POWER-LAW CORRECTIONS TO EXPONENTIAL DECAY OF CONNECTIVITIES AND CORRELATIONS IN LATTICE MODELS

KENNETH S. ALEXANDER

Abstract. Consider a translation-invariant bond percolation model on the integer lattice which has exponential decay of connectivities, that is, the probability of a connection $0 \leftrightarrow x$ by a path of open bonds decreases like $e^{-m(\theta)|x|}$ for some positive constant $m(\theta)$ which may depend on the direction $\theta = x/|x|$. In two and three dimensions, it is shown that if the model has an appropriate mixing property and satisfies a special case of the FKG property, then there is at most a power-law correction to the exponential decay—there exist A and C such that $e^{-m(\theta)|x|} \geq P(0 \leftrightarrow x) \geq A|x|^{-C}e^{-m(\theta)|x|}$ for all nonzero x. In four or more dimensions, a similar bound holds with $|x|^{-C}$ replaced by $e^{-C(\log|x|)^2}$. In particular the power-law lower bound holds for the Fortuin-Kasteleyn random cluster model in two dimensions whenever the connectivity decays exponentially, since the mixing property is known to hold in that case. Consequently a similar bound holds for correlations in the Potts model at supercritical temperatures.

1. Introduction and Statement of Results

Many quantities encountered in statistical mechanics decay at an approximately exponential rate as a function of distance. Typical finite-range spin systems have exponential decay of correlations at sufficiently high tempreatures, and many standard percolation models, such as the Fortuin-Kastelyn random cluster model [12], are known or believed to have exponential decay of connectivities for those parameter values (other than critical points) at which there is no percolation. For the modified correlation function

$$\rho(0, x) = \frac{q^2}{q-1} \operatorname{cov}(\delta_{[\sigma_0=i]}, \delta_{[\sigma_x=i]})$$

of the (free-boundary) q-state Potts model, or for the connectivity function

$$\rho(0, x) = P(0 \leftrightarrow x)$$

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of a translation-invariant percolation model having the FKG property, supermultiplicativity holds:

$$\rho(0, x+y) \geq \rho(0,x)\rho(0,y),$$

so $-\log \rho(0,x)$ is a subadditive function of x. (Here $[0 \leftrightarrow x]$ denotes the event that 0 is connected to x by a path of open bonds, and σ_x denotes the spin at site x.) From standard properties of subadditive sequences, this implies that the limit

$$m = m(x/|x|) = \lim_{n\to\infty} \frac{1}{n|x|} \log \rho(0, nx) \tag{1.1}$$

exists, and the exponential approximation is an upper bound for the actual correlation or connectivity function:

$$\rho(0,x) \leq e^{-m|x|}, \quad x \in \mathbb{Z}^d.$$

It is therefore of interest to find lower bounds, establishing results of the form

$$\rho(0,x) \geq f(x)e^{-m|x|}, \quad x \in \mathbb{Z}^d, \tag{1.2}$$

where ρ is the correlation or connectivity function, f decays subexponentially and $|\cdot|$ is the Euclidean norm.

Ornstein and Zernike [19] predicted for certain models that the analog of the correlation should behave like

$$|x|^{-(d-1)/2}e^{-m|x|} \tag{1.3}$$

as $|x| \to \infty$, for some constant m. This was verified for a wide class of models at very high temperatures by Bricmont and Fröhlich [9], for self-avoiding random walk by Chayes and Chayes [11] and Ioffe [16], and for Bernoulli percolation at arbitrary subcritical densities, with x "near an axis," by Campanino, Chayes and Chayes [10]. For the two-dimensional Ising model at supercritical temperatures, (1.3) can be obtained from the exact solution (see [18] or Section 7 of [20].) (An exception to (1.3) is found for the two-dimensional Ising model at subcritical temperatures under "plus" or under "minus" boundary conditions, where the correct exponent on $|x|$ is 2, not 1/2; see [18]. The heuristics for this are discussed in [9].) In the case of connectivity functions, the heuristic for (1.3) in general is as follows; see e.g. [10] or [16] for more. For simplicity take x on an axis, and let H_x be the hyperplane orthogonal to the axis at x. The sum of $P(0 \leftrightarrow y)$ over sites y in H_x should be approximately $e^{-m|x|}$ with nearly no correction. Given that there is a path from 0 to H_x, it should reach only a few close-together sites in H_x, and from the central limit theorem, since the transverse fluctuations of different segments of the path are approximately independent, the location of these sites in H_x should be approximately Gaussian distributed with variance of order $|x|$. This Gaussian distribution accounts for the factor $|x|^{-(d-1)/2}$. (Note constants have been omitted in this heuristic.)

Thus the form we should seek for the function $f(x)$ in (1.2) is an inverse power of $|x|$, that is, a power-law correction to exponential decay. We will not attempt to obtain the optimal power $(d-1)/2$. We will instead obtain results of form (1.2) with $f(x)$ an inverse power of $|x|$ when $d=2$ or 3, and with $f(x)=e^{-C(\log|x|)^2}$ for some constant C when $d \geq 4$. Analogous results for Bernoulli percolation at arbitrary subcritical densities are in [2] and [4]. Our results have suboptimal powes of $|x|$ for two reasons. First, we wish to work with quite general models and at arbitrary supercritical temperatures, which likely makes rigorous proof of precise behavior as in (1.3) a particularly difficult problem. Second, interesting applications of (1.2) do not always require the optimal power $(d-1)/2$. For example, power-law correction results from [2] are applied in [3] to study boundary fluctuations in the Wulff construction for Bernoulli percolation, and Pfister and Velenik [20] use only the existence of a power-law correction (obtained from the exact solution) for correlations in the two-dimensional Ising model in their study of the continuum limit of that model.

For simplicity we restrict attention to the integer lattice, but our results apply to more general lattices. For $\Lambda \subset \mathbb{Z}^d$ let $\mathcal{B}(\Lambda)$ denote the set of all nearest-neighbor bonds $\langle xy \rangle$ with $x, y \in \Lambda$, and let $\overline{\mathcal{B}}(\Lambda)$ denote the set of all nearest-neighbor lattice bonds $\langle xy \rangle$ with x or y in Λ. A *bond percolation model* on $\mathbb{Z}^d$ is a measure P on $\{0,1\}^{\mathcal{B}(\mathbb{Z}^d)}$. We consider here only translation-invariant models. A *bond configuration* is an element $\omega \in \{0,1\}^{\mathcal{B}(\mathbb{Z}^d)}$; when convenient we view ω as a subset of $\mathcal{B}(\mathbb{Z}^d)$. A bond e is *open* in a configuration ω if $\omega_e = 1$, and *closed* if $\omega_e = 0$. The configuration $\{\omega_e : e \in \mathcal{G}\}$ restricted to a set $\mathcal{G}$ of bonds is denoted $\omega_{\mathcal{G}}$. P has *positive connection correlations* if

$$P(0 \leftrightarrow x+y) \geq P(0 \leftrightarrow x) P(x \leftrightarrow x+y) \quad \text{for all } x, y;$$

this is a special case of the standard FKG property. We write $P_{\Lambda,\rho}$ for $P(\cdot \mid \omega_{\mathcal{B}(\Lambda^c)} = \rho_{\mathcal{B}(\Lambda^c)})$; we assume the latter is given by a regular conditional measure. Let $\mathcal{F}_\Lambda$ denote the σ-algebra generated by $\{\omega_e : e \in \mathcal{B}(\Lambda)\}$. P has the *weak mixing property* if for some $C, \lambda > 0$, for all finite sets Δ, Λ with $\Delta \subset \Lambda$,

$$\begin{aligned} \sup\{\mathrm{Var}(P_{\Lambda,\rho}(\omega_{\mathcal{B}(\Delta)} \in \cdot), P_{\Lambda,\rho'}(\omega_{\mathcal{B}(\Delta)} \in \cdot)) : \rho, \rho' \in \{0,1\}^{\mathcal{B}(\Lambda^c)}\} \\ \leq C \sum_{x \in \Delta, y \in \Lambda^c} e^{-\lambda|x-y|}, \end{aligned}$$

where $\mathrm{Var}(\cdot,\cdot)$ denotes total variation distance between measures. Roughly, the influence of the boundary condition on a finite region decays exponentially with distance from that region. Equivalently, for some $C, \lambda > 0$, for all sets $\Delta, \Gamma \subset \mathbb{Z}^d$,

$$\begin{aligned} \sup\{|P(E \mid F) - P(E)| : E \in \mathcal{F}_\Delta, F \in \mathcal{F}_\Gamma, P(F) > 0\} \\ \leq C \sum_{x \in \Delta, y \in \Gamma} e^{-\lambda|x-y|}. \end{aligned} \tag{1.4}$$

P has the *ratio weak mixing* property if for some $C, \lambda > 0$, for all sets $\Delta, \Gamma \subset \mathbb{Z}^d$,

$$\sup\left\{\left|\frac{P(E \cap F)}{P(E)P(F)} - 1\right| : E \in \mathcal{F}_\Delta, F \in \mathcal{F}_\Gamma, P(E)P(F) > 0\right\} \leq C \sum_{x \in \Delta, y \in \Gamma} e^{-\lambda|x-y|}, \tag{1.5}$$

whenever the right side of (1.5) is less than 1. For $\Lambda \subset \mathbb{Z}^d$ finite, $\rho \in \{0,1\}^{\mathcal{B}(\Lambda^c)}$, and $\Gamma \subset \Lambda^c$ finite, we call $\mathcal{B}(\Gamma)$ a *controlling region* for $\overline{\mathcal{B}}(\Lambda)$ and ρ if for every $\rho' \in \{0,1\}^{\mathcal{B}(\Lambda^c)}$ such that $\rho = \rho'$ on $\mathcal{B}(\Gamma)$, we have $P_{\Lambda,\rho} = P_{\Lambda,\rho'}$. We say P has *exponentially bounded controlling regions* if there exist constants $C, \lambda > 0$ such that for every choice of disjoint finite sets Λ and Γ,

$$\begin{aligned} &P(\{\rho \in \{0,1\}^{\mathcal{B}(\Lambda^c)} : \mathcal{B}(\Gamma) \text{ is not a controlling region for } \overline{\mathcal{B}}(\Lambda) \text{ and } \rho\}) \\ &\leq C \sum_{x \in \Lambda, y \in \Lambda^c \setminus \Gamma} e^{-\lambda|x-y|}. \end{aligned}$$

Note that when $P(E)$ is much smaller than the right side of (1.4), the weak mixing condition (1.4) allows $P(E \mid F)$ to be many times larger than $P(E)$, but the ratio weak mixing condition (1.5) does not allow this. Nonetheless, it is proved in [6] that if P has exponentially bounded controlling regions and the weak mixing property, then P has the ratio weak mixing property. We say P has *exponential decay of connectivities* if there exist $C, \lambda > 0$ such that for all x and y,

$$P(x \leftrightarrow y) \leq Ce^{-\lambda|x-y|}.$$

Writing θ for $x/|x|$, when the limit

$$\lim_{n\to\infty} \frac{1}{n|x|} \log P(0 \leftrightarrow nx)$$

exists for all $x \in \mathbb{Z}^d$, is finite and depends only on θ (as for example when P has positive connection correlations), we denote this limit by $m(\theta)$ and say P is *nondegenerate.*

Here is the main result of this paper.

Theorem 1.1. *Suppose*

(1.6) *P is a nondegenerate translation-invariant bond percolation model on $\mathbb{Z}^d$ which has positive connection correlations, exponential decay of connectivities and the ratio weak mixing property.*

(i) If $d = 2$ or 3, then there exist positive finite A, C and $m(\theta)$ such that

$$e^{-m(\theta)|x|} \geq P(0 \leftrightarrow x) \geq \frac{A}{|x|^C} e^{-m(\theta)|x|} \quad \text{for all nonzero } x \in \mathbb{Z}^d, \tag{1.7}$$

where $\theta = x/|x|$.

(ii) If $d \geq 4$, *then there exist positive finite* C *and* $m(\theta)$ *such that*

$$e^{-m(\theta)|x|} \geq P(0 \leftrightarrow x) \geq e^{-C(\log|x|)^2} e^{-m(\theta)|x|} \quad \textit{for all nonzero } x \in \mathbb{Z}^d, \tag{1.8}$$

where $\theta = x/|x|$.

The proof will be given in Sections 2 and 3. Theorem 1.1 applies to site percolation models as well; we restrict attention to bond percolation to keep the exposition simple.

The only obstacle to proving the superior result (i), instead of (ii), in dimension $d \geq 4$ is the purely geometric Proposition 2.7 of [4], which is proved only for $d = 2$ and 3; we believe this Proposition is true in all dimensions, and certainly we expect that (1.7) is true in all dimensions.

The Fortuin-Kasteleyn random cluster model (or simply, the *FK model*) with parameters (q, p) and free boundary, on a finite subgraph $(\Lambda, \mathcal{B}(\Lambda))$ of the lattice $\mathbb{Z}^d$, is the percolation model with probabilities given by the weights

$$p^{|\omega|}(1-p)^{|\mathcal{B}(\Lambda)|-|\omega|} q^{K(\omega)}, \quad \omega \in \{0,1\}^{\mathcal{B}(\Lambda)},$$

where $|\omega|$ denotes the number of open bonds in ω and $K(\omega)$ denotes the number of connected components in ω. Here $q > 0$ and $p \in [0,1]$. Taking the limit $\Lambda \nearrow \mathbb{Z}^d$ yields the FK model, with free boundary, on the full lattice (see [13].) This model was introduced in [12]; see also [1] and [13] for basic properties. For the q-state Potts model at a supercritical temperature T, for $\beta = 1/T, p = 1 - e^{-\beta}$ and the FK model at (p, q), the covariance in the Potts model and the connectivity in the FK model are related by

$$q^2 \operatorname{cov}(\delta_{[\sigma_0 = i]}, \delta_{[\sigma_x = i]}) = (q-1) P(0 \leftrightarrow x), \quad i = 1, .., q; \tag{1.9}$$

see [1] or [14]. Thus exponential decay of connectivities in the FK model is equivalent to exponential decay of correlations in the corresponding Potts model. Further, the critical inverse temperature $\beta_c(q, d)$ of the Potts model and the percolation critical point $p_c(q, d)$ of the FK model are related by

$$p_c(q, d) = 1 - e^{-\beta_c(q,d)};$$

again see [1] or [13]. For $q \geq 1$, the FK model has the FKG property [12] and hence has positive connection correlations. For the two-dimensional FK model, the following facts are known. For $q = 1, q = 2$, and $q \geq 25.72$, we have $p_c(q, 2) = \frac{\sqrt{q}}{1+\sqrt{q}}$ [17], and the connectivity decays exponentially for all $p < p_c(q, 2)$ [14]. This is believed to be true for all q; for $2 < q < 25.72$ the connectivity is known to decay exponentially at least for all $p < \frac{\sqrt{q-1}}{1+\sqrt{q-1}}$ [5]. For general $q \geq 1$ and $p < p_c(q, 2)$, if the connectivity decays exponentially then the model has the ratio weak mixing property [6]. With Theorem 1.1 and (1.9), these facts yield the following results.

Theorem 1.2. *Suppose that the FK model on $\mathbb{Z}^2$ with parameters (q,p), with $q \geq 1$ and $p < p_c(q,2)$, has exponential decay of connectivities. Then there exist positive finite A, C and $m(\theta)$, depending on p and q, such that*

$$e^{-m(\theta)|x|} \geq P(0 \leftrightarrow x) \geq \frac{A}{|x|^C} e^{-m(\theta)|x|} \quad \text{for all } x \in \mathbb{Z}^2, \tag{1.10}$$

where $\theta = x/|x|$. In particular (1.10) holds for all $p < p_c(q,2) = \frac{\sqrt{q}}{1+\sqrt{q}}$ if $q = 2$ or $q \geq 25.72$, and (1.10) holds for all $p < \frac{\sqrt{q-1}}{1+\sqrt{q-1}}$ if $2 < q < 25.72$.

Corollary 1.3. *Suppose that the q-state Potts model on $\mathbb{Z}^2$ at inverse temperature $\beta < \beta_c(q,2)$ has exponential decay of correlations. Then there exist positive finite A, C and $m(\theta)$, depending on β and q, such that*

$$e^{-m(\theta)|x|} \geq \operatorname{cov}(\delta_{[\sigma_0=i]}, \delta_{[\sigma_x=i]}) \geq \frac{A}{|x|^C} e^{-m(\theta)|x|} \quad \text{for all } x \in \mathbb{Z}^2, i \leq q, \tag{1.11}$$

where $\theta = x/|x|$.

For the FK model in general dimension, exponential decay of connectivities implies exponentially bounded controlling regions (see [6]), so that weak mixing and exponential decay of connectivities together imply ratio weak mixing. It is believed that weak mixing and exponential decay of connectivities hold whenever $p < p_c(q,d)$, in which case Theorem 1.1 gives a power-law correction for all subcritical p, for $d = 3$ and $q \geq 1$, and a correction as in (1.8) for all subcritical p, for $d \geq 4$ and $q \geq 1$.

It is of interest in certain contexts (see e.g. Lemma 4.3 and Theorem 4.1 of [3]) to have an analog of Theorem 1.1 for connections in halfspaces; this is our next result.

Theorem 1.4. *Assume (1.6). Let H be the intersection with $\mathbb{Z}^d$ of a closed halfspace in $\mathbb{R}^d$ containing 0.*

(i) If $d = 2$ or 3, then there exist positive finite A, C and $m(\theta)$ such that

$$e^{-m(\theta)|x|} \geq P(0 \leftrightarrow x \text{ in } \mathcal{B}(H)) \geq \frac{A}{|x|^C} e^{-m(\theta)|x|} \quad \text{for all nonzero } x \in H, \tag{1.12}$$

where $\theta = x/|x|$.

(ii) If $d \geq 4$, then there exist positive finite C and $m(\theta)$ such that

$$e^{-m(\theta)|x|} \geq P(0 \leftrightarrow x \text{ in } \mathcal{B}(H)) \geq e^{-C(\log|x|)^2} e^{-m(\theta)|x|} \quad \text{for all nonzero } x \in H, \tag{1.13}$$

where $\theta = x/|x|$.

2. Proof of Theorem 1.1(ii)

Throughout the paper, $C_1, C_2, ...$ and $c_1, c_2, ...$ denote constants which may depend on the model P, but not on x. Additional parameters on which these constants may depend are listed in parentheses after the constant, e.g. $C_9(C, K)$. Phrases such as "sufficiently large" or "small enough" implicitly mean "larger/smaller than a constant depending only on P," unless otherwise specified. Throughout the paper we tacitly assume in proofs that $|x|$ and C are sufficiently large, in this sense, and assume (1.6). To facilitate bookkeeping we will use C_i for constants appearing in statements of results, and c_i for constants which appear only in the course of proofs.

Define

$$h(x) = -\log P(0 \leftrightarrow x), \quad x \in \mathbb{Z}^d,$$

so that, by positive connection correlation, h is subadditive:

$$h(x + y) \leq h(x) + h(y).$$

In particular $\{h(nx) : n \geq 1\}$ is a subadditive sequence, so by standard methods the limit

$$m(x) = \lim_{n\to\infty} \frac{h(nx)}{n} \tag{2.1}$$

exists, extending the definition (1.1), and for all $x \in \mathbb{Z}^d$,

$$m(x) \leq h(x). \tag{2.2}$$

In fact for $x \in \mathbb{Q}^d$, if we restrict n to those values for which $nx \in \mathbb{Z}^d$, then the limit in (2.1) exists, so $m(\cdot)$ extends to $\mathbb{Q}^d$. By exponential decay of correlations, $m(x)$ is strictly positive for all $x \neq 0$. Further, from subadditivity, $m(x)$ is finite if and only if x is in the linear span of $\{e_i : P(\omega_{\langle 0e_i\rangle} = 1) > 0\}$, where e_i denotes the ith unit coordinate vector. Under positive connection correlations, nondegeneracy of P is equivalent to

$$P(\omega_{\langle 0e_i\rangle} = 1) > 0 \quad \text{for all } i. \tag{2.3}$$

By the arguments in [7], m is uniformly continuous and m extends to a function on $\mathbb{R}^d$ which is continuous, convex, and positive-homogeneous of order 1. In particular,

$$m(x) = m(\theta)|x|,$$

where $\theta = x/|x|$. Let

$$m_0 = \min_i m(e_i), \quad M_0 = \max_i m(e_i).$$

It follows from convexity that

$$m_0|x|_\infty \leq m(x) \leq M_0|x|_1, \tag{2.4}$$

where $|\cdot|_r$ denotes the l^r norm. We suppress the r in the notation for the Euclidean norm, $r = 2$.

Observe that (1.8) may be rewritten as

$$m(x) \leq h(x) \leq m(x) + C(\log|x|)^2 \quad \text{for all } x \in \mathbb{Z}^d \text{ with } |x| > 1,$$

which in the terminology of [4] is the *general approximation property*, or *GAP*, with exponent 0 and correction factor $(\log|x|)^2$, for the subadditive function h. It is proved in [4] that to establish this property, it is sufficient to establish what is called the *convex-hull approximation property*, or *CHAP*, with exponent 0 and correction factor $\log|x|$. So we give now a description of CHAP.

Let $B_1 = \{x \in \mathbb{R}^d : m(x) \leq 1\}$. For $x \in \mathbb{R}^d$ let T_x denote a hyperplane tangent to $\partial(m(x)B_1)$ at x; note that if ∂B_1 is not smooth, there is not necessarily a unique choice of T_x. Let T_x^0 denote the hyperplane through 0 parallel to T_x. There is a unique linear functional m_x on $\mathbb{R}^d$ satisfying

$$m_x(y) = 0 \quad \text{for all } y \in T_x^0, \quad m_x(x) = m(x).$$

The functional m_x is a linear approximation to m, for vectors nearly parallel to x. By convexity and symmetry of m we have

$$|m_x(y)| \leq m(y) \quad \text{for all } y \in \mathbb{R}^d. \tag{2.5}$$

For $y \in \mathbb{R}^d, m_x(y)$ is the m-length of a projection of y onto the line through 0 and x. The value $m_x(y)$ may therefore be thought of as the amount of progress (measured in the norm m) toward x made by a vector increment of y. Then for fixed x,

$$s_x(y) = h(y) - m_x(y)$$

is a measure of the error or inefficiency associated with an increment of y within a path from 0 to x. For $x \in \mathbb{R}^d$ and $C > 1$ we define a set of vector increments for which this "error" is of order at most $\log|x|$:

$$Q_x(C) = \{y \in \mathbb{Z}^d : m_x(y) \leq m(x), s_x(y) \leq C\log|x|\}.$$

Note that s_x is nonnegative and subadditive, by (2.2) and (2.5). For $M > 0$ and $C, t > 1$, we say that h satsfies CHAP(M, C, t) (with exponent 0 and correction factor $\log(\cdot)$) if

$$\frac{x}{\alpha} \in \mathrm{Co}(Q_x(C)) \quad \text{for some } \alpha \in [1, t], \quad \text{for all } x \in \mathbb{Z}^d \text{ with } |x| \geq M,$$

where $\mathrm{Co}(\cdot)$ denotes the convex hull. Roughly this says that, up to a bounded constant, every x is in the convex hull of some sites satisfying the desired power-law lower bound, except that m is replaced by the linear approximation m_x.

Remark 2.1. In [4] the definition of $Q_x(\cdot)$ requires in addition, for some constant K, that $|y| \leq K|x|$. No such requirement is needed here because of Lemma 2.4(i) below.

From ([4], Lemma 1.6), one way to establish $\mathrm{CHAP}(M, C, t)$ is to find a lattice path γ from 0 to nx for some n which can be cut up into at most tn increments, each in $Q_x(C)$. That is, there must exist sites $0 = u_0, u_1, .., u_k = nx$ in γ such that $k \leq tn$ and $u_i - u_{i-1} \in Q_x(C)$ for all $i \leq k$. This was the approach taken in [2], and our approach here is based somewhat on the methods employed there. Loosely the idea is to show that for large n, the probability that 0 is connected to nx by a path of open bonds which *fails* to have this "cutting-up" property is strictly less than $P(0 \leftrightarrow nx)$.

By a *path* we always implicitly mean a self-avoiding lattice path, that is, a sequence $x_0, \langle x_0 x_1 \rangle, x_1, \langle x_1 x_2 \rangle, x_2, .., x_n$ of alternating sites and bonds, with all x_i distinct. An *open path* is a path in which all bonds are open. Define

$$G_x = \{y \in \mathbb{Z}^d : m_x(y) \leq m(x)\}.$$

For $D \subset \mathbb{R}^d$ and $y \in \mathbb{R}^d$ we let $D + y$ denote the translate of the set D by the vector y. Let $d(\cdot, \cdot)$ denote Euclidean distance, $d(D, E) = \inf\{d(z, w) : z \in D, w \in E\}$ and $d(z, D) = d(\{z\}, D)$. For $D \subset \mathbb{Z}^d$ let

$$\partial D = \{x \in D^c : x \text{ adjacent to } D\}, \quad \overline{D} = D \cup \partial D, \quad \partial_{in} D = \partial(D^c).$$

For $D \subset \mathbb{R}^d$ and $y \in D \cap \mathbb{Z}^d$ let $\Gamma(y, D)$ denote the union of $\{y\}$ and all sites in open paths in $\mathcal{B}(D)$ which contain y; if $y \notin D$ we define $\Gamma(y, D)$ to be empty. Note that

$$[\Gamma(y, D) = R] \in \mathcal{F}_{\overline{R}} \quad \text{for all } \ y, D \text{ and } R. \tag{2.6}$$

Given x and C, we say a path γ is (x, C)-*clean* (or just *clean* if confusion is unlikely) if for every pair of sites u, v in γ with u preceding v, we have $s_x(v - u) < C \log |x|$. For sites $y, z \in G \subset \mathbb{Z}^d$ we say z is (x, C)-*cleanly reachable from* y *inside* G if there exists an (x, C)-clean path (not necessarily open!) from y to z having all sites in G. Note that clean reachability is a deterministic property, not dependent on the bond configuration. If z is (x, C)-cleanly reachable from y inside G, but is adjacent to some site in G which is not cleanly reachable from y inside G, we say z is *barely* (x, C)-*cleanly reachable from* y *inside* G. Define

$$\tilde{Q}_x(C) = \{y \in \mathbb{Z}^d : y \text{ is cleanly reachable from 0 inside } G_x\}$$

and observe that

$$\tilde{Q}_x(C) \subset Q_x(C). \tag{2.7}$$

Finally define

$$\Delta_{x,C}(y,D) = \{z \in \Gamma\big(y, (y+\tilde{Q}_x(C)) \cap D\big) \cap \partial_{in}(y+\tilde{Q}_x(C)) : \\ \langle zw \rangle \text{ is open for some } w \in (y+G_x)\backslash(y+\tilde{Q}_x(C))\}.$$

Note that every site in $\Delta_{x,C}(y,D)$ is connected to y by an open path (not necessarily clean!) with all sites in $(y+\tilde{Q}_x(C)) \cap D$, and is barely cleanly reachable from y inside $y+G_x$.

Remark 2.2. Let $u_0, .., u_n$ be sites of $\mathbb{Z}^d$. For Bernoulli bond percolation, from the FKG-Harris [15] and van den Berg-Kesten [8] inequalities one has

$$P(u_0 \leftrightarrow u_1 \leftrightarrow .. \leftrightarrow u_n) \geq \prod_{i=1}^{n} P(u_{i-1} \leftrightarrow u_i) \tag{2.8}$$

and

$$P(u_0 \leftrightarrow u_1 \leftrightarrow .. \leftrightarrow u_n \text{ via disjoint paths}) \leq \prod_{i=1}^{n} P(u_{i-1} \leftrightarrow u_i), \tag{2.9}$$

which together, roughly speaking, allow one to treat distinct segments of a path from u_0 to u_n as independent. Recall that such independence underlies the central limit theorem heuristic for Ornstein-Zernike behavior as in (1.3). The near-independence given by (2.8) and (2.9) was strongly exploited in [2], though not in the context of the central limit theorem, and the lack of an analog of (2.9) is perhaps the major difficulty in adapting the methods of [2] to other models. The ratio weak mixing property substitutes in part for (2.9), but its application requires in effect that one specify nonrandom disjoint sets of bonds on which the two events of interest are going to occur, which is not always feasible for pairs of events like $[u_{i-1} \leftrightarrow u_i]$ and $[u_{j-1} \leftrightarrow u_j]$ in the contexts we would like. Our solution, again roughly speaking, involves expressing an event $[u \leftrightarrow v]$ as a union $\cup_R [\Gamma(u,D) = R]$ for an appropriate choice of D, where the union is over an appropriate collection of sets R containing u and v. This is helpful because the event $[\Gamma(u,D) = R]$ necessarily takes place on the set of bonds $\overline{\mathcal{B}}(R)$ (cf. (2.6).)

For $D \subset \mathbb{Z}^d$ and $r \geq 0$ we define

$$D^r = \{x \in \mathbb{R}^d : d(x,D) \leq r\}.$$

Definition 2.3. For y and z sites in a path γ with y preceding z, we let $\gamma[y,z]$ denote the segment of γ from y to z. Suppose there is a path γ of open bonds in ω from 0 to z for some z. For $C > 1, x \in \mathbb{Z}^d$ and $r > 0$, we can then define the *gapped* (C,r,x)*-skeleton* derived from γ in ω, a finite sequence $\{(u_i, v_i, v_i', w_i), 0 \leq i \leq k\}$ of tuples of sites in γ, iteratively as follows. Let $u_0 = 0$ and $D_0 = \mathbb{Z}^d$. Having defined $u_0, .., u_i, v_0, .., v_{i-1}, v_0', .., v_{i-1}', w_0, .., w_{i-1}, D_0, .., D_i$,

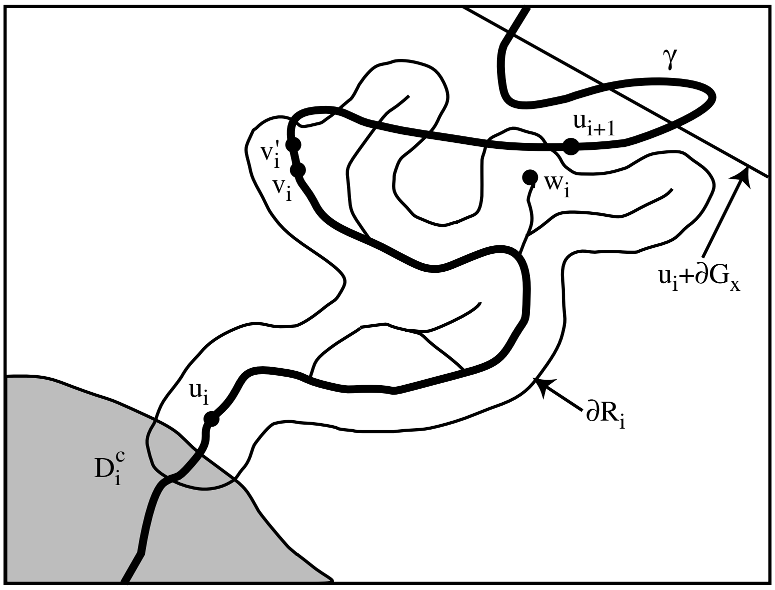

FIGURE 1. A short increment: $v_i \in \Delta_{x,C}(u_i, D_i)$. The site v_i' is not cleanly reachable from u_i. The path γ is the heavy line; lighter lines represent other paths in Γ_i, and boundaries of regions.

$\Gamma_0, .., \Gamma_{i-1}$, and $R_0, .., R_{i-1}$, let

$$\begin{aligned} \Gamma_i &= \Gamma(u_i, (u_i + \tilde{Q}_x(C)) \cap D_i) \\ R_i &= (\Gamma_i)^{r \log |x|} \\ D_{i+1} &= (R_0 \cup .. \cup R_i)^c. \end{aligned}$$

Then let v_i' be the first site of $\gamma[u_i, z]$ which is not in Γ_i, if such v_i' exists. If there is no such v_i', then $z \in \Gamma_i$ and we let $v_i' = v_i = w_i = z$ and end the construction; otherwise let v_i be the site in γ immediately preceding v_i'. Next let u_{i+1} be the first site of γ after v_i with the property that $\gamma[u_{i+1}, z]$ is contained in D_{i+1}, if such u_{i+1} exists. If no such u_{i+1} exists then $z \in R_i \backslash \Gamma_i$ and we let $v_i' = v_i = w_i = z$ and end the construction. Let w_i be the closest site to u_{i+1} in Γ_i. Note that $v_k' = v_k = w_k = z$. See Figures 1 and 2.

From the definition of u_{i+1}, the site u_{i+1}' immediately preceding u_{i+1} in γ must be in $\cup_{j=0}^{i} R_j$. Since $\gamma[u_i, z]$ does not intersect R_j for $j < i$, we must in fact have $u_{i+1}' \in R_i$. Therefore

$$r \log |x| \leq |u_{i+1} - w_i| \leq 1 + r \log |x|. \tag{2.10}$$

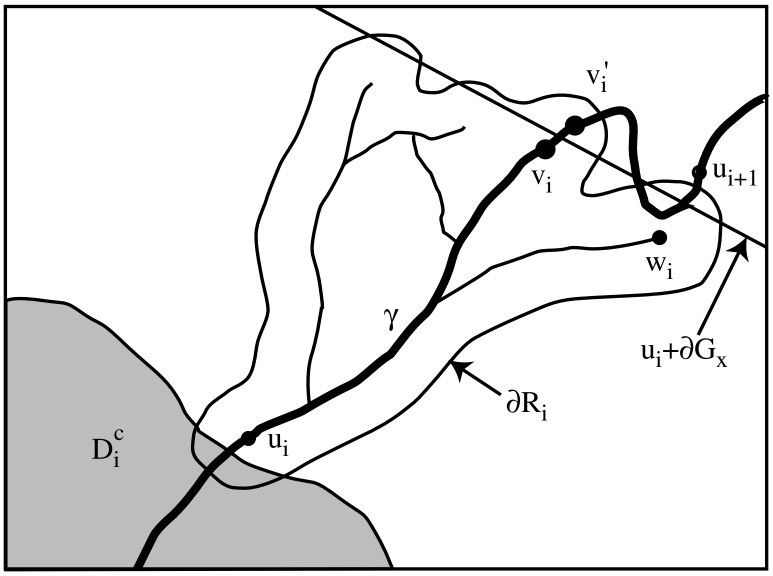

FIGURE 2. A long increment: $v_i \in u_i + \partial_{in} G_x$. The path γ stays in the cleanly reachable region all the way to $u_i + \partial G_x$.

The gapped (C, r, x)-skeleton then has the following properties:

(2.11) For each i there exist both an open path ψ_i from u_i to w_i, and the open path $\gamma[u_i, v_i]$ from u_i to v_i, each having all sites in D_i and all sites cleanly reachable from u_i inside $u_i + G_x$.

(2.12) For $i \neq j$, the clusters Γ_i and Γ_j are separated by a distance of at least $r \log |x|$.

(2.13) For each $i \leq k-1, v_i \in (u_i + \partial_{in} G_x) \cup \Delta_{x,C}(u_i, D_i)$.

Note that the paths ψ_i are not necessarily segments of the path γ, and we need not have $w_i \in \gamma$. For fixed C, from (2.13) we divide the indices into two classes, corresponding to "short" and "long" increments $v_i - u_i$, as follows (see Figures 1 and 2):

$$S\big((u_i, v_i, v_i', w_i)_{i \leq k}\big) = \{i : 0 \leq i \leq k-1, v_i \in \Delta_{x,C}(u_i, D_i) \backslash (u_i + \partial_{in} G_x)\}$$
$$L\big((u_i, v_i, v_i', w_i)_{i \leq k}\big) = \{i : 0 \leq i \leq k-1, v_i \in u_i + \partial_{in} G_x\}.$$

Set

$$p = \min_i P(\omega_{\{0e_i\}} = 1),$$

so $p > 0$ by (2.3), and note that by positive connection correlations,

$$P(0 \leftrightarrow x) \geq p^{|x|_1} \quad \text{for all } x. \tag{2.14}$$

The next lemma summarizes some basic properties of the quantities we have defined.

Lemma 2.4. *(i) Given $C > 1$ there exists a constant $C_1(C)$ such that if $y \in Q_x(C)$ and $|x| \geq C_1(C)$ then*

$$m(y) \leq 2m(x) \quad \text{and} \quad |y| \leq 2dM_0|x|/m_0.$$

(ii) For all $y \in \mathbb{Z}^d, 0 \leq s_x(y) \leq 2|y|_1 \log \frac{1}{p}$.

(iii) If $y \in \partial_{in} G_x$ then $m_x(y) \geq m(x) - M_0$.

Proof. (i) Suppose $m(y) > 2m(x)$ and $m_x(y) \leq m(x)$. Then from (2.2) and (2.5),

$$2m(x) < m(y) \leq h(y) = m_x(y) + s_x(y) \leq m(x) + s_x(y),$$

so from (2.4), $s_x(y) > m(x) > C \log|x|$, provided $|x|$ is large (depending on C.) Thus $y \notin Q_x(C)$ and the first inequality in (i) follows. The second inequality then follows from (2.4).

(ii) The fact that s_x is nonnegative has already been noted. From (2.2), (2.5) and (2.14) we have

$$s_x(y) \leq h(y) + |m_x(y)| \leq 2h(y) \leq 2|y|_1 \log \frac{1}{p}.$$

(iii) We have $z = y \pm e_i$ for some $z \notin G_x$ and $i \leq d$. Therefore using (2.5) we have $m_x(y) = m_x(z) - m_x(\pm e_i) \geq m(x) - M_0$. ☐

Let $\operatorname{diam}(B)$ denote the d-diameter of a set B. The following is immediate from the definition of ratio weak mixing.

Lemma 2.5. *Let P be a bond percolation model on $\mathbb{Z}^d$ with the ratio weak mixing property. There exists a constant C_2 as follows. Suppose $s > 3$ and $U, V \subset \mathbb{Z}^d$ with $\operatorname{diam}(U) \leq s$ and $d(U, V) \geq C_2 \log s$. Then for $D \in \mathcal{F}_U, E \in \mathcal{F}_V$ we have $P(D \cap E) \leq 2P(D)P(E)$.*

For $y \in \mathbb{Z}^d$ and $r > 0$, define $B(y, r) = \{z \in \mathbb{Z}^d : |z - y| \leq r\}$.

Lemma 2.6. *Assume (1.6). There exists C_3 such that for $r \geq C_3$,*

$$P(0 \leftrightarrow \partial B(0, r)) \leq e^{-m_0 r/2d}.$$

Proof. For $y \in \partial B(0, r)$ we have $m(y) > m_0 r/d$ by (2.4), so $P(0 \leftrightarrow y) \leq e^{-m_0 r/d}$. The result follows easily. ☐

We say there is an *r-near connection* from y to z in the configuration ω if there exist u, v such that $|u - v| \leq r, y \leftrightarrow u$ in ω, and $v \leftrightarrow z$ in ω.

Lemma 2.7. *Assume (1.6). There exist C_4 and C_5 such that if $|y| > 1, x \neq 0$ and $r \geq C_4 \log |y|$ then*

$$P(\text{there is an } r\text{-near connection from 0 to } y) \leq e^{-m_x(y)+C_5 r}.$$

Proof. By Lemma 2.6, (2.4) and (2.5) there exists c_1 such that

$$P(0 \leftrightarrow \partial B(0, c_1|y|)) \leq e^{-m_x(y)}. \tag{2.15}$$

Therefore we need only consider r-near connections in $\mathcal{B}(B(0, c_1|y|))$. Let $E = B(0, c_1|y|)$ and $\Gamma_0 = \Gamma(0, E \backslash B(y, r))$, and for $R \subset \mathbb{Z}^d$ let $F(R) = (R^r)^c \cap E$, so

$$\begin{aligned} [\Gamma_0 = R] \in \mathcal{F}_{\overline{R}}, \quad \operatorname{diam}(F(R)) &\leq \operatorname{diam}(E) \leq 2c_1|y| \\ \text{and} \quad d(\overline{R}, F(R)) &\geq C_4 \log |y| - 1. \end{aligned} \tag{2.16}$$

If there is an r-near connection, but not a connection, from 0 to y in $\mathcal{B}(E)$ in a configuration ω, let $v(\omega)$ be the closest site to Γ_0 which has an open path to y in $\mathcal{B}(F(\Gamma_0))$, and let $u(\omega)$ be the closest site to $v(\omega)$ in Γ_0; ties are broken arbitrarily. The existence of the r-near connection implies that $r \leq |u(\omega) - v(\omega)| \leq r + 1$. Note that

$$m_x(v - u) \leq c_2 r.$$

Using this, along with (2.16) and Lemma 2.5, we obtain

$$\begin{aligned} &P(\text{there is an } r - \text{near connection from 0 to } y \text{ in } \mathcal{B}(E)) \\ &\leq P(0 \leftrightarrow y) + \sum_{R,u,v} P\big(\Gamma_0 = R, u(\omega) = u, v(\omega) = v\big) \\ &\leq P(0 \leftrightarrow y) + \sum_{R,u,v} P\big(\Gamma_0 = R, v \leftrightarrow y \text{ in } \mathcal{B}(F(R))\big) \\ &\leq P(0 \leftrightarrow y) + \sum_{R,u,v} 2P\big(\Gamma_0 = R)P(v \leftrightarrow y) \\ &\leq P(0 \leftrightarrow y) + \sum_{u,v} 2P(0 \leftrightarrow u)P(v \leftrightarrow y) \\ &\leq e^{-m_x(y)} + \sum_{u,v} 2e^{-m_x(u)-m_x(y-v)} \\ &= e^{-m_x(y)} + \sum_{u,v} 2e^{-m_x(y)+m_x(v-u)} \\ &\leq e^{-m_x(y)} + 2|E|^2 e^{-m_x(y)+c_2 r} \end{aligned} \tag{2.17}$$

$$\le e^{-m_x(y)+c_3 r},$$

where the sums are over all $u, v \in E$ with $r \le |v-u| \le r+1$ and over all possible values R of Γ_0 containing u. Together (2.15) and (2.17) yield the lemma. $\square$

From (2.2) and (2.5), the probability of an open path $0 \leftrightarrow y$ is at most $e^{-m_x(y)}$. Consider for some C an (x, C)-unclean open path $0 \leftrightarrow u \leftrightarrow v \leftrightarrow y$ with $s_x(v-u) \ge C \log |x|$. One can ask whether the cost of such a path (measured by the negative log of the probability) is increased by an amount of order $C \log |x|$, meaning that the probability is at most $e^{-m_x(y)-cC\log|x|}$. For Bernoulli percolation the van den Berg-Kesten inequality [8] can be used to show there is always such a cost increase. But for dependent percolation the situation is more complex. Consider the situation in which u and v are approximately on the straight line $[0, y]$, with v closer to 0, so that the path of interest "doubles back" from u to v on the way to y. If the doubling back occurs in a narrow enough tube around the straight line $[0, y]$, then the three near-parallel segments of the path between approximately v and u are not far enough apart for (ratio) weak mixing to ensure that there is any extra cost. The next lemma, however, shows that if after doubling back from u to v (or otherwise traversing an expensive segment) the path does not return to a neighborhood of u, then an extra cost is indeed paid.

Lemma 2.8. *Assume (1.6). There exist C_6 with the following property: For every $a \ge 1$ and $0 < b < C_6$, there exist $C_7(a, b), C_8(a, b)$ and $C_9(a, b)$ such that if $C \ge C_7, |x| \ge C_8$ and $|y| \le a|x|$, then*

$$\begin{aligned}(2.18)\quad & P(\text{for some } u, v \in \mathbb{Z}^d \text{ with } s_x(v-u) \ge C \log |x|, 0 \leftrightarrow y \text{ via a path } \gamma \text{ which} \\ & \qquad \text{visits } u \text{ before } v \text{ and does not return to } B(u, bC \log |x|) \text{ after visiting } v) \\ & \le e^{-m_x(y)-C_9 C \log |x|}.\end{aligned}$$

Proof. By Lemma 2.6, (2.4) and (2.5), there exists $c_4(a)$ such that for $E = B(0, c_4(|x| + C \log |x|))$, we have

$$P(0 \leftrightarrow \partial E) \le e^{-m_x(y)-C\log|x|} \quad \text{for all } |y| \le a|x|, \tag{2.19}$$

so it is sufficient to consider paths γ within $\mathcal{B}(E)$.

For $u, v \in \mathbb{Z}^d$ let

$$\begin{aligned} B_u &= B(u, bC \log |x|), \quad \tilde{B}_u = B(u, \frac{1}{2} bC \log |x|), \\ S_v &= B(v, 4dc_6 C \log |x|), \quad \tilde{S}_v = B(v, 2dc_6 C \log |x|), \end{aligned}$$

where $0 < b < c_5$; here $c_5 < c_6$ are constants to be specified later. Provided c_6 is small enough, we obtain using Lemma 2.4 that for some $c_7 < 1/8$, for all u, v with $s_x(v-u) \geq C \log|x|$,

$$\begin{aligned} &m(w-t) \geq 2c_7 C \log|x| \quad \text{and} \quad s_x(w-t) \geq \frac{C}{2} \log|x| \\ &\text{for every } t \in \overline{B_u}, w \in \overline{S_v} \end{aligned} \tag{2.20}$$

so that in particular B_u and S_v are disjoint. Further, again provided c_6 is small enough, we have

$$|m_x(q-r)| \leq c_7 C \log|x| \quad \text{for all } q, r \in \overline{S_0} \tag{2.21}$$

and if also c_5 is small enough relative to c_6,

$$|m_x(t-s)| < \frac{m_0}{4} c_6 C \log|x| < c_7 C \log|x| \quad \text{for all } s, t \in \overline{B_0}. \tag{2.22}$$

Fix $y \in \mathbb{Z}^d$. For u, v with $s_x(v-u) \geq C \log|x|$, let $A(u,v)$ be the event that there exists an open path γ from 0 to y in $\mathcal{B}(E)$ which visits u before v and does not return to B_u after reaching v.

Case 1. $0, y \notin B_u \cup S_v$. We can then further decompose $A(u,v)$ as follows: for $s, t \in \partial B_u$ and $w, z \in \partial S_v$, let $A(u,v;s,t,w,z)$ be the event that there exists γ as above which first reaches ∂B_u at s, which last exits B_u via a step to t, which has $\gamma[t,y]$ first enter S_v at w, and which last exits S_v via a step to z.

Ideally, when $A(u,v;s,t,w,z)$ occurs we would like the three segments $\gamma[0,s]$ from 0 to ∂B_u, $\gamma[t,w]$ from ∂B_u to ∂S_v, and $\gamma[z,y]$ from ∂S_v to y to be well-separated from one another, so that Lemma 2.5 can be applied, but in fact there may be various unwanted connections or near-connections outside B_u and/or S_v which we must handle. Depending on the presence of these near-connections, the source of the extra cost $C_9 C \log|x|$ exhibited in (2.18) is either the expensive segment from u to v, or the connection from u to $\partial\tilde{B}_u$ inside B_u, or the connection from v to $\partial\tilde{S}_v$ inside S_v.

For $q, r \in A \subset \mathbb{Z}^d$, let $N(q,r,A)$ be the event that there is a $(2 + c_8 bC \log|x|)$-near connection from q to r in $\mathcal{B}(A)$, where c_8 is a (small) constant to be specified.

Note that the distance $2 + c_8 bC \log|x|$ quantifying "near-connections" is much less than the diameter of B_u; B_u, in turn, is much smaller than S_v.

First consider $A(u,v) \cap N(0,y,E \backslash B_u)$. When this event occurs we have a near-connection from 0 to y outside B_u, and a connection from u to $\partial\tilde{B}_u$. If c_8 is sufficiently small (depending on C_5), and C and x are sufficiently large (depending on a, b), then Lemmas 2.5, 2.6 and 2.7

give for some c_9

$$
\begin{aligned}
(2.23) \qquad & P(A(u,v) \cap N(0,y,E\backslash B_u)) \\
&\le P([u \leftrightarrow \partial \tilde{B}_u] \cap N(0,y,E\backslash B_u)) \\
&\le 2P(u \leftrightarrow \partial \tilde{B}_u) P(N(0,y,\mathbb{Z}^d)) \\
&\le 2e^{-m_x(y)+C_5 c_8 bC \log|x| - m_0 bC(\log|x|)/4d} \\
&\le 2e^{-m_x(y)-c_9 bC\log|x|}.
\end{aligned}
$$

Second, consider $A(u,v;s,t,w,z) \cap N(0,y,E\backslash B_u)^c \cap N(t,y,E\backslash(B_u \cup S_v))^c$. When this occurs we have clusters $\Gamma(0,E\backslash B_u)$ containing the sites of $\gamma[0,s]$, $\Gamma(t,E\backslash(B_u \cup S_v))$ containing the sites of $\gamma[t,w]$, and $\Gamma(y,E\backslash(B_u\cup S_v))$ containing the sites of $\gamma[z,y]$, these clusters being separated from each other by at least $2+c_8 bC\log|x|$. Therefore using Lemma 2.5, (2.6), (2.20) and (2.21), if C is sufficiently large (depending on b) and if $|x|$ is sufficiently large (depending on a,b), we obtain that for some c_{10},

(2.24)

$$
\begin{aligned}
&P(A(u,v;s,t,w,z) \cap N(0,y,E\backslash B_u)^c \cap N(t,y,E\backslash(B_u\cup S_v))^c) \\
&\quad \le \sum_{I,J,K} P\big(\Gamma(0,E\backslash B_u)=I, \Gamma(t,E\backslash(B_u\cup S_v))=J, \Gamma(y,E\backslash(B_u\cup S_v))=K\big) \\
&\quad \le \sum_{I,J,K} 4P\big(\Gamma(0,E\backslash B_u)=I\big)P\big(\Gamma(t,E\backslash(B_u\cup S_v))=J\big)P\big(\Gamma(y,E\backslash(B_u\cup S_v))=K\big) \\
&\quad \le 4P(0\leftrightarrow s)P(t\leftrightarrow w)P(z\leftrightarrow y) \\
&\quad \le 4e^{-[m_x(s)+m_x(w-t)+s_x(w-t)+m_x(y-z)]} \\
&\quad = 4e^{-[m_x(y)-m_x(t-s)-m_x(z-w)+s_x(w-t)]} \\
&\quad \le 4e^{-m_x(y)+2c_7 C\log|x| - \frac{C}{4}\log|x|} \\
&\quad \le e^{-m_x(y)-c_{10}C\log|x|}
\end{aligned}
$$

where the sums are over all $I,J,K \subset E$ with $0,s\in I, t,w\in J, z,y\in K$, and $\min(d(I,J), d(I,K), d(J,K)) > 2+c_8 bC\log|x|$.

Third, consider $A(u,v;s,t,w,z)\cap N(0,y,E\backslash B_u)^c \cap N(0,s,E\backslash(B_u\cup S_v))^c$. When this occurs and $\Gamma(0,E\backslash(B_u\cup S_v))=I, \Gamma(s,E\backslash(B_u\cup S_v))=J, \Gamma(0,E\backslash B_u)=K$, $\Gamma(y,E\backslash(B_u\cup S_v))=L$ for some I,J,K,L, we must have $I\cup J\subset K, d(I,J) > 2+c_8 bC\log|x|$ and $d(K,L) > 2+c_8 bC\log|x|$; I contains the sites of an open path from 0 to ∂S_v, J contains the sites of an open path from ∂S_v to s, and L contains the sites of $\gamma[z,y]$. (Here we do not make use of the cluster containing t and w.) Therefore using Lemma 2.5, (2.6), (2.20) and (2.21), if

C and x are sufficiently large (depending on a, b), we obtain that for some c_{11},

(2.25)

$$\begin{aligned}
&P\big(A(u,v;s,t,w,z) \cap N(0,y,E\backslash B_u)^c \cap N(0,s,E\backslash(B_u \cup S_v))^c\big) \\
&\quad \le \sum_{I,J,L} P\big(\Gamma(0,E\backslash(B_u \cup S_v)) = I, \Gamma(s,E\backslash(B_u \cup S_v)) = J, \Gamma(y,E\backslash B_u) = L\big) \\
&\quad \le \sum_{I,J,L} 4P\big(\Gamma(0,E\backslash(B_u \cup S_v)) = I\big)P\big(\Gamma(s,E\backslash(B_u \cup S_v)) = J\big)P\big(\Gamma(y,E\backslash B_u) = L\big) \\
&\quad \le 4P(0 \leftrightarrow \partial S_v)P(\partial S_v \leftrightarrow s)P(z \leftrightarrow y) \\
&\quad \le 4 \sum_{q,r \in \partial S_v} P(0 \leftrightarrow q)P(r \leftrightarrow s)P(z \leftrightarrow y) \\
&\quad \le 4 \sum_{q,r \in \partial S_v} e^{-[m_x(q)+m(s-r)+m_x(y-z)]} \\
&\quad = 4 \sum_{q,r \in \partial S_v} e^{-m_x(y)+m_x(z-q)-m(s-r)} \\
&\quad \le |\partial S_v|^2 e^{-m_x(y)-c_7 C \log|x|} \\
&\quad \le e^{-m_x(y)-c_{11} C \log|x|}
\end{aligned}$$

where the sum is over $I, J, L \subset E$ with $0 \in I, I \cap \partial S_v \neq \phi, s \in J, J \cap \partial S_v \neq \phi, y, z \in L$, and $\min(d(I,J), d(I,L), d(J,L)) > 2 + c_8 bC \log|x|$.

Fourth, consider

$$A(u,v;s,t,w,z) \cap N(0,y,E\backslash B_u)^c \cap N(0,s,E\backslash(B_u \cup S_v)) \cap N(t,y,E\backslash(B_u \cup S_v)).$$

When this occurs and $\Gamma(0,E\backslash(B_u \cup S_v)) \cup \Gamma(s,E\backslash(B_u \cup S_v)) = I, \Gamma(0,E\backslash B_u) = J, \Gamma(t,E\backslash(B_u \cup S_v)) \cup \Gamma(y,E\backslash(B_u \cup S_v)) = K$ and $\Gamma(y,E\backslash B_u) = L$ for some I, J, K, L, we must have $I \subset J, K \subset L$ and $d(J,L) \ge 2 + c_8 bC \log|x|$; I contains the sites of a near-connection from 0 to s, and K contains the sites of a near-connection from t to y. There is also an open path from v to $\partial\tilde{S}_v$. Therefore assuming b is small enough relative to c_6, using Lemmas 2.5 - 2.7, (2.6) and (2.22), if C and x are sufficiently large (depending on a, b), we obtain

(2.26)

$$\begin{aligned}
&P\big(A(u,v;s,t,w,z) \cap N(0,y,E\backslash B_u)^c \cap N(0,s,E\backslash(B_u \cup S_v)) \cap N(t,y,E\backslash(B_u \cup S_v))\big) \\
&\quad \le \sum_{I,K} P\big(\Gamma(0,E\backslash(B_u \cup S_v)) \cup \Gamma(s,E\backslash(B_u \cup S_v)) = I, \\
&\qquad\qquad \Gamma(t,E\backslash(B_u \cup S_v)) \cup \Gamma(y,E\backslash(B_u \cup S_v)) = K, v \leftrightarrow \partial\tilde{S}_v\big)
\end{aligned}$$

$$\leq \sum_{I,K} 4P\big(\Gamma(0, E\backslash(B_u \cup S_v)) \cup \Gamma(s, E\backslash(B_u \cup S_v)) = I\big)$$
$$P\big(\Gamma(y, E\backslash(B_u \cup S_v)) \cup \Gamma(t, E\backslash(B_u \cup S_v)) = K\big) P(v \leftrightarrow \partial\tilde{S}_v)$$
$$\leq 4P(N(0, s, \mathbb{Z}^d))P(N(t, y, \mathbb{Z}^d))P(v \leftrightarrow \partial\tilde{S}_v)$$
$$\leq 4e^{-m_x(s)-m_x(y-t)+2C_5c_8bC\log|x|-m_0c_6C\log|x|}$$
$$\leq e^{-m_x(y)+m_x(t-s)-\frac{1}{2}m_0c_6C\log|x|}$$
$$\leq e^{-m_x(y)-\frac{1}{4}m_0c_6C\log|x|}.$$

where the sum is over those $I \ni 0, s$ and $K \ni t, y$ consistent with the event appearing in the first sum in (2.26), with $d(I, K) \geq 2 + c_8bC\log|x|$. Combining (2.23), (2.24), (2.25) and (2.26)and summing over s, t, w, z, provided C and x are sufficiently large (depending on a, b), we obtain

$$P(A(u, v)) \leq e^{-m_x(y)-C_{14}bC\log|x|}. \tag{2.27}$$

It remains to consider cases with 0 and/or y in $B_u \cup S_v$. Note that when $A(u, v)$ occurs we cannot have $y \in B_u$. Also, the bound (2.23) is valid regardless of the locations of u and v (the left side is 0 if $0 \in B_u$.)

Case 2. $0 \in B_u,\ y \notin B_u \cup S_v$. Here there is no longer a site s but we can define t, w, z and $A(u, v; t, w, v)$ similarly to Case 1. Similarly to (2.24) we obtain

$$\begin{aligned} &P\big(A(u, v, t, w, z) \cap N(0, y, E\backslash B_u)^c \cap N(t, y, E\backslash(B_u \cup S_v))^c\big) \\ &\quad\leq 2P(t \leftrightarrow w)P(z \leftrightarrow y) \\ &\quad\leq 2e^{-[m_x(w-t)+s_x(w-t)+m_x(y-z)]} \\ &\quad= 2e^{-[m_x(y)-m_x(t)-m_x(z-w)+s_x(w-t)]} \\ &\quad\leq 2e^{-m_x(y)+2c_7C\log|x|-\frac{C}{4}\log|x|} \\ &\quad\leq e^{-m_x(y)-c_{10}C\log|x|}, \end{aligned} \tag{2.28}$$

while similarly to (2.26) we obtain

$$\begin{aligned} &P\big(A(u, v; t, w, z) \cap N(0, y, E\backslash B_u)^c \cap N(t, y, E\backslash(B_u \cup S_v))\big) \\ &\quad\leq 2P(N(t, y, \mathbb{Z}^d))P(v \leftrightarrow \partial\tilde{S}_v) \\ &\quad\leq 4e^{-m_x(y-t)+C_5c_8bC\log|x|-m_0c_6C\log|x|} \\ &\quad\leq e^{-m_x(y)+m_x(t)-\frac{1}{2}m_0c_6C\log|x|} \\ &\quad\leq e^{-m_x(y)-C_{14}C\log|x|}. \end{aligned} \tag{2.29}$$

Summing these over t, w, z and combining with (2.23) yields (2.27).

Case 3. $0 \in S_v, y \notin B_u \cup S_v$. Similarly to (2.25), using (2.20) and (2.21) we obtain

$$\begin{aligned} P\big(A(u,v;s,t,w,z) \cap N(0,y,E\backslash B_u)^c\big) \\ \leq 2P(0 \leftrightarrow s)P(z \leftrightarrow y) \\ \leq 2e^{-[m(s)+m_x(y-z)]} \\ \leq e^{-\frac{C}{2}\log|x| - m_x(y) + c_7 C \log|x|} \\ \leq e^{-m_x(y) - \frac{C}{4}\log|x|}. \end{aligned} \tag{2.30}$$

Summing over s, t, w, z and combining with (2.23) again yields (2.27).

Case 4. $0 \notin B_u \cup S_v, y \in S_v$. This time there is no longer a site z but we can define s, t, w and $A(u,v;s,t,w)$ similarly to Case 1. Similarly to (2.24) we obtain

$$\begin{aligned} P\big(A(u,v;s,t,w) \cap N(0,y,E\backslash B_u)^c\big) \\ \leq 2P(0 \leftrightarrow s)P(t \leftrightarrow w) \\ \leq 2e^{-[m_x(s)+m_x(w-t)+s_x(w-t)]} \\ = 2e^{-[m_x(y)-m_x(t-s)-m_x(y-w)+s_x(w-t)]} \\ \leq 2e^{-m_x(y)+2c_7 C\log|x| - \frac{C}{4}\log|x|} \\ \leq e^{-m_x(y)-c_{10}C\log|x|}. \end{aligned} \tag{2.31}$$

Once again, summing over s, t, w and combining with (2.23) yields (2.27).

Case 5. $0 \in B_u, y \in S_v$. Here

$$\begin{aligned} P(A(u,v)) &\leq P(u \leftrightarrow v) \\ &\leq e^{-[m_x(v-u)+s_x(v-u)]} \\ &= e^{-[m_x(y)-m_x(y-v)-m_x(u)+s_x(v-u)]} \\ &\leq e^{-m_x(y)+2c_7 C\log|x| - C\log|x|} \\ &\leq e^{-m_x(y)-\frac{C}{2}\log|x|}, \end{aligned} \tag{2.32}$$

so again (2.27) is valid.

Case 6. $0, y \in S_v$. Here, from (2.20) and (2.21),

$$\begin{aligned} P(A(u,v)) &\leq P(0 \leftrightarrow u) \\ &\leq e^{-m(u)} \\ &\leq e^{-2c_7 C \log |x|} \\ &\leq e^{-m_x(y) - c_7 C \log |x|}, \end{aligned}$$

so once more (2.27) is valid.

Thus (2.27) is valid for all $u, v \in E$ with $s_x(v-u) \geq C \log |x|$. Summing over such u, v and combining with (2.19) yields (2.18). □

As discussed in the remarks preceding Lemma 2.8, an unclean open path from 0 to some y need not cost more than a clean one, outside of the circumstances of (2.18). The next lemma shows that when *all* paths to y, open or not, are unclean, then every open path to y must be as in (2.18), so an extra cost is always paid. The lemma is valid for more general G than stated, but we only need the halfspace case.

Lemma 2.9. *Assume (1.6). There exist C_i such that if G is either $\mathbb{Z}^d$ or the intersection of a halfspace with $\mathbb{Z}^d$, $C \geq C_{11}, |x| \geq C_{12}$ and $y \in Q_x(C)$ is not (x, C)-cleanly reachable from 0 inside G, then*

$$P(0 \leftrightarrow y \text{ in } \mathcal{B}(G)) \leq e^{-m_x(y) - C_{10} C \log |x|}.$$

Proof. Suppose y is not (x, C)-cleanly reachable from 0 but $0 \leftrightarrow y$ via some path γ of open bonds in $\mathcal{B}(G)$. (If there is more than one such γ, we can choose one arbitrarily.) Define $0 = w_0, w_1, .., w_m = y$ inductively as follows: w_{i+1} is the first site in γ after w_i for which $\gamma[w_{i+1}, z] \subset B(w_i, C_{15} C \log |x|)^c$, where C_{15} is a constant to be specified; if there is no such w_{i+1} for some value $i = m-1$, then $y \in B(w_i, C_{15} C \log |x|)$ and we end the construction. Let $B_i = B(w_i, C_{15} C \log |x|)$.

Since $w_{i+1} \in \partial B_i$, it is easy to see that there exists a lattice path α from 0 to y in $\mathcal{B}(G)$ contained in $\cup_{i=0}^m B_i$, and α can be chosen so that once α leaves any of the balls B_i, it does not return to $\cup_{j \leq i} B_j$. (Since $G \cap B_i$ is "connected" for each i, one need only ensure that when α exits any ball B_i, it exits into the ball B_j of maximal index j for which $B_j \cap B_i \neq \phi$. Informally speaking, α is approximately γ with doublebacks erased.) Since y is not (x, C)-cleanly reachable from 0 inside G, there must exist sites u, v in α with u preceding v such that $s_x(v-u) \geq C \log |x|$. Let k and l be such that $u \in B_k, v \in B_l$. Provided C_{15} is small enough, we have by Lemma 2.4(ii):

$$|s_x(u - w_k)| \leq \frac{1}{4} C \log |x|, \qquad |s_x(v - w_l)| \leq \frac{1}{4} C \log |x|$$

and therefore

$$s_x(w_l - w_k) \geq \frac{1}{2} C \log |x|, \tag{2.33}$$

which by Lemma 2.4(ii) implies $|w_l - w_k| > 2C_{15}C \log |x|$. Thus B_k and B_l are disjoint. Since α visits u before v and does not visit $\cup_{j \leq k} B_j$ after leaving B_k, it follows that $k < l$ and so w_k precedes w_l in γ. But now we are in the situation of Lemma 2.8 (with $b = C_{15}$, $a = 2dM_0/m_0$ obtained from Lemma 2.4(i), and $C/2$ in place of C): we have an open path γ from 0 to y which visits w_k, then w_l, after which it does not return to B_k, and (2.33) holds; provided C_{15} is small enough, the lemma follows. □

Let $m_x^+(\cdot) = \max(m_x(\cdot), 0)$.

Lemma 2.10. *Assume (1.6). There exist constants C_i such that if $D \subset \mathbb{Z}^d, C \geq 1, |x| \geq C_{13}(C)$ and $v, w \in \tilde{Q}_x(C)$, then*

$$P(v \in \Delta_{x,C}(0, D) \text{ and } 0 \leftrightarrow w) \leq e^{-m_x^+(w) - C_{12}C \log |x|}. \tag{2.34}$$

Note that, in contrast to (2.34), from (2.2) and (2.5) one obtains that the probability of the event $0 \leftrightarrow w$ alone can be bounded above by $e^{-m(w)}$ or $e^{-m_x^+(w)}$. The significance of Lemma 2.10 is that because $v \in \Delta_{x,C}(0, D)$, so that v is barely cleanly reachable from 0, the additional presence of a path $0 \leftrightarrow v$ introduces an extra cost of $C_{12}C \log |x|$, even though the two paths need not be disjoint.

Proof of Lemma 2.10. There exist $C_{16}(C)$ and c_{15} such that $|x| \geq C_{16}$ and $w \in \tilde{Q}_x(C)$ imply $|w| \leq c_{15}|x|$. Hence as in (2.19), we have for some c_{16}:

$$P(0 \leftrightarrow \partial B(0, c_{16}|x|)) \leq e^{-m_x(w) - C \log |x|}. \tag{2.35}$$

Let $0 < c_{17} < c_{18}$ be constants to be specified later, let

$$E = B(0, c_{16}|x|), \qquad B_v = B(v, 4dc_{18}C \log |x|), \qquad \tilde{B}_v = B(v, 2dc_{18}C \log |x|),$$

and let N denote the event that there is a $(c_{17}C \log |x|)$-near connection from 0 to w in $\mathcal{B}(E \backslash B_v)$. If c_{17} is sufficiently small relative to c_{18}, and $|x|$ and C are sufficiently large, then by Lemmas 2.5, 2.6 and 2.7 we have

$$\begin{aligned} P\big([v \in \Delta_{x,C}(0, D)] \cap N\big) &\leq P([v \leftrightarrow \partial \tilde{B}_v] \cap N) \\ &\leq 2P(v \leftrightarrow \partial \tilde{B}_v) P(N) \\ &\leq 2e^{-m_0 c_{18} C \log |x|} e^{-m_x(w) + C_5 c_{17} C \log |x|} \\ &\leq 2e^{-m_x(w) - c_{19} C \log |x|}. \end{aligned} \tag{2.36}$$

Next, we suppose first that $w \notin \overline{B_v}$. Using Lemma 2.5, again assuming $|x|$ and C are sufficiently large,

$$\begin{aligned} P\big([v \in \Delta_{x,C}(0,D)] \cap [0 \leftrightarrow w] \cap N^c\big) \qquad (2.37)\\ \le P(0 \leftrightarrow \partial E) + \sum_{I,J} P\big(\Gamma(0, E\backslash B_v) = I, \Gamma(w, E\backslash B_v) = J\big) \\ \le P(0 \leftrightarrow \partial E) + \sum_{I,J} 2P\big(\Gamma(0, E\backslash B_v) = I\big) P\big(\Gamma(w, E\backslash B_v) = J\big) \\ \le P(0 \leftrightarrow \partial E) + 2P\big(0 \leftrightarrow \partial B_v \text{ in } \mathcal{B}(\tilde{Q}_x(C))\big) P(\partial B_v \leftrightarrow w), \end{aligned}$$

where the sum is over all I, J with $0 \in I, w \in J, J \cap \partial B_v \neq \phi, d(I,J) > c_{17} C \log|x|$ and with I containing the sites of a path from 0 to ∂B_v in $\mathcal{B}(\tilde{Q}_x(C))$. Presuming c_{18} is sufficiently small we have $s_x(u - z) \le (C \log|x|)/4$ for all $u, z \in \overline{B_v}$. Since v is barely (x, C)-cleanly reachable from 0 inside G_x, it follows readily that no $y \in \partial B_v$ is $(x, C/2)$-cleanly reachable from 0 inside G_x. Hence by Lemma 2.9, for some $c_{20} < 1$,

$$P\big(0 \leftrightarrow y \text{ in } \mathcal{B}(\tilde{Q}_x(C))\big) \le e^{-m_x(y) - 5c_{20} C \log|x|} \qquad \text{for all } y \in \partial B_v. \qquad (2.38)$$

Further, if $|x|$ is large and c_{18} is chosen sufficiently small, for $y \in \overline{B_v}$ we have from (2.5) and (2.4):

$$|m_x(y) - m_x(v)| \le c_{20} C \log|x|. \qquad (2.39)$$

Therefore if $|x|$ is sufficiently large, (2.38) yields

$$2P\big(0 \leftrightarrow \partial B_v \text{ in } \mathcal{B}(\tilde{Q}_x(C))\big) \le 2|\partial B_v| e^{-m_x(v) - 4c_{20} C \log|x|} \le e^{-m_x(v) - 3c_{20} C \log|x|} \qquad (2.40)$$

and similarly

$$P(\partial B_v \leftrightarrow w) \le e^{-m_x(w-v) + 2c_{20} C \log|x|}. \qquad (2.41)$$

If $w \in \overline{B_v}$, the same argument applies with $P(\partial B_v \leftrightarrow w)$ replaced by 1 throughout, by (2.39). Combining (2.35), (2.37), (2.40) and (2.41) shows that

$$P\big([v \in \Delta_{x,C}(0,D)] \cap [0 \leftrightarrow w] \cap N^c\big) \le 3e^{-m_x(w) - c_{20} C \log|x|},$$

which with (2.36) proves (2.34) with m_x in place of m_x^+. But since v is barely (x, C)-cleanly reachable from 0 inside G_x, for some c_{21} and c_{22} we must have $|v| \ge c_{21} C \log|x|$ and hence the left side of (2.34) is bounded by

$$P(0 \leftrightarrow v) \le e^{-c_{22} C \log|x|},$$

so we can replace m_x with m_x^+ in (2.34). □

Let $\mathcal{S}(C,r,x)$ denote the set of all gapped (C,r,x)-skeletons derived from all paths γ (starting from 0) in all configurations ω, and

$$\mathcal{S}_{jl}(C,r,x) = \{(u_i,v_i,v_i')_{i\leq k} \in \mathcal{S}(C,r,x) : v_k = z, \\ |S((u_i,v_i,v_i',w_i)_{i\leq k})| = j, |L((u_i,v_i,v_i',w_i)_{i\leq k})| = l\}.$$

The next result is the analog of Lemma 2.3 of [2].

Lemma 2.11. *Assume (1.6). There exist constants $C_i > 3$ such that if $C \geq C_{13}, r \geq C_{14}$ and $|x| \geq C_{17}(C)$, then for n sufficiently large, there exist a configuration ω and a path γ from 0 to nx for which the gapped (C,r,x)-skeleton consists of at most $3n$ tuples.*

Proof. Fix $x \in \mathbb{Z}^d$ and $C > 1$. The conclusion will follow if we can show that

$$P(0 \leftrightarrow nx \text{ via a path } \gamma \text{ for which the gapped } (C,r,x) - \text{skeleton consists of} \\ \text{more than } 3n \text{ tuples}) < P(0 \leftrightarrow nx), \quad \text{for } n \text{ large.} \tag{2.42}$$

From the definition of $m(x)$ we have

$$P(0 \leftrightarrow nx) \geq 2^{-n} e^{-nm(x)} \quad \text{for } n \text{ large.} \tag{2.43}$$

Fix j,l and $U = (u_i,v_i,v_i',w_i)_{i\leq k} \in \mathcal{S}_{jl}(C,r,x,z)$. Let $\Lambda_i, 0 \leq i \leq k$, be subsets of $\mathcal{B}(\mathbb{Z}^d)$ which are possible values of the clusters $\Gamma(u_i,(u_i+\tilde{Q}_x(C)) \cap D_i)$, subject to (2.12) and satisfying $v_i,w_i \in \Lambda_i$ whenever $\Lambda_i \neq \phi$, where $D_i = \mathbb{Z}^d \backslash \cup_{j<i} (\Lambda_j)^{r\log|x|}$ as in Definition 2.3. That is, we suppose there exists a configuation ω for which $v_i,w_i \in \Lambda_i$ whenever $\Lambda_i \neq \phi$, and $\Gamma(u_i,(u_i+\tilde{Q}_x(C)) \cap D_i,\omega) = \Lambda_i$ for all $i \leq k$, and (2.10) holds. (Note that we can have $\Lambda_i = \phi$ only for $i = k$.) We call such sequences of sets $\Lambda_i, 0 \leq i \leq k$, *allowable.* Then

$$\begin{aligned} &P\big(0 \leftrightarrow nx \text{ via a path } \gamma \text{ with gapped } (C,r,x) - \text{skeleton } U \\ &\qquad \text{and } \Gamma(u_i,(u_i+\tilde{Q}_x(C)) \cap D_i) = \Lambda_i \text{ for all } i \leq k\big) \\ &\quad \leq P\big(\Gamma(u_i,(u_i+\tilde{Q}_x(C)) \cap D_i) = \Lambda_i \text{ and } \langle v_iv_i'\rangle \text{ is open for all } i \leq k\big). \end{aligned} \tag{2.44}$$

Here, for convenience of notation, we define the event "$\langle v_kv_k'\rangle$ is open" to be the full probability space $\{0,1\}^{\mathcal{B}(\mathbb{Z}^d)}$, since $v_k = v_k' = z$. If $|x| \geq C_1(C)$ then by (2.7) and Lemma 2.4(i),

$$\operatorname{diam}(\overline{\Lambda}_i) \leq \operatorname{diam}(Q_x(C)) + 2 \leq c_{23}|x|. \tag{2.45}$$

Since the event $[\Gamma(u_i,(u_i+\tilde{Q}_x(C)) \cap D_i) = \Lambda_i$ and $\langle v_iv_i'\rangle$ is open$] \in \mathcal{F}_{\overline{\Lambda}_i}$, if $|x|$ and r are sufficiently large then from (2.12), (2.45) and Lemma 2.5 it follows that

$$\begin{aligned} &P\big(\Gamma(u_i,(u_i+\tilde{Q}_x(C)) \cap D_i) = \Lambda_i \text{ and } \langle v_iv_i'\rangle \text{ is open for all } i \leq k\big) \\ &\quad \leq 2^k \prod_{i\leq k} P\big(\Gamma(u_i,(u_i+\tilde{Q}_x(C)) \cap D_i) = \Lambda_i \text{ and } \langle v_iv_i'\rangle \text{ is open}\big). \end{aligned} \tag{2.46}$$

From (2.7) and Lemma 2.4(i), for a given value of u_i there are at most $c_{24}|x|^d$ choices for each of v_i, v_i' and w_i, for some $c_{24}(C)$. Therefore for some c_{25}, if $|x| \geq C_1(C)$ then

$$|\mathcal{S}_{jl}(C,r,x,nx)| \leq e^{c_{25}(j+l)\log|x|}. \tag{2.47}$$

If C is large enough and $|x| \geq c_{26}(C)$ then by (2.7), (2.44), (2.45), Lemma 2.10 and Lemma 2.4(iii), summing (2.46) over all allowable sequences $\{\Lambda_i, 0 \leq i \leq k\}$ gives

$$\begin{aligned} &P\big(0 \leftrightarrow nx \text{ via a path } \gamma \text{ with gapped } (C,r,x)-\text{skeleton } U\big) \\ &\quad \leq 2^k \prod_{i \leq k} P\big(v_i \in \Delta_{x,C}(u_i, D_i) \cup (u_i + \partial_{in} G_x); \\ &\qquad\qquad u_i \leftrightarrow v_i \text{ and } u_i \leftrightarrow w_i \text{ both in } \mathcal{B}(u_i + \tilde{Q}_x(C))) \\ &\quad \leq 2^k \exp\left(-\sum_{i \in S(U)} m_x^+(w_i - u_i) - C_{12}C|S(U)|\log|x| \right) \\ &\qquad \times \exp\left(-\max\left[\sum_{i \notin S(U)} m_x(w_i - u_i), \sum_{i \in L(U)} m_x(v_i - u_i) \right] \right) \\ &\quad \leq 2^{j+l} \exp\left(-\sum_{i \in S(U)} m_x^+(w_i - u_i) - C_{12}Cj\log|x| \right) \\ &\qquad \times \exp\left(-\max\left[\sum_{i \notin S(U)} m_x(w_i - u_i), l(m(x) - M_0) \right] \right). \end{aligned} \tag{2.48}$$

The remainder of the proof follows that of Lemma 2.3 of [2]. (It should be noted at this point that there is a significant misprint in that proof, corrected in Remark 2.13 below.) Choose C such that $C_{12}C \geq 4c_{25} + 6dM_0r$. We consider first $l \geq 3n$. If $|x|$ is large, then using (2.48), (2.47) and (2.43),

$$\begin{aligned} &\sum_{l \geq 3n} \sum_{j \geq 0} P\big(0 \leftrightarrow nx \text{ via a path } \gamma \text{ with gapped } (C,r,x)-\text{skeleton in } \mathcal{S}_{jl}(C,r,x,nx)\big) \\ &\quad \leq \sum_{l \geq 3n} \sum_{j \geq 0} 2^{j+l} e^{c_{25}(j+l)\log|x|} e^{-C_{12}Cj\log|x|} e^{-l(m(x)-M_0)} \\ &\quad \leq 2e^{-3n[m(x)-M_0-c_{25}\log|x|-\log 2]} \\ &\quad \leq e^{-2nm(x)} \\ &\quad = o(P(0 \leftrightarrow nx)) \quad \text{as } n \to \infty. \end{aligned} \tag{2.49}$$

Next we consider $n \le l < 3n$. Again from (2.48), (2.47) and (2.43), for $|x|$ large,

$$\begin{aligned}
(2.50)\qquad & \sum_{n\le l<3n}\sum_{j\ge 3n-l} P\big(0\leftrightarrow nx \text{ via a path } \gamma \text{ with gapped} \\
&\qquad\qquad (C,r,x)-\text{skeleton in } \mathcal{S}_{jl}(C,r,x,nx)\big) \\
&\le \sum_{n\le l<3n}\sum_{j\ge 3n-l} 2^{j+l}e^{c_{25}(j+l)\log|x|}e^{-C_{12}Cj\log|x|}e^{-l(m(x)-M_0)} \\
&\le 2e^{-2C_{12}Cn\log|x|}\sum_{l\ge n} e^{-l(m(x)-M_0-4c_{25}\log|x|)} \\
&\le 2e^{-nm(x)-n(C_{12}C\log|x|-M_0)} \\
&= o(P(0\leftrightarrow nx)) \quad \text{as } n\to\infty.
\end{aligned}$$

Finally we consider $l < n$. From (2.5), (2.4) and (2.10) we have

$$m_x(u_{i+1}-w_i) \le 2dM_0 r\log|x|$$

so

$$\sum_{i\le k} m_x(w_i-u_i) = m_x(nx) - \sum_{i\le k-1} m_x(u_{i+1}-w_i) \ge nm(x) - 2kdM_0 r\log|x|.$$

With (2.48), (2.47) and (2.43) this shows

$$\begin{aligned}
(2.51)\qquad & \sum_{0\le l<n}\sum_{j\ge 3n-l} P\big(0\leftrightarrow nx \text{ via a path } \gamma \text{ with gapped} \\
&\qquad\qquad (C,r,x)-\text{skeleton in } \mathcal{S}_{jl}(C,r,x,nx)\big) \\
&\le \sum_{0\le l<n}\sum_{j\ge 3n-l} 2^{j+l}e^{c_{25}(j+l)\log|x|}e^{-\sum_{i\le k} m_x(w_i-u_i)-C_{12}Cj\log|x|} \\
&\le \sum_{0\le l<n}\sum_{j\ge 3n-l} 2^{j+l}e^{c_{25}(j+l)\log|x|}e^{-nm(x)+2(j+l)dM_0 r\log|x|-C_{12}Cj\log|x|} \\
&\le 2n\cdot 2^{3n}e^{3nc_{25}\log|x|-nm(x)+6ndM_0 r\log|x|-2C_{12}Cn\log|x|} \\
&\le e^{-nm(x)-C_{12}Cn\log|x|} \\
&= o(P(0\leftrightarrow nx)) \quad \text{as } n\to\infty.
\end{aligned}$$

Statement (2.42) now follows from (2.49), (2.50) and (2.51). □

Theorem 1.1(ii) is an immediate consequence of Theorem 1.9 of [4] and the next Proposition, which proves slightly more than CHAP for the function h.

Proposition 2.12. *Assume (1.6). There exist C and M such that*

$$\frac{x}{\alpha} \in \mathrm{Co}(\tilde{Q}_x(C)) \quad \textit{for some } \alpha \in [2,6], \quad \textit{for all } x \in \mathbb{Z}^d \textit{ with } x \geq M. \tag{2.52}$$

Proof. In the notation of Lemma 2.11, let C, r satisfy $C \geq C_{13}$ and $C_{14} \leq r \leq c_{27}C$, where c_{27} is a constant to be specified later, and suppose $|x| \geq C_{17}(C)$. By Lemma 2.11 there exist n and a gapped (C,r,x)-skeleton $(u_i, v_i, v_i', w_i)_{i \leq k}$ corresponding to some path from 0 to nx, with $k < 3n$. From (2.11) we have $w_i - u_i \in \tilde{Q}_x(C)$. For each $i < k$ there is a path φ_i from w_i to u_{i+1} of length $|u_{i+1} - w_i|_1$. If z, y are vertices of φ_i then by Lemma 2.4(ii) and (2.10), provided c_{27} is chosen small enough,

$$s_x(z-y) \leq 2|z-y|_1 \log\frac{1}{p} \leq 4dr(log|x|)\log\frac{1}{p} \leq C\log|x|,$$

and by (2.5) and (2.4), if $|x| \geq c_{28}(C)$,

$$m_x(z-y) \leq 2M_0|u_{i+1} - w_i|_1 \log\frac{1}{p} \leq 4drM_0(\log|x|)\log\frac{1}{p} \leq m(x).$$

If follows that $u_{i+1} - w_i \in \tilde{Q}_x(C)$. Thus we have

$$nx = \sum_{i=0}^{k}(w_i - u_i) + \sum_{i=0}^{k-1}(u_{i+1} - w_i) = \sum_{y \in \tilde{Q}_x(C)} n(y)y, \tag{2.53}$$

where $n(y)$ is the number of times y appears in the first two sums in (2.53). Since

$$\sum_{y \in \tilde{Q}_x(C)} n(y) = 2k+1 \in [2n, 6n],$$

the conclusion (2.52) is obtained by dividing (2.53) by $\sum_{y \in \tilde{Q}_x(C)} n(y)$. ☐

Remark 2.13. In the proof of ([2] Lemma 2.3), the top three lines on page 1554 should read as follows:

First, for $||x|| \geq$ some c_{10}, by (2.12) and Lemma 2.2(ii),

$$\sum_{k \geq 3n} \sum_{j \geq 0} \sum_{(v_i) \in \Gamma^x_{jk}(nx)} \left(\prod_{i \notin L((v_i))} e^{-s_x(v_{i+1}-v_i) - \sigma g_x(v_{i+1}-v_i)} \right) \times \left(\prod_{i \in L((v_i))} e^{-\sigma g_x(v_{i+1}-v_i)} \right).$$

3. Proof of Theorem 1.1(i).

Theorem 1.1(i) is a consequence of Proposition 2.12 above, together with some results from [2] (Lemmas 2.6, 2.8, and 2.9 and Proposition 2.7) modified only slightly. Therefore we will give only a sketch of the proof.

We say that a path γ *x-backtracks by* t if there exist sites u, v in γ with u preceding v but $m_x(v-u) \le -t$. Since $0 \le h(v-u) = m_x(v-u) + s_x(v-u)$, this implies $s_x(v-u) \ge t$. Thus an (x,C)-clean path cannot x-backtrack by more than $C \log x$.

By Proposition 2.12, there exists C such that for $|x| \ge M$ we can express x as

$$x = \sum_{i=1}^{d+1} \alpha_i y_i \quad \text{with } \alpha_i \ge 0, \quad 2 \le \sum_{i=1}^{d+1} \alpha_i \le 6, \quad \text{and} \quad y_i \in \tilde{Q}_x(C).$$

In fact, since we can have $y_i = y_j$ for $i \ne j$, we have a similar statement with $\alpha_i \le 1$ for all i:

$$x = \sum_{i=1}^{d+6} \alpha_i y_i \quad \text{with } 0 \le \alpha_i \le 1 \quad \text{and } y_i \in \tilde{Q}_x(C). \tag{3.1}$$

For each y_i, there is an (x,C)-clean path from 0 to y_i, and consequently $s_x(y_i) \le C \log x$. We would like to find a constant b, depending only on P, such that

$$s_x(\alpha_i y_i) \le bC \log |x| \quad \text{for all } i. \tag{3.2}$$

Of course $\alpha_i y_i$ need not be in $\mathbb{Z}^d$, in which case $s_x(\alpha_i y_i)$ is not defined, but we can replace $\alpha_i y_i$ with an "adjacent" lattice site at the expense of an easily manageable error. If we can establish (3.2), then (3.1) and subadditivity of s_x give

$$s_x(x) \le \sum_{i=1}^{d+6} \alpha_i y_i \le (d+6) bC \log |x|. \tag{3.3}$$

Since $m_x(x) = m(x) = m(\theta)|x|$, we have $P(0 \leftrightarrow x) = e^{-m(\theta)|x| - s_x(x)}$, so (3.3) yields the desired conclusion (1.7). Thus it is enough to prove the following:

(3.4) there exists b such that if $y \in \tilde{Q}_x(C)$ and $0 \le \alpha \le 1$, then $s_x(\alpha y) \le bC \log |x|$.

(Again, αy should be interpreted as an adjacent lattice site if αy is not itself a lattice site.) To prove (3.4), let $y \in \tilde{Q}_x(C)$ and $0 \le \alpha \le 1$, and let $\gamma : [0,1] \to \mathbb{R}^d$ be an (x,C)-clean lattice path from 0 to y. Since γ does not x-backtrack by $C \log |x|$ or more, we can approximate γ to within $C \log |x|$ (measured in the norm $m(\cdot)$) by a curve $\tilde{\gamma}$ (not necessarily a lattice path) from 0 to y which does not x-backtrack at all; in fact we can have $m_x(\tilde{\gamma}(t))$ strictly increasing. Proposition 2.7 of [2] then states that there exist k_d, depending only on the dimension $d = 2$ or 3, and a collection of k_d subintervals $[s_j, t_j], j = 1, .., k_d$, of [0,1], such that

$$\alpha y = \sum_{i=1}^{k_d} (\tilde{\gamma}(t_j) - \tilde{\gamma}(s_j)) \tag{3.5}$$
$$= \sum_{i=1}^{k_d} [(\tilde{\gamma}(t_j) - \gamma(t_j)) + (\gamma(t_j) - \gamma(s_j)) + (\gamma(s_j) - \tilde{\gamma}(s_j))].$$

(The intervals $[s_j, t_j]$ may depend on γ here, and may overlap.) Let us assume all the points appearing in (3.5) are lattice sites; otherwise we again approximate them by adjacent lattice sites. Since γ is (x, C)-clean, we have

$$s_x(\gamma(t_j) - \gamma(s_j)) \leq C \log |x|.$$

From (2.4) and Lemma 2.4(ii), for some c_{29} depending only on P,

$$s_x(\tilde{\gamma}(t_j) - \gamma(t_j)) \leq c_{29} m(\tilde{\gamma}(t_j) - \gamma(t_j)) \leq c_{29} C \log |x|$$

and similarly for $s_x(\tilde{\gamma}(s_j) - \gamma(s_j))$. Together with (3.5), these bounds yield

$$s_x(\alpha y) \leq k_d(4c_{29} + 1) C \log |x|,$$

so (3.4) is proved.

4. Proof of Theorem 1.4

We will adapt the techniques of ([2], Lemma 4.3) to our present context.

We may assume that $0 \in \partial H$. Let n denote the outward unit normal to H. Given ω in which $0 \leftrightarrow x$, let γ_ω be an open path in ω from 0 to x (chosen arbitrarily if there is more than one such path) and let $X(\omega)$ be a site in γ_ω which maximizes $n \cdot y$ over $y \in \gamma_\omega$ (the first such site, say, if there is more than one.) Note that if the segment of γ_ω from 0 to $X(\omega)$ and the segment from $X(\omega)$ to x are interchanged, the result is a path from 0 to x in H. To make such an interchange possible, in a sense, we must show that the two segments are nearly independent.

By Lemma 2.7, provided $|x|$ is large and $c_{30} \geq C_4$, the event

$$N = \{\omega : \text{ there is a } (c_{30} \log |x|)\text{-near connection from 0 to } x \text{ in } \omega\}$$

satisfies

$$P(N) \leq e^{-m(x) + c_{30} C_5 \log |x|}. \tag{4.1}$$

By Lemma 2.6 and Theorem 1.1, there exists c_{31} such that for $B_0 = B(0, c_{31}|x|)$ the event

$$U = \{\omega : 0 \leftrightarrow \partial B_0\}$$

satisfies

$$P(U) \leq \frac{1}{2}P(0 \leftrightarrow x). \tag{4.2}$$

Therefore there exist $z \in B_0$ and c_{32} such that, letting H_z denote the translate of H with $z \in \partial H_z$,

$$P(0 \leftrightarrow z \leftrightarrow x \text{ in } \mathcal{B}(H_z \cap B_0)) \geq P(0 \leftrightarrow x, X = z, U^c) \geq \frac{c_{32}}{|x|^d}P(0 \leftrightarrow x). \tag{4.3}$$

Note that z is not in the interior of H, so $x - z \in H$. Using positive connection correlations,

$$\begin{aligned} P(0 \leftrightarrow x \text{ in } \mathcal{B}(H)) &\geq P(0 \leftrightarrow x - z \leftrightarrow x \text{ in } H) \\ &\geq P(0 \leftrightarrow x - z \text{ in } \mathcal{B}(H))P(x - z \leftrightarrow x \text{ in } \mathcal{B}(H)) \\ &= P(0 \leftrightarrow z \text{ in } \mathcal{B}(H_z))P(z \leftrightarrow x \text{ in } \mathcal{B}(H_z)). \end{aligned} \tag{4.4}$$

We need to compare the left side of (4.3) to the right side of (4.4). For $x \in \mathbb{Z}^d$ let $f_d(x)$ be $\log|x|$ for $d = 2, 3$ and $(\log|x|)^2$ for $d \geq 4$. By Lemma 2.6, there exists c_{33} such that, defining

$$B_z = B(z, 2 + c_{33}f_d(x)), \qquad \tilde{B}_z = B(z, c_{33}f_d(x)),$$

we have

$$P(z \leftrightarrow \partial\tilde{B}_z) \leq \frac{c_{32}A}{4}e^{-(d+c_{30}C_5+C)f_d(x)}, \tag{4.5}$$

where A, C are as in Theorem 1.1. ($A = 1$ if $d \geq 4$.) Assume first that $0, x \notin B_z$. Define

$$\tilde{N} = \{\omega : \text{ there is a } (c_{30}\log|x|)\text{-near connection from } 0 \text{ to } x \text{ outside } B_z \text{ in } \omega\}.$$

For $|x|$ large, Lemma 2.5, (4.5), (4.1), Theorem 1.1 and (4.3) give

$$\begin{aligned} &P([0 \leftrightarrow z \leftrightarrow x \text{ in } H_z] \cap \tilde{N}) \\ &\quad\leq P([z \leftrightarrow \partial\tilde{B}_z] \cap \tilde{N}) \\ &\quad\leq 2P(z \leftrightarrow \partial\tilde{B}_z)P(\tilde{N}) \\ &\quad\leq \frac{c_{32}A}{2}e^{-m(x)-(d+C)f_d(x)} \\ &\quad\leq \frac{1}{2}P(0 \leftrightarrow z \leftrightarrow x \text{ in } H_z \cap B_0). \end{aligned} \tag{4.6}$$

Therefore by Lemma 2.5, positive connection correlations and (2.14), for some c_{34}, provided c_{30} and $|x|$ are large,

$$
\begin{aligned}
(4.7)\qquad & P\big(0 \leftrightarrow z \leftrightarrow x \text{ in } \mathcal{B}(H_z \cap B_0)\big) \\
&\le 2P\big([0 \leftrightarrow z \leftrightarrow x \text{ in } \mathcal{B}(H_z \cap B_0)] \cap \tilde{N}^c\big) \\
&\le 2P\big([0 \leftrightarrow \partial B_z \text{ in } H_z \cap B_0] \cap [x \leftrightarrow \partial B_z \text{ in } H_z \cap B_0] \cap \tilde{N}^c\big) \\
&\le \sum_{I,J} P\big(\Gamma(0,(H_z \cap B_0)\backslash B_z) = I, \Gamma(x,(H_z \cap B_0)\backslash B_z) = J\big) \\
&\le \sum_{I,J} 2P\big(\Gamma(0,(H_z \cap B_0)\backslash B_z) = I\big) P\big(\Gamma(x,(H_z \cap B_0)\backslash B_z) = J\big) \\
&\le 2P\big(0 \leftrightarrow \partial B_z \text{ in } \mathcal{B}(H_z)\big) P\big(x \leftrightarrow \partial B_z \text{ in } \mathcal{B}(H_z)\big) \\
&\le \sum_{q,r \in H_z \cap \partial B_z} 2P(0 \leftrightarrow q \text{ in } \mathcal{B}(H_z)) P(x \leftrightarrow r \text{ in } \mathcal{B}(H_z)) \\
&\le e^{c_{34} f_d(x)} P(0 \leftrightarrow z \text{ in } \mathcal{B}(H_z)) P(z \leftrightarrow x \text{ in } \mathcal{B}(H_z))
\end{aligned}
$$

where the sum is over $I, J \subset (B_0 \cap H_z)\backslash B_z$ with $0 \in I, I \cap \partial B_z \neq \phi, x \in J, J \cap \partial B_z \neq \phi$, and $d(I,J) \ge c_{30} \log |x|$. Combining (4.3), (4.4) and (4.7) yields

$$
(4.8)\qquad P(0 \leftrightarrow x \text{ in } \mathcal{B}(H)) \ge e^{-c_{34} f_d(x)} \frac{c_{32}}{|x|^d} P(0 \leftrightarrow x).
$$

With Theorem 1.1 this completes the proof, when $0 \notin B_z$ and $x \notin B_z$.

When $0 \in B_z$ the proof is simpler. We have

$$
(4.9)\qquad P(0 \leftrightarrow z \text{ in } \mathcal{B}(H_z)) \ge e^{-c_{35} f_d(x)}
$$

and in place of (4.7),

$$
\begin{aligned}
(4.10)\qquad & e^{-c_{36} f_d(x)} P(0 \leftrightarrow z \leftrightarrow x \text{ in } \mathcal{B}(H_z \cap B_0)) \\
&\le P(0 \leftrightarrow z \text{ in } \mathcal{B}(H_z)) P(z \leftrightarrow x \text{ in } \mathcal{B}(H_z)).
\end{aligned}
$$

Combining (4.3), (4.4), (4.9) and (4.10) again yields (4.8). The proof when $x \in B_z$ is similar.

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Department of Mathematics DRB 155, University of Southern California, Los Angeles, CA 90089-1113

E-mail address: alexandr@math.usc.edu