PALINDROME COMPLEXITY BOUNDS FOR PRIMITIVE SUBSTITUTION SEQUENCES

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Abstract. We consider one-sided infinite words generated via iteration by primitive substitutions on finite alphabets and provide bounds on the palindrome complexity function as well as uniform bounds on the frequencies of palindromes in such words. As an application of these bounds, we prove that the strongly palindromic sequences in a primitive substitution dynamical system form a set of measure zero.

1. Introduction

In this paper we are concerned with a study of local reflectional symmetries in general infinite words generated by primitive substitutions. We shall study combinatorial issues, namely, the palindrome complexity function associated to such words, as well as dynamical issues, that is, the frequencies of palindromes occurring in these words. Before we present the results we shall prove, let us recall some basic notions.

Consider a finite set $\mathcal{A} = \{a_1, \dots , a_s\}$, called the *alphabet*. The elements of $\mathcal{A}^* = \cup_{n=0}^{\infty} \mathcal{A}^n$ are called (finite) *words over* $\mathcal{A}$. A word x has *length* n if it belongs to $\mathcal{A}^n$. We shall also consider one-sided (resp., two-sided) infinite words, that is, elements of $\mathcal{A}^{\mathbb{N}}$ (resp., $\mathcal{A}^{\mathbb{Z}}$). The set $\mathcal{A}^*$ naturally carries a free semigroup structure with respect to concatenation. Given a word $x \in \mathcal{A}^*$, $y \in \mathcal{A}^*$ is called a *factor* of x if there are words z_1, z_2 such that $x = z_1 y z_2$. Similarly, a word y is a factor of an infinite word $u \in \mathcal{A}^{\mathbb{N}}$ if there exist $z \in \mathcal{A}^*$ and $v \in \mathcal{A}^{\mathbb{N}}$ such that $u = zyv$. Given a word u (finite or infinite), we define $F_u = \{y : y$ is a factor of $u\}$ and, for $n \in \mathbb{N}$, $F_u(n) = F_u \cap \mathcal{A}^n$. The *complexity function* $f_u : \mathbb{N} \to \mathbb{N}_0$ corresponding to a word u is given by $f_u(n) = |F_u(n)|$, $|\cdot|$ denoting cardinality. A word $p \in \mathcal{A}^*$ is called a *palindrome* if it is the same when read backwards. Denote by $PAL_{\mathcal{A}}$ the set of all palindromes over $\mathcal{A}$ and define for a given word u, $P_u = F_u \cap PAL_{\mathcal{A}}$ and $P_u(n) = P_u \cap \mathcal{A}^n$. Thus, the set $P_u(n)$ consists of all palindromes which are a factor of u and have length n. The *palindrome complexity* *function* $p_u : \mathbb{N} \to \mathbb{N}_0$ corresponding to a word u is given by $p_u(n) = |P_u(n)|$. We refer the reader to the introductory articles [1, 3, 5, 6] for more information on issues related to combinatorial complexity.

A mapping $S : \mathcal{A} \to \mathcal{A}^*$ is called a *substitution*. Such a mapping S can be extended to $\mathcal{A}^*$ by $S(b_1 \dots b_k) = S(b_1) \dots S(b_k)$ and, similarly, to $\mathcal{A}^{\mathbb{N}}$ by $S(b_1 b_2 \dots) = S(b_1)S(b_2)\dots$, $b_i \in \mathcal{A}$. A substitution S is called *primitive* if there is some $k \in \mathbb{N}$ such that for every $a_i \in \mathcal{A}$, the word $S^k(a_i)$ contains all the elements of $\mathcal{A}$. Prominent examples of primitive substitutions are given by

$\mathcal{A} = \{a, b\}$,	$a \mapsto ab,\ b \mapsto a$	Fibonacci substitution,
$\mathcal{A} = \{a, b\}$,	$a \mapsto ab,\ b \mapsto aa$	Period doubling substitution,
$\mathcal{A} = \{a, b\}$,	$a \mapsto ab,\ b \mapsto aaa$	Binary non-Pisot substitution,
$\mathcal{A} = \{a, b\}$,	$a \mapsto ab,\ b \mapsto ba$	Thue-Morse substitution,
$\mathcal{A} = \{a, b, c, d\}$,	$a \mapsto ab,\ b \mapsto ac,\ c \mapsto db,\ d \mapsto dc$	Rudin-Shapiro substitution.

A fixed point $u \in \mathcal{A}^{\mathbb{N}}$ of a substitution S is called a *substitution sequence*. If S is primitive, a substitution sequence u always exists for a suitable power of S and the set of finite factors in a substitution sequence depends only on S, that is, if u_1, u_2 are fixed points of S, then $F_{u_1} = F_{u_2}$. Moreover, in the primitive case the a priori exponential upper bound on f_u can be improved considerably. In fact, a linear upper bound can be proven [16]. Now, the palindrome complexity function, which also has an exponential a priori upper bound, can therefore grow at most linearly since, obviously, $p_u(n) \le f_u(n)$ for every $n \in \mathbb{N}$. This can be improved as the following theorem shows.

Theorem 1.1. *Let S be a primitive substitution. Then there exists $C < \infty$ such that for every substitution sequence u associated to S, $p_u(n) \le C$ for every $n \in \mathbb{N}$.*

Remarks. 1. The palindrome complexity function has been investigated for the Fibonacci case [11, 12], the period doubling case [9], and the Rudin-Shapiro case [2] (see also [4]). These works provide explicit descriptions of the sets $P_u(n)$, $n \in \mathbb{N}$, in the respective cases.
2. The palindrome complexity function implicitly plays an important role in a paper by Hof et al. [13]. They study the eigenvalue problem of one-dimensional Schrödinger operators with potentials generated by primitive substitutions. Their criterion for absence of eigenvalues is applicable if and only if the palindrome complexity function does not eventually vanish and gives generic absence of eigenvalues.

Given a substitution sequence u of a primitive substitution S, we can consider the associated substitution dynamical system (Ω, T), where

$$\Omega = \{\omega \in \mathcal{A}^{\mathbb{Z}} \ : \ F_\omega = F_u\}$$

and

$$T : \Omega \to \Omega,\ (T\omega)_n = \omega_{n+1}.$$

If $\mathcal{A}$ is equipped with discrete topology and $\mathcal{A}^{\mathbb{Z}}$ is equipped with product topology, the topological dynamical system (Ω, T) is strictly ergodic, that is, the orbit of every $\omega \in \Omega$ is dense in Ω and there exists a unique T-invariant Borel measure μ which is necessarily ergodic [16]. In particular, all frequencies of factors in u exist uniformly and are strictly positive, that is, for every $v \in F_u$ there exists $d_u(v) > 0$

such that for every $k \in \mathbb{N}$, we have

$$\frac{\#_v(u_k \dots u_{k+n-1})}{n} \to d_u(v)$$

as $n \to \infty$, where for $x, y \in \mathcal{A}^*$, $\#_x(y)$ denotes the number of occurrences of x in y, and the convergence is uniform in k. The measure of a cylinder set is given by the frequency of the defining word, that is, for each $v \in F_u$ and $k \in \mathbb{N}$, we have

$$\mu(\{\omega \in \Omega : \omega_k \dots \omega_{k+|v|-1} = v\}) = d_u(v). \tag{1}$$

For palindromes in u, the following uniform bounds can be obtained for their frequencies.

Theorem 1.2. *Let S be a primitive substitution. Then there exist $0 < D, E < \infty$ such that for every aperiodic substitution sequence u associated to S, for every $n \in \mathbb{N}$, and for every $p \in P_n(u)$, we have*

$$\frac{D}{n} \le d_u(p) \le \frac{E}{n}. \tag{2}$$

Given a substitution dynamical system and a function $g : \mathcal{A} \to \mathbb{R}$, one can define an ergodic family of Schrödinger operators $(H_\omega)_{\omega \in \Omega}$ in $l^2(\mathbb{Z})$ by

$$(H_\omega \phi)(n) = \phi(n+1) + \phi(n-1) + V_\omega(n)\phi(n),$$

where $V_\omega(n) = g(\omega_n)$. A study of palindromic structures in substitution sequences is now connected to the eigenvalue problem for these operators as follows. Let

$$\Omega_c = \{\omega \in \Omega : \sigma_{\mathrm{pp}}(H_\omega) = \emptyset\},$$

that is, the bi-infinite sequences $\omega \in \Omega$ whose associated operators have no eigenvalues.

A two-sided sequence is called *strongly palindromic* if it contains palindromes p_n having length l_n and centers c_n such that $c_n \to \infty$ and

$$\frac{\exp(Bc_n)}{l_n} \to 0$$

for a certain constant $B \ge 1$ (which is determined by g). Define

$$\Omega_{\mathrm{sp}} = \{\omega \in \Omega : \omega \text{ is strongly palindromic}\}.$$

Then, Hof et al. have shown that

$$\Omega_{\mathrm{sp}} \subseteq \Omega_c.$$

Moreover, if the palindrome complexity function does not eventually vanish, then Ω_{sp} is not empty and, in fact, Ω_{sp} is uncountable. Using [17], it can even be shown that Ω_c is then a dense G_δ in Ω.

The set Ω_{sp} is invariant and hence has measure either 0 or 1. With the bounds from the above theorems we are able to show that the measure is always zero.

Theorem 1.3. *Let S be a primitive substitution which admits an aperiodic substitution sequence and Ω the induced substitution dynamical system. Then, $\mu(\Omega_{\mathrm{sp}}) = 0$.*

Remark. This result should be contrasted with the following. By different methods it has been shown that $\mu(\Omega_c) = 1$ for a class of substitutions including period doubling and binary non-Pisot [8]; see also [14] (resp., [7]) for a direct proof in the

Fibonacci (resp., period doubling) case. In the Fibonacci case it can even be shown that $\Omega_c = \Omega$ [10].

The organization of this article is as follows. In Section 2 we prove a rather useful tool which concerns the repetitivity of the finite factors in primitive substitution sequences. We use this lemma in Section 3 to relate powers and palindromes in such sequences and to prove Theorems 1.1 and 1.2. We show in Section 4 how the bounds from these two theorems imply Theorem 1.3. Finally, Section 5 contains some concluding remarks.

2. The window lemma

This section is concerned with a proof of what could be called the window lemma, a result saying that given a primitive substitution sequence u, one can find all subwords of a given length n by inspecting an arbitrary subword of length Cn, where the constant C only depends on the substitution. That is, one may put a window of length Cn on some arbitrary portion of u and be sure that one "sees" all the words from $F_u(n)$. In fact, this property has been termed *linearly repetitive* by Lagarias and Pleasants in [15]. They show that strict ergodicity of the associated dynamical system (Ω, T) follows from this property. The result presented in the lemma below seems to be known. However, since we could not find a suitable reference, we include a proof for the reader's convenience.

Lemma 2.1. *Let S be a primitive substitution. Then there exists $C_S < \infty$ such that for every one-sided infinite fixed point u of S and every $n \in \mathbb{N}$, every subword of u of length n is a factor of every subword of u of length $C_S n$.*

Remark. This recovers the linear bound on $f_u(n)$. In fact, our proof will refine the argument from the proof of this bound given in [16].

Before giving the proof of Lemma 2.1, let us recall some standard facts which can all be found in [16]. Given a substitution S on some alphabet $\mathcal{A}$ having s symbols, we can associate an $s \times s$-matrix $M = (m_{ij})$, the *substitution matrix*, by $m_{ij} =$ number of occurrences of a_i in $S(a_j)$. Primitivity of S is then equivalent to primitivity of M (i.e., existence of $k \in \mathbb{N}$ such that all entries of M^k are strictly positive). In the primitive case, the Perron-Frobenius theorem ensures the existence of a dominant eigenvalue θ of M which can be regarded as a uniform asymptotic blow-up factor. That is, for each $a \in \mathcal{A}$, there is $c_a \in (0, \infty)$ such that

$$\frac{|S^k(a)|}{\theta^k} \to c_a.$$

Define

$$\mathrm{lmax}(k) = \max_{a \in \mathcal{A}} |S^k(a)|, \ \mathrm{lmin}(k) = \min_{a \in \mathcal{A}} |S^k(a)|,$$

that is, the maximum and minimum lengths of k-fold iterations of the substitution S upon letters. Then there exist $z_1, z_2 \in (0, \infty)$ such that for all $k \in \mathbb{N}$, we have

$$z_1 \theta^k \leq \mathrm{lmin}(k) \leq \mathrm{lmax}(k) \leq z_2 \theta^k.$$

Another concept we shall need is the following. The associated substitutions S_l treat $\mathcal{A}_l = F_u(l)$ as a new alphabet and map from $\mathcal{A}_l$ to $\mathcal{A}_l^*$. Given some $b_1 \dots b_l \in \mathcal{A}_l$, $S_l(b_1 \dots b_l)$ is given by

$$S_l(b_1 \dots b_l) = (x_1 \dots x_l)(x_2 \dots x_{l+1}) \dots (x_{|S(b_1)|} \dots x_{|S(b_1)|+l-1}),$$

where the x_i are defined by

$$S(b_1 \dots b_l) = x_1 \dots x_{|S(b_1)|} x_{|S(b_1)|+1} \dots x_{|S(b_1 \dots b_l)|}.$$

Then, it is easy to check that $|S_l(b_1 \dots b_l)| = |S(b_1)|$ and that S_l admits the sequence

$$U_l = (u_1 \dots u_l)(u_2 \dots u_{l+1})(u_3 \dots u_{l+2}) \dots$$

as a fixed point. Moreover, u satisfies the window property with window length m for words of length l if and only if U_l satisfies the window property with window length $m - l + 1$ for words of length 1.

Proof of Lemma 2.1. The proof consists of two steps. One treats the case $n = 2$ first and then handles the general case. Consider the substitution S_2. By primitivity of S_2, every window of length $C_1 = 2\mathrm{lmax}(k)$ on U_2, k being the primitivity constant of S_2, contains all the symbols from $\mathcal{A}_2$ and thus all the words in $F_u(2)$. By the above remarks, every window of length $C_2 = C_1 + 1$ on u contains all the words from $F_u(2)$.

Now let $n \geq 3$ be arbitrary. We claim that with $C_S = \frac{(C_2+2)z_2\theta}{z_1}$, the assertion of the lemma holds. So let $i \in \mathbb{N}$ be arbitrary and consider the window $w = u_i \dots u_{i+C_S n-1}$. We have to show that this window contains all the words from $F_u(n)$ as factors. By $S(u) = u$ we have $u_1u_2u_3 \dots = S^p(u_1)S^p(u_2)S^p(u_3) \dots$ for each $p \geq 1$, and we choose p to be that value for which

$$\mathrm{lmin}(p-1) < n \leq \mathrm{lmin}(p). \tag{3}$$

Consider now the partition of u into blocks of the type $S^p(b)$, $b \in \mathcal{A}$. Let $S^p(u_r) \dots S^p(u_{r+j-1})$ be the blocks that are fully contained in this window. We can now infer the following:

$$\begin{aligned} j &\geq \frac{C_S n - 2\mathrm{lmax}(p)}{\mathrm{lmax}(p)} \\ &\geq \frac{\frac{(C_2+2)z_2\theta}{z_1} n}{z_2\theta^p} - 2 \\ &\geq C_2 + 2 - 2. \end{aligned}$$

Thus, $u_r \dots u_{r+j-1}$ contains all the words from $F_u(2)$. The claim now follows from (3) since any word in $F_u(n)$ is contained in some $S^p(v)$ with a suitably chosen $v \in F_u(2)$. □

3. Powers and palindromes

In this section we use the window lemma to relate powers and palindromes occurring in substitution sequences and to prove Theorems 1.1 and 1.2. Given a word u (finite or infinite), denote by $\pi(u) \in \mathbb{N} \cup \{\infty\}$ the largest power occurring in u, that is,

$$\pi(u) = \sup\{k \in \mathbb{N} : v^k \in F_u, v \in \mathcal{A}^*\}.$$

If u is a substitution sequence associated to a primitive substitution, then either $\pi(u)$ is bounded by C_S or u is ultimately periodic as the following lemma shows.

Lemma 3.1. *Let S be a primitive substitution. Suppose u is an aperiodic substitution sequence associated to S. Then $\pi(u) < C_S$.*

Proof. It is well known that if u is aperiodic, then $f_u(n) \geq n+1$ for every $n \in \mathbb{N}$; see, for example, [16]. The claim now follows from Lemma 2.1 as follows. Suppose $\pi(u) \geq C_S$. Then there exists $v \in \mathcal{A}^*$ such that $v^{C_S} \in F_u$. Let $m = |v|$. Now $f_{v^{C_S}}(m) \leq m$ and hence $f_u(m) \leq m$, a contradiction. □

The next lemma deduces the existence of powers from overlapping palindromes in u. Together with the preceding lemma this allows us to bound the distance between consecutive palindromes from below. Given an occurrence of a palindrome $p = u_{k_1} \dots u_{k_2} \in P_u$ in u, we say that p is *centered* at $c(p)$, where $c(p) = \lfloor \frac{k_2+k_1}{2} \rfloor$. This is slightly unnatural for palindromes of even length, but will prevent us from having to treat the even and odd length palindromes separately.

Lemma 3.2. *Let u be a word and suppose $p_1, p_2 \in \mathcal{A}^{2n}$ (resp., $\mathcal{A}^{2n+1}$) are palindromes occurring in u centered at the sites $c(p_1), c(p_2)$, respectively. Let $d = |c(p_2) - c(p_1)|$. Then u contains a k-th power, where $k = \lfloor \frac{n}{2d} \rfloor$.*

Proof. Suppose that the occurrences of p_1, p_2 overlap, since otherwise there is nothing to prove. We can reflect at the respective centers of the palindromes to obtain relations between the symbols in the part of u where the palindromes overlap. In particular this yields a power of a word of length $2d$. The power that can be deduced before hitting the boundary is just $\lfloor \frac{n}{2d} \rfloor$. □

Proof of Theorem 1.1. Fix u and n. It is sufficient to consider the case of aperiodic u since any ultimately periodic sequence has bounded complexity function. Choose any window in u of length $C_S n$. All words of length n in u, and in particular, all palindromes from $P_u(n)$ are contained in this window. By the last two lemmas, the centers of these palindromes are separated by at least $\frac{n}{2(\pi(u)+1)}$. This yields the bound

$$p_u(n) \leq \frac{C_S n}{\left(\frac{n}{2(\pi(u)+1)}\right)} \leq 2C_S(C_S+1)$$

from which the claim follows. □

Proof of Theorem 1.2. The lower bound is immediate from the window lemma and the upper bound can be proved by an argument similar to the above, namely, that there is a linear lower bound with uniform constant for the distance between two consecutive occurrences of a given palindrome of length n in u. □

4. Strongly palindromic sequences

In this section we provide a proof of Theorem 1.3, that is, we show that for aperiodic primitive substitution dynamical systems, the set of strongly palindromic sequences has always measure zero. Given a subshift (Ω, T) which derives from a primitive substitution S, we define for $n \in \mathbb{N}$ the following set which is easily seen to be a Borel set,

$$\Omega_n = \{\omega \in \Omega : \omega \text{ contains a palindrome of length } \geq \exp(n) \text{ centered at } n\}.$$

Lemma 4.1. *With the above definition, we have*

$$\Omega_{\rm sp} \subseteq \limsup_{n\to\infty} \Omega_n, \tag{4}$$

that is, every $\omega \in \Omega_{\rm sp}$ is contained in infinitely many of the Ω_n.

Proof. Given $\omega \in \Omega_{\mathrm{sp}}$, we can find in ω palindromes p_n of length l_n centered at $c_n \to \infty$ such that for a certain $B \geq 1$, $\exp(Bc_n)/l_n \to 0$ as $n \to \infty$. Thus for a suitable $n_0 \in \mathbb{N}$, we have $\exp(c_n) \leq l_n$ for all $n \geq n_0$. In particular, $\omega \in \Omega_{c_n}$ for all $n \geq n_0$. $\square$

Lemma 4.2. *There exists a constant $F < \infty$ such that for every $n \in \mathbb{N}$,*

$$\mu(\Omega_n) \leq \frac{F}{\exp(n)}. \tag{5}$$

Proof. Obviously, if $\omega \in \Omega$ contains a palindrome of length $l \geq 3$ centered at c, then it contains a palindrome p' of length $l-2$ centered at c. Thus, we have for every $n \in \mathbb{N}$,

$$\begin{aligned}\Omega_n = \{&\omega \in \Omega : \omega \text{ contains a palindrome of length } \lceil \exp(n) \rceil \text{ centered at } n\} \cup \\ \{&\omega \in \Omega : \omega \text{ contains a palindrome of length } \lceil \exp(n) \rceil + 1 \text{ centered at } n\}.\end{aligned}$$

With the constants C, E from Theorems 1.1 and 1.2 along with equation (1), we obtain the bound

$$\mu(\Omega_n) \leq \frac{2CE}{\exp(n)}$$

and hence the assertion. $\square$

Proof of Theorem 1.3. The claim follows from equations (4) and (5) together with the standard Borel-Cantelli lemma. $\square$

5. Concluding remarks

We have considered palindromes in infinite words generated by substitution sequences. By exhibiting linear repetitivity, that is, the window lemma property, we were able to prove bounds on the number of palindromes of given length as well as bounds on their frequency. These bounds in turn implied that the set of strongly palindromic sequences in primitive substitution dynamical systems always has measure zero with respect to the unique ergodic measure. In this section we want to make a few remarks on some more or less obvious extensions of the above theorems.

First, the bound (2) in Theorem 1.2 holds for all words in $F_u(n)$, not only for words in $P_u(n)$. The lower bound was in fact proven for all words since it directly follows from the window lemma. The upper bound can be proven by a method similar to the proof of Lemma 3.2. Namely, the distance between two consecutive occurrences of some word $w \in F_u(n)$ has a linear lower bound with a uniform constant since an overlap also implies the occurrence of a power. This together with Lemma 3.1 yields the claimed lower bound and thus the upper bound on the frequency.

Second, note that the proofs only employ the window lemma property. Thus all the results immediately extend to infinite words which are linearly repetitive. These include, for example, Sturmian words $s_{\alpha,\beta} = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor$, $\alpha \in (0,1)$ irrational and $\beta \in (0,1)$ arbitrary, in the case where the coefficients a_n in the continued fraction expansion of α are bounded.

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