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OPERATOR-NORM APPROXIMATION OF SEMIGROUPS BY SECTORIAL CONTRACTIONS

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Abstract

We extend the Chernoff theory for approximation of contraction semigroups à la Trotter. We show that the Trotter-Neveu-Kato convergence theorem holds in operator norm for a family of uniformly m-sectorial generators in a Hilbert space. Then we obtain a Chernoff-type approximation theorem for sectorial contractions on a Hilbert space in the operator norm. As examples we give necessary and sufficient conditions for the operator-norm convergence of Trotter-type product formulæ.

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1 Introduction

In his article [4], Chernoff has proved the following proposition:

Proposition 1 *Let $F(t)$ be a strongly continuous function from $[0,\infty)$ to the linear contractions on a Banach space $\mathcal{X}$ such that $F(0) = I$. Suppose that the closure C of the strong derivative $F'(0)$ is the generator of a contraction semigroup. Then $F(t/n)^n$ converges to e^{tC} in the strong operator topology.*

This result describes a method for the approximation of a general contraction semigroup. It leads in particular to a simple proof of the strong convergence for the Trotter product formula (see [5, §3.4]). For self-adjoint families of operators on a Hilbert space there exists a generalization of the above result in the operator-norm topology [13], [15]:

Proposition 2 *Let $\{\Phi(s)\}_{s\geq 0}$ be a family of self-adjoint non-negative contractions on a separable Hilbert space $\mathcal{H}$, and let X_0 be a self-adjoint operator in the closed subspace $\mathcal{H}_0 \subseteq \mathcal{H}$. Define $X(s) = s^{-1}(I - \Phi(s))$, $s \geq 0$. Then the family $\{X(s)\}_{s\geq 0}$ converges in the uniform resolvent sense to X_0 as $s \to +0$ if and only if the sequence $\{\Phi(t/r)^r\}_{r\geq 1}$, $t \geq 0$, converges in operator-norm to $e^{-tX_0}P_0$ as $r \to +\infty$, uniformly on any compact t-interval in $(0,\infty)$. Here P_0 denotes the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$.*

The aim of the present paper is to show that the condition of self-adjointness in this result can be replaced by a much weaker condition on the corresponding numerical range. Let $\mathcal{H}$ be a separable Hilbert space and let T be a linear operator in $\mathcal{H}$. We denote by $\Theta(T)$ the numerical range of T :

$$\Theta(T) = \{(Tu, u), u \in \mathcal{D}(T) \text{ and } \|u\| = 1\}.$$

Below we use some important properties of this set (cf [10, Ch.V Theorem 3.2]):

Proposition 3 *If T is a closed operator in $\mathcal{H}$, then for any complex number $z \notin \overline{\Theta(T)}$, $T - z$ has closed range, it is injective and its deficiency $\operatorname{def}(T - z)$ is constant (in each connected component). If $\operatorname{def}(T - z) = 0$, then the spectrum of T is a subset of $\overline{\Theta(T)}$ and*

$$\|(T - z)^{-1}\| \leq \frac{1}{dist(z, \Theta(T))}. \tag{1}$$

For $0 < \alpha \leq \pi$, we denote by S_α the open sector

$$S_\alpha = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \alpha\}.$$

Definition 1 *Operator T in $\mathcal{H}$ is said to be sectorial with a semi-angle $\alpha \in (0, \pi/2)$ and with vertex at 0 if $\Theta(T) \subseteq S_\alpha$. If, in addition, T is closed and there exists $z \notin S_\alpha$ that belongs to the resolvent set of T, then T is said to be m-sectorial.*

We recall the following result (see [10, Ch.IX Theorem 1.24]):

Proposition 4 *Let T be an m-sectorial operator in a Hilbert space $\mathcal{H}$, with a semi-angle $\alpha \in (0, \pi/2)$ and with vertex at 0. Then the semigroup e^{-tT} is holomorphic in $S_{\pi/2-\alpha}$ and it is bounded by $\|e^{-tT}\| \leq 1$ for $t \in S_{\pi/2-\alpha}$.*

We define for $0 \leq \alpha \leq \pi/2$

$$D_\alpha = \{z \in \mathbb{C} : |z| \leq \sin\alpha\} \cup \{z \in \mathbb{C} : |\arg(1-z)| \leq \alpha \text{ and } |z-1| \leq \cos\alpha\}. \tag{2}$$

Definition 2 *We shall say that the contraction C on a Hilbert space $\mathcal{H}$ is sectorial with a semi-angle $0 \leq \alpha < \pi/2$ (and with vertex at 1), if its numerical range $\Theta(C)$ is a subset of D_α. Notice that the case $\alpha = 0$ corresponds to a non-negative self-adjoint contraction.*

Let $\{\Phi(s)\}_{s\geq 0}$ be a family of sectorial contractions on a Hilbert space $\mathcal{H}$ such that $\Theta(\Phi(s)) \subseteq D_\alpha$ for some $0 \leq \alpha < \pi/2$ and for all $s \geq 0$. Let $X(s) = (I - \Phi(s))/s$, and let X_0 be a closed operator in a closed subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ with non-empty resolvent set. Then the main result of this paper (see Theorem 3) can be formulated as the equivalence between two limits:

$$\|(\zeta + X(s))^{-1} - (\zeta + X_0)^{-1}P_0\| \longrightarrow 0 \text{ as } s \to +0,\ \mathrm{Re}\,\zeta > 0, \tag{3}$$

$$\|\Phi(t/n)^n - e^{-tX_0}P_0\| \longrightarrow 0 \text{ as } n \to \infty,\ t > 0. \tag{4}$$

Our method is an operator-norm generalization of the Chernoff theory [4]. Definition 2 will be illustrated in the next section. Then we generalize Lemma 2 of [4] to the operator-norm topology. In Section 4 we prove the operator-norm analogue of the Trotter-Neveu-Kato convergence theorem (see [18, Theorem 5.1], or e.g. [5, Theorem 3.17], [6, Ch. 1.7]) and the main result (3),(4) formulated above. Finally, we give some applications concerning Trotter-Kato formulæ.

2 Sectorial contractions

In this section we give some important examples of operator families corresponding to Definition 2, i.e. sectorial contractions. Let A be an m-sectorial operator with a semi-angle α and with vertex at 0, in a Hilbert space $\mathcal{H}$. Then there are two families of sectorial contractions that are generated by A in a natural way.

2.1 Resolvent of a sectorial operator

The family of resolvents $F(t) = (I + tA)^{-1}$, $t \geq 0$, has the following properties:

(i) Since $F(t = 0) = I$ and since for $t > 0$

$$\|F(t)\| \leq \frac{1}{t\,\mathrm{dist}(1/t, -S_\alpha)} = 1, \tag{5}$$

$F(t)$ is a family of contractions;

(ii) $\Theta(F(t)) \subseteq S_\alpha$, because for any $u \in \mathcal{H}$:

$$(u, F(t)u) = (v, v) + t(Av, v) \in S_\alpha, \tag{6}$$

where $v = (I + tA)^{-1}u$.

(iii) $\Theta(I - F(t)) \subseteq S_\alpha$, since for any $u \in \mathcal{H}$

$$(u, (I - F(t))u) = ((I + tA)v, tAv) = t(v, Av) + t^2(Av, Av) \in S_\alpha, \tag{7}$$

where $v = (I + tA)^{-1}u$.

Therefore, we have $\Theta(F(t)) \subseteq S_\alpha \cap (1 - S_\alpha) \subseteq D_\alpha$, i.e. $F(t)$ are sectorial contractions for $t \geq 0$.

2.2 Semigroup generated by a sectorial operator

The semigroup generated by the m-sectorial operator A is holomorphic and contractive in the sector $S_{\pi/2-\alpha}$ (cf Proposition 4). We shall show that in fact $\Theta(e^{-tA}) \subseteq D_\alpha$ for $t \geq 0$. It is not as simple as for the resolvent. The key statement is the following mapping theorem for the numerical range due to Kato [9]:

Proposition 5 *Let $f(z)$ be a rational function with $f(\infty) = \infty$. Let E' be a compact convex set in the complex plane, let $E = f^{-1}(E')$ and K be the convex kernel of E. If A is an operator with $\Theta(A) \subseteq K$, then $\Theta(f(A)) \subseteq E'$. In particular if D is a compact convex subset of $\mathbb{C}$ with $f(D) \subseteq D$ and $\Theta(A) \subseteq D$, then $\Theta(f(A)) \subseteq D$.*

Lemma 1 *Let $f(z) = z^n$ where $n \in \mathbb{N}$, then $f(D_\alpha) \subseteq D_\alpha$.*

Proof. If $|z| \leq \sin\alpha$, then $|z^n| \leq \sin\alpha$ and $z^n \in D_\alpha$. Thus it remains the case $z \in D_\alpha$, $|z| > \sin\alpha$. Let us consider the family of the straight lines $z(t) = 1 - te^{i\beta}$ in D_α, parametrized by $0 \leq t \leq \cos\alpha$ and $-\alpha \leq \beta \leq \alpha$. We study their images defined by $\zeta(t) = z(t)^n$ to prove that they also lie in D_α. They form a regular plane curve, i.e. $\zeta'(t) = -ne^{i\beta}(1 - te^{i\beta})^{n-1} \neq 0$. Then for each path $\zeta(t)$ we can define unit tangent vectors $T(t)$:

$$T(t) = \frac{\zeta'(t)}{|\zeta'(t)|} = -e^{i\beta}\left(\frac{1 - te^{i\beta}}{|1 - te^{i\beta}|}\right)^{n-1}. \tag{8}$$

The curvature $c(t)$ of a regular plane curve is defined by $T'(t) = ic(t)T(t)$, and we find by (8):

$$T'(t) = -e^{i\beta}(n-1)\left(\frac{z}{|z|}\right)^{n-2}\frac{d}{dt}\left(\frac{z}{|z|}\right) = i(n-1)\frac{\sin\beta}{|z|^2}e^{i\beta}\left(\frac{z}{|z|}\right)^{n-1}, \tag{9}$$

here $z = z(t) = 1 - te^{i\beta}$. Hence the expression for the curvature is $c(t) = -(n-1)\sin\beta/|z(t)|^2$. It does not vanish for any $0 \leq t \leq \cos\alpha$ and it has the sign opposite to β. Therefore, the curve $\zeta(t)$ lies in the half-plane defined by any tangent and containing the corresponding curvature center. Let $t = 0$, then the tangent is $z(t)$, and the curvature center is at $1 + ie^{i\beta}(n-1)\sin\beta$. Finally, all images $\zeta(t) = z(t)^n$, for $0 \leq t \leq \cos\alpha$ and $-\alpha \leq \beta \leq \alpha$, are squeezed between two extreme tangent lines corresponding to $\beta = \pm\alpha$. This finishes the proof.

□

Theorem 1 *If A is an m-sectorial operator with a semi-angle α and with vertex at 0, then $\Theta(e^{-tA})$ is a subset of D_α for any $t \geq 0$.*

Proof. By virtue of example from Section 2.1 we have that $\Theta((I+tA/n)^{-1}) \subseteq D_\alpha$ for any positive real t and any non-zero integer n. According to Proposition 5 and to Lemma 1 for the rational function $f(z) = z^n$ ($f(\infty) = \infty$), the operator $(I+tA/n)^{-1}$, and the compact convex set D_α, we obtain that $\Theta((I + tA/n)^{-n}) \subseteq D_\alpha$ for any $t \geq 0$ and $n \in \mathbb{N} \setminus \{0\}$.

On the other hand it is known (see [10, Ch.IX]) that the sequence $(I + tA/n)^{-n}$ converges strongly to the semigroup e^{-tA} as $n \to \infty$. Therefore, $\lim_{n\to\infty}((I+tA/n)^{-n}u, u) = (e^{-tA}u, u) \in D_\alpha$ for any unit vector $u \in \mathcal{H}$, which proves the assertion.

□

3 Generalization of the Chernoff lemma

In this section we show that an operator-norm analogue of the main Lemma 2 of [4] is valid if one supposes that the contraction C is sectorial in the sense of Definition 2.

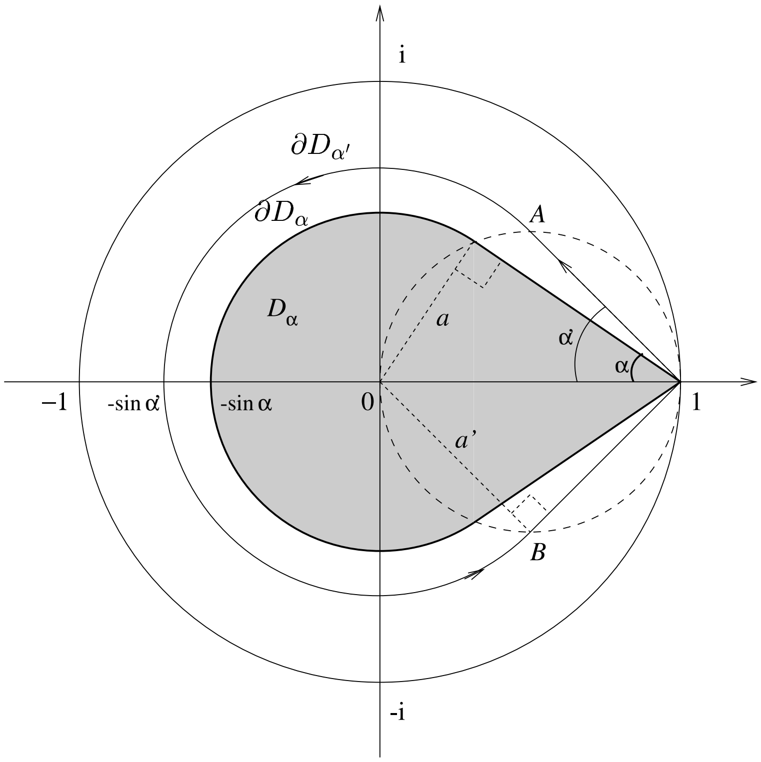

Figure 1: Illustration of the set D_α (shaded domain) with boundary ∂D_α, $a = \sin\alpha$, as well as of our choice of the contour $\partial D_{\alpha'}$ in the resolvent set $\rho(C)$, where $a' = \sin\alpha' > a$. The contour consists of two segments of tangent straight lines $(1, A)$ and $(1, B)$ and the arc (A, B) of radius a'. The dotted circle corresponds to the set of tangency points for the different values of $\alpha \in [0, \pi/2]$.

Lemma 2 *If C is a contraction on a Hilbert space $\mathcal{H}$, and there exists $0 \le \alpha < \pi/2$ such that the numerical range $\Theta(C)$ is a subset of the domain D_α, then*

$$\|C^n(I - C)\| \le \frac{K}{n+1}, \;\; n \in \mathbb{N}. \tag{10}$$

Proof.

Since the operator C is bounded, its spectrum is a subset of the closure of the numerical range $\overline{\Theta(C)}$. Then taking $\alpha < \alpha' < \pi/2$ we can choose a contour $\partial D_{\alpha'}$ outside D_α, but inside the unit circle (see Figure 1), such that by the Dunford-Taylor formula one has:

$$C^n(I - C) = \frac{1}{2\pi i}\int_{\partial D_{\alpha'}} \frac{z^n(1-z)}{z - C}dz. \tag{11}$$

By Proposition 3 (see (1)) we obtain: $\|(z-C)^{-1}\| \leq \mathrm{dist}(z, D_\alpha)^{-1}$. Thus $\|(z+C)^{-1}\| \leq (\cos\alpha' \sin(\alpha'-\alpha))^{-1}$ for $|\arg z| \geq \pi/2 - \alpha'$, and $\|(z+C)^{-1}\| \leq (|1-z|\sin(\alpha'-\alpha))^{-1}$ for $|\arg z| \leq \pi/2 - \alpha'$. We consider the following parametrization of $\partial D_{\alpha'}$: for the arc AB, we take $z(t) = e^{it}\sin\alpha'$ with $\pi/2 - \alpha' \leq t \leq 3\pi/2 + \alpha'$, and for the straight lines $(1, A)$, $(1, B)$, we put correspondingly $z_{\mp}(s) = 1 - se^{\mp i\alpha'}$ with $0 \leq s \leq \cos\alpha'$. Then

$$\begin{aligned} \|C^n(I-C)\| &\leq \frac{1}{2\pi}\int_{\frac{\pi}{2}-\alpha'}^{\frac{3\pi}{2}+\alpha'} \frac{|\sin\alpha'|^{n+1}|1-e^{it}\sin\alpha'|}{\cos\alpha'\sin(\alpha'-\alpha)}dt + \\ &\quad + \frac{1}{\pi}\int_0^{\cos\alpha'} \frac{|(1-e^{i\alpha'}s)^n e^{i\alpha'}s|}{s(\sin(\alpha'-\alpha))}ds \\ &\leq \frac{2(\sin\alpha')^{n+1}}{\cos\alpha'\sin(\alpha'-\alpha)} + \int_0^{\cos\alpha'} \frac{((1-s\cos\alpha')^2 + s^2(\sin\alpha')^2)^{n/2}}{\pi\sin(\alpha'-\alpha)}ds. \end{aligned} \tag{12}$$

By convexity (for $0 \leq s \leq \cos\alpha'$) one gets that

$$(1-s\cos\alpha')^2 + s^2(\sin\alpha')^2 \leq 1 - s\cos\alpha',$$

which leads to the inequality:

$$\begin{aligned} \int_0^{\cos\alpha'} ((1-s\cos\alpha')^2 + s^2(\sin\alpha')^2)^{n/2}\, ds &\leq \int_0^{\cos\alpha'} (1-s\cos\alpha')^{n/2} ds \\ \int_{(\sin\alpha')^2}^1 u^{n/2}\frac{du}{\cos\alpha'} &\leq \frac{1-(\sin\alpha')^{n+2}}{(n/2+1)\cos\alpha'}. \end{aligned}$$

Therefore, by (12) we get the estimate (10)

$$\|C^n(I-C)\| \leq \frac{2(\sin\alpha')^{n+1}}{\cos\alpha'\sin(\alpha'-\alpha)} + 2\frac{1-(\sin\alpha')^{n+2}}{\pi(n+2)\cos\alpha'\sin(\alpha'-\alpha)} \leq \frac{K}{n+1}, \tag{13}$$

where

$$K = \frac{2}{\cos\alpha'\sin(\alpha'-\alpha)}\left(\frac{1}{\pi} - \frac{1}{e\ln(\sin\alpha')}\right). \tag{14}$$

□

Lemma 3 *If C is a contraction on a Hilbert space $\mathcal{H}$ that satisfies the estimate (10), then:*

$$\|C^n - e^{n(C-I)}\| \leq \frac{2K+2}{n^{1/3}},\ n \in \mathbb{N}\setminus\{0\}. \tag{15}$$

Proof.

Since the operator C is bounded, we have the representation:

$$C^n - e^{n(C-I)} = e^{-n}\sum_{m=0}^{\infty}\frac{n^m}{m!}(C^n - C^m). \tag{16}$$

Let $\epsilon_n = n^{2/3}$, $n \geq 1$. We divide the sum (16) into two parts: the central part for $|m-n| \leq \epsilon_n$ and tails for $|m-n| > \epsilon_n$. We estimate tails by using the Tchebychev inequality (see e.g. [17]). Let X be a Poisson random variable of parameter n, i.e. $\mathrm{P}(X =$

$m) = n^m e^{-n}/m!$. Then expectation $\mathrm{E}(X) = n$ and variance $\mathrm{Var}(X) = n$. Therefore, by the Tchebychev inequality:

$$\forall \epsilon > 0,\ \mathrm{P}(|X - n| > \epsilon) \leq \frac{n}{\epsilon^2}, \tag{17}$$

and hence

$$e^{-n} \sum_{|m-n|>\epsilon_n} \frac{n^m}{m!} \|C^n - C^m\| \leq \frac{2n}{\epsilon_n^2} = \frac{2}{n^{1/3}}. \tag{18}$$

To estimate the central part of the sum (16), where $|m - n| \leq \epsilon_n$, we use (here $[\cdot]$ denotes the integer part):

$$\begin{aligned}\|C^n - C^m\| &= \left\|C^{n-[\epsilon_n]}\left(C^{[\epsilon_n]} - C^{m-n+[\epsilon_n]}\right)\right\| \\ &\leq |m-n| \|C^{n-[\epsilon_n]}(I - C)\| \\ &\leq \epsilon_n \, \frac{K}{n - [\epsilon_n] + 1} \leq \frac{2K}{n^{1/3}},\end{aligned}$$

and we estimate the sum by 1. Then, together with (18), we obtain (15):

$$\|C^n - e^{n(C-I)}\| \leq \frac{2K+2}{n^{1/3}} \tag{19}$$

for $n \in \mathbb{N} \setminus \{0\}$.

□

The next statement is an extension of the famous "$n^{1/2}$-Lemma" by Chernoff [4, Lemma 2]. It follows directly from Lemmata 2, 3 and it gives an operator-norm estimate instead of the well-known strong convergence [4].

Theorem 2 *Let C be a sectorial contraction on $\mathcal{H}$ with numerical range $\Theta(C) \subseteq D_\alpha$, $0 \leq \alpha < \pi/2$. Then*

$$\left\|C^n - e^{n(C-I)}\right\| \leq \frac{M}{n^{1/3}},\ n = 1, 2, 3, \ldots \tag{20}$$

where $M = 2K + 2$ and K is defined by (14).

Remark 1 *If no estimate is required, then the operator-norm convergence of the difference (20) to zero follows from the Lebesgue dominated convergence theorem for the Cauchy integral for $C^n - e^{n(C-I)}$ on the contour $\partial D_{\alpha'}$, cf (11).*

Remark 2 *If C is self-adjoint and non-negative (i.e. $\alpha = 0$), then directly by the spectral theorem one obtains (cf [13],[15]):*

$$\|C^n(I - C)\| \leq \frac{1}{n+1} \text{ and } \|C^n - e^{n(C-I)}\| \leq \frac{1}{n}. \tag{21}$$

4 Approximation Theorem

In this section we present the proof of our main Theorem. The first step was the Theorem 2, the second step is an operator-norm generalization of the Trotter-Neveu-Kato convergence theorem, see e.g. [5, Ch. 3.3], [6, Ch. 1.7]. In the self-adjoint case, such a generalization in the operator-norm topology is obtained in [15, Lemma 2.1]. For m-sectorial operators we prove it below.

Lemma 4 *Let $\{X(s)\}_{s>0}$ be a family of m-sectorial operators in a Hilbert space $\mathcal{H}$ with $\Theta(X(s)) \subseteq S_\alpha$ for some $0 < \alpha < \pi/2$ and for all $s > 0$. Let X_0 be an m-sectorial operator defined in a closed subspace $\mathcal{H}_0 \subseteq \mathcal{H}$, with $\Theta(X_0) \subseteq S_\alpha$. Then the following conditions are equivalent:*

$$(a) \qquad \lim_{s\to+0} \left\|(\zeta + X(s))^{-1} - (\zeta + X_0)^{-1}P_0\right\| = 0, \ \textit{for} \ \mathrm{Re}\,\zeta > 0; \tag{22}$$

$$(b) \qquad \lim_{s\to+0} \left\|e^{-tX(s)} - e^{-tX_0}P_0\right\| = 0, \ \textit{for} \ t > 0. \tag{23}$$

Here P_0 denotes the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_0$.

Proof.
$(a) \Rightarrow (b)$

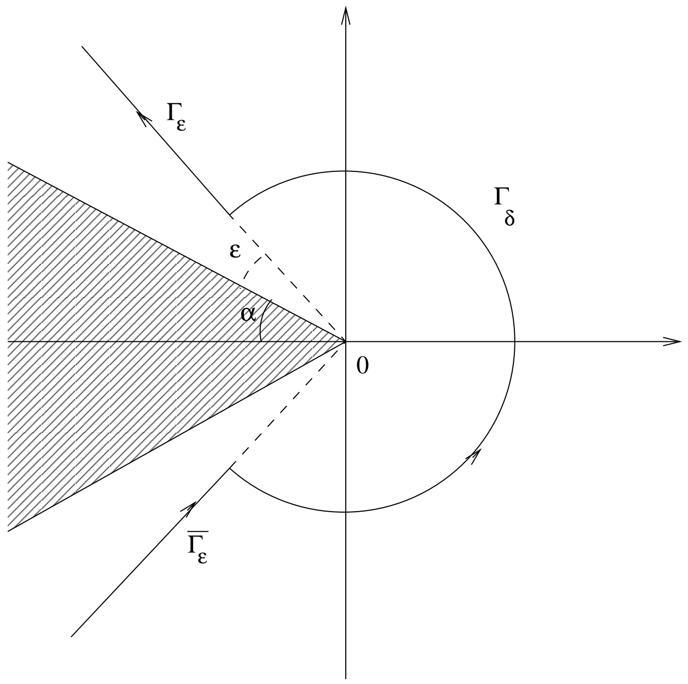

Figure 2: Illustration of the path Γ.

Since $\{X(s)\}_{s>0}$ are m-sectorial with $\Theta(X(s)) \subseteq S_\alpha$, the Dunford-Taylor formula

$$e^{-tX(s)} = \frac{1}{2\pi i}\int_\Gamma d\zeta \frac{e^{t\zeta}}{\zeta + X(s)} \tag{24}$$

defines a family of holomorphic semigroups $\{e^{-tX(s)}\}_{s>0}$ with $t \in S_{\pi/2-\alpha}$. Here $\Gamma \subset S_{\pi-\alpha}$ is a positively-oriented closed (at infinity) path around $-S_\alpha$, i.e. with the spectrum $\sigma(-X(s)) \subseteq \overline{\Theta(-X(s))}$ in its interior (see Figure 2). The same is true for the operator X_0:

$$e^{-tX_0}P_0 = \frac{1}{2\pi i}\int_\Gamma d\zeta \frac{e^{t\zeta}}{\zeta + X_0}P_0. \tag{25}$$

We set $\Gamma = \Gamma_\epsilon \cup \Gamma_\delta \cup \overline{\Gamma_\epsilon}$, where the arc $\Gamma_\delta = \{z \in \mathbb{C} : |z| = \delta > 0, |\arg z| \le \pi - \alpha - \epsilon\}$ (for $0 < \epsilon < \pi/2 - \alpha$) and $\Gamma_\epsilon, \overline{\Gamma_\epsilon}$ are two conjugate straight lines, $\Gamma_\epsilon = \{z \in \mathbb{C},\ \arg z = \pi - \alpha - \epsilon,\ |z| \ge \delta\}$. Then for $t > 0$ one gets

$$\left\|e^{-tX(s)} - e^{-tX_0}P_0\right\| \le \frac{1}{2\pi}\int_\Gamma |d\zeta||e^{t\zeta}| \left\|(\zeta + X(s))^{-1} - (\zeta + X_0)^{-1}P_0\right\|. \tag{26}$$

Since operators $X(s)$ and X_0 are m-sectorial, by Proposition 3 one gets the estimate

$$\left\|(\zeta + X(s))^{-1} - (\zeta + X_0)^{-1}P_0\right\| \le \frac{2}{\operatorname{dist}(\zeta, -S_\alpha)}, \tag{27}$$

which implies

$$\sup_{\zeta\in\Gamma}\left\|(\zeta + X(s))^{-1} - (\zeta + X_0)^{-1}P_0\right\| \le \frac{2}{\delta \sin\epsilon}. \tag{28}$$

Since for the path $\Gamma = \Gamma_\epsilon \cup \Gamma_\delta \cup \overline{\Gamma_\epsilon}$, the integral $I_t = \int_\Gamma |d\zeta||e^{t\zeta}|$ converges for any $t > 0$, then condition (a), estimates (26), (28) and the Lebesgue dominated convergence theorem proves (b).

$(b) \Rightarrow (a)$

By the Laplace transform for semigroups $e^{-tX(s)}$ and e^{-tX_0}, one gets for $\operatorname{Re}\zeta > 0$:

$$(\zeta + X(s))^{-1} - (\zeta + X_0)^{-1}P_0 = \int_0^\infty dt e^{-t\zeta}\left(e^{-tX(s)} - e^{-tX_0}P_0\right). \tag{29}$$

Since the generators are m-sectorial with vertex at 0, the corresponding semigroups are contractions: $\|e^{-tX(s)}\| \le 1$, $\|e^{-tX_0}P_0\| \le 1$ for $t \in S_{\pi/2-\alpha}$, $s > 0$. Then the Lebesgue dominated convergence theorem ensures (a) for $\operatorname{Re}\zeta > 0$, due to the estimate

$$\begin{aligned}\left\|(\zeta + X(s))^{-1} - (\zeta + X_0)^{-1}P_0\right\| \le \\ \int_0^\infty dt e^{-t\operatorname{Re}\zeta}\left\|e^{-tX(s)} - e^{-tX_0}P_0\right\|\end{aligned} \tag{30}$$

and the pointwise $\|.\|$-convergence (b). □

Remark 3 *The Vitali theorem concerning the convergence of holomorphic (operator-valued) functions (see e.g. [7, Theorem 3.14.1]) allows to extend the pointwise convergences in (a) and in (b) because the corresponding functions are holomorphic and uniformly bounded. In particular, we can replace the conditions (a) and (b) respectively by*

$(\tilde{a})$ $\lim_{s\to+0}\left\|(\zeta + X(s))^{-1} - (\zeta + X_0)^{-1}P_0\right\| = 0,$
uniformly on compacts in $S_{\pi-\alpha}$;

$(\tilde{b})$ $\lim_{s\to+0}\left\|e^{-tX(s)} - e^{-tX_0}P_0\right\| = 0,$
uniformly on compacts in $S_{\pi/2-\alpha}$.

In fact, we can prove the equivalence in Lemma 4 under weaker conditions:

Corollary 1 *Let $\{X(s)\}_{s>0}$ be a family of m-sectorial operators with $\Theta(X(s)) \subseteq S_\alpha$ for some $0 < \alpha < \pi/2$ and for all $s > 0$. Let X_0 be a closed operator in $\mathcal{H}_0 \subseteq \mathcal{H}$ with non-empty resolvent set. Then the following conditions are equivalent:*

$$(a') \qquad \lim_{s\to+0} \left\|(\zeta + X(s))^{-1} - (\zeta + X_0)^{-1}P_0\right\| = 0, \ \textit{for some } \zeta \in S_{\pi-\alpha};$$

$$(b') \qquad \lim_{s\to+0} \left\|e^{-tX(s)} - e^{-tX_0}P_0\right\| = 0, \ \textit{for } t \textit{ in a subset of } (0, +\infty)$$

having a limit point, if X_0 is generator of a semigroup, i.e. e^{-tX_0} is well defined at least for $t \geq 0$.

Proof.
It is sufficient to check that condition (a') implies: (a) holds and the operator X_0 is such that $\Theta(X_0) \subseteq S_\alpha$ for some $\alpha \in (0, \pi/2)$, and that condition (b') implies (b) for the same operator X_0.

By [10, Ch.IV, Theorem 2.25], the condition (a') is sufficient to have a generalized convergence in the sense of Kato. On the other hand, this generalized convergence implies that (a') holds for any ζ in the resolvent set of X_0. Since it is non-empty and open, it has a limit point. Then we can apply the theorem of Vitali, since the norms $\|(\zeta + X(s))^{-1}\|$ are uniformly bounded in s for $\zeta - \epsilon \in S_{\pi-\alpha}$ (for any fixed $\epsilon > 0$). Thus we obtain the pointwise convergence for any $\zeta \in S_{\pi-\alpha}$, and in particular, (a) holds for the operator X_0.

In order to show that X_0 is necessarily m-sectorial with $\Theta(X_0) \subseteq S_\alpha$, we use that the norm-convergence (22) implies:

$$\|(\zeta + X_0)^{-1}P_0\| = \lim_{s\to 0} \|(\zeta + X(s))^{-1}\| \leq \frac{1}{\operatorname{dist}(\zeta, -S_\alpha)}. \tag{31}$$

In the case $\operatorname{Re}\zeta > 0$, we obtain $\|(\zeta + X_0)^{-1}\|_0 \leq 1/\operatorname{Re}\zeta$, where $\|\cdot\|_0$ is the operator-norm in $\mathcal{H}_0$. This estimate implies that X_0 generates a contraction semigroup, and then is m-accretive. By the same way one checks that $e^{i\varphi}X_0$ is m-accretive for any $\alpha - \pi/2 \leq \varphi \leq \pi/2 - \alpha$, which proves that X_0 is m-sectorial with semi-angle α and with vertex at 0.

The implication $(b') \Rightarrow (b)$ is a direct consequence of the Vitali theorem. By $\|e^{-tX(s)}\| \leq 1$ for any $t \in S_{\pi/2-\alpha}$ and $s > 0$, and by (b') one gets that the functions $\{e^{-tX(s)}\}_{s>0}$ converge in operator-norm as $s \to +0$ for any $t \in S_{\pi/2-\alpha}$. Therefore, it follows that the limit e^{-tX_0} is a holomorphic semigroup with $\|e^{-tX_0}\|_0 = \|e^{-tX_0}P_0\| \leq 1$ for $t \in S_{\pi/2-\alpha}$, and then operator X_0 is m-sectorial with semi-angle α and with vertex at 0, in $\mathcal{H}_0 = P_0\mathcal{H}$. □

It is well-known that the convergence in the strong resolvent sense is well-adapted to the problem of semigroup approximation in the strong topology (see e.g. [5, Ch. 3.3] or [6, Ch. 1.7]). Similarly, the above results show that the convergence in the uniform resolvent sense (i.e. the generalized convergence in the sense of Kato [10, Ch. IV.6]) is well-adapted to the semigroup approximation in the operator-norm topology. This topology is natural for holomorphic contraction semigroups, which are continuous in the operator norm outside 0. We show that the corresponding class of generators (the m-sectorial operators) is in some sense stable for the convergence we consider.

Now we can prove our main Theorem:

Theorem 3 *Let $\{\Phi(s)\}_{s\geq 0}$ be a family of sectorial contractions on a Hilbert space $\mathcal{H}$. Let there exists $0<\alpha<\pi/2$ such that $\Theta(\Phi(s))\subseteq D_\alpha$, for all $s\geq 0$. Let $X(s)=(I-\Phi(s))/s$, and let X_0 be a closed operator with non-empty resolvent set, defined in a closed subspace $\mathcal{H}_0\subseteq\mathcal{H}$. Then the family $\{X(s)\}_{s>0}$ converges in the uniform resolvent sense to X_0 as $s\to+0$ if and only if*

$$\lim_{n\to\infty}\left\|\Phi(t/n)^n - e^{-tX_0}P_0\right\| = 0,\ t>0. \tag{32}$$

Here P_0 denotes the orthogonal projection onto $\mathcal{H}_0$.

Proof.

Necessity: Notice that $\Theta(\Phi(s))\subseteq D_\alpha$ means that $\Theta(X(s))\subseteq S_\alpha$ for any $s>0$. Thus, the convergence of $X(s)$ in the uniform resolvent sense coincides with the condition (a) of Lemma 4. Since

$$\begin{aligned}\|\Phi(t/n)^n - e^{-tX_0}P_0\| \ &\leq \\ &\|\Phi(t/n)^n - e^{-n(I-\Phi(t/n))}\| + \|e^{-n(I-\Phi(t/n))} - e^{-tX_0}P_0\|,\end{aligned} \tag{33}$$

the assertion follows from Theorem 2 for $C=\Phi(t/n)$ and part $(a)\Rightarrow(b)$ Lemma 4.

Sufficiency: We have to estimate the difference

$$\begin{aligned}\|e^{-tX(t/n)} - e^{-tX_0}P_0\| \ &\leq \\ &\|e^{-tX(t/n)} - \Phi(t/n)^n\| + \|\Phi(t/n)^n - e^{-tX_0}P_0\|,\ t>0.\end{aligned} \tag{34}$$

The first term in the right-hand side of (34) is estimated by Theorem 2 for $C=\Phi(t/n)$. The second term is supposed to tend to zero by (32). Then the assertion follows from the part $(b)\Rightarrow(a)$ of Lemma 4.

□

Remark 4 *In fact, it is sufficient to have (a'), in order to obtain (32). On the other hand, it is sufficient to have (32) for t in a subset of $(0,+\infty)$ having a limit point, in order to obtain the convergence (a). This, as well as the fact that convergence (32) implies that e^{-tX_0} is a holomorphic contraction semigroup for $t\in S_{\pi/2-\alpha}$, follows from the Corollary 1.*

5 Applications.

5.1 Error estimate for the Euler approximation of semigroups

Let A be an m-sectorial operator with a semi-angle α and with vertex at 0, $0<\alpha<\pi/2$. Then the operators $F(t)=(I+tA)^{-1}$, $t\geq 0$ are sectorial contractions, i.e. $\Theta(F(t))\subseteq D_\alpha$ (cf Section 2.1). Let $X(s)=(I-F(s))/s$, $s>0$, and $X_0=A$. Then $X(s)$ converges when $s\to+0$ to X_0 in the uniform resolvent sense:

$$\|(\zeta+X(s))^{-1}-(\zeta+X_0)^{-1}\| = s\left\|\frac{A}{\zeta+A+\zeta sA}\cdot\frac{A}{\zeta+A}\right\| = O(s),\ \zeta\in S_{\pi-\alpha}, \tag{35}$$

since we have the estimate

$$\left\|\frac{A}{\zeta+A+\zeta sA}\cdot\frac{A}{\zeta+A}\right\| \leq \left(1+\frac{|\zeta|}{\operatorname{dist}\left(\zeta(1+s\zeta)^{-1},-S_\alpha\right)}\right)\left(1+\frac{|\zeta|}{\operatorname{dist}(\zeta,-S_\alpha)}\right). \tag{36}$$

Therefore, the family $\{F(t)\}_{t\geq 0}$ satisfies conditions of Theorem 3. This gives the operator-norm approximation of the exponential function (semigroup for m-sectorial generator) by powers of resolvent (Euler's approximation):

Theorem 4 *If A is an m-sectorial operator in a Hilbert space $\mathcal{H}$, with semi-angle $\alpha \in (0, \pi/2)$ and with vertex at 0, then*

$$\lim_{n\to\infty} \left\|(I + tA/n)^{-n} - e^{-tA}\right\| = 0, \ t \in S_{\pi/2-\alpha}. \tag{37}$$

Moreover, if 0 belongs to the resolvent set of A, then uniformly in $t \geq 0$ one has the error estimate:

$$\left\|(I + tA/n)^{-n} - e^{-tA}\right\| \leq O\left(\frac{\ln n}{n}\right). \tag{38}$$

Proof. The convergence (37) follows directly from Theorem 3. To obtain (38), we use the representation:

$$\begin{aligned}&(I + tA/n)^{-n} - e^{-tA} = \\ &\sum_{m=0}^{n-1}(I + tA/n)^{-(n-m-1)}\left((I + tA/n)^{-1} - e^{-tA/n}\right) e^{-mtA/n}.\end{aligned} \tag{39}$$

By Lemmata 1-3 of [1], we have the estimates $\|A^{-1}\left((I + tA/n)^{-1} - e^{-tA/n}\right)\| \leq 2t/n$ and $\|A^{-1}\left((I + tA/n)^{-1} - e^{-tA/n}\right)A^{-1}\| \leq 3/2(t/n)^2$. Since by Proposition 4 the operator A generates a holomorphic semigroup, a standard result (see e.g. [10, Ch. IX, §1.6]) says that $\|Ae^{-\tau A}\| \leq C_A/\tau$ for some constant $C_A > 0$ and any $\tau > 0$. Finally, for any integer $k \geq 1$ we have the identity:

$$(I + tA/n)^{-k}A = \frac{n}{t}\left(I - (I + tA/n)^{-1}\right)(I + tA/n)^{-k+1}, \tag{40}$$

which leads to the estimate $\|(I + tA/n)^{-k}A\| \leq nK/kt$ by Lemma 2 for the sectorial contraction $C = (I + tA/n)^{-1}$.

By these estimates and (39) one gets for $n \geq 3$:

$$\begin{aligned}\|(I + tA/n)^{-n} - e^{-tA}\| \ &\leq \ \frac{nK}{(n-1)t}(2t/n) + (2t/n)\frac{C_A n}{(n-1)t} \\ &+ \sum_{m=1}^{n-2}\frac{nK}{(n-m-1)t}\frac{3t^2}{2n^2}\frac{C_A n}{mt}.\end{aligned} \tag{41}$$

Therefore, uniformly in $t \geq 0$ we obtain:

$$\|(I + tA/n)^{-n} - e^{-tA}\| \leq \frac{4(K + C_A)}{n} + 6KC_A\frac{\ln n}{n}, \tag{42}$$

which proves (38).

□

5.2 Arithmetic mean approximation

Let $A_1, \ldots, A_k$ be m-sectorial operators in $\mathcal{H}$, such that $\Theta(A_j) \subseteq S_\alpha$ for some $\alpha \in (0, \pi/2)$ and for any $1 \leq j \leq k$. By $a_1, \ldots, a_k$ we denote the corresponding closed sectorial forms. Similar to the Trotter product formula, there are expressions of the type $F_k(t/n)^n$ that approximate the semigroup generated by the form-sum of $A_1, \ldots, A_k$. $F_k(t)$ can be taken as a product $f_1(tA_1)\ldots f_k(tA_k)$ or as an arithmetic mean $k^{-1}(f_1(ktA_1) + \ldots + f_k(ktA_k))$, where $\{f_l\}_{1\leq l\leq k}$ are so-called Kato-functions (see e.g. [11]-[13]). Section 2 shows that our condition on the numerical range for sectorial contractions (cf Definition 2) is easily satisfied if $F_k(t)$ is the arithmetic mean of resolvents or semigroups generated by $\{A_j\}_{1\leq j\leq k}$. Let us consider these families of contractions (cf [12]):

$$F_k^{res}(t) = \frac{1}{k}\sum_{l=1}^{k}(I + ktA_l)^{-1}, t \geq 0, \tag{43}$$

$$F_k^{sem}(t) = \frac{1}{k}\sum_{l=1}^{k} e^{-ktA_l}, t \geq 0. \tag{44}$$

Notice that $f(x) = (1+x)^{-1}$ or $f(x) = e^{-x}$ are Kato-functions. Since by conditions on $\{A_j\}_{1\leq j\leq k}$ the operators $\{(I + ktA_l)^{-1}\}_{1\leq j\leq k}$ or $\{e^{-ktA_l}\}_{1\leq j\leq k}$ are sectorial contractions with a semi-angle α (see Section 2), and since D_α is convex, $F_k^{res,sem}(t)$ are families of sectorial contractions with a semi-angle α. Moreover, it is shown [12] that the families $F_k^{res,sem}(t/n)^n$ converge strongly to $e^{-tX_0}P_0$ for any $t \geq 0$ and $k \geq 1$, as $n \to \infty$, where X_0 is the form-sum of $A_1, \ldots, A_k$ and P_0 is the orthogonal projector on the closure of $\bigcap_{l=1}^{k} \mathcal{D}(a_l)$. Therefore, by Theorem 3 we obtain a necessary and sufficient condition for the operator-norm convergence of the Trotter-type formula with arithmetic mean of resolvents (43) or semigroups (44):

Theorem 5 *If $A_1, \ldots, A_k$ are m-sectorial operators in $\mathcal{H}$ such that for $1 \leq j \leq k$, $\Theta(A_j) \subseteq S_\alpha$ for some $\alpha \in (0, \pi/2)$, then $X_k(t) = (I - F_k^{res,sem}(t))/t$ converges (as $t \to +0$) in the uniform resolvent sense to $X_0 = A_1 \dot{+} \ldots \dot{+} A_k$ defined in the closed subspace $\mathcal{H}_0 = \overline{\bigcap_{l=1}^{k} \mathcal{D}(a_l)}$, if and only if*

$$\lim_{n\to\infty} \left\| F_k^{res,sem}(t/n)^n - e^{-tX_0}P_0 \right\| = 0. \tag{45}$$

Here P_0 is the orthogonal projector on $\mathcal{H}_0$.

If $k = 1$, Theorem 5 reduces to Theorem 4, where the condition of convergence in the uniform resolvent sense is ensured by (35).

5.3 Trotter-Kato product formula

Let A be a non-negative self-adjoint operator and let B be m-sectorial with $\Theta(B) \subseteq S_\alpha$ for some angle $0 < \alpha < \pi/2$. Then our main Theorem 3 gives necessary and sufficient conditions for the operator-norm convergence of a generalized Trotter product formula.

Let us consider the family of operators

$$\Phi(s) = f(sA)^{1/2} g(sB) f(sA)^{1/2}, s \geq 0, \tag{46}$$

where $f(t)$ and $g(t)$ are the Kato-functions $(1+t)^{-1}$ or e^{-t}. Since $f(sA)^{1/2}$ is self-adjoint and $\|f(sA)^{1/2}\| \leq 1$, by results of Section 2 one has:

$$(u, \Phi(s)u) = (f(sA)^{1/2}u, g(sB)f(sA)^{1/2}u) \in D_\alpha, \tag{47}$$

$$\begin{aligned}(u, (I - \Phi(s))u) = (u, (I - f(sA))u) + \\ +(f(sA)^{1/2}u, (I - g(sB))f(sA)^{1/2}u) \in S_\alpha,\end{aligned} \tag{48}$$

for any $u \in \mathcal{H}$, $\|u\| = 1$, which proves that $\Theta(\Phi(s)) \subseteq D_\alpha$. Therefore, Theorem 3 gives necessary and sufficient conditions for the operator-norm convergence of $\Phi(t/n)^n$ as $n \to \infty$. At this point we still not identify the limit X_0 via operators A and B.

But the case is similar to that of [11, Addendum]. In [11], Kato proved that for $n \to \infty$ the generalized Trotter product formula $(f(tA/n)g(Bt/n))^n$ converges strongly to $e^{-tC}P_0$, where $C = A\dot{+}B$ is the form-sum defined in the closure $\mathcal{H}_0$ of $\mathcal{D} = \mathcal{D}(A^{1/2}) \cap \mathcal{D}(B^{1/2})$ and P_0 is the orthogonal projection onto $\mathcal{H}_0$, for any non-negative self-adjoint operators A and B. One can easily check that this result is also valid for the case of the symmetrized formula $f(tA)^{1/2}g(tB)f(tA)^{1/2}$. Moreover in the Addendum of [11], it is shown that for $f(t) = g(t) = e^{-t}$ the strong convergence holds also for m-sectorial generators A and B. Let a and b be the associated closed m-sectorial forms. Then $A\dot{+}B$ is defined in the closure $\mathcal{H}_0$ of $\mathcal{D}(a) \cap \mathcal{D}(b)$. By the same arguments one checks, that this result is valid for $f(tA)^{1/2}g(tB)f(tA)^{1/2}$, when $f(t)$, $g(t)$! are $(1+t)^{-1}$ or e^{-t}. Therefore, $\Phi(t/n)^n$ (46) converges strongly to $e^{-t(A\dot{+}B)}P_0$, where P_0 is the orthogonal projector onto $\mathcal{H}_0$, as $n \to \infty$. Taking into account these observations and our Theorem 3, we obtain the following

Theorem 6 *Let A be a non-negative self-adjoint operator, and let B be an m-sectorial operator in $\mathcal{H}$. Let the functions $f(t)$ and $g(t)$ be $(1+t)^{-1}$ or e^{-t}. For $s \geq 0$ we put $X(s) = s^{-1}(I - f(sA)^{1/2}g(sB)f(sA)^{1/2})$. Let $C = A\dot{+}B$ be the form-sum of A and B defined in the closure $\mathcal{H}_0$ of $\mathcal{D}(a) \cap \mathcal{D}(b)$, and let P_0 be the orthogonal projection onto $\mathcal{H}_0$. Then the following assertions are equivalent:*

(i) $X(s)$ converges in the uniform resolvent sense to C as $s \to +0$,

$$(ii)\ \lim_{n\to\infty} \left\| \left(f(tA/n)^{1/2}g(tB/n)f(tA/n)^{1/2}\right)^n - e^{-tC}P_0 \right\| = 0. \tag{49}$$

Notice that in contrast to Theorem 4, see (38), Theorem 6 gives no error bound estimate for the rate of convergence in (49). For self-adjoint semigroups the operator-norm convergence of the Trotter-Kato product formula without error bound estimate is established for a general case of Kato-function f and g, see [15]. In particular, there it is shown that compactness of $(I + A)^{-1}$ or $(I + A)^{-1}(I + B)^{-1}$ is a sufficient condition of this convergence for non-negative self-adjoint generators A and B. Recently [2] we generalized these results to m-sectorial A and B for the Trotter product formula, i.e. for $f(t) = g(t) = e^{-t}$.

Another strategy to prove the Trotter-Kato product formula for semigroups is based on operator-norm error-bound estimates. After pioneering papers [16], [8], [13] for couples of self-adjoint generators A and B, these results have been recently generalized to nonself-adjoint semigroups. In our paper [1] it is done in Banach and Hilbert spaces for holomorphic contraction semigroups under Ichinose-Tamura conditions on generators, cf [8]. Another result of this kind is established in [3]. Let A be a non-negative self-adjoint operator. If B is m-accretive, then sufficient conditions for the operator-norm convergence of the Trotter product formula with the error bound $O(\ln n/n)$ are the following: $\|Bu\| \leq a\|Au\|$ and $\|B^*u\| \leq a_*\|Au\|$ for $u \in \mathcal{D}(A)$, $a < 1$, $a_* < 1$. Notice that methods of [3] can be adapted without supplementary conditions to the generalized formula $f(tA/n)^{1/2}g(tB/n)f(tA/n)^{1/2}$ for !$f(t)$ and $g(t)$ be $(1+t)^{-1}$ or e^{-t}. Fractional powers conditions on self-adjoint generators ensuring the operator-norm convergence of the Trotter-Kato with error-bound estimates are discussed in [14]. For a generalization to m-sectorial generators see [2].

6 Conclusion

In the present paper we extend the Chernoff theory to approximation of semigroups in the operator-norm topology in a Hilbert space. The condition for this operator-norm convergence is that the operators $\{\Phi(s)\}_{s\geq 0}$ are not only contractions, but sectorial contractions: they have their numerical ranges $\Theta(\Phi(s))$ in a domain D_α, $0 \leq \alpha < \pi/2$, uniformly in $s \geq 0$, see (2). This allows us to find the $1/n^{1/3}$-estimate in operator-norm for an analogue of the Chernoff lemma (see Theorem 2). Notice that the original Chernoff's estimate in this lemma is not convergent: $\|(C^n - e^{n(C-I)})u\| \leq n^{1/2}\|(C-I)u\|$, for any vector u. The next step is the analogue of the Trotter-Neveu-Kato approximation theorem in the operator-norm, see Lemma 4. We consider uniformly m-sectorial families of generators, which leads to uniformly bounded holomorphic semigroups. The Corollary 1 shows that the convergence in! the uniform resolvent sense (i.e. in the generalized convergence in the sense of Kato [10, Ch. IV]) is well-adapted to approximate semigroups in the operator-norm topology. On the other hand, this topology is natural for holomorphic contraction semigroups: they are continuous in this topology for $t > 0$, and generators of these semigroups are exactly m-sectorial operators with semi-angles $\alpha < \pi/2$. A first consequence of our main Theorem 3 is that the Euler approximation formula for the exponential function converges in operator-norm for any m-sectorial generator (Theorem 4).

This operator-norm approximation theory leads to necessary and sufficient conditions for convergence of Trotter-type formulæ. We formulate these conditions for the formula with arithmetic mean of resolvents or semigroups (see Theorem 5) and for the symmetrized product formula $f(tA)^{1/2}g(tB)f(tA)^{1/2}$ where $f(t), g(t) = (1+t)^{-1}$ or e^{-t}, see Theorem 6. Our condition on the numerical range $\Theta(\Phi(s)) \subseteq D_\alpha$ (sectorial contraction) seems to be naturally related to the m-sectorial operators. In fact we have verified it for the resolvent $(I+tA)^{-1}$, for the semigroup e^{-tA} generated by an m-sectorial operator A (see Section 2), and for the family $\Phi(s) = f(sA)^{1/2}g(sB)f(sA)^{1/2}$, see (46)-(48).

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