Resonances from perturbed symmetry in open quantum dots

Pierre Duclos[a,b], Pavel Exner[c,d], Bernhard Meller[e*]

a) Centre de Physique Théorique, C.N.R.S., F-13288 Marseille Luminy,
b) PHYMAT, Université de Toulon et du Var, F-83957 La Garde Cedex,
`duclos@univ-tln.fr`
c) Nuclear Physics Institute, Academy of Sciences, CZ–25068 Řež near Prague,
d) Doppler Institute, Czech Technical University, Břehová 7, CZ-11519 Prague,
`exner@ujf.cas.cz`
e) Facultad de Física, P. Universidad Católica de Chile, Casilla 306, Santiago 22, Chile
`bmeller@chopin.fis.puc.cl`

Abstract

We discuss the Nöckel model of an open quantum dot, *i.e.*, a straight hard-wall channel with a potential well. If this potential depends on the longitudinal variable only, there are embedded eigenvalues which turn into resonances if the symmetry is violated, either by applying a magnetic field or by deformation of the well. For a weak symmetry breaking we derive the perturbative expansion of these resonances. We also deduce a sufficient condition under which the discrete spectrum of such a system (without any symmetry) survives the presence of a strong magnetic field. It is satisfied, in particular, if the dot potential is purely attractive.

1 Introduction

A rapid progress of mesoscopic physics sheds a new light on some traditional aspects of quantum dynamics. A particular situation which we will consider in this paper concerns resonances which owe their existence to a weak violation of a certain symmetry. Such situations are common in nuclear and particle physics. In mesoscopic systems, however, the symmetry in question refers typically rather to

*Current address: Donaustr. 105, 12043 Berlin, Germany

geometry of the problem than to internal quantum numbers. Hence its violation is better accessible to an experimental investigation.

Quantum wire systems offer variety of situations with bound states embedded into a continuum. An illustration of this claim – aesthetically perfect – is the second eigenvalue in a cross–shape structure [SRW, EŠŠ]. There are many more examples. For instance, it is easy to see using a variational argument that *any* symmetric protrusion of a strip modelling a quantum wire will possess at least one embedded eigenvalue; to this aim one has to consider a "half" of it, *i.e.*, a strip whose one boundary is Dirichlet and deformed while the other is Neumann and straight, and mimick the argument of [BGRS]. Similar embedded eigenstates can be produced by other interactions as well [EGŠT]. Moreover, the effect is not restricted to quantum wires: in acoustic waveguides where the discrete spectrum is absent automatically, trapped modes of a similar origin have been studied recently [DP, ELV],[1] related states were found in elasticity [RVW], etc.

Since the mentioned eigenvalues exist only due to a particular symmetry, they turn into resonances once the symmetry is violated. Naturally, there is a spectral concentration: the resonance width measures the departure from the ideal state. A fresh discussion of this effect with a numerical analysis and experimental results can be found in [AVD] for wires with double stubs,[2] another recent example are magnetoresonances in a wire containing a circular potential well [NN]. The idea, however, is not new. The influence of a magnetic field on the cross–structure embedded eigenstate state was treated already in [SWR]. Later Nöckel [Nö] investigated the case of a wire with "quantum dot" modeled by a potential well and the resonances which appear in this setting when a magnetic field is applied. In distinction to [NN] he considered a rectangular well independent of the transverse variable. In such a case there are many more resonances because instead of the even–odd decomposition of the wire wavefunctions now every "longitudinal" bound state gives rise to a series of embedded eigenvalues.

The numerical analysis used in these papers is illustrative in revealing basic features of such resonances. At the same time, it remains somewhat hidden in this approach that the effect exhibits general properties and allows for a sensible perturbation theory. In this paper we want to present a discussion which puts emphasis on this aspect of the problem. For the sake of simplicity, we shall do it in the Nöckel model which is mathematically more accessible; a treatment of the protruded wire case is left to a subsequent publication.

Our method is based on a complex scaling the idea of which comes from the seminal paper of Aguilar and Combes [AC] (see also [RS, Sec. XII.6]). It has to be combined with the transverse–mode decomposition as in [DEM, DEŠ]. However, the present case is easier. The embedded eigenvalue turns into a resonance again

[1] When the symmetry is violated, these trapped modes change into resonances similar to those studied in the present paper. This effect was recently discussed in [APV].

[2] For completeness, let us recall that double–stub wires and similar systems have also "non-magnetic" resonances — see [BJ, BBJ, ISY] and a recent paper in the same vein [NR]. In our model the latter are included in eq. (3.7) below.

as a consequence of the tunneling between the transverse modes, but since the latter is controlled now by an "external" parameter such as a magnetic field or a deformation potential, it gives rise to a power series in the appropriate parameter which is at weak coupling dominated by the lowest term representing the Fermi golden rule. We will describe the used complex scaling transformation in Section 3 below. Before doing that we shall describe the model in Section 2, list the hypotheses and describe necessary preliminaries. As in [DEŠ, DEM] we restrict ourselves to the situation when the embedded eigenvalues are simple. The results can be extended to the degenerate case without any principal difficulty, but the resulting formulae are rather cumbersome and not very illustrative.

In the concluding section we are going to discuss another aspect of such a model. It concerns the opposite extremal situation: we shall ask whether the discrete spectrum persists in strong magnetic fields. We will derive a sufficient condition for that which provides an affirmative answer, in particular, if the potential describing the quantum dot is purely attractive.

2 Description of the model

Let us first describe the model. The quantum wire is regarded as an infinite planar strip. The corresponding state Hilbert space is $\mathcal{H} := L^2(\Omega)$, where $\Omega := \mathbb{R} \times S$, where either $S = (-a, a)$, or $S = \mathbb{R}$ in the case when the lateral confinement is realized by a potential alone. Given real–valued measurable functions V on $\mathbb{R}$, W on S, and U on Ω, we use

$$H(B, \lambda) := (-i\partial_x - By)^2 + V(x) - \partial_y^2 + W(y) + \lambda U(x, y) \tag{2.1}$$

as the model Hamiltonian describing an electron in the quantum wire subject to a homogeneous magnetic field B perpendicular to the plane. As usual we put $\hbar = 2m = 1$, the chosen sign of the vector potential corresponds to an electron, $e = -1$, and the field intensity $\vec{B}$ pointing up.

The Landau gauge employed in (2.1) suits to our situation in which a preferred direction exists. If $S = (-a, a)$ the transverse kinetic energy $-\partial_y^2$ operator is supposed to satisfy the Dirichlet condition at the strip boundary,

$$\psi(x, \pm a) = 0 \qquad \text{for any} \qquad x \in \mathbb{R}, \tag{2.2}$$

unless, of course, $W(y)$ tends to $+\infty$ at $x = \pm a$ fast enough to ensure the essential self–adjointness on $C_0^\infty(S)$; with the usual abuse of notation we identify $-\partial_y^2$ with $I \otimes (-\partial_y^2)$, *etc.* Assumptions about the potentials will be collected below.

If $B = 0$ or $\lambda = 0$ we drop the corresponding symbol at the *lhs* of (2.1), in particular, $H(0) := H(0, 0)$ is the unperturbed Hamiltonian. Since the latter is of the form $h^V \otimes I + I \otimes h^W$, its spectrum is the "sum" of the corresponding component spectra. This is a typical situation when embedded eigenvalues due to symmetry may arise.

Our aim is to analyze how such eigenvalues turn into resonances if the symmetry of the system is violated either by switching in the magnetic field, or geometrically by the potential U which does not decompose, $U(x,y) \neq U_1(x)+U_2(y)$ for some U_1, U_2. We shall show the existence of resonances, derive the corresponding perturbation series, and compute explicitly the lowest non–real term given by the Fermi golden rule.

Let us first list the assumptions about the potentials. Our goal is to illustrate the mechanism of resonance production rather than proving a most general result, and therefore we adopt relatively restrictive hypotheses. The lateral confinement is supposed to be "strong enough":

(i) $W(y) \geq cy^2$ for some $c \geq 0$. In particular, c is strictly positive if $S = I\!R$; without loss of generality we may assume that $c \geq 1$.

We make the potential part of the assumption with Section 4 in mind; here it would be enough to suppose $\lim_{|x|\to\infty} W(x) = +\infty$ which ensures that the spectrum of h_W, denoted by $\{\nu_j\}_{j=1}^{\infty}$, is discrete and simple, $\nu_{j+1} > \nu_j$. It may be convenient, however, to impose a stronger hypothesis, *e.g.*,

(i') $W(y) \geq y^2$ and $W'(y)\,\mathrm{sgn}\, y \geq c_0|y|$ if $|y| \geq y_0$ for some $c_0, y_0 > 0$,

which guarantees the existence of a uniform lower bound on the eigenvalue spacing,

$$\inf_j\, (\nu_{j+1} - \nu_j) \,>\, 0\,. \tag{2.3}$$

The longitudinal potential V which simulates the local deformation of the quantum wire responsible for the appearance of the discrete spectrum will be short–ranged and non–repulsive in the mean,

(ii) $|V(x)| \leq \mathrm{const}\, \langle x\rangle^{-2-\varepsilon}$ for some $\varepsilon > 0$, with $\int_{I\!R} V(x)\, dx \leq 0$,

where we have denoted conventionally $\langle x\rangle := \sqrt{1+x^2}$. Under this assumption, the longitudinal part $h^V := -\partial_x^2 + V(x)$ of the unperturbed Hamiltonian has a non–empty discrete spectrum,

$$\mu_1 < \mu_2 < \cdots \mu_N < 0\,, \tag{2.4}$$

which is simple and finite [BGS, Kl, Ne, Si1, Se]; the corresponding normalized eigenfunctions ϕ_n, $n = 1,\dots,N$, are exponentially decaying. It is convenient for analyzing the resonance behaviour to adopt an analyticity requirement. In the present case we assume that

(iii) the potential V extends to a function analytic in a sector $\mathcal{M}_{\alpha_0} := \{\, z \in \mathbb{C} : |\arg z| \leq \alpha_0\}$ for some $\alpha_0 > 0$ and obeys there the bound of *(ii)*.

The resonances we want to study have to be distinguished from those of the unperturbed problem coming from the operator h^V. Fortunately this can be achieved under the last assumption: by [AC] the resonances of h^V contained in $\mathcal{M}_{\alpha_0}$ do not accumulate, except possibly at the threshold. The last named possibility does not occur if V would decay exponentially, but in fact can be ruled out with a dilation analytic potential — see [Je, Lemma 3.4].

Finally, the magnetic part of the perturbation is governed by a single parameter, while the deformation is described by the potential U. We shall again restrict ourselves to the short–range case and require analyticity:

(iv) $|U(x,y)| \le \text{const}\,\langle x\rangle^{-2-\varepsilon}$ for some $\varepsilon > 0$ and all $(x,y) \in \Omega$. Moreover, $U(\cdot,y)$ extends for each fixed $y \in S$ into an analytic function in $\mathcal{M}_{\alpha_0}$ and satisfies there the same bound.

Since $\sigma_c(h^V) = [0,\infty)$, the spectrum of the unperturbed Hamiltonian consists of a continuous part,

$$\sigma_c(H(0)) \;=\; \sigma_{ess}(H(0)) \;=\; [\nu_1,\infty)\,, \tag{2.5}$$

and the infinite family of eigenvalues

$$\sigma_p(H(0)) \;=\; \{\,\mu_n+\nu_j\,:\; n=1,\dots,N\,,\; j=1,2,\dots\,\}\;. \tag{2.6}$$

Among these a finite subset is isolated or situated at the threshold, while the rest satisfying the condition

$$\mu_n+\nu_j \;>\; \nu_1 \tag{2.7}$$

is embedded in the continuous spectrum. We suppose also that they do not coincide with the higher thresholds,

$$\mu_n+\nu_j \;\ne\; \nu_k \tag{2.8}$$

for any k.

Finally, the transverse–mode decomposition is introduced in analogy with [DEŠ]. The transverse eigenfunctions are denoted as $\chi_j,\; h^W\chi_j = \nu_j\chi_j$. Then we define the embedding operators $\mathcal{J}_j$ and their projection adjoints by

$$\begin{aligned} \mathcal{J}_j \;&:\; L^2(I\!\!R) \to L^2(\Omega)\,, \qquad && \mathcal{J}_j f = f\otimes\chi_j\,, \\ \mathcal{J}_j^* \;&:\; L^2(\Omega) \to L^2(I\!\!R)\,, \qquad && (\mathcal{J}_j^* g)(x) = (\chi_j, g(x,\cdot))_{L^2(S)}\,. \end{aligned}$$

They allow us to replace the Hamiltonian $H(B,\lambda)$ by the matrix differential operator $\{H_{jk}(B,\lambda)\}_{j,k=1}^{\infty}$ with

$$H_{jk}(B,\lambda) \;:=\; \mathcal{J}_j^* H(B,\lambda)\mathcal{J}_k \;=\; \left(-\partial_x^2+V(x)+\nu_j\right)\delta_{jk}+\mathcal{U}_{jk}(B,\lambda)\,, \tag{2.9}$$

$$\mathcal{U}_{jk}(B,\lambda) \;:=\; 2iB\,m_{jk}^{(1)}\partial_x + B^2 m_{jk}^{(2)} + \lambda U_{jk}(x)\,, \tag{2.10}$$

where $m_{jk}^{(r)}$ are the transverse momenta,

$$m_{jk}^{(r)} \;:=\; \int_S y^r\overline{\chi}_j(y)\chi_k(y)\,dy\,,$$

and $U_{jk}(x) := \int_S U(x,y)\overline{\chi}_j(y)\chi_k(y)\,dy\,.$

3 Resonances

3.1 Complex scaling

In analogy with [DEM, DEŠ] we use a complex deformation in the longitudinal variable: we begin with the unitary operator

$$\mathcal{S}_\theta : \ (\mathcal{S}_\theta \psi)(x,y) \;=\; e^{\theta/2}\psi(e^\theta x, y)\,, \quad \theta \in I\!R\,, \tag{3.1}$$

and extend this scaling transformation analytically to $\mathcal{M}_{\alpha_0}$. This is possible because the transformed Hamiltonians are of the form

$$H_\theta(B,\lambda) \;:=\; \mathcal{S}_\theta H(B,\lambda)\mathcal{S}_\theta{}^{-1} \;=\; H_\theta(0) + \mathcal{U}_\theta(B,\lambda)\,, \tag{3.2}$$

with $V_\theta(x) := V(e^{\theta x})$ and

$$H_\theta(0) \;:=\; -e^{-2\theta}\partial_x^2 - \partial_y^2 + V_\theta(x) + W(y) \tag{3.3}$$

where the last named operators clearly form in view of the assumptions *(ii)* and *(iii)* a type (A) analytic family of m–sectorial operators in the sense of [Ka] for $|\mathrm{Im}\,\theta| < \min\{\alpha_0, \pi/4\}$. Furthermore

$$\mathcal{U}_\theta(B,\lambda) \;:=\; 2i\,e^{-\theta}By\,\partial_x + B^2y^2 + \lambda U_\theta(x,y) \tag{3.4}$$

with $U_\theta(x,y) := U(e^{\theta x}, y)$. Defining $R_\theta(z) := (H_\theta(0) - z)^{-1}$ we prove in Proposition A.1 the following estimate

$$\|\mathcal{U}_\theta(B,\lambda)R_\theta(\nu_1 + \mu_1 - 1)\| \le c(|B| + |B|^2 + |\lambda|)\,. \tag{3.5}$$

for $|\mathrm{Im}\,\theta| < \min\{\alpha_0, \pi/4\}$. Thus the "full" operators $H_\theta(B,\lambda)$ are also a type (A) analytic family for suitably small B and λ.

The transformed free part (3.3) separates variables, so its spectrum is

$$\sigma\left(H_\theta(0)\right) \;=\; \bigcup_{j=1}^{\infty} \left\{ \nu_j + \sigma\left(h_\theta^V\right) \right\}\,, \tag{3.6}$$

where $h_\theta^V := -e^{-2\theta}\partial_x^2 + V_\theta(x)$. Since the potential is dilation analytic by assumption, the discrete spectrum of h_θ^V is independent of θ; we have

$$\sigma\left(h_\theta^V\right) \;=\; e^{-2\theta}I\!R_+ \,\cup\, \{\mu_1, \dots, \mu_N\} \,\cup\, \{\rho_1, \rho_2 \dots\}\,, \tag{3.7}$$

where ρ_r are the "intrinsic" resonances of h^V. In view of *(iii)* and (2.8) the supremum of $\mathrm{Im}\,\rho_k$ over any finite region of the complex plane which does not contain any of the points ν_k is negative, so each eigenvalue $\mu_n + \nu_j$ has a neighbourhood containing none of the points $\rho_k + \nu_{j'}$. Consequently, when θ has a positive imaginary part, the eigenvalues of $H_\theta(0)$ are isolated. Thus due to the relative boundedness of $\mathcal{U}_\theta(B,\lambda)$ — *cf.* (3.5) — we can draw a contour around an unperturbed eigenvalue and apply perturbation theory. As indicated above we shall consider only the non-degenerate case when $\mu_n + \nu_j \neq \mu_{n'} + \nu_{j'}$ for different pairs of indices.

3.2 Perturbation expansion

Let us first introduce some notation. The unperturbed eigenvalue $\mu_n + \nu_j$ will be in the following labeled as e_0. Let $\theta = i\beta$ with an appropriately chosen $\beta > 0$; then in view of (3.6) and (3.7) we may chose a contour Γ belonging to the resolvent set of $H_\theta(0)$ and encircling e_0 such that this eigenvalue is the only one of $H_\theta(0)$ contained inside of Γ. As usual, let P_θ denote the eigenprojection referring to the eigenvalue e_0 and set

$$S_\theta^{(p)} := \frac{1}{2\pi i}\int_\Gamma \frac{R_\theta(z)}{(e_0-z)^p}\,dz$$

for $p \geq 0$. Then $P_\theta = -S_\theta^{(0)}$ and $\hat{R}_\theta(z) := S_\theta^{(1)}$ is the reduced resolvent of $H_\theta(0)$ at the point z. We will need the following estimates:

Proposition 3.1 *If* $\operatorname{Im}\theta \in (0,\alpha_0)$, *there exists a positive constant* c_θ *such that*

(i) $\max_{z\in\Gamma} \|\mathcal{U}_\theta(B,\lambda)R_\theta(z)\| \leq c_\theta(|B|+|B|^2+|\lambda|)$.

(ii) $\left\|\mathcal{U}_\theta(B,\lambda)S_\theta^{(p)}\right\| \leq c_\theta \frac{|\Gamma|}{2\pi}\left(\operatorname{dist}(\Gamma,e_0)\right)^{-p}(|B|+|B|^2+|\lambda|)$ *holds for* $p \geq 0$.

Proof: (i) is proved in Appendix A, the second claim follows immediately. ∎

Now we are in position to write the perturbation expansion. By assumption, $e_0 = \mu_n + \nu_j$ holds for a unique pair of the indices j and n. Following [Ka, Sec. II.2] we obtain the convergent series

$$e(B,\lambda) = \mu_n + \nu_j + \sum_{m=1}^{\infty} e_m(B,\lambda), \tag{3.8}$$

where

$$e_m(B,\lambda) = \sum_{p_1+\dots+p_m=m-1} \frac{(-1)^m}{m}\operatorname{Tr}\prod_{i=1}^{m}\mathcal{U}_\theta(B,\lambda)S_\theta^{(p_i)} \tag{3.9}$$

Using the preceding proposition we infer that

$$e_m(B,\lambda) = \sum_{l=0}^{m}\mathcal{O}\left(B^l\lambda^{m-l}\right); \tag{3.10}$$

in particular, $e_m(B) = \mathcal{O}(B^m)$ and $e_m(\lambda) = \mathcal{O}(\lambda^m)$ in cases of the pure magnetic and pure potential perturbation, respectively.

Let us compute first the lowest–order terms of the expansion (3.8) in the non-degenerate case, $\dim P_\theta = 1$. We denote by ϕ_n the corresponding eigenvector of h^V, *i.e.*, $h^V\phi_n = \mu_n\phi_n$. Then

$$\begin{aligned} e_1^{j,n}(B,\lambda) &= \operatorname{Tr}\left(\mathcal{U}_\theta(B,\lambda)P_\theta\right) = \left(\overline{\phi_n^\theta\otimes\chi_j}, \mathcal{U}_\theta(B,\lambda)\phi_n^\theta\otimes\chi_j\right) \\ &= \left(\phi_n\otimes\chi_j, \mathcal{U}(B,\lambda)\phi_n\otimes\chi_j\right) \\ &= 2iBm_{jj}^{(1)}(\phi_n,\phi_n') + B^2 m_{jj}^{(2)} + \lambda(\phi_n, U_{jj}\phi_n). \end{aligned}$$

Moreover, $i\,(\phi_n,\phi_n') \;=\; (\phi_n, i\partial_x\phi_n)$ is (up to the sign) the group velocity of the wavepacket, which is zero in a stationary state; notice that ϕ_n is real-valued up to a phase factor. In other words,

$$e_1(B,\lambda) \;=\; B^2\int_S y^2\left|\chi_j(y)\right|^2\,dy \;+\;\lambda\int_{I\!R\times S} U(x,y)\left|\phi_n(x)\chi_j(y)\right|^2\,dx\,dy \tag{3.11}$$

with the magnetic part independent of n. As usual in such situations the first–order correction is real and does not contribute to the resonance width. The second term in (3.9) can be computed in the standard way — see, *e.g.*, [RS, Sec.XII.6] — taking the limit $\mathrm{Im}\,\theta\to 0$ we get

$$\begin{aligned} e_2(B,\lambda) \;&=\; -\,\mathrm{Tr}\,\left(P_{j,n}\mathcal{U}(B,\lambda)\hat R_{\theta=0}(e_0-i0)\,\mathcal{U}(B,\lambda)P_{j,n}\right) &(3.12)\\ &=\; -\sum_{k=1}^{\infty}\left(\mathcal{U}_{jk}(B,\lambda)\phi_n,\,\left(\left(h^V\!-e_0+\nu_k\!-i0\right)^{-1}\right)^{\hat{}}\,\mathcal{U}_{jk}(B,\lambda)\phi_n\right)\,. \end{aligned}$$

We shall restrict our attention to the imaginary part of $e_2(B,\lambda)$ which determines the resonance width to leading order.

Notice first that the imaginary part of the last series is in fact a finite sum. Denote $k_{e_0}:=\max\{k\,:\,e_0-\nu_k>0\}$. If the unperturbed eigenvalue is embedded one has $k_{e_0}\ge 1$ by eq. (2.7); otherwise the set is empty and we put $k_{e_0}=0$ by definition. Introducing the symbol $\mathcal{R}_k:=\left(\left(h^V\!-e_0+\nu_k\!-i0\right)^{-1}\right)^{\hat{}}$ we have clearly $\mathcal{R}_k^\star=\mathcal{R}_k$ for $k>k_{e_0}$. Consequently, the corresponding terms in the series are real and

$$\mathrm{Im}\,e_2(B,\lambda) \;=\; \sum_{k=1}^{k_{e_0}}\left(\mathcal{U}_{jk}(B,\lambda)\phi_n,\,(\mathrm{Im}\,\mathcal{R}_k)\,\mathcal{U}_{jk}(B,\lambda)\phi_n\right)\,. \tag{3.13}$$

To make use of this formula, we have to express $\mathrm{Im}\,\mathcal{R}_k$. We follow [DEŠ] and employ the relations

$$\mathrm{Im}\,(h^V-E-i0)^{-1} \;=\; \omega(E+i0)^\star\mathrm{Im}\,(-\partial_x^2-z)^{-1}\omega(E+i0)$$

for $E>0$, where

$$\omega(z) \;:=\; \left[I+|V|^{1/2}(-\partial_x^2-z)^{-1}|V|^{1/2}\mathrm{sgn}\,(V)\right]^{-1}, \tag{3.14}$$

i.e., the inverse to the operator

$$\left(\omega^{-1}(z)f\right)(x) \;:=\; f(x)+\frac{i|V(x)|^{1/2}}{2\sqrt{z}}\int_{I\!R} e^{i\sqrt{z}|x-x'|}|V(x')|^{1/2}\mathrm{sgn}\,(V(x'))f(x')dx'\,.$$

The quantity $\omega(E+i0)$ is well defined in view of the assumptions *(ii)* and *(iii)*. In particular, the latter ensures the absence of positive eigenvalues for h^V. Furthermore,

$$\mathrm{Im}\,(-\partial_x^2-E-i0)^{-1} = \frac{\pi}{2\sqrt{E}}\sum_{\sigma=\pm}(\tau_E^\sigma)^\star\tau_E^\sigma$$

holds for $E > 0$, where $\tau^\sigma_E : \mathcal{H}^1 \to \mathbb{C}$ is the trace operator which acts on the first Sobolev space $\mathcal{H}^1$ in the following way,

$$\tau^\sigma_E := \hat{\phi}(\sigma\sqrt{E}), \qquad \sigma = \pm 1, \quad E > 0, \tag{3.15}$$

and as usual $\hat{\phi}$ is the Fourier transform of ϕ. Using the relations (3.14) and (3.15), we can thus rewrite the expression (3.13) as follows

$$\begin{aligned}
\operatorname{Im} e_2(B,\lambda) &= \sum_{k=1}^{k_{e_0}} \sum_{\sigma=\pm} \frac{\pi}{2\sqrt{e_0-\nu_k}} \left|\tau^\sigma_{e_0-\nu_k}\omega(e_0-\nu_k+i0)\,\mathcal{U}_{jk}(B,\lambda)\phi_n\right|^2 \qquad (3.16)\\
&= \sum_{k=1}^{k_{e_0}} \sum_{\sigma=\pm} \frac{\pi}{\sqrt{e_0-\nu_k}} \Big\{ -2B^2|m^{(1)}_{jk}|^2 \left|\tau^\sigma_{e_0-\nu_k}\omega(e_0-\nu_k+i0)\phi'_n\right|^2 \\
&\quad +2\lambda B m^{(1)}_{jk} \operatorname{Im}\left(\tau^\sigma_{e_0-\nu_k}\omega(e_0-\nu_k+i0)\phi'_n\,,\tau^\sigma_{e_0-\nu_k}\omega(e_0-\nu_k+i0)U_{jk}\phi_n\right) \\
&\quad -\frac{\lambda^2}{2}\left|\tau^\sigma_{e_0-\nu_k}\omega(e_0-\nu_k+i0)\,U_{jk}\phi_n\right|^2 \Big\} + \mathcal{O}(B^3) + \mathcal{O}(B^2\lambda)\,,
\end{aligned}$$

where we have used (2.10) and written explicitly only the lowest order terms. Let us summarize the results:

Theorem 3.2 *Assume* (i)–(iv) *and suppose that an unperturbed eigenvalue* $e_0 = \mu_n + \nu_j$ *is simple and satisfies (2.7), (2.8). Then the perturbation with small enough* B, λ *changes it into a resonance; the correponding pole position is given by (3.8)–(3.12). In particular, the Fermi golden rule (3.16) holds.*

4 Bound states in a strong magnetic field

In the final section we are going to address a different question. Since the separation of variables is not important in the following, we shall absorb the potential V into λU and denote $H(B,\lambda)$ as $H_{\lambda U}(B)$.

In the absence of a magnetic field a potential well in straight waveguide produces a non–empty discrete spectrum no matter how shallow it is. In fact, $H_{\lambda U}(0)$ has an isolated eigenvalue below the bottom of the essential spectrum for any small positive λ as long as $\int_{\mathbb{R}} U_{11}(x)\,dx \le 0$, where $U_{11}(x) := \int_S U(x,y)|\chi_1(y)|^2 dy$ is the projection onto the lowest transverse mode — the proof is given in [DE] for the hard–wall case and modifies easily to the situation when $S = \mathbb{R}$ and the lateral confinement is realized by the potential W.

Switching in the magnetic field changes the bound state energies; for small B the eigenvalue shift is given by (an appropriate modification of) the perturbation theory developed above. The eigenfunctions become complex so the bound states exhibit a nontrivial probability current, examples are worked out in [AVD, Nö]. If the magnetic field is made stronger, however, it may happen that some of the eigenvalues disappear in the continuum; it is not apriori clear whether the increase of B cannot destroy the discrete spectrum at all.

To answer this question, we have to find first the continuum threshold in the unperturbed case, $U=0$. Let F be the Fourier–Plancherel operator on $L^2(I\!R)$. The tensor product $F\otimes I$ means the Fourier transformation in the longitudinal variable; it takes $\mathcal{H}$ to $\hat{\mathcal{H}}=L^2(\Omega)$ which can be also written as $\int_{I\!R}^{\oplus} L^2(S)\,dp$ in the sense of [RS, Sec. XIII.16]. The operator $H_0(B)$ transforms into a decomposable operator $\hat H_0(B)=\{\,h_B^W(p)\,:\,p\in I\!R\}$ with the fiber

$$h_B^W(p)\;:=\;-\partial_y^2+W(y)+(By-p)^2 \tag{4.1}$$

on $L^2(S)$; if $S=(-a,a)$ the boundary conditions (2.2) are imposed.

In view of the assumption *(i)* and the minimax principle the spectra of the operators (4.1) are purely discrete an simple. We denote the corresponding eigenvalues and eigenfunctions by $\nu_j^B(p)$ and $\chi_j^B(\cdot;p)$, respectively. It is obvious that

$$\nu_j^B(p)\;\ge\;\nu_j^0(0)\;=\;\nu_j\,, \tag{4.2}$$

and moreover, an easy perturbation–theory argument shows that the functions $\nu_j^B(\cdot)$ are continuous. If $S=(-a,a)$, another straightforward lower bound,

$$\nu_j^B(p)\;\ge\;\inf_{y\in S}(By-p)^2+\left(\frac{\pi j}{2a}\right)^2, \tag{4.3}$$

implies

$$\lim_{|p|\to\infty}\nu_j^B(p)\;=\;\infty\,. \tag{4.4}$$

This argument does not work for $S=I\!R$ but (4.4) holds again, because

$$W(y)+(By-p)^2\;\ge\;\frac{cp^2}{c+B^2} \tag{4.5}$$

holds in view of the assumption *(i)*. Hence for given j and B there is one or more values of p at which the function $\nu_j^B(\cdot)$ reaches its minimum. We stress that the minimum may not be unique; a simple example is a strip with the mirror symmetry with respect to $y=0$, so $\nu_j^B(p)=\nu_j^B(-p)$, and W of a double–well form.

Since $H_0(B)$ has the direct integral decomposition, its spectrum is the union of the fiber spectra over the longitudinal momentum p. In particular,

$$\inf\sigma_{\rm ess}(H_0(B))\;=\;\min_{p\in I\!R}\,\nu_1^B(p)\,. \tag{4.6}$$

Now we are in position to formulate a sufficient condition for existence of the discrete spectrum.

Theorem 4.1 Assume *(i), (ii)* and denote $U_{11}^B(x;p)\;:=\;\int_S U(x,y)|\chi_1^B(y;p)|^2dy$. Suppose that the potential U satisfies $\lim_{|x|\to\infty}\sup_{y\in S}|U(x,y)|=0$ and

$$\int_{I\!R}U_{11}^B(x,p)\,dx\;<\;0 \tag{4.7}$$

holds true for some minimizing value p of ν_1^B; then the discrete spectrum of $H_U(B)$ is non–empty.

Proof: Due to the decay assumption, the potential does not change the essential spectrum,

$$\inf \sigma_{\rm ess}(H_U(B)) \,=\, \inf \sigma_{\rm ess}(H_0(B)) \,=\, \min_{p\in I\!\!R} \nu_1^B(p)\,. \tag{4.8}$$

To see that notice that the inequality $H_0(B) \ge H_0(0)$ holds[3] in view of (4.2), and therefore $(H_0(B)-z)^{-1} \le (H_0(0)-z)^{-1}$ for any $z < \nu_1$. It is thus sufficient to check that $U\,(H_0(0)-z)^{-1}$ is compact which is straightforward [RS, Sec. XIII.4]. In view of (4.6) the claim will be valid once we find a trial function $\Phi \in L^2(\Omega)$ which would make the quadratic form

$$q[\Phi] \,:=\, \|(-i\partial_x - By)\Phi\|^2 + \|\partial_y\Phi\|^2 + (\Phi, W\Phi) \,+\, (\Phi, U\Phi) - \nu_1^B(p)\,\|\Phi\|^2 \tag{4.9}$$

negative for a minimizing value p of ν_1^B; the above argument demonstrates that such values exist.

We choose $\Phi(x,y) := \phi(x)\chi_1^B(y;p)$ where ϕ is a function of $\mathcal{S}(I\!\!R)$ to be specified later. Notice that without loss of generality we may assume that $p=0$, because a shift in p is equivalent to a gauge transformation which consists of shifting By by an appropriate constant, $By \to B(y-y_0)$. The form (4.9) can be then rewritten as follows

$$q[\phi\times\chi_1^B] \,=\, \|\dot\phi\|^2_{L^2(I\!\!R)} - 2B\,{\rm Im}\,(\phi, \langle y\rangle_{11}^B\dot\phi)_{L^2(I\!\!R)} + \left(\phi, U_{11}^B\phi\right)_{L^2(I\!\!R)}\,, \tag{4.10}$$

where $\langle y\rangle_{11}^B := \int_S y\,|\chi_1^B(y;p)|^2 dy$.

The middle term at the *rhs* is absent if ϕ is real–valued. We choose a real $g \in \mathcal{S}(I\!\!R)$ such that $g(x)=1$ in $[-d,d]$ for some $d>0$ and employ the scaling trick in analogy with [GJ, DE] putting

$$\phi_\epsilon(x) \,:=\, \begin{cases} g(x) & \dots \quad |x| \le d \\ g(\pm d + \epsilon(x \mp d)) & \dots \quad \pm x > d \end{cases}$$

Then the kinetic term in (4.10) can be made arbitrarily small if ϵ is appropriately chosen,

$$q[\phi_\epsilon\times\chi_1^B] \,=\, \epsilon\|\dot g\|^2_{L^2(I\!\!R)} + \left(\phi, U_{11}^B\phi\right)_{L^2(I\!\!R)}\,. \tag{4.11}$$

On the other hand, the potential term in (4.11) tends to $\int_{I\!\!R} U_{11}^B(x,p)\,dx$ as $\epsilon\to 0$ by dominated convergence, so the *rhs* is negative for ϵ small enough. ∎

Corollary 4.2 In addition, let U be purely attractive, in other words, nonzero with $U(x,y) \le 0$ for any $(x,y)\in\Omega$; then $\sigma_{disc}(H_U(B))$ is non–empty for any B.

[3] In the case $S = I\!\!R$ this inequality can be alternatively proved by using the circular gauge.

A Appendix

Proposition A.1 *Assume (i)–(iv). If* $\operatorname{Im}\theta \in (0, \alpha_0)$ *and* Γ *is the contour described before Proposition 3.1, there exists a positive constant* c_θ *such that*

$$\max_{z\in\Gamma} \|\mathcal{U}_\theta(B,\lambda)R_\theta(z)\| \le c_\theta(|B| + |B|^2 + |\lambda|)\,.$$

If z *is replaced by* $z_0 = \nu_1 + \mu_1 - 1$, *the constant is independent of* θ *and the estimate is valid for all* $|\operatorname{Im}\theta| < \alpha_0$.

Proof: In view of the structure of $\mathcal{U}_\theta(B,\lambda)$ given by (3.4), we can treat the magnetic and the potential parts, $\mathcal{U}_\theta^B$ and $\mathcal{U}_\theta^\lambda$, of the perturbation separately. Furthermore, the contour Γ is by assumption contained in the resolvent set of $H_\theta(0)$. Since $R_\theta(\cdot)$ is bounded and continuous and Γ is compact, there exists a constant $\tilde{c}_\theta$ such that

$$\max_{z\in\Gamma} \|R_\theta(z)\| \le \tilde{c}_\theta\,.$$

Thus $\max_{z\in\Gamma} \|\mathcal{U}_\theta^\lambda R_\theta(z)\| \le |\lambda| c_U \tilde{c}_\theta$, where c_U denotes a bound on the norm of U_θ which is independent of θ by the assumption (iv).

As for the norm $\|\mathcal{U}_\theta^B R_\theta(z)\|$, it is clearly sufficient to consider $\theta = i\beta$. We have the following estimate

$$\left|\mathcal{U}_{i\beta}^B\right|^2 = \left|2iBye^{-i\beta}\partial_x + B^2y^2\right|^2 \le 8|B|^2y^2(-\partial_x^2) + 2|B|^4y^4\,. \tag{A.1}$$

in the form sense. The two terms at the *rhs* will be treated separately. Let us check first that the second one is relatively bounded. Using a simple commutation we get the estimate

$$\begin{aligned}
\left|y^2R_\theta(z)\right|^2 &\le R_\theta(z)^\star\Big(-\partial_y^2 + y^2\Big)^2 R_\theta(z) + 2\,|R_\theta(z)|^2 \\
&\le \left\|h^W R_\theta(z)\right\|^2 + 2\tilde{c}_\theta^2\,.
\end{aligned}$$

Since $R_\theta(z)$ maps $\mathcal{H}$ into the domain of $H_\theta(0)$ which is contained in the domain of $I\otimes h^W$, the map $z \mapsto h^W R_\theta(z)$ is bounded and continuous on the compact Γ, and therefore uniformly bounded by some constant $c_{\Gamma,\theta}$.

To show that the first term at the *rhs* of eq. (A.1) is relatively bounded, it remains to find a bound to $-\partial_x^2 R_{i\beta}(z)$. One has, uniformly for $z\in\Gamma$,

$$\begin{aligned}
\left\|-\partial_x^2 R_{i\beta}(z)\right\| &\le \left\|-e^{-2i\beta}\partial_x^2 R_{i\beta}(z)\right\| \\
&\le \|H_{i\beta}(0)R_{i\beta}(z)\| + \|h^W R_{i\beta}(z)\| + \|V_{i\beta}\|\tilde{c}_\theta \\
&\le 1 + (e_0 + r + \|V_{i\beta}\|)\tilde{c}_\theta + \max\{\nu_{k_0-1}\tilde{c}_\theta, d_1\}\,,
\end{aligned}$$

where r is the radius of Γ and we have employed the assumption (iii) about V.

It is straightforward to put these estimates together to obtain the first claim of the proposition. For the second statement note that z_0 is to the left of the numerical range of $H_\theta(0)$ at the unit distance. Thus $\|R_\theta(z_0)\| = 1$ and the constant $\tilde{c}_\theta$ in the above estimates may be replaced by 1. Furthermore, one may bound $\|h^W R_\theta(z_0)\|$ independently of θ (as long as $\operatorname{Im}\theta < \alpha_0$) since

$$\|h^W R_\theta(z_0)\| = \max_{k\in I\!N} \|\nu_k(h_\theta^V - z + \nu_k)^{-1}\| \le \max_{k\in I\!N} \frac{\nu_k}{1+\nu_k} < \nu_1\,.$$

Acknowledgment

The research has been partially supported by GA AS under the contract 1048801. During the research for this paper B.M. was supported by FONDECYT, Proyecto #3970026.

References

[AC] J. Aguilar, J.–M. Combes: A class of analytic perturbation for one–body Schrödinger Hamiltonians, *Commun. Math. Phys.* **22** (1971), 269–279.

[AVD] R. Akis, P. Vasilopoulos, R. Debray: Resonant tunnelling via quantum bound states in a classically unbound system of crossed, narrow channels, *Phys. Rev.* **B56** (1997), 9594–9602.

[APV] A. Aslanyan, L. Parnowski, D. Vassiliev: Complex resonances in acoustic waveguides, *mp_arc 99–52.*

[BJ] K.–F. Berggren, Zhen–li Ji: Bound states and transmission antiresonances in parabolically confined cross structures: influence of weak magnetic fields, *Phys. Rev.* **B43** (1991), 4760–4764.

[BBJ] K.–F. Berggren, Zhen–li Ji: Transition from laminar to vortex flow in a model semiconductor nanostructure, *Phys. Scripta* **T42** (1992),141–148

[BGS] R. Blanckenbecler, M.L. Goldberger, B. Simon: The bound states of weakly coupled long–range one–dimensional quantum Hamiltonians, *Ann. Phys.* **108** (1977), 69–78.

[BCD] Ph. Briet, J.–M. Combes, P. Duclos: On the location of resonances for Schrödinger operators in the semiclassical limit I: Resonance free domains, *J. Math. Anal.* **125** (1987), 90–99.

[BGRS] W. Bulla, F. Gesztesy, W. Renger, B. Simon: Weakly coupled bound states in quantum waveguides, *Proc. Am. Math. Soc.* **125** (1997), 1487–1495.

[DP] E.B. Davies, L. Parnowski: Trapped modes in acoustic waveguides, *J. Mech. Appl. Math.* **51** (1998), 477–492.

[DE] P. Duclos, P. Exner: Curvature–induced bound states in quantum waveguides in two and three dimensions, *Rev.Math.Phys.* **7** (1995), 73–102.

[DEM] P. Duclos, P. Exner, B. Meller: Exponential bounds on curvature–induced resonanaces in a two–dimensional Dirichlet tube, *Helv. Phys. Acta* **71** (1998), 133–162.

[DEŠ] P. Duclos, P. Exner, P. Šťovíček: Curvature–induced resonances in a two–dimensional Dirichlet tube, *Ann. Inst. H. Poincaré* **62** (1995), 81–101.

[ELV] D.V. Evans, D.M. Levitin, D. Vassiliev: Existence theorem for trapped modes, *J. Fluid Mech.* **261** (1994), 21–31.

[EGŠT] P. Exner, R. Gawlista, P. Šeba, M. Tater: Point interactions in a strip, *Ann. Phys.* **252** (1996), 133–179.

[EŠŠ] P. Exner, P. Šeba, P. Šťovíček: On existence of a bound state in an L–shaped waveguide, *Czech. J. Phys.* **B39** (1989), 1181–1191.

[GJ] J. Goldstone, R.L. Jaffe: Bound states in twisting tubes, *Phys. Rev.* **B45** (1992), 14100–14107.

[ISY] T. Itoh, N. Sano, A. Yoshii: Effects of width increase in the ballistic quantum wires, *Phys. Rev.* **B45** (1992), 1431–1435.

[Je] A. Jensen: Local decay in time of solutions to Schrödinger equation with a dilation-analytic interaction, *manuscripta mathematica* **25** (1978), 61–77.

[Ka] T. Kato: *Perturbation Theory for Linear Operators*, Springer, Heidelberg 1966.

[Kl] M. Klaus: On the bound state of Schrödinger operators in one dimension, *Ann.Phys.* **108** (1977), 288–300.

[NR] K.S. Na, L.E. Reichl: Electron conductance and lifetimes in a ballistic electron waveguide, *J. Stat. Phys.* **92** (1998), 519–542.

[Ne] R.G. Newton: Bounds on the number of bound states for the Schrödinger equation in one or two dimensions, *J. Operator Theory* **10** (1983), 119–125.

[NN] A. Nakamura, S. Nonoyama: Quantum interference effects on the electron conduction through negative–potential regions, *Phys. Rev.* **B56** (1997), 9649–9656.

[Nö] J.U. Nöckel: Resonances in quantum–dot transport, *Phys. Rev.* **B46** (1992), 15348–15356.

[OKN] A. Okiji, H. Kasai, A. Nakamura: Ballistic transport in mesoscopic systems, *Progr. Theor. Phys.* **106** (1991), 209–224.

[RS] M. Reed, B. Simon: *Methods of Modern Mathematical Physics, IV.Analysis of Operators*, Academic Press, New York 1978.

[RVW] I. Roitberg, D. Vassiliev, T. Weidl: Edge resonances in an elastic semi-strip, *Quarterly J. Mech. Appl. Math.* **51** (1998), 1–13.

[Si1] B. Simon: The bound state of weakly coupled Schrödinger operators in one and two dimensions, *Ann.Phys.* **97** (1976), 279–288.

[SRW] R.L.Schult, D.G.Ravenhall, H.W.Wyld: Quantum bound state in classically unbound system of crossed wires, *Phys. Rev.* **B39** (1989), 5476–5479.

[SWR] R.L. Schult, H.W. Wyld, D.G. Ravenhall: Quantum Hall effect and general narrow–wire circuits, *Phys. Rev.* **B41** (1990), 12760–12780.

[Se] N. Seto: Bargmann's inequalities in spaces of an arbitrary dimension, *Publ. RIMS* **9** (1974), 429–461.