Approximate renormalization for the breakup of invariant tori with three frequencies

C. Chandre[1] and R.S. MacKay[2]
[1]*Service de Physique Théorique, CEA Saclay, F-91191 Gif-sur-Yvette Cedex, France*
[2]*Mathematics Institute, University of Warwick, Coventry CV4 7AL, U.K.*

We construct an approximate renormalization transformation for Hamiltonian systems with three degrees of freedom in order to study the break-up of invariant tori with three incommensurate frequencies which belong to the cubic field $Q(\tau)$, where $\tau^3+\tau^2-2\tau-1=0$. This renormalization has two fixed points : a stable one and a hyperbolic one with a codimension one stable manifold. We compute the associated critical exponents that characterize the universality class for the break-up of the invariant tori we consider.

I. INTRODUCTION

In this paper, we define an approximate renormalization scheme for Hamiltonians with three degrees of freedom in order to study the break-up of invariant tori with the frequency vector $\boldsymbol{\omega}_0=(\tau^2+\tau,\tau,1)$ where $\tau=2\cos(2\pi/7)\approx 1.2469796$ is the root of modulus larger than one of the polynomial :

$$\tau^3+\tau^2-2\tau-1=0.$$

This approximate renormalization is defined for the following family of Hamiltonians which are quadratic in the actions $\boldsymbol{A}=(A_1,A_2,A_3)$:

$$H(\boldsymbol{A},\boldsymbol{\varphi})=H_0(\boldsymbol{A})+V(\boldsymbol{A},\boldsymbol{\varphi}), \tag{1.1}$$

where the Hamiltonian H_0 is quadratic :

$$H_0(\boldsymbol{A})=\boldsymbol{\omega}_0\cdot\boldsymbol{A}+\frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2, \tag{1.2}$$

and the perturbation is described by two scalar functions of the angles $\boldsymbol{\varphi}=(\varphi_1,\varphi_2,\varphi_3)$:

$$V(\boldsymbol{A},\boldsymbol{\varphi})=g(\boldsymbol{\varphi})\boldsymbol{\Omega}\cdot\boldsymbol{A}+f(\boldsymbol{\varphi}). \tag{1.3}$$

For H_0, the invariant torus with frequency vector $\boldsymbol{\omega}_0$ is located at $\boldsymbol{A}$ such that $\boldsymbol{\Omega}\cdot\boldsymbol{A}=0$ and $\boldsymbol{\omega}_0\cdot\boldsymbol{A}=E$ where E is the total energy of the system. Since $\boldsymbol{\omega}_0$ satisfies a diophantine condition, the KAM theorem for Hamiltonians (1.1) states that for a sufficiently small and smooth perturbation V, this invariant torus persists [1]. For a sufficiently large perturbation, this invariant torus is broken (converse KAM [2]).
The purpose of this paper is to construct a renormalization transformation in order to investigate the properties of invariant tori with the frequency vector $\boldsymbol{\omega}_0$ at criticality. The idea of the renormalization approach is to iterate a transformation in the space of Hamiltonians. For Hamiltonians that have a smooth invariant torus with frequency vector $\boldsymbol{\omega}_0$, the iteration should converge to a trivial fixed point. The set of Hamiltonians that have a non-smooth invariant torus with this frequency vector form a surface (called critical surface) which is invariant under the renormalization, and is expected to be a codimension one stable manifold of a non-trivial fixed set of the renormalization.
The aim is to define a renormalization as was done for Hamiltonians with two degrees of freedom in Refs. [3–5]. This renormalization is a combination of a partial elimination of the perturbation (we eliminate the Fourier modes of the perturbation which are sufficiently far from resonance) and a rescaling transformation which consists of a shift of the resonances, and a rescaling of time and of the actions.
An approximate renormalization scheme for the frequency vector $\boldsymbol{\omega}_0$ was defined in Ref. [6] following the renormalization defined for the spiral mean torus constructed in Ref. [7]. The main result of Ref. [6] was that the renormalization has a fixed point on the critical surface but it is not hyperbolic. This renormalization dynamics is structurally unstable.
We propose to improve on Ref. [6], just as was done for Ref. [7] in Refs. [8,9] for the spiral mean torus. The main difference with the scheme constructed in Ref. [6] is that we use a different normalization condition for the rescaling in the actions and we include a term that was previously neglected. The main result we find is that a large part of the critical surface is the codimension-one stable manifold of a hyperbolic fixed point of this renormalization. We expect the dynamics of the exact renormalization to be the same as this approximate renormalization, at least locally.
Following the approach of Escande and Doveil [10,11] for systems with two degrees of freedom, we perform two approximations in this renormalization :
(1) A three resonance approximation : we keep only the three main Fourier modes of the perturbation V at each step of the transformation.
(2) A mean-value quadratic approximation : we neglect the dependence on the angles of the quadratic term in the actions.

The frequency vector $\boldsymbol{\omega}_0$ is an eigenvector of the matrix $\tilde{N}$ with the eigenvalue $(\tau+1)^{-1}\approx 0.445$, where $\tilde{N}$ is the transposed matrix of the following matrix N with integer coefficients and determinant -1 :

$$N=\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{1.4}$$

From $\tilde{N}$ one generates a sequence of periodic orbits (with frequency vector $\{\boldsymbol{\Omega}_k\}$) approximating the motion with frequency vector $\boldsymbol{\omega}_0$, i.e. $\boldsymbol{\varphi}(t) = \boldsymbol{\omega}_0 t + \boldsymbol{\varphi}_0 \bmod 2\pi$:

$$\boldsymbol{\Omega}_k = (p_k/r_k, q_k/r_k, 1) \to_{n\to\infty} \boldsymbol{\omega}_0,$$

where p_k, q_k and r_k are determined by the following recursion relations :

$$\begin{aligned} p_{k+1} &= p_k + 2q_k + r_k, \\ q_{k+1} &= p_k, \\ r_{k+1} &= q_k + r_k, \end{aligned}$$

and $p_0 = 3$, $q_0 = 1$, $r_0 = 1$, i.e. $\boldsymbol{\Omega}_k = \tilde{N}^{-k}\boldsymbol{\Omega}_0$. These relations define a sequence of simultaneous rational approximations $(p_k/r_k, q_k/r_k)$ to the pair $(\tau^2+\tau, \tau)$.
The sequence of resonances is generated by the matrix N from the vector $\boldsymbol{\nu}_1 = (1,0,0)$:

$$\boldsymbol{\nu}_k = N^{k-1}\boldsymbol{\nu}_1.$$

The small denominators decrease geometrically to zero with the ratio $(\tau+1)^{-1} \approx 0.445$:

$$\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_k = \tau(\tau+1)^{2-k}.$$

The modes $\boldsymbol{\nu}_k$ are in resonance with the periodic motion with frequency vector $\boldsymbol{\Omega}_k$ (with period r_k) :

$$\boldsymbol{\Omega}_k \cdot \boldsymbol{\nu}_{k+4} = 0.$$

The matrix N can be decomposed in the following way :

$$N = LPL^2, \tag{1.5}$$

where

$$L = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \tag{1.6}$$

and

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.7}$$

We notice that $\det L = -\det P = 1$ and that P is a permutation of two elements of the basis; in particular, it satisfies $P^2 = 1$. This decomposition follows from the construction of the Farey sequence for an incommensurate vector $\boldsymbol{\omega} \in \mathbb{R}^3$ [12,7,6]. With this decomposition of the matrix N, we construct the renormalization from two operators (one associated with L and the other one to P) in a similar way as MacKay did for Hamiltonian systems with two degrees of freedom in Ref. [13] in order to investigate the break-up of invariant tori with arbitrary winding ratio. The approximate renormalization transformation we define for $\boldsymbol{\omega}_0$ can be constructed for a more general frequency vector from these two operators given the Farey decomposition of the frequency vector. For instance, concerning the spiral mean torus [8,9], the renormalization is equal to one of the operators (the one which is associated with L).

II. DEFINITION OF THE RENORMALIZATION TRANSFORMATION

The renormalization transformation $\mathcal{R}$ we define for the frequency vector $\boldsymbol{\omega}_0$ is composed of two operators : one associated with the matrix P and the other one with L. We denote $\mathcal{R}_P$ and $\mathcal{R}_L$ these operators. The transformation $\mathcal{R}$ is defined for a *fixed* frequency vector $\boldsymbol{\omega}_0$ and is given by

$$\mathcal{R} = \mathcal{R}_L \circ \mathcal{R}_L \circ \mathcal{R}_P \circ \mathcal{R}_L.$$

A. Definition of $\mathcal{R}_P$

The renormalization operator $\mathcal{R}_P$ acts on the following Hamiltonians :

$$H(\boldsymbol{A}, \boldsymbol{\varphi}) = H_0(\boldsymbol{A}) + h(\boldsymbol{\Omega}\cdot\boldsymbol{A}, \boldsymbol{\varphi}), \tag{2.1}$$

where

$$\begin{aligned} H_0(\boldsymbol{A}) &= \boldsymbol{\omega}\cdot\boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2, \\ h(\boldsymbol{\Omega}\cdot\boldsymbol{A}, \boldsymbol{\varphi}) &= \sum_{i=1}^{3} h_i(\boldsymbol{A})\cos\varphi_i = \sum_{i=1}^{3}(f_i + g_i\boldsymbol{\Omega}\cdot\boldsymbol{A})\cos\varphi_i. \end{aligned}$$

It contains a shift of the Fourier modes and a rescaling of time and of the actions. The shift of the Fourier modes is constructed such that it exchanges the mode $(1,0,0)$ and $(0,0,1)$ without changing the mode $(0,1,0)$. More precisely, we require that the new angles $\boldsymbol{\varphi}'$ satisfy :

$$\begin{aligned} \cos\varphi'_1 &= \cos\varphi_3, \\ \cos\varphi'_2 &= \cos\varphi_2, \\ \cos\varphi'_3 &= \cos\varphi_1. \end{aligned}$$

This is performed by the following linear canonical transformation :

$$(\boldsymbol{A}, \boldsymbol{\varphi}) \mapsto (\boldsymbol{A}', \boldsymbol{\varphi}') = (P\boldsymbol{A}, P\boldsymbol{\varphi}),$$

where we recall that P is symmetric and orthogonal. The vectors $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are changed into $P\boldsymbol{\omega}$ and $P\boldsymbol{\Omega}$. We impose that the images $\boldsymbol{\omega}'$ and $\boldsymbol{\Omega}'$ of the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ by the renormalization $\mathcal{R}_P$ satisfy the following normalization conditions : $\boldsymbol{\Omega}'$ must be of (euclidean) norm one, and the third component of $\boldsymbol{\omega}'$ must be equal to one. Since P is orthogonal, $P\boldsymbol{\Omega}$ is again of norm one, and thus $\boldsymbol{\Omega}' = P\boldsymbol{\Omega}$. The third component of $P\boldsymbol{\omega}$ is equal to ω_1. We rescale the time by a factor ω_1, i.e. we multiply the Hamiltonian by $1/\omega_1$. Then $\boldsymbol{\omega}'$ is given by $\boldsymbol{\omega}' = \boldsymbol{\omega}/\omega_1 = (\omega_3/\omega_1, \omega_2/\omega_1, 1)$.
The quadratic part of the Hamiltonian H_0 becomes $(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2/(2\omega_1)$. In order that this quadratic term is equal to $(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2/2$, we rescale the actions by a factor $\lambda_P = 1/\omega_1$, i.e. we change the Hamiltonian H into $\lambda_P H(\boldsymbol{A}/\lambda_P, \boldsymbol{\varphi})$.

In summary, a Hamiltonian H given by Eq. (2.1) is mapped into

$$H'(\boldsymbol{A},\boldsymbol{\varphi}) = \boldsymbol{\omega}'\cdot\boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2 + \sum_{i=1}^{3}(f_i' + g_i'\boldsymbol{\Omega}'\cdot\boldsymbol{A})\cos\varphi_i,$$

where $\boldsymbol{\omega}' = (\omega_3/\omega_1, \omega_2/\omega_1, 1)$, $\boldsymbol{\Omega}' = (\Omega_3, \Omega_2, \Omega_1)$, and

$$\begin{aligned}
f_1' &= f_3/\omega_1^2,\\
f_2' &= f_2/\omega_1^2,\\
f_3' &= f_1/\omega_1^2,\\
g_1' &= g_3/\omega_1,\\
g_2' &= g_2/\omega_1,\\
g_3' &= g_1/\omega_1.
\end{aligned}$$

The renormalization operator $\mathcal{R}_P$ is equivalent to the 10-dimensional map given by the above equations (we recall that we impose a normalization condition on $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$) :

$$(\boldsymbol{\omega}, \boldsymbol{\Omega}, f_i, g_i; i=1,2,3) \mapsto (\boldsymbol{\omega}', \boldsymbol{\Omega}', f_i', g_i'; i=1,2,3).$$

B. Definition of $\mathcal{R}_L$

The renormalization operator $\mathcal{R}_L$ is associated to the matrix L and acts on the family of Hamiltonians (2.1). It contains an elimination of the mode $\boldsymbol{\nu}_1$ of the perturbation, and a rescaling procedure (shift of the resonances, rescaling of time and of the actions) such that the image of a Hamiltonian H given by Eq. (2.1) is of the same general form as H and describes the system on a smaller scale in phase space and at a longer time scale. This operator is similar to the approximate renormalization constructed for the spiral mean torus in Refs. [8,9].

We eliminate the mode $\boldsymbol{\nu}_1$ of the scalar functions f and g, by a near-identity canonical transformation. We perform a Lie transformation generated by a function $S(\boldsymbol{A},\boldsymbol{\varphi})$. The image of a Hamiltonian H is given by :

$$H' = \exp(\hat{S})H = H + \{S,H\} + \frac{1}{2}\{S,\{S,H\}\} + \cdots, \tag{2.2}$$

where $\{\, ,\, \}$ denotes the Poisson bracket :

$$\{f,g\} = \frac{\partial f}{\partial\boldsymbol{\varphi}}\cdot\frac{\partial g}{\partial\boldsymbol{A}} - \frac{\partial g}{\partial\boldsymbol{\varphi}}\cdot\frac{\partial f}{\partial\boldsymbol{A}},$$

and the operator $\hat{S}$ acts on H like $\hat{S}H = \{S,H\}$. The generating function is chosen linear in the actions and of the form :

$$S(\boldsymbol{A},\boldsymbol{\varphi}) = (z + y\boldsymbol{\Omega}\cdot\boldsymbol{A})\sin\varphi_1. \tag{2.3}$$

We choose z and y in the following way :

$$\begin{aligned}
z &= -f_1/\omega_1,\\
y &= -(g_1 - f_1\Omega_1/\omega_1)/\omega_1.
\end{aligned}$$

Then the generating function S satisfies :

$$\{S, H_0\} = (Q(\boldsymbol{A}) - h_1(\boldsymbol{A}))\cos\varphi_1,$$

with $Q(\boldsymbol{A}) = y\Omega_1(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2$. The image of a Hamiltonian H given by Eq. (2.1) is :

$$\begin{aligned}
H' = {} & H_0 + h_1\cos\varphi_1 + h_2\cos\varphi_2 + h_3\cos\varphi_3\\
& +\{S,H_0\} + \{S, h_1\cos\varphi_1\} + \{S, h_2\cos\varphi_2\}\\
& +\{S, h_3\cos\varphi_3\} + \frac{1}{2}\{S,\{S,H_0\}\} + O(\varepsilon^3).
\end{aligned}$$

We neglect the term $Q(\boldsymbol{A})\cos\varphi_1$ produced by the terms $\{S,H_0\} + h_1\cos\varphi_1$, i.e. we neglect the dependence on the angles of the quadratic terms in the actions. The term $\{S, h_1\cos\varphi_1\}$ is of degree one in the actions and contains the modes $\mathbf{0}$ and $2\boldsymbol{\nu}_1$. We neglect the mode $2\boldsymbol{\nu}_1$ and we eliminate the mode $\mathbf{0}$ of the linear term in the actions by a shift : $\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{a}$ where $\boldsymbol{a}$ is of order ε^2 (in the direction of $\boldsymbol{\Omega}$). We neglect the modes with frequency vectors $\boldsymbol{\nu}_1 \pm \boldsymbol{\nu}_2$ produced by the term $\{S, h_2\cos\varphi_2\}$. The term $\{S, h_3\cos\varphi_3\}$ generates the modes $\boldsymbol{\nu}_1 \pm \boldsymbol{\nu}_3$. We neglect the mode $\boldsymbol{\nu}_1 + \boldsymbol{\nu}_3$ and keep the next relevant Fourier mode $\boldsymbol{\nu}_1 - \boldsymbol{\nu}_3$ whose amplitude is denoted $h_3' = f_3' + g_3'\boldsymbol{\Omega}\cdot\boldsymbol{A}$, where

$$f_3' = (zg_3\Omega_1 + yf_3\Omega_3)/2, \tag{2.4}$$

$$g_3' = (\Omega_1 + \Omega_3)yg_3/2. \tag{2.5}$$

The term $\{S, Q\cos\varphi_1\}/2$ produced by $\{S,\{S,H_0\}\}/2$ gives a contribution to the quadratic part of the Hamiltonian. The new quadratic part is equal to $m(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2/2$ where m is given by :

$$m = 1 + \frac{3}{2}y^2\Omega_1^2. \tag{2.6}$$

Then after the elimination of the mode $\boldsymbol{\nu}_1$, the new Hamiltonian is equal to :

$$\begin{aligned}
H' = {} & \boldsymbol{\omega}\cdot\boldsymbol{A} + m(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2/2 + h_2\cos\varphi_2 + h_3\cos\varphi_3\\
& + h_3'\cos(\varphi_1 - \varphi_3),
\end{aligned} \tag{2.7}$$

where h_3' is given by Eqs. (2.4) and (2.5).

We shift the Fourier modes according to the following linear canonical transformation :

$$(\boldsymbol{A},\boldsymbol{\varphi}) \mapsto (\boldsymbol{A}',\boldsymbol{\varphi}') = (L^{-1}\boldsymbol{A}, \tilde{L}\boldsymbol{\varphi}),$$

such that $\cos\varphi_2 = \cos\varphi_1'$, $\cos\varphi_3 = \cos\varphi_2'$ and $\cos(\varphi_1 - \varphi_3) = \cos\varphi_3'$. The vectors $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ are changed in the following way :

$$\begin{aligned}
\boldsymbol{\Omega}' &= \frac{\tilde{L}\boldsymbol{\Omega}}{\|\tilde{L}\boldsymbol{\Omega}\|} = (\Omega_2, \Omega_3, \Omega_1 - \Omega_3)/(1 + \Omega_3(\Omega_3 - 2\Omega_1)),\\
\boldsymbol{\omega}' &= \frac{\tilde{L}\boldsymbol{\omega}}{(\tilde{L}\boldsymbol{\omega})_3} = (\omega_2/(\omega_1 - 1), 1/(\omega_1 - 1), 1).
\end{aligned}$$

We rescale time in such a way that the linear term in the actions of H_0 is equal to $\boldsymbol{\omega}'$: we multiply the Hamiltonian

H' by a factor $1/(\omega_1-1)$. The quadratic term of the new Hamiltonian is equal to $m\|\tilde{L}\mathbf{\Omega}\|^2(\mathbf{\Omega}'\cdot\boldsymbol{A})^2/(2\omega_1-2)$. In order to map this quadratic part into $(\mathbf{\Omega}'\cdot\boldsymbol{A})^2/2$, we rescale the actions by a factor

$$\lambda_L = m\|\tilde{L}\mathbf{\Omega}\|^2/(\omega_1-1),$$

by changing the Hamiltonian H' into $\lambda_L H'(\boldsymbol{A}/\lambda_L,\boldsymbol{\varphi})$. After this elimination and rescaling procedures, a Hamiltonian H is mapped into

$$H'' = \boldsymbol{\omega}'\cdot\boldsymbol{A} + \frac{1}{2}(\mathbf{\Omega}'\cdot\boldsymbol{A})^2 + h_1''\cos\varphi_1 + h_2''\cos\varphi_2 + h_3''\cos\varphi_3,$$

where $h_i'' = f_i'' + g_i''\mathbf{\Omega}'\cdot\boldsymbol{A}$ and

$$\begin{aligned} f_1'' &= f_2 c_f, \\ f_2'' &= f_3 c_f, \\ f_3'' &= f_3' c_f, \\ g_1'' &= g_2 c_g, \\ g_2'' &= g_3 c_g, \\ g_3'' &= g_3' c_g, \end{aligned}$$

with $c_f = m\|\tilde{L}\mathbf{\Omega}\|^2/(\omega_1-1)^2$ and $c_g = \|\tilde{L}\mathbf{\Omega}\|/(\omega_1-1)$. The renormalization transformation is equivalent to the 10-dimensional map

$$(\boldsymbol{\omega},\mathbf{\Omega},f_i,g_i;i=1,2,3)\mapsto(\boldsymbol{\omega}',\mathbf{\Omega}',f_i'',g_i'';i=1,2,3),$$

given by the above equations.

C. Renormalization transformation $\mathcal{R}$

In summary, the renormalization $\mathcal{R} = \mathcal{R}_L^2\mathcal{R}_P\mathcal{R}_L$ acts in the following way : a canonical transformation eliminates the three main Fourier modes and produce the next three resonances which are the main Fourier modes at a smaller scale in phase space. A rescaling procedure shift these resonances into the original Fourier modes, and normalize the new Hamiltonian by rescaling the time and the actions.
The rescaling procedure acts on $\boldsymbol{\omega}_0$ in the following way : $\boldsymbol{\omega}_0$ is changed into $\boldsymbol{\omega}^{(1)} = (1/(3-\tau^2), 1/(3\tau-\tau^3), 1)$ by $\mathcal{R}_L$, then into $\boldsymbol{\omega}_0^{(2)} = (3-\tau^2, \tau^{-1}, 1)$ by $\mathcal{R}_P$, then into $\boldsymbol{\omega}_0^{(3)} = (1/(2\tau-\tau^3), 1/(2-\tau^2), 1)$ by $\mathcal{R}_L$, and after the final step $\mathcal{R}_L$, into $\boldsymbol{\omega}_0^{(4)} = \boldsymbol{\omega}_0$. Thus the renormalization $\mathcal{R}$ is defined for a *fixed* frequency vector $\boldsymbol{\omega}_0$. In the same way, the action of $\mathcal{R}$ on $\mathbf{\Omega}$ reduces to the map :

$$\mathbf{\Omega}' = \frac{\tilde{N}\mathbf{\Omega}}{\|\tilde{N}\mathbf{\Omega}\|}. \tag{2.8}$$

The spectrum of $\tilde{N}$ is real and consists of the following eigenvalues $-\tau \approx -1.2469$, $(\tau+1)^{-1} \approx 0.4450$, $1+\tau^{-1} \approx 1.8019$. Thus by iterating the map (2.8), the vector $\mathbf{\Omega}$ converges to the eigenvector (denoted $\mathbf{\Omega}^{(3)}$, with euclidean norm one) associated with the largest eigenvalue of $\tilde{N}$. With this reduction, the renormalization $\mathcal{R}$ reduces to a 6-dimensional map $(f_i,g_i;i=1,2,3)\mapsto(f_i',g_i';i=1,2,3)$.
The renormalization transformation we define can be also constructed for the following family of Hamiltonians :

$$H(\boldsymbol{A},\boldsymbol{\varphi}) = \boldsymbol{\omega}_0\cdot\boldsymbol{A} + \frac{1}{2}\boldsymbol{A}\cdot M\boldsymbol{A} + \boldsymbol{g}(\boldsymbol{\varphi})\cdot\boldsymbol{A} + f(\boldsymbol{\varphi}), \tag{2.9}$$

where M is a 3×3 symmetric matrix with non-zero mean-value, and $\boldsymbol{g}$ is a three dimensional vector. Applying the renormalization changes the matrix M into :

$$\mathcal{R}(M) = \frac{\tilde{N}MN}{\mathrm{tr}\ (\tilde{N}MN)}, \tag{2.10}$$

where tr $(\tilde{N}MN)$ is the trace of the matrix $\tilde{N}MN$. By iterating the map (2.10), the matrix converges to $\mathbf{\Omega}^{(3)}\otimes\mathbf{\Omega}^{(3)}$. Moreover, $\boldsymbol{g}$ is renormalized into $\tilde{N}\boldsymbol{g}$ which tends to be aligned to $\mathbf{\Omega}^{(3)}$ by iteration. Then the Hamiltonians (2.9) tend to the degenerate Hamiltonians (1.1) under the iterations of renormalization. We expect that the Hamiltonians (2.9) belong to the same universality class as the Hamiltonians (1.1).

III. RENORMALIZATION FLOW

The renormalization transformation $\mathcal{R}$ has the following properties : $\mathcal{R}$ has an attractive integrable fixed point H_0 given by

$$H_0(\boldsymbol{A}) = \boldsymbol{\omega}_0\cdot\boldsymbol{A} + \frac{1}{2}(\mathbf{\Omega}^{(3)}\cdot\boldsymbol{A})^2,$$

and another non-integrable fixed point H_* which lies on the boundary of the domain of attraction of H_0. Outside the closure of the domain of attraction of H_0, the iterations of renormalization diverge to infinity. The renormalization dynamics has the same qualitative features as the renormalization for the golden mean torus for Hamiltonian systems with two degrees of freedom [4]. The properties of critical tori are given by the analysis of the renormalization around the non-trivial fixed point H_*. In particular, the existence of a non-trivial fixed point for an exact renormalization would imply self-similarity of critical invariant tori. The stable manifold of H_* is of codimension one; the linearized renormalization around H_* has only one eigenvalue of modulus larger than one : $\delta \approx 3.4414$. The value of the total rescaling coefficient in the actions is $\lambda \approx 11.2726$, and the rescaling coefficient of time is $\tau+1 \approx 2.2469$.
We also find a critical fixed cycle with period 7 on the critical surface. This periodic cycle is obtained from the non-trivial fixed point H_* by changing the sign of the Fourier coefficients f_i and g_i. The existence of this cycle is explained by symmetry reasons as it was done for the critical cycle with period 3 for the golden mean case [14].

In particular, it involves the same critical exponents and scaling factors as the non-trivial fixed point, and belongs to the same universality class.

IV. CONCLUSION

We have defined two elementary operators $\mathcal{R}_P$ and $\mathcal{R}_L$. With these operators, we have defined an approximate renormalization in order to study invariant tori with frequency vector $\boldsymbol{\omega}_0 = (\tau^2+\tau, \tau, 1)$. The renormalization has a hyperbolic fixed point with codimension one stable manifold. Consequently, we expect critical invariant tori with this frequency vector to be self-similar at criticality. In order to give a firm basis of this statement, it will be interesting to build an exact renormalization transformation without the drastic approximations we used.
We notice also that the approximate renormalization we define in this paper can be generalized to an arbitrary incommensurate frequency vector $\boldsymbol{\omega}_0$ given by its Farey sequence, with the operators $\mathcal{R}_P$ and $\mathcal{R}_L$. For a periodic Farey sequence the hyperbolic invariant sets are expected to be fixed points, periodic orbits or strange non-chaotic attractors (such as the spiral mean torus [15]). For a non-periodic Farey sequence, this fixed sets can be strange chaotic attractors, but this point has not been investigated yet.

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