Center Manifold for Nonintegrable Nonlinear Schrödinger Equations on the Line*

Ricardo Weder†
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas,
Universidad Nacional Autónoma de México,
Apartado Postal 20-726, México D.F. 01000
E-Mail: weder@servidor.unam.mx

Abstract

In this paper we study the following nonlinear Schrödinger equation on the line,

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{d^2}{dx^2}u(t,x) + V(x)u(t,x) + f(x,|u|)\frac{u(t,x)}{|u(t,x)|}, u(0,x) = \phi(x),$$

where f is real-valued, and it satisfies suitable conditions on regularity, on grow as a function of u and on decay as $x \to \pm\infty$. The *generic* potential, V, is real-valued and it is chosen so that the spectrum of $H := -\frac{d^2}{dx^2} + V$ consists of one simple negative eigenvalue and absolutely-continuous spectrum filling $[0,\infty)$. The solutions to this equation have, in general, a localized and a dispersive component. The nonlinear bound states, that bifurcate from the zero solution at the energy of the eigenvalue of H, define an invariant center manifold that consists of the orbits of time-periodic localized solutions . We prove that all small solutions approach a particular periodic orbit in the center manifold as $t \to \pm\infty$. In general, the periodic orbits are different for $t \to \pm\infty$. Our result implies also that the nonlinear bound states are asymptotically stable, in the sense that each solution with initial data near a nonlinear bound state is asymptotic as $t \to \pm\infty$ to the periodic orbits of nearby nonlinear bound states that are, in general, different for $t \to \pm\infty$.

*AMS classification 35P, 35Q, 35R and 81U
†Fellow Sistema Nacional de Investigadores

1 Introduction

We study below the small solutions to the nonintegrable, nonlinear Schrödinger equation,

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{d^2}{dx^2}u(t,x) + V(x)u(t,x) + f(x,|u|)\frac{u(t,x)}{|u(t,x)|}, u(0,x) = \phi(x), \tag{1.1}$$

where u is a complex-valued function defined for $t, x \in \mathbf{R}$. For each fixed $x \in \mathbf{R}$, $f(x,\cdot) \in C^1(\mathbf{R},\mathbf{R})$, $\frac{\partial}{\partial x}f(x,\cdot) \in C(\mathbf{R},\mathbf{R})$, $f(x,0) = 0$ and,

$$\left|\frac{\partial}{\partial u}f(x,u)\right| \leq C|u|^{p-1},$$

$$\left|\frac{\partial}{\partial x}f(x,u)\right| \leq C|u|^p, \text{for some}\, p > 2. \tag{1.2}$$

The potential, V, is a real-valued function. For any $\gamma \in \mathbf{R}$ we denote by L^1_γ the Banach space of all complex-valued measurable functions, ϕ, defined on $\mathbf{R}$ and such that

$$\|\phi\|_{L^1_\gamma} := \int |\phi(x)|\,(1+|x|)^\gamma\, dx < \infty. \tag{1.3}$$

If $V \in L^1_1$, $-\frac{d^2}{dx^2} + V$ has a unique self-adjoint realization in L^2, that we denote by H . Moreover, H has no singular-continuous spectrum, and its absolutely-continuous spectrum is $[0,\infty)$. H has no positive or zero eigenvalues and, in general, it has a finite number of negative eigenvalues that are simple. For these results see [5] and [26]. We assume that H has only one negative eigenvalue, that we denote by E_0. We also suppose that,

$$N(V) := \sup_{x\in\mathbf{R}} \int_x^{x+1} |V(y)|^2\, dy < \infty. \tag{1.4}$$

It follows from Theorem 2.7.1 in page 35 of [20] that $D(H) = H_2$, where, $H_n, n = 1, 2, \cdots$, denote the Sobolev spaces [1].

To explain our results let us consider first the associated linear Schrödinger equation with $f \equiv 0$,

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{d^2}{dx^2}u(x,t) + V(x)u(t,x), u(0,x) = \phi(x). \tag{1.5}$$

Let Ψ_0 be the eigenvector associated to the eigenvalue E_0, with L^2 norm equal to one. Equation (1.5) has an invariant center manifold given by,

$$\mathcal{M}_0 := \left\{re^{i\theta}\Psi_0 : r \geq 0, 0 \leq \theta < 2\pi\right\}. \tag{1.6}$$

The invariant manifold, $\mathcal{M}_0$, consists of the orbits of periodic localized solutions to (1.5) of the form $e^{-itE_0} r e^{i\theta}\Psi_0 : r \geq 0, 0 \leq \theta < 2\pi$.

Every solution to (1.5), $u = e^{-itH}\phi$, with initial data in $\phi \in L^2$, can be decomposed as follows,

$$e^{-itH}\phi = e^{-itE_0} P_0 \phi + e^{-itH} P_c \phi, \tag{1.7}$$

where P_0 denotes the orthogonal projector in L^2 onto the one-dimensional subspace generated by Ψ_0, $P_0\phi := (\phi, \Psi_0)\Psi_0$. Moreover, $P_c := I - P_0$ is the projector onto the subspace of continuity of H, $\mathcal{H}_c := P_c L^2$. The component $e^{-itH}P_c\phi$ is a scattering state that propagates out to infinity as $t \to \pm\infty$. In fact, it is an easy consequence of the RAGE theorem [19] that,

$$\lim_{t\to\pm\infty} \left\| e^{-itH} P_c \phi \right\|_{L^2_{-s}} = 0, \ \ s > 0, \tag{1.8}$$

where $L^2_s, s \in \mathbf{R}$, denotes the weighted L^2 space consisting of all functions, f, that are locally in L^2 and such that $(1+|x|^2)^{s/2} f(x) \in L^2$, with norm,

$$\|f\|_{L^2_s} := \left\| (1+|x|^2)^{s/2} f(x) \right\|_{L^2}. \tag{1.9}$$

Equation (1.8) is an expression of the fact that, as the scattering state propagates to infinity the local energies tend to zero. Note that (1.9) does not holds with $s = 0$ because e^{-itH} is unitary in L^2. By (1.7) and (1.8) every solution, u, to (1.5) approaches a periodic orbit in the invariant center manifold in the sense that for any $s > 0$,

$$\lim_{t\to\pm\infty} \left\| u(t) - e^{-itE_0} P_0 \phi \right\|_{L^2_{-s}} = 0. \tag{1.10}$$

For functions $u(t,x)$ defined for $t, x \in \mathbf{R}$, we denote $u(t)$ for $u(t,\cdot)$. Equation (1.10) gives us also a time-dependent characterization of the stability of the bound state Ψ_0. It tells us that given any initial state in L^2, $\Psi = re^{i\theta}\Psi_0 + \phi$ with $\phi \in \mathcal{H}_c$, the solution to (1.5) is the sum of the periodic orbit of the bound state, $e^{-itE_0} r e^{i\theta}\Psi_0$, and a dispersive solution, $e^{-itH}\phi$, that propagates out to infinity as $t \to \pm\infty$ and whose local energies tend to zero.

As we show below, this situation persists in the nonlinear case. There is an invariant center manifold, consisting of the orbits of periodic localized solutions, such that all small solutions to (1.1) approach particular orbits in the center manifold as $t \to \pm\infty$. Invariant manifold theorems have been extensively used in the analysis of the time evolution of dissipative equations. See for example, [4] and [9]. Equation (1.1) is however, dispersive.

A periodic localized solution to (1.1) is a solution of the type $u(t,x) := e^{-itE}\Psi_E$, where the nonlinear bound state, Ψ_E, is a solution to the following nonlinear eigenvalue problem,

$$H\Psi_E + f(x, |\Psi_E|)\frac{\Psi_E}{|\Psi_E|} = E\Psi_E, \Psi_E \in H_2. \tag{1.11}$$

It is a consequence of standard bifurcation theory (see Theorem 3.2.2 in page 77 of [15]) that the set of nontrivial solutions to (1.11) near the trivial solution $\Psi_{E_0} = 0$ consists of exactly one continuous curve such that $|E - E_0| < \epsilon$ for some $\epsilon > 0$, and,

$$\lim_{E\to 0} \|\Psi_E\|_{H_2} = 0. \tag{1.12}$$

In other words, the nonlinear bound states bifurcate from the zero solutions at $E = E_0$. We prove in Appendix 1 (see (1.24) below)that for some constant C

$$|\Psi_0(x)| \leq Ce^{-\sqrt{|E_0|}|x|};\ |\Psi_E(x)| \leq Ce^{-\sqrt{|E|}|x|}, |E - E_0| < \epsilon. \tag{1.13}$$

If follows from (1.13) that for any $s > 0$ there is a constant C_s such that,

$$\|\Psi_E\|_{L^2_s} \leq C_s, |E - E_0| < \epsilon. \tag{1.14}$$

The invariant center manifold for the nonlinear Schrödinger equation (1.1) is given by

$$\mathcal{M} := \left\{e^{i\theta}\Psi_E : |E - E_0| < \epsilon, 0 \leq \theta < 2\pi\right\}. \tag{1.15}$$

$\mathcal{M}$ consists of the orbits of periodic solutions to (1.1) of the form $e^{-itE}e^{i\theta}\Psi_E$. On the spirit of the central manifold theorem, [4], [9], [16], let us write $\mathcal{M}$ as the graph of a function from P_0L^2 into its orthogonal complement ,$\mathcal{H}_c$. The proof given in [16] in the case of three or more dimensions applies also in one dimension, and it follows that there is a $\delta > 0$ and a function, h, from $\{z \in \mathbf{C} : |z| < \delta\}$ into $\mathcal{H}_c \cap H_2 \cap L^2_s, s > 0$, such that,

$$\mathcal{M} = \{\Psi = z\Psi_0 + h(z) : |z| < \delta\}. \tag{1.16}$$

Moreover, h is a C^1 in the real sense, i.e., as a function from $\mathbf{C}$, as a two-dimensional real space, into $\mathcal{H}_c \cap H_2 \cap L^2_s$. Furthermore, $h(0) = 0$, and $h(e^{i\theta}z) = e^{i\theta}h(z)$. See Proposition 2.2 of [16] and its proof.

Let us denote by $F(x,u) := \int_0^u f(x,v)\,dv$. It follows from (1.2) and since $f(x,0)=0$, that

$$F(x,|u|) \leq C|u|^{p+1}, \tag{1.17}$$

for some constant C. It is a consequence of standard standard results (see [10] and [8]) that there is a $\rho > 0$ such that the initial value problem (1.1) has a unique solution in $C(\mathbf{R}, H_1)$ for every $\phi \in H_1$ such that $\|\phi\|_{H_1} < \rho$. If moreover,

$$F(x,|u|) \geq -C(|u|^2 + |u|^{q+1}), \text{ for some } 1 < q < 5, \tag{1.18}$$

then the initial value problem (1.1) has a unique solution in $C(\mathbf{R}, H_1)$ for every $\phi \in H_1$. In both cases the L^2 norm and the energy are conserved quantities,

$$\|u(t)\|_{L^2} = \|\phi\|_{L^2}, \tag{1.19}$$

$$\int dx \left[\frac{1}{2}\left(\left| \frac{\partial}{\partial x} u(t,x) \right|^2 + V(x)|u(t,x)|^2 \right) + F(x,|u|) \right] =$$

$$\int dx \left[\frac{1}{2}\left(\left| \frac{d}{dx} \phi(x) \right|^2 + V(x)|\phi(x)|^2 \right) + F(x,|\phi|) \right]. \tag{1.20}$$

Moreover, for any $\epsilon > 0$ there is a $\nu > 0$ such that

$$\|u(t)\|_{H_1} < \epsilon, t \in \mathbf{R}, \text{if} \|\phi\|_{H_1} < \nu. \tag{1.21}$$

Before we state our main result we need to introduce some more standard notations. For any pair u, v of solutions to the stationary Schrödinger equation:

$$-\frac{d^2}{dx^2} u + Vu = k^2 u,\ k \in \mathrm{C}, \tag{1.22}$$

let $[u,v]$ denotes the Wronskian of u, v:

$$[u,v] := \left(\frac{d}{dx} u \right) v - u \frac{d}{dx} v. \tag{1.23}$$

Let $f_j(x,k), j = 1,2, \Im k \geq 0$, be the Jost solutions to (1.22) (see [6], [7], [5] and [24]). A potential V is said to be *generic* if $[f_1(x,0), f_2(x,0)] \neq 0$ and V is said to be *exceptional* if $[f_1(x,0), f_2(x,0)] = 0$. If V is *exceptional* there is a bounded solution to (1.22) with $k^2 = 0$ (a half-bound state or a zero-energy resonance). Note that the trivial potential, $V = 0$, is *exceptional.* We prove in Appendix 2 that the *generic* potentials are a dense open set in L^1_1.

Our main result is the following theorem.

THEOREM 1.1. *Suppose that $V \in L^1_2$, that $N(V) < \infty$, that V is a* generic *potential and that $H := -\frac{d^2}{dx^2} + V(x)$ has only one negative eigenvalue. We assume that for each fixed $x \in \mathbf{R}$, $f(x,\cdot) \in C^1(\mathbf{R},\mathbf{R})$, $\frac{\partial}{\partial x} f(x,\cdot) \in C(\mathbf{R},\mathbf{R})$, $f(x,0) = 0$ and, for some $p > 2$,*

$$\left| \frac{\partial}{\partial u} f(x,u) \right| \leq q(x) |u|^{p-1}, \tag{1.24}$$

where $(1+|x|)^{2s+4\beta} q(x) \in L^\infty$, for some $s > 1$, and , $1/2 < \beta \leq 1$. Moreover,

$$\left| \frac{\partial}{\partial x} f(x,u) \right| \leq C |u|^p. \tag{1.25}$$

Then there is a $\eta > 0$, such that for all $\phi \in H_1 \cap L^2_{s+2\beta}$ with $\|\phi\|_{H_1} < \eta$, there exist functions, $E(t)$ and $\theta(t)$, in $C^1(\mathbf{R},\mathbf{R})$, such that for some constant C,

$$\left\| u(t) - e^{-i\int_0^t E(\rho) d\rho} e^{i\theta(t)} \Psi_{E(t)} \right\|_{L^2_{-s-2\beta}} \leq C(1+|t|)^{-1/2-\beta} \| P_c\phi - h((\phi, \Psi_0)) \|_{L^2_{1+2\beta}}, \tag{1.26}$$

where $u(t)$ is the solution to (1.1) with initial value ϕ. Moreover, the following limits exists,

$$\lim_{t\to\pm\infty} E(t) := E_\pm; \ \lim_{t\to\pm\infty} \theta(t) := \theta_\pm. \tag{1.27}$$

Equation (1.26) tells us that u tends to the periodic orbit of $e^{i\theta_\pm}\Psi_{E_\pm}$. In particular, solutions with initial data near a nonlinear bound state are asymptotic as $t \to \pm\infty$ to the periodic orbits of nearby nonlinear bound states. Note that the dispersive part, $u(t) - e^{-i\int_0^t E(\rho)d\rho} e^{i\theta(t)} \Psi_{E(t)}$, tends to zero in $L^2_{-s-2\beta}$, as $t \to \pm\infty$, with the same rate that as the dispersive solutions to the associated linear Schrödinger equation (1.5) (see Theorem 1.2 below). A result as in Theorem 1.1 was proven in three or more dimensions in [21], [22] and [16]. For results on the asymptotic stability of stationary (time independent) solutions to nonlinear evolution equations see [11], [13] , [12] and the references mentioned in these papers. For results on the asymptotic stability of solitons see [2] and [3]. For studies of the orbital stability of solitons see [27] and [28].

The proof of Theorem 1.1 is based in ideas from center manifold theory, [4], [9] and [16]. The basic dynamical imput of the proof is the following $L^2_{s+2\beta} - L^2_{-s-2\beta}$ estimate. For any pair of Banach spaces, X, Y, $\mathcal{B}(X,Y)$ denotes the Banach space of all bounded operators from X into Y. In the Theorem below we consider the general case where H has a finite number of eigenvalues.

THEOREM 1.2. *Suppose that $V \in L^1_2$, and that V is* generic*. Then, for any $s > 1$ and $0 \leq \beta \leq 1$, there is a constant C such that,*

$$\left\|e^{-itH}P_c\right\|_{\mathcal{B}\left(L^2_{s+2\beta},L^2_{-s-2\beta}\right)} \leq C(1+|t|)^{-1/2-\beta}. \tag{1.28}$$

A decay estimate as (1.28) was proven in Theorem 7.6 of [14] for potentials such that V is a compact operator from H_l into L^2_ρ for some $l<1$ and some $\rho>4$. Murata's condition rougly means that V decays as $|x|^{-\rho}$ as $|x|\to\infty$, $\rho>4$, whereas our condition, $V\in L^1_2$, requires $\rho>3$. We give below a rather simple proof of Theorem 1.2, quite different from the one in [14], based on our results in [24]. The key issue of estimate (1.28) is that it is integrable for large times if $\beta>1/2$. For *exceptional* potentials the decay rate is $(1+|t|)^{-1/2}$. We can prove this result with our methods, but we do not consider this issue here. See [14] on this point.

The paper is organized as follows. We prove Theorem 1.1 in Section 2. Theorem 1.2 is proven in Section 3. In Appendix 1 we prove estimate (1.13) and in Appendix 2 we prove that the *generic* potentials are a dense open set in L^1_1.

2 The Center Manifold

In this Section we prove Theorem 1.1. Let us project the solution to (1.1) along P_0 and P_c, i.e., $u(t)=u_p(t)\Psi_0+u_c(t)$, where $u_c(t):=P_cu(t)$. Then, (1.1) is equivalent to the following system,

$$i\frac{d}{dt}u_p=E_0u_p+g_p(u_p,u_c);\ i\frac{\partial}{\partial t}u_c=Hu_c+g_c(u_p,u_c), \tag{2.1}$$

where, denoting $g(x,u):=f(x,|u|)\frac{u}{|u|}$, we have that,

$$g_p(u_p,u_c):=P_0g(x,u_p\Psi_0+u_c)=(g(x,u_p\Psi_0+u_c),\Psi_0)\Psi_0;\ g_c(u_p,u_c):=P_cg(x,u_p\Psi_0+u_c). \tag{2.2}$$

In a similar way, any point on the center manifold is written as, $e^{i\theta}\Psi_E=u_p\Psi_0+h(u_p)$, $h(u_p)\in\mathcal{H}_c$, where $u_p,h(u_p)$ are the solution to the following system (see [16]),

$$E_0-E=-\frac{g_p(u_p,h(u_p))}{u_p};\ h(u_p)=-(H-E)^{-1}g_c(u_p,h(u_p)). \tag{2.3}$$

We first prove that $u(t)$ approaches the center manifold. For this purpose, let us consider the vector in $\mathcal{M}$ that has the same projection along P_0 that $u(t)$, i.e., $\Psi(t):=u_p(t)\Psi_0+h(u_p(t))$. We prove below that the difference, $v(t):=u(t)-\Psi(t)=u_c(t)-h(u_p(t))$ satisfies the estimate,

$$\|v(t)\|_{L^2_{-s_1}}\leq C(1+|t|)^{-1/2-\beta}\|v(0)\|_{L^2_{s_1}}, \tag{2.4}$$

where, $s_1 := s + 2\beta$. By (2.1), $v(t)$ is a solution of the following equation,

$$i\frac{\partial}{\partial t}v(t) = Hv(t) + Q(u_p(t), v(t)), \tag{2.5}$$

where,

$$Q(u_p, u_c) := g_c(u_p, h(u_p) + v) - g_c(u_p, h(u_p)) - (Dh)(u_p)[g_p(u_p, h(u_p) + v) - g_p(u_p, h(u_p))], \tag{2.6}$$

with (Dh) the Frechét derivative of h. To verify (2.6) we must prove that,

$$(Dh)(u_p)\left[E_0 u_p + g_p(u_p, h(u_p))\right] = Hh(u_p) + g_c(u_p, h(u_p)). \tag{2.7}$$

We prove below this equation at $t = t_0$, for any $t_0 \in \mathbf{R}$. We denote, $E := E(u_p(t_0))$. Note that by (2.3), $[e^{-itE}u_p(t_0), h(e^{-itE}u_p(t_0))]$ is a solution to (2.1) (recall that $h(e^{-itE}u_p) = e^{-itE}h(u_p)$). Then, using the equation for u_p in (2.1),

$$i\frac{\partial}{\partial t}h(e^{-itE}u_p(t_0)) = (Dh)(e^{-itE}u_p(t_0))\left[E_0\, e^{-itE}u_p(t_0) + e^{-itE}g_p(u_p(t_0), h(u_p(t_0)))\right]. \tag{2.8}$$

Moreover, by the equation for u_c in (2.1),

$$i\frac{\partial}{\partial t}h(e^{-itE}u_p(t_0)) = Hh(e^{-itE}u_p(t_0)) + e^{-itE}g_c(u_p(t_0), h(u_p(t_0))). \tag{2.9}$$

Equation (2.7) follows taking $t = 0$ in (2.8) and (2.9). By (1.24), $|g(x, u + v) - g(x, u)| \leq Cq(x)(|u|^{(p-1)} + |v|^{(p-1)})|v|$, and we have that,

$$\|g(x, u_p\Psi_0 + h(u_p)) - g(x, u_p\Psi_0 + h(u_p) + v)\|_{L^2_{s_1}} \leq$$

$$C\|(1+|x|)^{2s_1}q(x)\|_{L^\infty}\left(\|(u_p\Psi_0 + h(u_p)\|_{H_1}^{(p-1)} + \|v\|_{H_1}^{(p-1)}\right)\|v\|_{L^2_{-s_1}}, \tag{2.10}$$

where we used Sobolev's [1] theorem to bound the L^∞ norms by the H_1 norms. By (1.13) P_0 and $P_c = I - P_0$ are bounded operators on $L^2_s, s \in \mathbf{R}$, and it follows from (2.10) that

$$\|g_p(u_p, h(u_p)) - g_p(u_p, h(u_p) + v)\|_{L^2_{s_1}} \leq C\left(\|u_p\Psi_0 + h(u_p)\|_{H_1}^{(p-1)} + \|v\|_{H_1}^{(p-1)}\right)\|v\|_{L^2_{-s_1}}, \tag{2.11}$$

$$\|g_c(u_p, h(u_p)) - g_c(u_p, h(u_p) + v)\|_{L^2_{s_1}} \leq C\left(\|u_p\Psi_0 + h(u_p)\|_{H_1}^{(p-1)} + \|v\|_{H_1}^{(p-1)}\right)\|v\|_{L^2_{-s_1}}. \tag{2.12}$$

By (1.21) given any $\epsilon_1 > 0$ we can take η so small that if $\|\phi\|_{H_1} < \eta$, we have that, $|u_p(t)| = |(u(t), \Psi_0)| \leq \|u(t)\|_{H_1} < \epsilon_1$. Moreover, since h is C^1 and $h(0) = 0$,

$$\|h(u_p(t))\|_{H_1} \leq C|u_p| \leq C\epsilon_1, \tag{2.13}$$

and we conclude that,

$$\|v(t)\|_{H_1} \leq C\epsilon_1. \tag{2.14}$$

By (2.6), (2.10), (2.11) and (2.12),

$$\|Q(u_p(t), v(t))\|_{L^2_{s_1}} \leq C\epsilon_1 \|v(t)\|_{L^2_{-s_1}}, \ \text{ if } \|\phi\|_{H_1} < \eta. \tag{2.15}$$

We write (2.5) as an integral equation,

$$v(t) = e^{-itH} v(0) + \frac{1}{i} \int_0^t ds\, e^{-i(t-s)H} Q(u_p(s), v(s)). \tag{2.16}$$

Let us denote $v_T := \max_{|t| \leq T} (1+|t|)^{1/2+\beta} \|v(t)\|_{L^2_{-s_1}}$. By Theorem 1.2, and (2.15), for $|t| \leq T$,

$$\begin{aligned} \|v(t)\|_{L^2_{-1/2-\beta}} &\leq C(1+|t|)^{-1/2-\beta} \|v(0)\|_{L^2_{s_1}} + C\epsilon_1 \,(\text{sign t}) \int_0^t ds\, (1+|t-s|)^{-1/2-\beta} (1+|s|)^{-1/2-\beta} v_T \\ &\leq C(1+|t|)^{-1/2-\beta} [\|v(0)\|_{L^2_{s_1}} + C\epsilon_1 v_T]. \end{aligned} \tag{2.17}$$

Taking η so small that $C\epsilon_1 < 1/2$, we obtain that,

$$v_T \leq C \|v(0)\|_{L^2_{s_1}}, \tag{2.18}$$

and since the constant C is independent of T equation (2.4) follows. Equation (1.26) follows from (2.4) by the argument given in Section 4 of [16].

3 The $L^2_{s+2\beta} - L^2_{-s-2\beta}$ Estimate

The results on the spectral theorem for H that we state below follow from the Weyl–Kodaira-Titchmarsch theory. See for example [5]. For a version of the Weyl–Kodaira–Titchmarsch theory adapted to our situation see Appendix 1 of [29] and also the proof of Theorem 6.1 in page 78 of [29]. Let us denote for any $k \in \mathbf{R}$

$$\Psi_+(x,k) := \begin{cases} \frac{1}{\sqrt{2\pi}} T(k) f_1(x,k), & k \geq 0, \\ \\ \frac{1}{\sqrt{2\pi}} T(-k) f_2(x,-k), & k < 0, \end{cases} \tag{3.1}$$

and $\Psi_-(x,k) := \overline{\Psi_+(x,-k)}$. Recall that $\mathcal{H}_c$ is the subspace of continuity of H. Then the following limits,

$$\hat{\phi}_\pm(k) := s - \lim_{N \to \infty} \int_{-N}^{N} \overline{\Psi_\pm(x,k)}\, \phi(x)\, dx, \tag{3.2}$$

exist in the strong topology in L^2 for every $\phi \in \mathcal{H}_c$ and the operators

$$(F_{\pm}\phi)(k) := \hat{\phi}_{\pm}(k), \tag{3.3}$$

are unitary from $\mathcal{H}_c$ onto L^2. Moreover, the $F^*_{\pm}$ are given by

$$\left(F^*_{\pm}\phi\right)(x) = s - \lim_{N\to\infty} \int_{-N}^{N} \Psi_{\pm}(x,k)\,\phi(k)\,dk, \tag{3.4}$$

where the limits exist in the strong topology in L^2. Furthermore, the operators $F^*_{\pm}\,F_{\pm}$ are the orthogonal projector onto $\mathcal{H}_c$. For each eigenvalue of H, let $\Psi_j, 0 = 1, 2, \cdots, N$, be the corresponding eigenfunction normalized to one, i.e. $\|\Psi_j\|_{L^2} = 1$. The operators:

$$F_j\phi := (\phi, \Psi_j)\Psi_j, j = 0, 2, \cdots, N, \tag{3.5}$$

are unitary from the eigenspace generated by Ψ_j onto C. The following operators

$$F^{\pm} = F_{\pm} \oplus_{j=0}^{N} F_j, \tag{3.6}$$

are unitary from L^2 onto $L^2 \oplus_{j=0}^{N} C$ and for any $\phi \in D(H)$:

$$F^{\pm}H\phi = \left\{k^2(F_{\pm}\phi)(k), E_0F_0\phi, \cdots, E_NF_N\phi\right\}. \tag{3.7}$$

Moreover, for any bounded Borel function , Φ, defined on $\mathbf{R}$

$$F^{\pm}\Phi(H)\phi = \left\{\Phi(k^2)(F_{\pm}\phi)(k), \Phi(E_0)F_0\phi, \cdots, \Phi(E_N)F_N\phi\right\}. \tag{3.8}$$

The projector, P_0, onto the subspace of L^2 generated by the eigenvectors of H is given by

$$P_0\phi := \sum_{j=0}^{N}(\phi, \Psi_j)\Psi_j. \tag{3.9}$$

Since H has no singular–continuous spectrum the projector onto the continuous subspace of H is given by: $P_c := I - P_0$. It follows from (3.6) that

$$e^{-itH}P_c = F^*_{\pm}e^{-ik^2t}F_{\pm}. \tag{3.10}$$

Equation (3.10) is the starting point of our proof of Theorem 1.2.

Proof of Theorem 1.2: It follows from (3.10) that for any $\phi \in L^2 \cap L^1$,

$$e^{-itH}P_c\phi(x) = \int \Phi_t(x,y)\phi(y)dy, \tag{3.11}$$

where

$$\Phi_t(x,y) := \int e^{-ik^2 t}\Psi_+(x,k)\overline{\Psi_+(y,k)}\, dk. \tag{3.12}$$

The proofs of Lemmas 2.4 and 2.5 of [24] imply that

$$|\Phi_t(x,y)| \leq C\,\frac{1}{\sqrt{|t|}}. \tag{3.13}$$

We prove below that,

$$(1+|x|)^{-2}\ |\Phi_t(x,y)(1+|y|)^{-2}\ \leq C\,\frac{1}{|t|^{3/2}}. \tag{3.14}$$

By (3.13) and (3.14), for any $s>1$ and $0\leq \beta \leq 1$,

$$(1+|x|)^{-s-2\beta}\ |\Phi_t(x,y)|\,(1+|y|)^{-s-2\beta}\ \leq C\,\frac{1}{|t|^{1/2+\beta}}. \tag{3.15}$$

Equation (1.28) follows from (3.15), from Schur's criterion and from the unitarity in L^2 of $e^{-itH}P_c$.

We now prove (3.14). Let $\chi_1 \in C_0^\infty(\mathbf{R})$ satisfy , $\chi_1(k^2)=1$ for $|k|\leq 1$ and let us denote $\chi_2 := 1-\chi_1$. Changing the variable of integration in (3.12) to $\lambda := k^2$ we obtain that,

$$\Phi_t = \Phi_{1,t} + \Phi_{2,t}, \tag{3.16}$$

where

$$\Phi_{j,t}(x,y) := \int_0^\infty \frac{1}{\sqrt{\lambda}}\,\chi_j(\lambda)\,e^{-i\lambda t}\,\Re\left(\Psi_+(x,\sqrt{\lambda})\overline{\Psi_+(y,\sqrt{\lambda})}\right)d\lambda,\ j=1,2, \tag{3.17}$$

where we used that $\Psi_+(x,-k)=\overline{\Psi_+(x,k)}$. Let us denote,

$$h_j(\lambda,x,y) := \frac{1}{\sqrt{\lambda}}\,\chi_j(\lambda)\ \Re\left(\Psi_+(x,\sqrt{\lambda})\overline{\Psi_+(y,\sqrt{\lambda})}\right),\ j=1,2. \tag{3.18}$$

We first estimate $\Phi_{1,t}(x,y)$. The key issue is that in the *generic* case the transmission coefficient satisfies $T(k)=\alpha k + o(k)$, as $k\to 0$, where $\alpha\neq 0$, and the reflection coefficients satisfy $R_j(0)=-1, j=1,2$ (see [5]). Then, it follows from [24]: equation (2.5), Lemma 2.1, equations (2.38) and (2.40-2.42), that

$$|h_1(\lambda,x,y)| \leq C\frac{\sqrt{\lambda}}{1+\sqrt{\lambda}};\ \left|\frac{\partial}{\partial\lambda}h_1(\lambda,x,y)\right| \leq C\frac{1}{\sqrt{\lambda}}(1+|x|)(1+|y|). \tag{3.19}$$

Let us extend $h_1(\lambda,x,y)$ to a function defined for $\lambda\in\mathbf{R}$ by setting $h_1(\lambda,x,y):=0$ for $\lambda\leq 0$. Then,

$$\Phi_{1,t}(x,y) = \frac{1}{2}\left[\int e^{-it\lambda}h_1(\lambda,x,y)d\lambda - \int e^{-it(\lambda-\pi/t)}h_1(\lambda,x,y)d\lambda\right] =$$

$$\frac{1}{2}\int e^{-it\lambda}[h_1(\lambda,x,y)-h_1(\lambda+\pi/t,x,y)]d\lambda. \tag{3.20}$$

Hence,

$$|\Phi_{1,t}(x,y)|\leq C\int |h_1(\lambda,x,y)-h_1(\lambda+\pi/t,x,y)|\,d\lambda. \tag{3.21}$$

If $t>0$,

$$\int |h_1(\lambda,x,y)-h_1(\lambda+\pi/t,x,y)|\,d\lambda \leq 2\int_0^{2\pi/t}|h_1(\lambda,x,y)|d\lambda+\int_{\pi/t}^{\infty}d\lambda\int_{\lambda}^{\lambda+\pi/t}\left|\frac{\partial}{\partial\rho}h_1(\rho,x,y)\right|\,d\rho$$

$$\leq \frac{C}{t^{3/2}}(1+|x|)(1+|y|)+\frac{\pi}{t}\int_{\pi/t}^{\infty}\left|\frac{\partial}{\partial\rho}h_1(\rho,x,y)\right|\,d\rho\leq\frac{C}{t^{3/2}}(1+|x|)\,(1+|y|), \tag{3.22}$$

where we used (3.19). If $t<0$ we change the variable of integration in (3.21) to $\acute{\lambda}:=\lambda+\pi/t$, and we proceed as above.

Let us denote, $m_1(x,k):=e^{-ikx}f_1(x,k)$ and $m_2(x,k):=e^{ikx}f_2(x,k)$. We designate, $\ddot{m}_j(x,k):=\frac{\partial^2}{\partial k^2}m_j(x,k), j=1,2$, and $\acute{m}_j:=\frac{\partial}{\partial x}m_j(x,k), j=1,2$. We prove below that if $V\in L^1_2$, then, for any $x_0\in \mathbf{R}$,

$$|\ddot{m}_1(x,k)|\leq C_{x_0}\frac{1+|k|}{|k|^2}, x\geq x_0;\;\; |\ddot{m}_2(x,k)|\leq C_{x_0}\frac{1+|k|}{|k|^2}, x\leq x_0, \tag{3.23}$$

$$\left|\ddot{\acute{m}}_1(x,k)\right|\leq C_{x_0}\left[1+\frac{1+|k|}{|k|^2}\right], x\geq x_0;\;\; \left|\ddot{\acute{m}}_2(x,k)\right|\leq C_{x_0}\left[1+\frac{1+|k|}{|k|^2}\right],\, x\leq x_0. \tag{3.24}$$

Integrating by parts twice with respect to λ in (3.17) with $j=2$ and using (3.23), (3.24) and [24]: equation (2.5), Lemma 2.1, equations (2.38) and (2.40-2.42), we obtain that,

$$|\Phi_{2,t}(x,y)|\leq C\frac{1}{t^2}(1+|x|)^2\,(1+|y|)^2. \tag{3.25}$$

Equation (3.14) follows from (3.16), (3.21), (3.22) and (3.25).

We prove now (3.23) and (3.24) for m_1. The case of m_2 follows in a similar way. As is well known (see [5], [24]),

$$m_1(x,k)=\lim_{n\to\infty}m_{1,n}(x,k), \tag{3.26}$$

where $m_{1,0}(x,k):=1$ and the $m_{1,n}$ satisfy the following equation for $n=0,1,\cdots$

$$m_{1,n+1}(x,k)=1+\int_x^{\infty}D_k(y-x)V(y)m_{1,n}(y,k)dy, \tag{3.27}$$

where,

$$D_k(x) := \int_0^x e^{2iky} dy = \begin{cases} \frac{1}{2ik}(e^{2ikx}-1), & k \neq 0, \\ x, & k = 0. \end{cases} \tag{3.28}$$

Clearly,

$$\left|\ddot{D}_k(x)\right| \leq C \frac{(1+|x|)^2\,(1+|k|)}{|k|^2}. \tag{3.29}$$

By (3.27),

$$\ddot{m}_{1,n+1}(x,k) = \int_x^\infty D_k(y-x) V(y) \ddot{m}_{1,n}(y,k)\, dy + A_n(x,k), \tag{3.30}$$

where,

$$A_n(x,k) := 2\int_x^\infty \dot{D}_k(y-x) V(y) \dot{m}_{1,n}(y,k)\, dy + \int_x^\infty \ddot{D}_k(y-x) V(y) m_{1,n}(y,k)\, dy. \tag{3.31}$$

Then by (3.29) and [24]: equations (2.10), (2.18) and (2.21) with $\gamma = 2$, for $|k| \leq 1$ and $\gamma = 1$ for $|k| \geq 1$,

$$|A_n(x,k)| \leq C_{x_0} \frac{1+|k|}{|k|^2} \|V\|_{L^1_2},\ x \geq x_0, \tag{3.32}$$

and it follows from (3.29) and (3.30) that,

$$|\ddot{m}_{1,n+1}(x,k)| \leq C_{x_0} \frac{1+|k|}{|k|^2} + C_{x_0} \frac{1+|k|}{|k|^2} \int_x^\infty |y-x|^2 |V(y)| |\ddot{m}_{1,n}(y,k)|\, dy. \tag{3.33}$$

Iterating (3.33) $n+1$ times we prove that

$$|\ddot{m}_{1,n+1}(x,k)| \leq C_{x_0} \frac{1+|k|}{|k|^2},\ x \geq x_0, \tag{3.34}$$

and (3.23) for m_1 follows taking the limit as $n \to \infty$. Equation (3.24) folows from (3.23) and from [24]: Lemma 2.1 and equation (2.23).

4 Appendix 1

In this Appendix we prove equation (1.13). For this purpose it is enough to assume that $V \in L^1$, and that $N(V) < \infty$. Note that as $\Psi_E \in H_2$ if follows from Sobolev's theorem [1] that $\|\Psi_E\|_{L^\infty} \leq C \|\Psi_E\|_{H_2}$. Then $q_e := V + f(x, |\Psi_E|)/|\Psi_E| \in L^1$ and $N(q_e) < \infty$. By Theorem 2.7.1 in page 35 of [20] the differential expression $-\frac{d^2}{dx^2}+q_e$ is essentially-self-adjoint on C_0^∞. We denote by H_e the unique self-adjoint realization in L^2. Moreover, $D(H_e)=H_2$. Let $g_j\left(x, i\sqrt{|E|}\right), j=1,2$, be the Jost solutions for the potential q_e at energy E. They satisfy (see Lemma 1 of [5]),

$$\left|g_1\left(x, i\sqrt{|E|}\right)\right| \leq Ce^{-\sqrt{|E|}|x|}, x\geq 0; \; \left|g_2\left(x, i\sqrt{|E|}\right)\right| \leq Ce^{-\sqrt{|E|}|x|}, x\leq 0. \tag{4.1}$$

But the differential expression $-\frac{d^2}{dx^2}+q_e$ is in the limit-point case at $\pm\infty$ [26]. Hence, $g_j(x, i\sqrt{|E|}), j=1,2$, are, respectively, the only independent solutions to the eigenvalue equation

$$-\frac{d^2}{dx^2}g+q_e g=Eg, \tag{4.2}$$

that are square integrable on a neighborhood of $\pm\infty$. By (1.11) Ψ_E is a solution in H_2 to the linear eigenvalue equation (4.2). Then, g_1 and g_2 are linearly dependent, and

$$\Psi_E(x)=\alpha g_1\left(x, i\sqrt{|E|}\right)=\beta g_2\left(x, i\sqrt{|E|}\right), \tag{4.3}$$

for some constants α, β. Equation (1.13) follows from (4.1) and (4.3). Note that the argument above also implies that,

$$|\Psi_0(x)|\leq Ce^{-\sqrt{|E_0|}|x|}. \tag{4.4}$$

5 Appendix 2

In this Appendix we prove that the *generic* potentials are a dense open set in L^1_1. For each fixed $x\in\mathbf{R}$ the functions $V\in L^1_1\hookrightarrow f_j(x,k), j=1,2$, are continuous [5]. It follows that the set of *generic* potentials is an open set in L^1_1. Suppose that $V\in L^1_1$ and denote,

$$W(\lambda):=\left[f_{1,\lambda}(x,0), f_{2,\lambda}(x,0)\right], \tag{5.1}$$

where $f_{j,\lambda}, j=1,2$, are the Jost solutions for λV given by [24] equations (2.11-2.13)) for $j=1$ and similar formulas for $j=2$. $W(\lambda)$ is an entire analytic function of λ and there are two posibilities. (a) $W(\lambda)$ is not identically zero. Then the set of zeros of $W(\lambda)$ is discrete and there exists a sequence λ_n, with $W(\lambda_n)\neq 0$ and $\lim_{n\to\infty}\lambda_n=1$. It follows that $\lambda_n V$ is *generic* and that $\lim_{n\to\infty}\lambda_n V=V$ strongly in L^1_1. (b) $W(\lambda)$ is identically zero . In this case it follows from [24]: equations (2.11-2.13) and (2.23) that

$$-\int V(y)\,dy=\frac{d}{d\lambda}W(\lambda)\Big|_{\lambda=0}=0. \tag{5.2}$$

Take any $q(x) \in L^1_1$ with $q(x) > 0$ and any sequence, $\epsilon_n > 0$, with $\lim_{n\to\infty} \epsilon_n = 0$. As $V + \epsilon_n q$ does not satisfies (5.2) there are sequences $\lambda_{n,m}$ with $\lim_{m\to\infty} \lambda_{n,m} = 1$ such that $V_{n,m} := \lambda_{n,m}(V + \epsilon_n q)$ are *generic*. We can always take a subsequence of the $V_{n,m}$ that converges strongly to V in L^1_1. It follows that the set of *generic* potentials is dense in L^1_1.

References

[1] R.A. Adams, "Sobolev Spaces", Academic Press, New York, 1975.

[2] V.S. Buslaev and G.S. Perelman, Scattering states for the nonlinear Schrödinger equation: states close to a soliton, Algebra i Analiz **4** (1992), 63-102 [english translation in St. Petersburg Math. J. **4** (1993), 1111-1142].

[3] V. S. Buslaev and G.S. Perelman, On the stability of solitary waves for nonlinear Schroödinger equation, Amer. Math. Soc. Transl. (2) **164** (1995), 75-98.

[4] J. Carr, " Applications of Centre Manifold Theory", Applied Mathematical Sciences **35**, Springer-Verlag, New York, 1981.

[5] P. Deift and E. Trubowitz, Inverse scattering on the line, Commun. Pure Appl. Math. **XXXII** (1979), 121-251.

[6] L.D. Faddeev, Properties of the S matrix of the one–dimensional Schrödinger equation, Trudy Math. Inst. Steklov **73** (1964), 314-333 [english translation American Mathematical Society Translation Series 2 **65** (1964), 139-166].

[7] L.D. Faddeev, Inverse problems of quantum scattering theory, II, Itogi Nauki i Tekhniki Sovremennye Problemy Matematiki **3** (1974), 93-180 [english translation J. Soviet Math. **5** (1976), 334-396].

[8] J. Ginibre, "Introduction aux Équations de Schrödinger non Linéaires", Onze Édition, Paris, 1998.

[9] D. Henry, "Geometric Theory of Semilinear Parabolic Equations", Lecture Notes in Mathematics, **840**, Springer-Verlag, Berlin, 1981.

[10] T. Kato, Nonlinear Schrödinger equations, *in* “Schrödinger Operators”, pp. 218–263, H. Holden and A. Jensen, eds., Lecture Notes in Physics **345**, Springer–Verlag, Berlin, 1989.

[11] A. Komech and B. Vainberg, On asymptotic stability of stationary solutions to nonlinear wave and Klein-Gordon equations, Arch. Rational Mechanics and Analysis, **134** (1996), 227-248.

[12] A. Komech, On transitions to stationary states in Hamiltonia nonlinear wave equations, Physics Letters A **241** (1998), 311-322.

[13] A. Komech, H. Spohn and M. Kunze, Long-time asymptotics for a classical particle interacting with a scalar field, Comm. Part. Diff. Equations, **22** (1997),307-335.

[14] M. Murata, Asymptotic expansions in time for solutions of Schrödinger-type equations, J. Funct. Analysis, **49** (1982), 10-56.

[15] L. Nirenberg, ”Topics in Nonlinear Functional Analysis”, Courant Institute of Mathematical Sciences Lecture Notes, New York University, New York, 1974.

[16] C.-A. Pillet and C.E. Wayne, Invariant manifolds for a class of dispersive, hamiltonian, partial differential equations, J. Differential Equations, **141** (1997), 310-326.

[17] M. Reed and B. Simon, “Methods of Modern Mathematical Physics I: Functional Analysis”, Academic Press, New York, 1972

[18] M. Reed and B. Simon, “Methods of Modern Mathematical Physics II: Fourier Analysis, Self–Adjointness”, Academic Press, New York, 1975.

[19] M. Reed and B. Simon, “Methods of Modern Mathematical Physics III: Scattering Theory”, Academic Press, New York, 1978.

[20] M. Schechter, “Operator Methods in Quantum Mechanics”, North Holland, New York, 1981.

[21] A. Soffer and M.I. Weinstein, Multichannel nonlinear scattering for nonintegrable equations, Commun. Math. Phys. **133** (1990), 119-146.

[22] A. Soffer and M.I. Weinstein, Multichannel nonlinear scattering for nonintegrable equations II. The case of anisotropic potentials and data, J. Differential Equations, **98** (1992), 376-390.

[23] R. Weder, Inverse scattering for the nonlinear Schrödinger equation, Commun. Part. Diff. Equations **22** (1997), 2089-2103.

[24] R. Weder, $L^p - L^{\acute{p}}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential, Preprint, 1998, 28 pages, to appear in J. Funct. Analysis.

[25] R. Weder, The $W_{k,p}$-continuity of the Schrödinger wave operators on the line, Preprint, 1999, 16 pages, to appear in Comm. Math. Phys.

[26] J. Weidmann, "Spectral Theory of Ordinary Differential Operators", Lecture Notes in Mathematics **1258**, Springer–Verlag, Berlin, 1987.

[27] M.I. Weinstein, Modulation stability of ground states of nonlinear Schrödinger equations , SIAM J. Math. Anal. **16** (1985), 472-491.

[28] M.I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, Commun. Pure Appl. Math. **39** (1986), 51-67.

[29] C. H. Wilcox, "Sound Propagation in Stratified Fluids", Applied Mathematical Sciences **50**, Springer–Verlag, New York, 1984.