NO OR INFINITELY MANY A.C.I.P. FOR PIECEWISE EXPANDING C^r MAPS IN HIGHER DIMENSIONS

Jérôme Buzzi

Centre de Mathématique de l'Ecole Polytechnique / C.N.R.S.

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Abstract. We modify Tsujii's example [8] to show that in contrast with the one-dimensional case, piecewise uniformly expanding and C^r maps of the plane may:

(1) either have **no** absolutely continuous invariant probability measures (a.c.i.p. for short) and be such that **every point** is statistically attracted to a fixed repelling point;
(2) or have **infinitely many ergodic a.c.i.p.**

Introduction

A piecewise expanding C^2 map on the interval is a map $F : [0,1] \to [0,1]$ such that $[0,1]$ is the union of finitely many intervals on each of which F is C^2 and satisfies $\inf |F'| > 1$. It is well-known (Lasota-Yorke [4]) that such a map admits a **finite, non-zero number of a.c.i.p.** (ergodic absolutely continuous invariant probability measures) $\mu_1, \dots, \mu_r$, such that for Lebesgue almost every point $x \in [0,1]$,

$$\frac{1}{n} \sum_{k=0}^{n} \delta_{F^k(x)} \to \mu_i$$

in the weak star topology, for some i depending on x (one says that x is statistically attracted to μ_i).

On the interval, very little smoothness is required (say C^1 plus summability of the modulus of continuity of F' (see [6,1])). In dimension 2, the corresponding result [2,7] requires **real-analyticity**.

M. Tsujii [8] proved that, at any rate, finite differentiability is not enough: for arbitrary $r < \infty$, there exist piecewise C^r and expanding maps of the plane such that for a set of points x with positive measure, $\frac{1}{n} \sum_{k=0}^{n} \delta_{F^k(x)} \to \delta_p$ as $n \to \infty$, for some $p \in X$ independent of x. Thus there is a large set of points not statistically attracted to an a.c.i.p., in contrast with the Lasota-Yorke situtation recalled above.

Question: Can one arrange this example so as to have no a.c.i.p.? infinitely many a.c.i.p.?

In this note, we answer this question positively.

Results.

Let us recall that a piecewise C^r map is $(X, \mathcal{Z}, F)$ such that:

(1) X is a compact subset of $\mathbb{R}^d$,
(2) $\mathcal{Z}$ is a **finite** collection of pairwise disjoint open subsets of X, the union of which is dense in X. Moreover the subsets in $\mathcal{Z}$ have C^r **boundaries**, i.e., the boundaries are finite unions of compact subsets of open C^r submanifolds,
(3) F restricted to each subset in $\mathcal{Z}$ extends to a C^r diffeomorphism between a neighborhood of the closure of the subset and some open set in $\mathbb{R}^d$.

F is expanding if $\|F'(x).v\| \geq \lambda\|v\|$ for all $x \in \bigcup_{Z\in\mathcal{Z}} Z$, $v \in \mathbb{R}^d$ and some constant $\lambda > 1$.

Theorem.

1. For any $r < \infty$, there exists a piecewise C^r expanding map of the square with no a.c.i.p.

2. For any $r < \infty$, there exists a piecewise C^r expanding map of the square with infinitely many a.c.i.p.

Remark. It was already known [3] that one can have any (finite, non-zero) number of ergodic a.c.i.p. for maps with 10 pieces (i.e., elements of the partition $\mathcal{Z}$).

Tsujii's Counter-Example

We recall the basic structure of Tsujii's construction for the convenience of the reader.

Choose the differentiability $r < \infty$. It fixes $\gamma := r + 1$, $\epsilon > 0$ small enough and M large enough.

The map F is defined on $[0, 1] \times [-1, 1]$. It is made of 8 nearly rectangular pieces (see Figure 1), only three of which being important (they are labelled A, B and C on the figure). Observe the role of the functions f (defining the discontinuity line near $y = 0$) and η (defining the downward push in piece B).

The key property of F is the existence of a countable collection of disjoint rectangles $R(k, m)$, for $m > M$ and $0 \leq k < \gamma^m$. Each rectangle is mapped into the next one: $F(R(k, m)) = R(k+1, m)$ if $k < \gamma^m - 1$, and $F(R(\gamma^m - 1, m)) = R(0, m + 1)$.

The rectangles are organized into lines $R(0, m), R(1, m), \ldots, R(\gamma^m - 1, m)$ and into columns: $R(k, m) \subset J_{\gamma^m - k} \times [0, 1]$ with $J_\ell = [e^{-\epsilon\ell}, e^{-\epsilon\ell} + (e^{-\epsilon(\ell-1)} - e^{-\epsilon\ell})/6]$.

To each column $J_\ell \times [0, 1]$ with $\ell > M$, corresponds a "bump" in the functions f and η. These functions are constant on J_ℓ with values such that the discontinuity line cuts in the middle the highest rectangle in the column[1] (i.e., f on J_ℓ is the y-coordinate of the middle of the rectangle) and F maps the higher half of the rectangle to the same place as the lower half (i.e., η on J_ℓ is half the height of the rectangle).

[1] the highest rectangle is $R(\gamma^{m(\ell)} - \ell, m(\ell))$ with $m(\ell)$ minimum with $\gamma^m > \ell$.

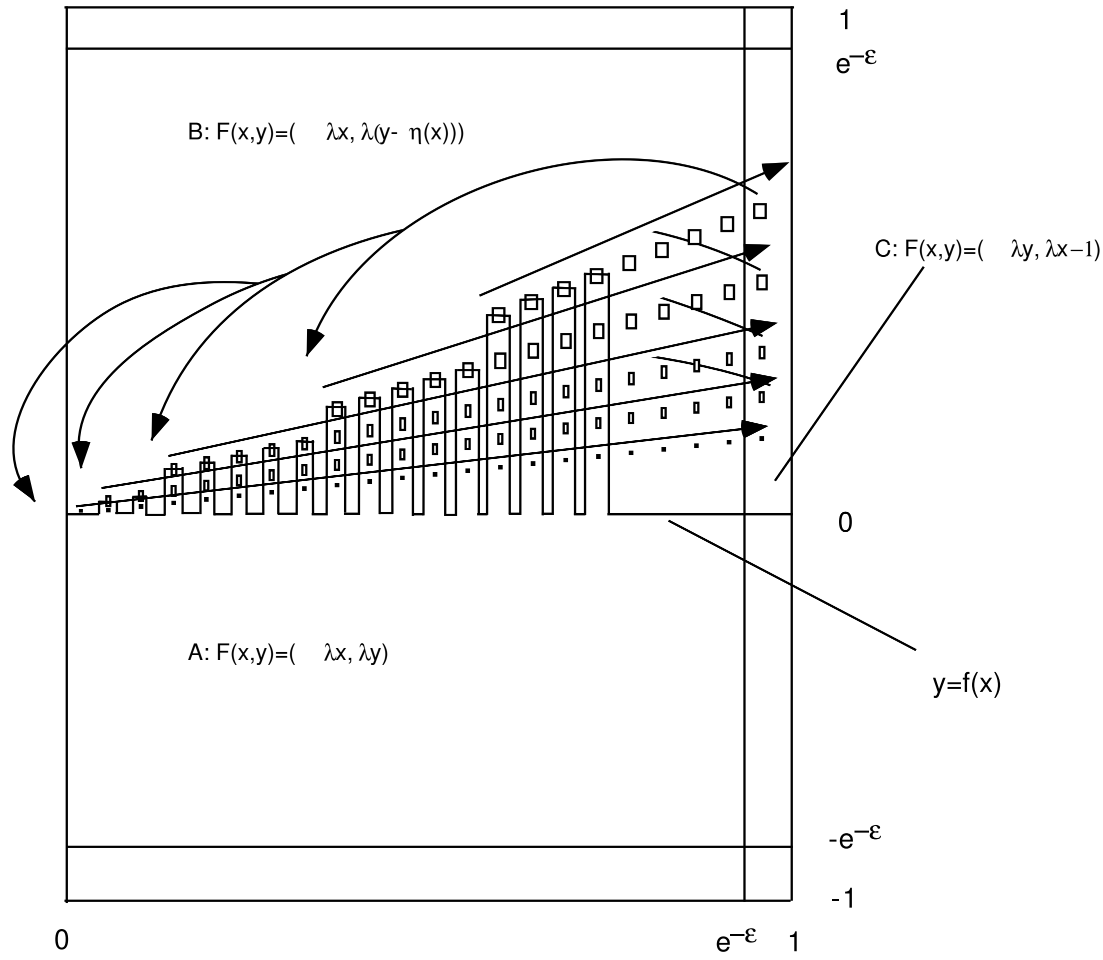

Figure 1.

Remarks: 1. $\lambda = e^{\epsilon}$. 2. The smooth bumps in the discontinuity lines are drawn as rectangles.

No a.c.i.p.

We start with Tsujii's map $F : [0,1] \times [-1,1] \to [0,1] \times [-1,1]$ and modify it by carving some new pieces. The idea is to capture in these new pieces every orbit which would not eventually enter some rectangle $R(k,m)$ under the iteration of F so that the modification of F can send it into some rectangle. To do this we are going to specify a subset $E \subset [0,1] \times [-1,1]$ with C^r **boundary** such that every F-orbit which does not eventually enter some rectangle eventually enters E. This will solve our problem: we shall define F' on $[0,1] \times [-1,1]$ to coincide with F outside of E and to be a piecewise affine and expanding map on E with values inside $R(0, M+1)$. Thus, **every** orbit of F' will enter the rectangles and therefore be statistically attracted to $p = (0,0)$, excluding the existence of any a.c.i.p. (or any invariant probability measure indeed).

Let $E_0 = [0,1] \times [-1,1] \setminus ([0, e^{-\epsilon}] \times [0, e^{-\epsilon}])$.

Consider the pre-images in $[e^{-\epsilon},1]\times[0,e^{-\epsilon}]$ of the higher rectangles. Make these rectangles into a strip S, as pictured in figure 2 taking vertically some margin (i.e., enlarge the rectangles by replacing the horizontal side of $R(\gamma^{m(\ell)}-\ell,m(\ell))$ by J_ℓ). Each of the two lateral boundaries of the strip can be assumed to be a C^r curve: it has essentially the same shape as the discontinuity line $y=f(x)$.

Define E to be E_0 minus the strip S. The boundary of E is C^r. The strip containing all the rectangles in E_0 (i.e., $R(\gamma^m-1,m)$, $m>M$), E does not meet any rectangle. It remains to see that any orbit of F eventually enters E.

First, observe that on the complement of E_0, F multiplies the y-coordinate by e^ϵ. Thus, any point eventually enters E_0.

Consider a point, which never enters the rectangles. We have just seen that it enters E_0. If it does not enter E, this means that it enters the strip S. At this time, the point must be in the strip between two higher rectangles.

If it is far from either rectangles, in the sense that it does not belong to a column of the form $J_\ell\times[0,1]$, then it will eventually be mapped into $[e^{-\epsilon},1]\times[-1,1]\setminus J_1\times[-1,1]$. But this is a subset of E.

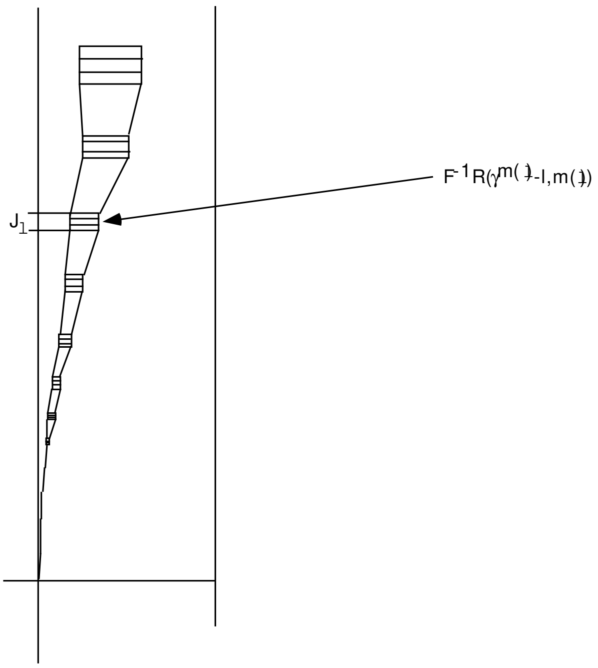

Figure 2.

Otherwise the point is near the highest rectangle $R(\gamma^m-\ell,m)$ in the column $J_\ell\times[0,1]$. It is then easy to see that, under iteration of F, the point will remain on the side of the rectangle. Hence, when it will come back to column $J_1\times[0,1]$, it will be outside the strip S and therefore in E.

We have seen that all points not entering the rectangles are eventually caught in E. This concludes the construction of the map with no a.c.i.p. $\square$

Infinitely many a.c.i.p.

We start againg from Tsujii's map $F:[0,1]\times[-1,1]\to[0,1]\times[-1,1]$. This time we modify the cutting (i.e., functions f and η) so that, instead of having a single sequence of rectangles, one mapped into the next, we shall have infinitely many cycles of rectangles, each supporting an a.c.i.p.

Fix $\gamma = 2r$, $0 < \epsilon < (1-\gamma^{-1})\log 2/(1+\gamma)$. Set M so large that $\epsilon\gamma^M > 1000$.

For each even integer $m \geq M$, we shall have the following cycle: the rectangles $R(0,m), R(1,m), \ldots, R(\gamma^m - 1, m), R(0, m+1), \ldots, R(\gamma^{m+1} - 1, m+1)$ are each mapped into the next one (the last one being mapped into the first one).

This is represented in Figure 3, together with the necessary cutting. The rectangles $R(k,m)$ for $0 \leq k < \gamma^m - 2\gamma^{m-2}$ and the rectangles $R(k, m+1)$ for $\gamma^{m+1} - 2\gamma^m \leq k < \gamma^{m+1} - \gamma^m$ are cut in the middle by a horizontal segment which is part of the discontinuity line $y = f(x)$. What is above the discontinuity line is pushed down by half the height of the rectangle. In this way both upper and lower halves of the rectangle have the same image. Observe that in each column $J_\ell \times [0,1]$ we do this at most once.

We realize this cutting in the same way as Tsujii by making bumps in the functions f and η. To ensure that we get globally C^r functions it is enough to see that the height h and the width w of these bumps satisfy: $h/w^r \to 0$ as $m \to \infty$.

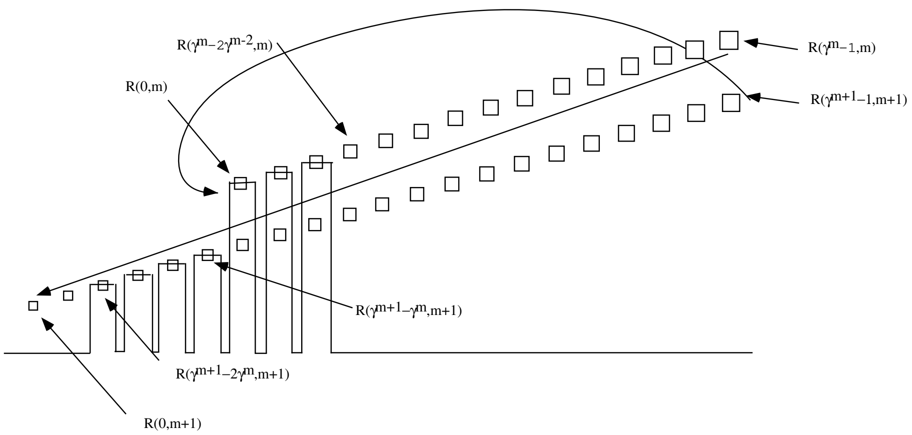

Figure 3.

Remark: The smooth bumps are represented as large, vertical rectangles.

We compute the position of the bottom left corner (x,y) of $R(0,m)$ by writing that it comes back to itself under iteration of F:

$$x = e^{\epsilon\gamma^m + \epsilon\gamma^{m+1}} x - e^{\epsilon\gamma^{m+1}} \text{ and } y = e^{\epsilon\gamma^m + \epsilon\gamma^{m+1}} y - 1.$$

Thus:

$$x = \frac{1}{e^{\epsilon\gamma^m} - e^{-\epsilon\gamma^{m+1}}} \text{ and } y = \frac{1}{e^{\epsilon\gamma^m + \epsilon\gamma^{m+1}} - 1}.$$

We define the height and width of $R(0,m)$ to be $e^{-\epsilon\gamma^{m+10}}$ (any small number would do). One checks easily that the rectangles so defined remain inside the columns $J_\ell \times [0,1]$.

Observe that $F^{\gamma^m + \gamma^{m+1}}(R(0,m)) \subset R(0,m)$ because the height of $R(0,m)$ is multiplied by:

$$e^{\epsilon(\gamma^m + \gamma^{m+1})}/2^{\gamma^m - \gamma^{m-1}} = e^{\gamma^m(\epsilon(1+\gamma) - (1-1/\gamma)\log 2)} < 1,$$

and its width by:

$$e^{\epsilon(\gamma^m+\gamma^{m+1})}/2^{\gamma^m} = e^{\gamma^m(\epsilon(1+\gamma)-\log 2)} < 1.$$

Finally we check the condition for C^r smoothness. Observe that the bumps associated to rectangles $R(k,m)$, resp. $R(k,m+1)$, are magnified versions of the one associated to $R(0,m)$, resp. $R(\gamma^{m+1}-2\gamma^m, m+1)$. Hence it is enough to check the condition for these two rectangles.

The width of the bump associated to $R(0,m) \subset J_m \times [0,1]$ is about $e^{-\epsilon\gamma^m}$, its height is y which is approximately $e^{-\epsilon\gamma^m-\epsilon\gamma^{m+1}}$, the relevant quotient is therefore $e^{-\epsilon\gamma^m(\gamma+1-r)}$. As $\gamma > r-1$, this goes to zero and it is ok.

For the second bump, the relevant quotient is similarly $(e^{\epsilon\gamma^m}x - 1)/e^{-2\epsilon\gamma^m\cdot r}$ which is equivalent to $e^{-\epsilon\gamma^m(\gamma+1-2r)}$. This is also ok as $\gamma > 2r-1$.

$H := F^{\gamma^m+\gamma^{m+1}} : R(0,m) \to R(0,m)$ is a piecewise expanding and affine map preserving the two axis directions together and made of rectangular pieces with sides aligned with the axis. It is well-known that such a map has an a.c.i.p. (see, for instance [5], using that any iterate is piecewise affine on rectangles aligned with the axis and therefore at most 4 rectangles can touch a given point: $Y(H^n) \leq 4$ in the language of that paper).

We therefore have built a piecewise expanding and C^r map with infinitely many a.c.i.p. $\square$

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CMAT / Ecole Polytechnique / 91128 Palaiseau cedex / France

E-mail address: buzzi@math.polytechnique.fr