot p_v + p_v \cdot v)\right\} + w_2 p_{v_2} - \frac{1}{qB} p_{w_2} p_{v_1} \\ &+ \frac{m_1(m_1+2m_2)}{2M^2} v_2 p_{w_2} + \frac{m_1(m_1+2m_2)qB}{2M^2} v_1 w_2. \end{aligned} \tag{2.9}$$

The self-adjointness of this operator $\hat{A}$ can be guaranteed by the Nelson's commutator theorem (see [RS]). This operator $\hat{A}$ has some good properties. By a straightforward computation, we may check that the following equality holds:

$$\begin{aligned} i[\hat{H}_0, \hat{A}] =& 2\left(\hat{H}_0 - \frac{1}{2m_2}\left\{\left(-p_{v_1} + \frac{m_2 qB}{M} w_2 - \frac{m_1(m_1+2m_2)qB}{2M^2} v_2\right)^2\right.\right. \\ &\left.\left. + \left(-p_{v_2} + \frac{m_2}{M} p_{w_2} + \frac{m_1(m_1+2m_2)qB}{2M^2} v_1\right)^2\right\}\right), \end{aligned} \tag{2.10}$$

where $\hat{H}_0 = \hat{H} - V$. Now we should note that

$$\begin{aligned} &U^*\left(\frac{1}{2m_2}(p_y - q\boldsymbol{A}(y))^2\right)U \\ =& \frac{1}{2m_2}\left\{\left(-p_{v_1} + \frac{m_2 qB}{M} w_2 - \frac{m_1(m_1+2m_2)qB}{2M^2} v_2\right)^2\right. \\ &\left. + \left(-p_{v_2} + \frac{m_2}{M} p_{w_2} + \frac{m_1(m_1+2m_2)qB}{2M^2} v_1\right)^2\right\}. \end{aligned} \tag{2.11}$$

Using the commutation property (2.10) analogous to (2.7), we have the following Mourre estimate.

Theorem 2.1. (The Mourre estimate) *Suppose that the potential V satisfies the conditions $(V1)$ and $(V2)$. Put*

$$d(\lambda) = \mathrm{dist}(\lambda, \tau \cap (-\infty, \lambda])$$

for $\lambda \geq \inf \tau = |q|B/(2m_2)$. Then for any $\lambda \geq \inf \tau$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any real-valued $f \in C_0^\infty(\boldsymbol{R})$ supported in the open interval $(\lambda - \delta, \lambda + \delta)$, there exists a compact operator K on $L^2(\boldsymbol{R}^4)$ such that

$$f(\hat{H}) i[\hat{H}, \hat{A}] f(\hat{H}) \geq 2(d(\lambda) - \varepsilon) f(\hat{H})^2 + K \tag{2.12}$$

holds.

Using the abstract Mourre theory (see e.g. [CFKS]), the following corollary follows from Theorem 2.1 and the fact that under the condition $(V1)$ V is $\hat{H}_0$-compact, where $\hat{H}_0 = \hat{H} - V$ (see [AHS2]). The latter fact implies that $\sigma_{\mathrm{ess}}(\hat{H}) = \sigma_{\mathrm{ess}}(\hat{H}_0)$.

Corollary 2.2. *Assume the same conditions as in Theorem 2.1. Then eigenvalues of $\hat{H}$ can accumulate only at τ. Moreover, $\tau \cup \sigma_{\mathrm{pp}}(\hat{H})$ is a closed countable set.*

Proof of Theorem 2.1. First we consider the term $i[\hat{H}_0, \hat{A}]$. We should note that $f(\hat{H}) - f(\hat{H}_0)$ is compact on $L^2(\boldsymbol{R}^4)$. This fact follows from the fact that

$$(\hat{H} - \zeta)^{-1} - (\hat{H}_0 - \zeta)^{-1} = -(\hat{H} - \zeta)^{-1} V (\hat{H}_0 - \zeta)^{-1}$$

is compact on $L^2(\boldsymbol{R}^4)$, where $\zeta \in \boldsymbol{C} \setminus \boldsymbol{R}$. We also notice that $i[\hat{H}_0, \hat{A}]$ is $\hat{H}_0$-bounded by virtue of (2.10). Then we have

$$f(\hat{H}) i[\hat{H}_0, \hat{A}] f(\hat{H}) = f(\hat{H}_0) i[\hat{H}_0, \hat{A}] f(\hat{H}_0) + K_1 \tag{2.13}$$

for some compact operator K_1 on $L^2(\boldsymbol{R}^4)$. Letting λ, ε, δ and f be as in the statement of the theorem, if we take $\delta > 0$ sufficiently small, we have

$$f(\hat{H}_0) i[\hat{H}_0, \hat{A}] f(\hat{H}_0) \geq 2 \left(d(\lambda) - \frac{1}{2}\varepsilon \right) f(\hat{H}_0)^2, \tag{2.14}$$

by virtue of (2.10). By using the compactness of $f(\hat{H}) - f(\hat{H}_0)$ again, the following estimate follows from (2.13) and (2.14):

$$f(\hat{H}) i[\hat{H}_0, \hat{A}] f(\hat{H}) \geq 2 \left(d(\lambda) - \frac{1}{2}\varepsilon \right) f(\hat{H})^2 + K_2 \tag{2.15}$$

for some compact operator K_2 on $L^2(\boldsymbol{R}^4)$.

Next we consider the term $i[V, \hat{A}]$. By the definition of the operator $\hat{A}$, we have only to consider the term

$$i[V, \hat{A}] = - \left(z \partial_z V + \frac{m_2}{M} v \cdot \nabla_v V + w_2 \partial_{v_2} V - \frac{1}{qB} p_{w_2} \partial_{v_1} V \right).$$

Now we may rewrite the Hamiltonian $\hat{H}$ as follows (see [GL1,2,3] and [Sk1,2]):

$$\begin{aligned}\hat{H} =& \frac{1}{2\mu}\left\{D_z^2 + \left(p_{v_1} + \frac{m_1^2 qB}{2M^2}v_2\right)^2 + \left(p_{v_2} - \frac{m_1^2 qB}{2M^2}v_1\right)^2\right\} \\ &+ \frac{1}{2M}\left\{\left(p_{w_2} + \frac{m_1 qB}{M}v_1\right)^2 + \left(qBw_2 - \frac{m_1 qB}{M}v_2\right)^2\right\} + V. \end{aligned} \tag{2.16}$$

By virtue of this equality (2.16), we see that the operators

$$p_{w_2} + \frac{m_1 qB}{M}v_1, \quad qBw_2 - \frac{m_1 qB}{M}v_2$$

are $\hat{H}$-bounded. By using the fact that $F(|x-y|/R \le 1)$ is $\hat{H}$-compact (see [AHS2]), inserting $1 = F(|x-y|/R \le 1) + (1 - F(|x-y|/R \le 1))$ and letting R sufficiently large, we have

$$f(\hat{H})\left\{-\left(w_2 - \frac{m_1}{M}v_2\right)\partial_{v_2}V - \left(-\frac{1}{qB}p_{w_2} - \frac{m_1}{M}v_1\right)\partial_{v_1}V\right\}f(\hat{H}) \ge -\frac{\varepsilon}{2}f(\hat{H})^2 + K_3 \tag{2.17}$$

for some compact operator K_3 on $L^2(\boldsymbol{R}^4)$. Then we have to consider the term

$$-\left(z\partial_z V + \frac{m_2}{M}v\cdot\nabla_v V + \frac{m_1}{M}v_2\partial_{v_2}V + \frac{m_1}{M}v_1\partial_{v_1}V\right) = -r\cdot\nabla_r V(r),$$

where we write $r = x - y$. In the same way as above, we have

$$f(\hat{H})\{-r\cdot\nabla_r V\}f(\hat{H}) \ge -\frac{\varepsilon}{2}f(\hat{H})^2 + K_4 \tag{2.18}$$

for some compact operator K_4 on $L^2(\boldsymbol{R}^4)$. Combining (2.15) and (2.17) with (2.18), we obtain the theorem. □

§3. Propagation estimates

In this section, we prove some propagation estimates for the evolution of the Hamiltonian H, not $\hat{H}$.

First we need the Mourre estimate for the Hamiltonian H. As we have seen in the previous section, we have the following relation between H and $\hat{H}$:

$$U^*HU = \hat{H}\otimes I \tag{3.1}$$

on $L^2(\boldsymbol{R}^5) = L^2(\boldsymbol{R}^4)\otimes L^2(\boldsymbol{R})$. Here I is the identity operator. As for the Hamiltonian $\hat{H}$, the following proposition holds, by virtue of Theorem 2.1 and Corollary 2.2:

Proposition 3.1. *Suppose that the potential V satisfies the conditions $(V1)$ and $(V2)$. Let $\lambda \geq \inf \tau$ be such that $\lambda \notin \tau \cup \sigma_{\rm pp}(\hat{H})$. Then there exist $\delta > 0$ and $c > 0$ such that for any real-valued $f \in C_0^\infty(\boldsymbol{R})$ supported in the open interval $(\lambda - \delta, \lambda + \delta)$,*

$$f(\hat{H}) i[\hat{H}, \hat{A}] f(\hat{H}) \geq c f(\hat{H})^2 \tag{3.2}$$

holds.

We shall translate this proposition into the one for the Hamiltonian H, by using the relation (3.1). We define the self-adjoint operator A on $L^2(\boldsymbol{R}^5)$ by

$$A = U(\hat{A} \otimes I) U^*. \tag{3.3}$$

Then, by virtue of the relation (3.1), we obtain the following proposition immediately.

Proposition 3.2. *Suppose that the potential V satisfies the conditions $(V1)$ and $(V2)$. Let $\lambda \geq \inf \tau$ be such that $\lambda \notin \tau \cup \sigma_{\rm pp}(H)$. Then there exist $\delta > 0$ and $c > 0$ such that for any real-valued $f \in C_0^\infty(\boldsymbol{R})$ supported in the open interval $(\lambda - \delta, \lambda + \delta)$,*

$$f(H) i[H, A] f(H) \geq c f(H)^2 \tag{3.4}$$

holds.

Remark 3.1. We should note that eigenvalues of H can accumulate only at τ and that $\tau \cup \sigma_{\rm pp}(H)$ is a closed countable set, which follows from Corollary 2.2 and the relation (3.1).

Then we have the following propagation estimate associated with the observable A, by following the standard argument in the N-body scattering theory originated with the works of Sigal-Soffer (see e.g. [SS1,2]).

Proposition 3.3. *Assume the same condition as in Proposition 3.2. Let λ, δ, c and f be also as in Proposition 3.2. Then for any real-valued $g \in C_0^\infty(\boldsymbol{R})$ supported in $(-\infty, c)$, there exists $C > 0$ such that*

$$\int_1^\infty \left\| g\left(\frac{A}{t}\right) f(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2 \tag{3.5}$$

for any $\psi \in L^2(\boldsymbol{R}^5)$.

Proof. The proof is done in exactly the same way as in [SS2]. We sketch the proof (see also [GŁ3]).

Let G be defined by

$$G(s) = \int_{-\infty}^s g(u)^2 \, du,$$

so that $G'(s) = g(s)^2 \in C_0^\infty(\boldsymbol{R})$ with g being real-valued. We use

$$\Phi_1(t) = G\left(\frac{A}{t}\right)$$

as a propagation observable. We note that $\Phi_1(t)$ is uniformly bounded in $t \geq 1$. The Heisenberg derivative of this observable is calculated as

$$\mathbf{D}_H(\Phi_1(t)) = \partial_t \Phi_1(t) + i[H, \Phi_1(t)]. \tag{3.6}$$

If we take $f_1 \in C_0^\infty(\boldsymbol{R})$ such that $f_1 = 1$ on the support of f, then

$$f(H)i[H, \Phi_1(t)]f(H) = f(H)i[f_1(H)H, \Phi_1(t)]f(H).$$

By using the almost analytic extension method due to Helffer-Sjöstrand [HeSj] in order to calculate several commutators (see also [G]), we have

$$f(H)i[H, \Phi_1(t)]f(H) = g\left(\frac{A}{t}\right) f(H) i \left[H, \frac{A}{t}\right] f(H) g\left(\frac{A}{t}\right) + O(t^{-2}).$$

Here we should note that $[(H+i)^{-1}, A]$ and $[[(H+i)^{-1}, A], A]$ are bounded on $L^2(\boldsymbol{R}^5)$ by assumption. Hence it follows from (3.4) that

$$\begin{aligned} f(H)i[H, \Phi_1(t)]f(H) &\geq \frac{c}{t} g\left(\frac{A}{t}\right) f(H)^2 g\left(\frac{A}{t}\right) + O(t^{-2}) \\ &\geq \frac{c}{t} f(H) g\left(\frac{A}{t}\right)^2 f(H) + O(t^{-2}) \end{aligned}$$

On the other hand, the first term on the right-hand side of (3.6) is estimated as

$$f(H)\partial_t \Phi_1(t) f(H) \geq -\frac{c_0}{t} f(H) g\left(\frac{A}{t}\right)^2 f(H)$$

for some c_0 such that $0 < c_0 < c$, which depends on the support of g. Thus we obtain

$$f(H)\mathbf{D}_H(\Phi_1(t))f(H) \geq \frac{c - c_0}{t} f(H) g\left(\frac{A}{t}\right)^2 f(H) + O(t^{-2}).$$

This proves the proposition. $\square$

Next we shall consider the observable $\Xi = (x_\perp, z)$ instead of the observable A. To this end, we need the localization of the total pseudomomentum k as well as the localization of the energy. Following the argument of [AHS2], we introduce the creation operator a^* by using the total pseudomomentum k as follows:

$$a^* = \frac{1}{\sqrt{2|q|B}} \left(\frac{q}{|q|} k_2 - i k_1 \right). \tag{3.7}$$

Here we used the relation (2.4). Thus we use the localization of the number operator $N = a^* a$ in addition to the localization of the energy. Then we obtain the following lemma.

Lemma 3.4. *Assume the same condition as in Proposition 3.2. Let λ, δ, c and f be also as in Proposition 3.2. Let c_0 be such that $0 < c_0 < c$. Then for any real-valued $h \in C_0^\infty(\boldsymbol{R})$, there exists $\varepsilon_0 > 0$ such that*

$$\left\| F\left(\frac{|A|}{t} \geq c_0\right) F\left(\frac{|\Xi|}{t} \leq \varepsilon_0\right) h(N) f(H) \right\| = O(t^{-1})$$

as $t \to \infty$.

Proof. First of all, we should note that the following equality holds, which may be checked by a straightforward computation.

$$\begin{aligned}\tilde{A} - A =& \frac{1}{2}\left\{\left(\frac{m_1}{2M}w_1 - \frac{m_1^2}{2M^2}v_1 - \frac{1}{qB}p_{v_2}\right)\left(p_{w_1} - \frac{qB}{2}w_2 + \frac{m_1 qB}{2M}v_2\right)\right.\\ &\left. + \left(p_{w_1} - \frac{qB}{2}w_2 + \frac{m_1 qB}{2M}v_2\right)\left(\frac{m_1}{2M}w_1 - \frac{m_1^2}{2M^2}v_1 - \frac{1}{qB}p_{v_2}\right)\right\}\\ &+ \frac{1}{2}\left\{\left(\frac{m_1}{2M}w_2 - \frac{m_1^2}{2M^2}v_2 + \frac{1}{qB}p_{v_1}\right)\left(p_{w_2} + \frac{qB}{2}w_1 - \frac{m_1 qB}{2M}v_1\right)\right.\\ &\left. + \left(p_{w_2} + \frac{qB}{2}w_1 - \frac{m_1 qB}{2M}v_1\right)\left(\frac{m_1}{2M}w_2 - \frac{m_1^2}{2M^2}v_2 + \frac{1}{qB}p_{v_1}\right)\right\},\end{aligned} \tag{3.8}$$

where $\tilde{A}$ was defined in (2.6). We should also notice that

$$k_1 = p_{w_1} - \frac{qB}{2}w_2 + \frac{m_1 qB}{2M}v_2, \quad k_2 = p_{w_2} + \frac{qB}{2}w_1 - \frac{m_1 qB}{2M}v_1, \tag{3.9}$$

by virtue of $y_\perp = w - (m_1/M)v$. On the other hand, we may rewrite H as follows:

$$H = \frac{1}{2\mu}\left\{D_z^2 + \left(p_v + \frac{m_1 q}{M}\boldsymbol{A}(y)\right)^2\right\} + \frac{1}{2M}(p_w - q\boldsymbol{A}(y))^2 + V. \tag{3.10}$$

Thus the operators

$$\begin{aligned} p_{v_1} - \frac{m_1 qB}{2M}y_2 &= p_{v_1} + \frac{m_1^2 qB}{2M^2}v_2 - \frac{m_1 qB}{2M}w_2,\\ p_{v_2} + \frac{m_1 qB}{2M}y_1 &= p_{v_2} - \frac{m_1^2 qB}{2M^2}v_1 + \frac{m_1 qB}{2M}w_1\end{aligned}$$

are H-bounded. Noting that

$$\begin{aligned}\frac{m_1}{2M}w_1 - \frac{m_1^2}{2M^2}v_1 - \frac{1}{qB}p_{v_2} &= -\frac{1}{qB}\left(p_{v_2} - \frac{m_1^2 qB}{2M^2}v_1 + \frac{m_1 qB}{2M}w_1\right) + \frac{m_1}{M}w_1 - \frac{m_1^2}{M^2}v_1\\ &= -\frac{1}{qB}\left(p_{v_2} - \frac{m_1^2 qB}{2M^2}v_1 + \frac{m_1 qB}{2M}w_1\right) + \frac{m_1}{M}y_1,\\ \frac{m_1}{2M}w_2 - \frac{m_1^2}{2M^2}v_2 + \frac{1}{qB}p_{v_1} &= \frac{1}{qB}\left(p_{v_1} + \frac{m_1^2 qB}{2M^2}v_2 - \frac{m_1 qB}{2M}w_2\right) + \frac{m_1}{M}w_2 - \frac{m_1^2}{M^2}v_2\\ &= \frac{1}{qB}\left(p_{v_1} + \frac{m_1^2 qB}{2M^2}v_2 - \frac{m_1 qB}{2M}w_2\right) + \frac{m_1}{M}y_2,\end{aligned}$$

if we will obtain the boundedness of $y_j h(N)f(H)$ $(j = 1, 2)$, we will see $(\tilde{A} - A)h(N)f(H)$ is bounded on $L^2(\boldsymbol{R}^5)$. Taking account of the boundedness of the operators $kh(N)$, $p_x f(H)$ and $(p_y - q\boldsymbol{A}(y))f(H)$, and the relation $k - p_x - (p_y - q\boldsymbol{A}(y)) = 2q\boldsymbol{A}(y)$, we see that $\boldsymbol{A}(y)h(N)f(H)$ is bounded on $L^2(\boldsymbol{R}^5)$, which implies the boundedness of $y_j h(N)f(H)$ $(j = 1, 2)$.

We set $F_A(t) = F(|A|/t \geq c_0)$ and $F_\Xi(t) = F(|\Xi|/t \leq \varepsilon_0)$. And for $u_1 \in L^2(\boldsymbol{R}^5)$, we put $u = F_A(t)F_\Xi(t)h(N)f(H)u_1$. Since $(\tilde{A} - A)h(N)f(H)$ is bounded on $L^2(\boldsymbol{R}^5)$, controlling some commutators, we have

$$\||A|u\| \leq C_1(\||\tilde{A}|u\| + \|u_1\|) \tag{3.11}$$

with some $C_1 > 0$ independent of $t \geq 1$. On the other hand, introducing $G_\Xi(t) = F(|\Xi| \leq 2\varepsilon_0)$ such that $G_\Xi(t)F_\Xi(t) = F_\Xi(t)$, taking account of the definition of the operator $\tilde{A}$, and controlling some commutators, we obtain

$$\begin{aligned} \||A|u\| &\geq c_0\|F_A(t)F_\Xi(t)h(N)f(H)u_1\| - C_2\|u_1\|, \\ \||\tilde{A}|u\| &\leq C_3\varepsilon_0 t\|F_A(t)F_\Xi(t)h(N)f(H)u_1\| + C_4\|u_1\| \end{aligned}$$

with some $C_j > 0$ $(j = 2, 3, 4)$ independent of $t \geq 1$. Hence, combining these inequalities with (3.11), if we take $\varepsilon_0 > 0$ so small that $C_1C_3\varepsilon_0 < c_0$, we obtain the lemma. □

Combining Proposition 3.3 with Lemma 3.4, we obtain the following propagation estimate immediately.

Proposition 3.5. (The minimal velocity estimate) *Assume the same condition as in Proposition 3.2. Let λ, δ, c and f be also as in Proposition 3.2. Then for any real-valued $h \in C_0^\infty(\boldsymbol{R})$, there exists $\varepsilon_0 > 0$ such that*

$$\int_1^\infty \left\| F\left(\frac{|\Xi|}{t} \leq \varepsilon_0\right) h(N)f(H)e^{-itH}\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2 \tag{3.12}$$

for any $\psi \in L^2(\boldsymbol{R}^5)$, with $C > 0$ independent of ψ.

By virtue of the above minimal velocity estimate, we conclude the following propagation property.

Proposition 3.6. *Assume the same condition as in Proposition 3.2. Let λ, δ, c and f be also as in Proposition 3.2. Then for any real-valued $h \in C_0^\infty(\boldsymbol{R})$, there exists $\varepsilon > 0$ such that*

$$\mathrm{s}\text{-}\lim_{t\to\infty} F\left(\frac{|\Xi|}{t} \leq \varepsilon\right) h(N)f(H)e^{-itH} = 0.$$

Proof. First we claim that the following strong limit exists:

$$\mathrm{s}\text{-}\lim_{t\to\infty} e^{itH}F\left(\frac{|\Xi|}{t} \leq \varepsilon\right) h(N)f(H)e^{-itH} \tag{3.13}$$

with $\varepsilon = \varepsilon_0/2$, where $\varepsilon_0 > 0$ is as in Proposition 3.5. In fact, taking $f_1 \in C_0^\infty(\boldsymbol{R})$ such that $f_1 f = f$ with the same properties as f and $h_1 \in C_0^\infty(\boldsymbol{R})$ such that $h_1 h = h$ with the same properties as h, and noting that $[F(|\Xi|/t \le \varepsilon), f_1(H)] = O(t^{-1})$ and $[F(|\Xi|/t \le \varepsilon), h_1(N)] = O(t^{-1})$ it suffices to show the existence of the strong limit of

$$\tilde{W}(t) = e^{itH} f_1(H) h_1(N) F\left(\frac{|\Xi|}{t} \le \varepsilon\right) h(N) f(H) e^{-itH}$$

as $t \to \infty$. By Proposition 3.5, we have

$$|(\varphi, \tilde{W}(s_1)\psi) - (\varphi, \tilde{W}(s_2)\psi)| = o(1)\|\varphi\|, \quad s_1,\, s_2 \to \infty,$$

for $\varphi,\ \psi \in L^2(\boldsymbol{R}^5)$, by virtue of the Cook-Kuroda method. This implies that $\{\tilde{W}(t)\psi\}_{t \ge 1}$ is a Cauchy sequence and hence the existence of (3.13) is proved. By using Proposition 3.5 again, we see that for $\psi \in L^2(\boldsymbol{R}^5)$, there exists a subsequence $\{t_n\}_{n \in \boldsymbol{N}}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n \to \infty} F\left(\frac{|\Xi|}{t_n} \le \varepsilon\right) h(N) f(H) e^{-it_n H}\psi = 0, \tag{3.14}$$

where the choice of subsequence $\{t_n\}_{n \in \boldsymbol{N}}$ depends on ψ. By the existence of (3.13) and (3.14), we have for $\psi \in L^2(\boldsymbol{R}^5)$,

$$\lim_{t \to \infty} F\left(\frac{|\Xi|}{t} \le \varepsilon\right) h(N) f(H) e^{-itH}\psi = 0.$$

Thus the proposition is obtained. $\square$

Next we shall consider the observable $r = x - y = (x_\perp - y_\perp, z)$ instead of the observable Ξ. First we need the following lemma.

Lemma 3.7. *Assume the same condition as in Proposition 3.2. Let $f,\, h \in C_0^\infty(\boldsymbol{R})$. Then for any $\varepsilon > 0$, we have*

$$\left\| F\left(\frac{|y_\perp|}{t} \ge \varepsilon\right) h(N) f(H) \right\| = O(t^{-1})$$

as $t \to \infty$.

Proof. First we recall that $|y_\perp| h(N) f(H)$ is bounded on $L^2(\boldsymbol{R}^5)$, which was seen in the proof of Lemma 3.4.

For $u_1 \in L^2(\boldsymbol{R}^5)$, we put $u = F(|y_\perp|/t \ge \varepsilon) h(N) f(H) u_1$. Then we have

$$\varepsilon t\|u\| \le \||y_\perp| u\| \le C\|u_1\|$$

with some $C > 0$ independent of $t \ge 1$. This implies the lemma. $\square$

Noting that $|\Xi| \le |\Xi - (y_\perp, 0)| + |(y_\perp, 0)|$, the following proposition follows from Proposition 3.5 and Lemma 3.7 immediately.

Proposition 3.8. (The minimal velocity estimate) *Assume the same condition as in Proposition 3.2. Let λ, δ, c and f be also as in Proposition 3.2. Then for any real-valued $h \in C_0^\infty(\boldsymbol{R})$, there exists $\varepsilon_0 > 0$ such that*

$$\int_1^\infty \left\| F\left(\frac{|r|}{t} \leq \varepsilon_0\right) h(N) f(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2 \tag{3.15}$$

for any $\psi \in L^2(\boldsymbol{R}^5)$, with $C > 0$ independent of ψ.

By virtue of the above minimal velocity estimate, we conclude the following propagation property. Since it may be proved in the same way as in the proof of Proposition 3.6, we omit the proof.

Proposition 3.9. *Assume the same condition as in Proposition 3.2. Let λ, δ, c and f be also as in Proposition 3.2. Then for any real-valued $h \in C_0^\infty(\boldsymbol{R})$, there exists $\varepsilon > 0$ such that*

$$\text{s - }\lim_{t\to\infty} F\left(\frac{|r|}{t} \leq \varepsilon\right) h(N) f(H) e^{-itH} = 0.$$

Now, for simplicity of computation, we introduce another metric $\langle\, , \rangle$ on $\boldsymbol{R}^3_\Xi$ as follows:

$$\langle \Xi, \Xi' \rangle = m_1(x_1 x_1' + x_2 x_2') + \mu z z', \quad |\Xi|_1 = \sqrt{\langle \Xi, \Xi \rangle} \tag{3.16}$$

for $\Xi = (x_1, x_2, z)$ and $\Xi' = (x_1', x_2', z')$. We should note that the norm $|\cdot|_1$ is equivalent with the Euclidean norm $|\cdot|$. We denote p_Ξ by

$$p_\Xi = \left(\frac{-i\partial_{x_1}}{m_1}, \frac{-i\partial_{x_2}}{m_1}, \frac{-i\partial_z}{\mu}\right), \tag{3.17}$$

which should be called the velocity operator associated with Ξ. This convention has been often used in the N-body scattering theory.

Next we need the maximal velocity estimate, which will be used in the study of long-range scattering.

Proposition 3.10. (The maximal velocity estimate) *Suppose that the potential V satisfies the conditions $(V1)$ and $(V2)$. Then for any real-valued $f \in C_0^\infty(\boldsymbol{R})$ there exists $M > 0$ such that for any $M_2 > M_1 \geq M$,*

$$\int_1^\infty \left\| F\left(M_1 \leq \frac{|\Xi|_1}{t} \leq M_2\right) f(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} \leq C \|\psi\|^2 \tag{3.18}$$

for any $\psi \in L^2(\boldsymbol{R}^5)$, with $C > 0$ independent of ψ. Moreover, for any $\psi \in L^2(\boldsymbol{R}^5)$ such that $(1 + |\Xi|_1)^{1/2}\psi \in L^2(\boldsymbol{R}^5)$

$$\int_1^\infty \left\| F\left(\frac{|\Xi|_1}{t} \geq M_1\right) f(H) e^{-itH} \psi \right\|^2 \frac{dt}{t} < \infty. \tag{3.19}$$

Proof. The proof is done in the way similar to the one in [SS2] (see also [D], [DG], [Gr]). We sketch the proof.

Let G_1 be defined by

$$G_1(s) = \int_{-\infty}^{s} g_1(u)^2\, du,$$

with real-valued $g_1 \in C_0^\infty(\boldsymbol{R})$ supported in $[M_0, \infty)$ such that $g_1 = 1$ on $[M_1, M_2]$. We use

$$\Phi_1(t) = -G_1\left(\frac{|\Xi|_1}{t}\right)$$

as a propagation observable. We note that $\Phi_1(t)$ is uniformly bounded in $t \geq 1$. The Heisenberg derivative of this observable is calculated as follows:

$$\begin{aligned}\mathbf{D}_H(\Phi_1(t)) =& \frac{|\Xi|_1}{t^2} g_1\left(\frac{|\Xi|_1}{t}\right)^2 \\ & - \frac{1}{2t}\left\{ g_1\left(\frac{|\Xi|_1}{t}\right)^2 \left\langle \frac{\Xi}{|\Xi|_1}, p_\Xi \right\rangle + \left\langle p_\Xi, \frac{\Xi}{|\Xi|_1}\right\rangle g_1\left(\frac{|\Xi|_1}{t}\right)^2 \right\}.\end{aligned}$$

By virtue of the boundedness of $p_\Xi f(H)$, we have

$$f(H)\mathbf{D}_H(\Phi_1(t))f(H) \geq \frac{M_0 - C}{t} f(H) g\left(\frac{|\Xi|_1}{t}\right)^2 f(H) + O(t^{-2})$$

for some $C > 0$ which depends on f. Then if we take $M_0 > 0$ so large that $M_0 > C$, we obtain the estimate (3.18).

Next we put

$$G_2(s) = \int_{-\infty}^{s} g_2(u)^2\, du$$

with real-valued $g_2 \in C_0^\infty(\boldsymbol{R})$ which is supported in $[M_0, M_0 + 1]$ and satisfies $\int_{-\infty}^{\infty} g_2(u)^2\, du = 1$. We use

$$\Phi_2(t) = -\left(\frac{|\Xi|_1}{t} - M_0\right) G_2\left(\frac{|\Xi|_1}{t}\right)$$

as a propagation observable. We note that $(1+|\Xi|_1)^{-1/2} e^{itH} f(H)\Phi_2(t) f(H) e^{-itH} (1+|\Xi|_1)^{-1/2}$ is uniformly bounded in $t \geq 1$. In the same way as above, we have

$$\begin{aligned}& f(H)\mathbf{D}_H(\Phi_2(t))f(H) \\ \geq & f(H)\left\{ \frac{M_0 - C_1}{t} G_2\left(\frac{|\Xi|_1}{t}\right) - \frac{C_2}{t} g_2\left(\frac{|\Xi|_1}{t}\right)^2 + O(t^{-2}) \right\} f(H).\end{aligned}$$

Thus if we take M_0 so large that $M_0 > C_1$ and (3.18) holds, we obtain the estimate (3.19). $\square$

Noting that $|\Xi|_1 \geq \big| |\Xi - (y_\perp, 0)|_1 - |(y_\perp, 0)|_1 \big|$, we obtain the following proposition immediately, by virtue of Lemma 3.7.

Proposition 3.11. (The maximal velocity estimate) *Suppose that the potential V satisfies the conditions $(V1)$ and $(V2)$. Then for any real-valued f, $h \in C_0^\infty(\boldsymbol{R})$ there exists $M > 0$ such that for any $M_2 > M_1 \geq M$,*

$$\int_1^\infty \left\| F\left(M_1 \leq \frac{|r|_1}{t} \leq M_2\right) h(N) f(H) e^{-itH}\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2 \tag{3.20}$$

for any $\psi \in L^2(\boldsymbol{R}^5)$, with $C > 0$ independent of ψ. Moreover, for any $\psi \in L^2(\boldsymbol{R}^5)$ such that $(1 + |\Xi|_1)^{1/2}\psi \in L^2(\boldsymbol{R}^5)$

$$\int_1^\infty \left\| F\left(\frac{|r|_1}{t} \geq M_1\right) h(N) f(H) e^{-itH}\psi \right\|^2 \frac{dt}{t} < \infty. \tag{3.21}$$

These two propositions imply the following corollary. Since the proof is similar to the one of Proposition 3.6, we omit the proof.

Corollary 3.12. *Assume the same condition as in Proposition 3.10. Then for any real-valued f, $h \in C_0^\infty(\boldsymbol{R})$, there exists $M > 0$ such that*

$$\begin{aligned} &\mathrm{s}\text{-}\lim_{t\to\infty} F\left(\frac{|\Xi|_1}{t} \geq M\right) f(H) e^{-itH} = 0, \\ &\mathrm{s}\text{-}\lim_{t\to\infty} F\left(\frac{|r|_1}{t} \geq M\right) h(N) f(H) e^{-itH} = 0. \end{aligned}$$

The next propagation estimate is analogue to the one derived by Graf [Gr] (see also [D], [DG]).

Proposition 3.13. *Assume the same condition as in Proposition 3.2. Let λ, δ, c and f be also as in Proposition 3.2. Then for any real-valued $h \in C_0^\infty(\boldsymbol{R})$, there exist $c_1 > 0$ and $c_2 > 0$ such that*

$$\int_1^\infty \left\| \left|\frac{\Xi}{t} - p_\Xi\right|_1 F\left(c_1 \leq \frac{|r|_1}{t} \leq c_2\right) h(N) f(H) e^{-itH}\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2 \tag{3.22}$$

for any $\psi \in L^2(\boldsymbol{R}^5)$, with $C > 0$ independent of ψ.

Proof. Following the idea of Dereziński [D], we put

$$\Phi(t) = -\frac{|\Xi|_1^2}{t^2} + \frac{\langle \Xi, p_\Xi\rangle + \langle p_\Xi, \Xi\rangle}{t}. \tag{3.23}$$

By a straightforward computation, we see that

$$\mathbf{D}_{H_0}(\Phi(t)) = \frac{2}{t}\left|p_\Xi - \frac{\Xi}{t}\right|_1^2. \tag{3.24}$$

We use

$$\Phi_1(t) = F_r(t)\Phi(t)F_r(t)$$

as a propagation observable, where

$$F_r(t) = F\left(c_1 \le \frac{|r|_1}{t} \le c_2\right).$$

We take $c_1 > 0$ so small that for any $\psi \in L^2(\boldsymbol{R}^5)$

$$\int_1^\infty \left\| F\left(\frac{|r|_1}{t} \le 2c_1\right) h(N)f(H)e^{-itH}\psi \right\|^2 \frac{dt}{t} \le C\|\psi\|^2$$

holds, by virtue of Proposition 3.8. We also take $c_2 > 0$ so large that for any $\psi \in L^2(\boldsymbol{R}^5)$

$$\int_1^\infty \left\| F\left(c_2 - 1 \le \frac{|r|_1}{t} \le c_2 + 1\right) f(H)e^{-itH}\psi \right\|^2 \frac{dt}{t} \le C\|\psi\|^2$$

holds, by virtue of Proposition 3.11.

We compute the Heisenberg derivative of $\Phi_1(t)$:

$$\mathbf{D}_H(\Phi_1(t)) = \mathbf{D}_H(F_r(t))\Phi(t)F_r(t) + F_r(t)\mathbf{D}_H(\Phi(t))F_r(t) + F_r(t)\Phi(t)\mathbf{D}_H(F_r(t)). \tag{3.25}$$

Noting that $\|\Phi(t)F_r(t)(H+i)^{-1}\| = O(1)$ and $\|\mathbf{D}_H(F_r(t))(H+i)^{-1}\| = O(t^{-1})$, and taking account of the support of $F_r'(t)$, we may apply Propositions 3.8 and 3.11 to the first and last terms in (3.23). Thus we have only to consider the second term in (3.23). Noting that (3.22) holds, we have to consider the term $F_r(t)i[V, \Phi(t)]F_r(t)$. We should note that $i[V, \Phi(t)]$ is decomposed as

$$i[V, \Phi(t)] = -\frac{2}{t}\{(r \cdot \nabla_r V) + (y_\perp \cdot \nabla_{r_\perp} V)\}. \tag{3.26}$$

Noting that $|y_\perp|_1 h(N)f(H)$ is bounded, by the condition $(V2)$, we see that

$$\|f(H)h(N)F_r(t)i[V, \Phi(t)]F_r(t)h(N)f(H)\| = O(t^{-1-\mu_1}).$$

Therefore the proposition is obtained. $\square$

§4. Short-range case

In this section, we shall prove the asymptotic completeness of the system under the short-range assumption (SR).

Proof of Theorem 1.1. We shall prove the existence of the following limit:

$$\mathrm{s}\text{-}\lim_{t\to\infty} e^{itH_0}e^{-itH}P_c(H), \tag{4.1}$$

where $P_c(H)$ is the spectral projection onto the continuous spectral subspace $L^2_c(H)$ of the Hamiltonian H. Since the existence of the wave operator W^+ may be proved quite similarly, if we will see that the above strong limit exists, then we will obtain the theorem by a standard argument in the scattering theory.

Noting that N commutes with H, by a density argument, for $\psi \in L^2_c(H)$ such that

$$\psi = h(N)\psi, \quad \psi = f(H)\psi$$

with $h \in C_0^\infty(\boldsymbol{R})$ and $f \in C_0^\infty(\boldsymbol{R})$ which is as in Proposition 3.2, we have only to show the existence of the limit

$$\lim_{t\to\infty} e^{itH_0}e^{-itH}\psi. \tag{4.2}$$

Here we used the fact mentioned in Remark 3.1 (see also Corollary 2.2). By virtue of Proposition 3.9, it suffices to prove that

$$\lim_{t\to\infty} e^{itH_0}F\left(\frac{|r|}{t}\geq\varepsilon\right)e^{-itH}\psi \tag{4.3}$$

exists, where $\varepsilon > 0$ is sufficiently small as in Proposition 3.9. Now taking $f_1 \in C_0^\infty(\boldsymbol{R})$ such that $f_1 f = f$ with the same properties as f, and $h_1 \in C_0^\infty(\boldsymbol{R})$ such that $h_1 h = h$, as in the proof of Proposition 3.6, we have only to show the existence of the limit

$$\lim_{t\to\infty} e^{itH_0}f_1(H_0)h_1(N)F\left(\frac{|r|}{t}\geq\varepsilon\right)e^{-itH}\psi. \tag{4.4}$$

Here we should note that

$$f_1(H_0)F\left(\frac{|r|}{t}\geq\varepsilon\right) - F\left(\frac{|r|}{t}\geq\varepsilon\right)f_1(H) = O(t^{\max(-\mu_S,-1)}) = O(t^{-1}).$$

We also notice that both H and H_0 have the same total pseudomomentum. Then noting that $\|V(r)F(|r|/t\geq\varepsilon)(-\Delta_r+1)^{-1}\| = O(t^{-\mu_S})$ is integrable in $t\geq 1$ by virtue of $\mu_S > 1$, we may prove the existence of the limit (4.4) in the way quite similar to the one in the proof of Proposition 3.6, by virtue of Proposition 3.8 and the same propagation property of the free evolution e^{-itH_0}. Thus we obtain the theorem. □

§5. Long-range case

In this section, we shall prove the asymptotic completeness of the system under the long-range assumption (LR).

Proof of Theorem 1.2. First of all, we should note that in the same way as in the proof of Theorem 1.1, we may prove the following statement: Defining the Hamiltonian H_L acting on $L^2(\boldsymbol{R}^5)$ by

$$H_L = \frac{1}{2\mu}D_z^2 + \frac{1}{2m_1}p_x^2 + \frac{1}{2m_2}(p_y - e\boldsymbol{A}(y))^2 + V_L(x-y), \tag{5.1}$$

and the wave operator Ω^+ by

$$\Omega^+ = \text{s - }\lim_{t\to\infty} e^{itH}e^{-itH_L}P_c(H_L), \tag{5.2}$$

where $P_c(H_L)$ is the spectral projection onto the continuous spectral subspace $L^2_c(H_L)$ of the Hamiltonian H_L, the wave operator Ω^+ exists and $Ran\,\Omega^+ = L^2_c(H)$.

By virtue of the above statement, we have only to prove the existence of the following limit:

$$\text{s - }\lim_{t\to\infty} U_D(t)^* e^{-itH_L}P_c(H_L), \tag{5.3}$$

where

$$U_D(t) = e^{-itH_0 - i\int_0^t V_L(sp_\Xi)\,ds}.$$

We should note that the existence of the wave operator W_D^+ may be proved in the same way as below. By a density argument, for $\psi \in L^2_c(H_L)$ such that

$$\psi = h(N)\psi, \quad \psi = f(H_L)\psi$$

with $h \in C_0^\infty(\boldsymbol{R})$ and $f \in C_0^\infty(\boldsymbol{R})$ which is as in Proposition 3.2, we have only to show the existence of the limit

$$\lim_{t\to\infty} U_D(t)^* e^{-itH_L}\psi. \tag{5.4}$$

Before doing that, we need the following propagation property due to Proposition 3.13.

Proposition 5.1. *Assume the same condition as in Theorem 1.2. Let λ, δ, c and f be also as in Proposition 3.2. Then for any real-valued $h \in C_0^\infty(\boldsymbol{R})$, there exist $c_1 > 0$ and $c_2 > 0$ such that*

$$\text{s - }\lim_{t\to\infty} \left|\frac{\Xi}{t} - p_\Xi\right|_1 F\left(c_1 \le \frac{|r|_1}{t} \le c_2\right) h(N) f(H_L) e^{-itH_L} = 0.$$

Proof. Following the idea of [DG], we use

$$\Phi(t) = -F_r(t)\left|\frac{\Xi}{t} - p_\Xi\right|_1^2 F_r(t)$$

as a propagation observable, where $F_r(t)$ is as in the proof of Proposition 3.13. We should note that

$$\mathbf{D}_{H_0}\left(\left|\frac{\Xi}{t} - p_\Xi\right|_1^2\right) = -\frac{2}{t}\left|\frac{\Xi}{t} - p_\Xi\right|_1^2. \tag{5.5}$$

We compute the Heisenberg derivative of $\Phi(t)$:

$$\begin{aligned}\mathbf{D}_{H_L}(\Phi(t)) = &- \mathbf{D}_{H_L}(F_r(t))\left|\frac{\Xi}{t} - p_\Xi\right|_1^2 F_r(t) - F_r(t)\mathbf{D}_{H_L}\left(\left|\frac{\Xi}{t} - p_\Xi\right|_1^2\right)F_r(t)\\ &- F_r(t)\left|\frac{\Xi}{t} - p_\Xi\right|_1^2 \mathbf{D}_{H_L}(F_r(t)).\end{aligned} \tag{5.6}$$

Noting that $\||\Xi/t - p_\Xi|_1^2 F_r(t)(H_L+i)^{-1}\| = O(1)$ and $\|\mathbf{D}_{H_L}(F_r(t))(H_L+i)^{-1}\| = O(t^{-1})$, and taking account of the support of $F_r'(t)$, we may apply Propositions 3.8 and 3.11 to the first and last terms in (5.6). Thus we have only to consider the second term in (5.6). Noting that (5.5) holds, we may apply Proposition 3.13 to the term in which $\mathbf{D}_{H_0}(|\Xi/t - p_\Xi|_1^2)$ appears. Thus we have to consider the term $F_r(t)i[V_L, |\Xi/t - p_\Xi|_1^2]F_r(t)$. By the condition (LR) and Proposition 3.13, for $\psi \in L^2(\boldsymbol{R}^5)$, we see that

$$\left|\left(e^{-itH_L}\psi, f(H_L)h(N)F_r(t)i\left[V, \left|\frac{\Xi}{t} - p_\Xi\right|_1^2\right] F_r(t)h(N)f(H_L)e^{-itH_L}\psi\right)\right|$$

is integrable in $t \geq 1$. Thus there exists the limit

$$\lim_{t\to\infty}\left\|\left|\frac{\Xi}{t} - p_\Xi\right|_1 F_r(t)h(N)f(H_L)e^{-itH_L}\psi\right\|^2$$

for $\psi \in L^2(\boldsymbol{R}^5)$. But, by Proposition 3.13, the limit equals zero. This implies the proposition. $\square$

Continuation of the proof of Theorem 1.2. By Proposition 3.6, we have

$$\lim_{t\to\infty} F\left(\frac{|\Xi|}{t} \leq \varepsilon\right) F\left(c_1 \leq \frac{|r|}{t} \leq c_2\right) e^{-itH_L}\psi = 0, \tag{5.7}$$

for some $\varepsilon > 0$, with any $c_2 > c_1 > 0$. We may take $\varepsilon > 0$ so small that

$$\lim_{t\to\infty} F\left(\frac{|r|}{t} \leq \varepsilon\right) e^{-itH_L}\psi = 0 \tag{5.8}$$

holds. By virtue of Proposition 5.1 and the Baker-Campbell-Hausdorff formula (see e.g. [DG]), that is, putting $G(\Xi) = F(|\Xi| \leq \varepsilon)$,

$$\begin{aligned}&F(|p_\Xi| \leq \varepsilon) - F\left(\frac{|\Xi|}{t} \leq \varepsilon\right)\\ &= \int_0^1 (\nabla G)\left(sp_\Xi + (1-s)\frac{\Xi}{t}\right)\cdot\left(p_\Xi - \frac{\Xi}{t}\right) ds + \frac{i}{2t}\int_0^1 (\Delta G)\left(sp_\Xi + (1-s)\frac{\Xi}{t}\right) ds,\end{aligned}$$

it follows from (5.7) that

$$\lim_{t\to\infty} F(|p_\Xi| \leq \varepsilon) F\left(c_1 \leq \frac{|r|}{t} \leq c_2\right) e^{-itH_L}\psi = 0, \tag{5.9}$$

where $c_1 > 0$ is sufficiently small and $c_2 > 0$ is sufficiently large. Using Proposition 3.9 and Corollary 3.12, we see that

$$\lim_{t\to\infty} F(|p_\Xi| \leq \varepsilon)e^{-itH_L}\psi = 0 \tag{5.10}$$

by (5.9). Thus it suffices to prove the existence of the limit

$$\lim_{t\to\infty} U_D(t)^* \bar{F}(|p_\Xi| \geq \varepsilon) e^{-itH_L}\psi, \tag{5.11}$$

where $\bar{F}(s \geq \varepsilon) = 1 - F(s \leq \varepsilon)$.

Now we introduce a time-dependent potential:

$$V(t, r) = V_L(r) F\left(\frac{|r|}{t} \geq \frac{\varepsilon}{2}\right). \tag{5.12}$$

$V(t, r)$ satisfies

$$|\partial_r^\alpha V(t, r)| \leq C_\alpha (t + \langle r \rangle)^{-|\alpha| - \mu_L} \tag{5.13}$$

for $t \geq 1$. Then we define two time-dependent Hamiltonians:

$$H(t) = H_0 + V(t, \Xi),$$
$$H_0(t) = H_0 + V(t, tp_\Xi).$$

$H(t)$ and $H_0(t)$ generate the propagators $U(t)$ and $U_0(t)$, respectively. Noting that

$$(H_0 + V(tp_\Xi))\bar{F}(|p_\Xi| \geq \varepsilon) - \bar{F}(|p_\Xi| \geq \varepsilon) H_0(t) = 0,$$

we see that the strong limit

$$\text{s - }\lim_{t\to\infty} U_D(t)^* \bar{F}(|p_\Xi| \geq \varepsilon) U_0(t) \tag{5.14}$$

exists.

By the way, by virtue of the scattering theory for time-dependent Hamiltonians (see e.g. [D] and [DG]), we know that the strong limit

$$\text{s - }\lim_{t\to\infty} U_0(t)^* U(t) \tag{5.15}$$

exists under the condition $\mu_L > 1/2$. We used (5.13) and the fact that $U(t)$ and $U_0(t)$ are represented by

$$U(t) = \hat{U}(t) \otimes e^{-itH_2}, \quad U_0(t) = \hat{U}_0(t) \otimes e^{-itH_2}$$

on $L^2(\boldsymbol{R}^5) = L^2(\boldsymbol{R}^3_\Xi) \otimes L^2(\boldsymbol{R}^2_{y_\perp})$, where

$$H_2 = \frac{1}{2m_2}(p_y - q\boldsymbol{A}(y))^2$$

acting on $L^2(\boldsymbol{R}^2_{y_\perp})$, and $\hat{U}(t)$ and $\hat{U}_0(t)$ are the propagators generated by the time-dependent Hamiltonians

$$\hat{H}(t) = \frac{1}{2\mu} D_z^2 + \frac{1}{2m_1} p_x^2 + V(t, \Xi), \quad \hat{H}_0(t) = \frac{1}{2\mu} D_z^2 + \frac{1}{2m_1} p_x^2 + V(t, tp_\Xi),$$

respectively. We sketch the proof. We have only to show the existence of the strong limit

$$\text{s - }\lim_{t\to\infty} \hat{U}_0^*(t)\hat{U}(t).$$

Since $\|\mathbf{D}_{\hat{H}(t)}(tp_\Xi - \Xi)\| = O(t^{-\mu_L})$, we have

$$\|(tp_\Xi - \Xi)\hat{U}(t)\phi\| = O(t^{1-\mu_L})$$

for $\phi \in C_0^\infty(\boldsymbol{R}^3)$. Here we assumed $1/2 < \mu_L < 1$ for simplicity. In order to prove the above limit, we have to deal with $V(t, tp_\Xi) - V(t, \Xi)$. By using the Baker-Campbell-Hausdorff formula and the above estimate, we have for $\phi \in C_0^\infty(\boldsymbol{R}^3)$

$$\left\| \frac{d}{dt}(\hat{U}_0(t)^*\hat{U}(t)\phi) \right\| = O(t^{-2\mu_L}).$$

Since $\mu_L > 1/2$, this is integrable in $t \geq 1$. This implies the existence of the above strong limit by virtue of the Cook-Kuroda method.

Since (5.14) and (5.15) exist, we have only to show the existence of the limit

$$\lim_{t\to\infty} U(t)^*\bar{F}\left(\frac{|r|}{t} \geq \varepsilon\right) e^{-itH_L}\psi. \tag{5.16}$$

In order to show that (5.16) exists, we should notice that for the propagator $U(t)$ the statement similar to the one of Proposition 3.5 holds. In fact,

$$\int_1^\infty \left\| F\left(\frac{|\Xi|}{t} \leq \epsilon_0\right) f(H_0)U(t)\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2$$

holds, where $\varepsilon_0 > 0$ and f are as in Proposition 3.5. The proof is similar to the one of Proposition 3.5. Then, as in §4, we have only to show the existence of the limit

$$\lim_{t\to\infty} U(t)^* f_1(H_0)\bar{F}\left(\frac{|r|}{t} \geq \varepsilon\right) e^{-itH_L}\psi, \tag{5.17}$$

where $f_1 \in C_0^\infty(\boldsymbol{R})$ such that $f_1 f = f$ which has the same properties as f. Now we have to deal with

$$f_1(H_0)\partial_t\left\{\bar{F}\left(\frac{|r|}{t} \geq \varepsilon\right)\right\} + iH(t)f_1(H_0)\bar{F}\left(\frac{|r|}{t} \geq \varepsilon\right) - f_1(H_0)\bar{F}\left(\frac{|r|}{t} \geq \varepsilon\right) iH_L. \tag{5.18}$$

As for the first term in (5.18), taking account of the support of $\bar{F}'(|r|/t \geq \varepsilon)$, by virtue of Lemma 3.7, we may apply Proposition 3.8 and the minimal velocity estimate for the propagator $U(t)$ mentioned above to it. As for the remainder term in (5.18), noting that $\|[V(t,\Xi), f_1(H_0)]\| = O(t^{-1-\mu_L})$ which is integrable in $t \geq 1$ and that the term concerned with $[H_0, \bar{F}(|r|/t \geq \varepsilon)]$ may be controlled as the first term, we have to deal with the term $f(H_0)(V(t,\Xi) - V_L(r))\bar{F}(|r|/t \geq \varepsilon)$. Here we should notice that

$$V_L(r)\bar{F}\left(\frac{|r|}{t} \geq \varepsilon\right) = V(t,r)\bar{F}\left(\frac{|r|}{t} \geq \varepsilon\right).$$

Thus, taking account of

$$V(t,\Xi) - V(t,r) = \int_0^1 \nabla_{r_\perp} V(t, s\Xi + (1-s)r)\, ds \cdot y_\perp,$$

noting that (5.13) holds and that $|y_\perp| h(N) f(H)$ is bounded, the term concerned with $f(H_0)(V(t,\Xi) - V_L(r))\bar{F}(|r|/t \geq \varepsilon)$ is also integrable in $t \geq 1$. Therefore by the Cook-Kuroda method, we see that (5.14) exists. This implies the theorem. $\square$

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