SINGULAR BOHR-SOMMERFELD RULES FOR 2D INTEGRABLE SYSTEMS

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Abstract

In this paper, we describe Bohr-Sommerfeld rules for semi-classical completely integrable systems with 2 degrees of freedom with non degenerate singularities (Morse-Bott singularities) under the assumption that the energy level of the first Hamiltonian is non singular. The more singular case of *focus-focus* singularities is studied in [27] and [26]. The case of 1 degree of freedom has been studied in [10].

Our theory is applied to some famous examples: the geodesics of the ellipsoid, the 1 : 2-resonance, and Schrödinger operators on the sphere S^2. A numerical test shows that the semi-classical Bohr-Sommerfeld rules match very accurately the "purely quantum" computations.

Keywords: eigenvalues, Bohr-Sommerfeld rules, semi-classics, completely integrable systems, microlocal analysis, saddle point, Morse lemma, normal forms.
Mathematical classification: 34E20 - 34L25 - 81Q20 - 58F07 - 58C40 - 58C27

Introduction

In this paper, we describe extensions of results of [10] to completely integrable semi-classical systems with 2 degrees of freedom. If $\hat{H}_1, \hat{H}_2$ are two commuting h-pseudo-differential operators on a 2-manifold X, we introduce the *momentum map* $F = (H_1, H_2) : T^\star X \to \mathbb{R}^2$ where H_j $(j = 1, 2)$ is the principal symbol of $\hat{H}_j$; we assume that F is a proper map. If $o = (0, 0)$ is not a critical value of H, the existence and the construction of solutions of the system $\hat{H}_j u = O(h^\infty)$ has a long history and existence of solutions is well known to depend on the *Bohr-Sommerfeld* rules involving action integrals and Maslov indices of loops generating the homology of the fibre of F which is a 2-torus.

We will assume that o is a critical value and more precisely that the critical points are of Morse-Bott type. A very simple classification of such points with the corresponding normal forms is given in [25]. The *focus-focus* case have been described in [27]. We will be interested in the case where the set of critical points of H is a 2-dimensional manifold with a transversally hyperbolic (saddle) singularity: it means that 0 is not a critical value of H_1 and that H_2 restricted to Poincaré sections of the flow of H_1 admits critical points of saddle type. The set of critical values of F is then a 1-dimensional submanifold of $\mathbb{R}^2$. The main result of our paper is a description of *singular Bohr-Sommerfeld rules* in this situation. These rules give necessary and sufficient conditions for existence of solutions of the system $\hat{H}_j u = O(h^\infty)$, $j = 1, 2$ and approximations of the solutions. More precisely, we show the existence near the singular fibre $\Lambda_o = F^{-1}(o)$ of a Hamiltonian H_p with periodic flow which allows to *reduce* the classical and also the semi-classical study to the 1-dimensional case on the reduced phase space. This reduced phase space can be *singular* because the S^1 action induced by the flow of H_p is not principal in general; non trivial isotropy group isomorphic to $\mathbb{Z}/2\mathbb{Z}$ may appear.

We provide a description of Λ_o: the topological type can be rather unusual like a Klein bottle!

The precise description of the quantisation rules is given in theorem 2.7. There is one rule giving a quantum number associated to the periodic orbits of H_p and another rule given in terms of the graph G which is the quotient of Λ_o by the S^1-action associated with H_p. With a viewpoint similar to [27] but somewhat different from [10], we interpret these rules as a universal *regularisation* of the usual Bohr-Sommerfeld rules for tori. This, in addition, allows us to have a description of the joint spectrum inside a fixed neighbourhood of o, but raises some technical difficulties due to the possible non-connectedness of the fibres of F, especially in the C^∞ category.

At the end we describe 3 examples for which we provide explicit calculation and numerical checking:

1. High energy limit for eigenvalues of the Laplacian on ellipsoids.
2. Semi-excited states for anharmonic oscillators with a resonance 2 : 1.
3. High energy limit for the Schrödinger spectrum on the 2-sphere.

For the last two examples, numerical computations of eigenvalues of large matrices are compared with the eigenvalues obtained from the singular Bohr-Sommerfeld rules. We observe a very good accuracy of the results even for not very big quantum numbers.

1 Classics

The goal of this section is to give a semi-global description of the Lagrangian fibres of a 2-degrees-of-freedom integrable system having non-degenerate rank-one singularities of hyperbolic type.

Let (M, ω) be a symplectic manifold of dimension 4, and let H_1, H_2 be two Poisson commuting Hamiltonian functions in $C^\infty(M, \mathbb{R})$. The corresponding momentum map will be denoted by $F = (H_1, H_2)$; we shall always assume F to be proper, H_1 to be non-singular on the level set $\Lambda_o := F^{-1}(o)$ (for some point $o \in \mathbb{R}^2$). Moreover, we assume that Λ_o is connected and that the critical points of F on Λ_o are transversally non-degenerate, in the following sense:

for any Poincaré section Σ of $\mathcal{X}_1$, the restriction of H_2 to Σ is a Morse function.

The main results of this section are

- The description of the topology of the fibre Λ_o (section 1.2).
- The construction of partial action-angle coordinates in a full neighbourhood of Λ_o; we show in particular that there exists a Hamiltonian H_p defined in some neighbourhood of Λ_o that Poisson commutes with H_j, $j = 1, 2$, and all orbits of which are periodic (theorem 1.6). Moreover, up to finite covering, a neighbourhood of Λ_o is symplectomorphic to a product of T^*S^1 by a "global" Poincaré section.
- The construction of normal forms for the system near each connected component of the critical set of F (theorem 1.13).

The core of these results can be obtained from Fomenko's classification [14], and the first two points are actually due to Nguyên Tiên Dung [23]. However, we felt that these results deserved a detailed description with independent arguments.

1.1 Notations

For convenience of the reader we group here some of the notations: our integrable system is given by the momentum map $F = (H_1, H_2) : M \to \mathbb{R}^2$. Because F is proper, it is a momentum map for a Hamiltonian action of $\mathbb{R}^2$ on M. $o = (a_o, b_o)$ is a critical value of F, U small disk around o in $\mathbb{R}^2$. The smooth energy level is $S_o = H_1^{-1}(a_o)$. Ω is $F^{-1}(U)$; $\Lambda_o = F^{-1}(o)$. $\gamma = \cup_{i=1}^N \gamma_i$ is the critical set of F in Λ_o. $\Gamma = \cup \Gamma_i$ is the critical set in Ω. $\Lambda_o \setminus \gamma = \cup \Lambda^k_{\{i,j\}}$ where $\Lambda^k_{\{i,j\}}$ is a smooth Lagrangian cylinder whose closure is $\Lambda^k_{\{i,j\}} \cup \gamma_i \cup \gamma_j$. For any Hamiltonian $H_{something}$ we will denote by $\mathcal{X}_{something}$ the associated Hamiltonian vector field. G is a graph with N vertices associated to Λ_o. The Hessian of H_2 restricted to Σ is $\mathcal{H}_\Sigma(H_2)$. The absolute value of its determinant (with respect to the density induced by the symplectic for on Σ) is independent on Σ and denoted by $|\mathcal{H}_\Sigma(H_2)|$.

1.2 Topology of Λ_o

Proposition 1.1 *The critical set γ of F in S_o is a compact submanifold of dimension 1 of Λ_o which is a finite union of disjoint periodic orbits γ_i ($i =$*

$1, \cdots, N)$ *of* $\mathcal{X}_1$. *The* γ_i *admits* orbit-cylinders Γ_i *which consists of* $\gamma_i(a)$, *a close to* a_o.

Proof. Locally the reduced manifold of $S_o := \{H_1 = a_o\}$ is symplectomorphic to a Poincaré section Σ of $\mathcal{X}_1$. Because $\mathcal{X}_1$ is transversal to Σ in S_o, a neighbourhood V of Σ in S_o is diffeomorphic to $\Sigma \times I$, for a small interval I, in such a way that the trajectories of $\mathcal{X}_1$ are of the form $\{\sigma\} \times I$, $\sigma \in \Sigma$. Because H_2 is constant under the $\mathcal{X}_1$-flow, $H_2(\{\sigma\} \times I) = (H_2)_{\restriction\Sigma}(\sigma)$. By hypothesis, H_2 restricted to Σ admits a non degenerated critical point at $\sigma := \gamma \cap \Sigma$. Since the rank of F if invariant by the flow of $\mathcal{X}_1$, we must have $\gamma \cap V = \{\sigma\} \times I$, which says that γ is a smooth manifold of dimension 1. Because, by definition, γ is closed in the compact set Λ_o and hence is compact, it must be a union of circles $\gamma = \bigcup \gamma_i$. Then each of these circles must be an $\mathcal{X}_1$-orbit, and only a finite number N of such critical circles arises due to the properness of F.

The non-degeneracy of $(H_2)_{\restriction\Sigma}$ implies that the isolated critical point $\sigma = \sigma(a_o)$ depends smoothly on a close to a_o. Therefore the above description extends to any leaf of the foliation $H_1 = a$, a close to a_o, yielding a smooth family of circles $\gamma_i(a)$. Since a small neighbourhood of S_o in M is diffeomorphic to $S_o \times \mathbb{R}$ such that a is a coordinate for the second factor, the union $\Gamma_i = \bigcup_{a \text{ close to } a_o} \gamma_i(a)$ is diffeomorphic to a cylinder $S^1 \times (\mathbb{R}, a_o)$. $\square$

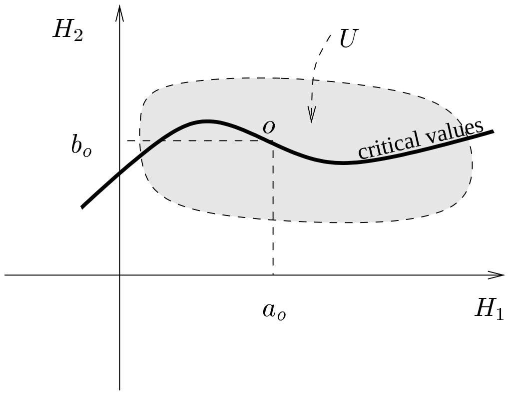


Figure 1: Image of the momentum map. The set of critical values near o is a smooth curve parameterised by H_1. If the critical point is a saddle point, the regular values are on both sides of the curve of critical values. If it is a maximum or a minimum, only one side is occupied.

If the critical point σ is a local maximum or minimum of H_2, Λ_o reduces to one elliptic periodic orbit, a situation which was studied a long time ago by several people (see [7]). We will from now assume that *this critical point is a saddle point.* Then one can have several critical circles in γ; we show now how they can be connected to each other inside Λ_o.

Proposition 1.2 $\Lambda_o \backslash \gamma$ *is a union of* $\mathbb{R}^2$*-orbits that are cylinders. Each cylinder contains in its closure 1 or 2* γ_i*'s. We will denote by* $\Lambda_{\{i,j\}} = \bigcup_k \{\Lambda^k_{\{i,j\}}\}$ *the set of cylinders that connect* γ_i *and* γ_j.

Proof. Since γ is $\mathbb{R}^2$-invariant, $\Lambda_o \setminus \gamma$ is a union of orbits, on all of which the action is non-singular. These orbits are therefore 2-dimensional quotients of $\mathbb{R}^2$, hence tori, cylinders or planes. Because Λ_o is connected and contains singular points, any orbit contains critical points in its closure, which excludes tori. The local structure near each γ_i will show independently the existence of periodic sub-orbits, leading to the non-triviality of the stabilisers of the $\mathbb{R}^2$-action, which excludes planes. Hence all orbits are cylinders.

Now, the closure of such a cylinder consists of singular orbits: there are 1 or 2 of them. $\square$

Definition 1.1 *We define the graph G as follows: G has N vertices (where N is the number of critical circles in Λ_o), and there are exactly $|\Lambda_{\{i,j\}}|$ different edges connecting the vertices i and j.*

Let Σ be a Poincaré section at a point $m \in \gamma_i$. By hypothesis, $\Sigma \cap \Lambda_o$ is diffeomorphic to a "hyperbolic cross", the union the local stable and unstable manifolds $W^{\pm}(m)$ for the flow of $\mathcal{X}_{H_{2\upharpoonright\Sigma}}$. Let Ω be a small neighbourhood of γ_i, and define $W^{\pm}(\gamma_i)$ as the union of the connected components of $(\Lambda_o \setminus \gamma_i) \cap \Omega$ intersecting $W^{\pm}(m)$. These manifolds do not depend on the choice of m.

Proposition 1.3 *Either $W^+(\gamma_i)$ and $W^-(\gamma_i)$ are diffeomorphic to the disjoint union of 2 cylinders or both are diffeomorphic to 1 cylinder. In the first case the vertex i of G has degree 4 while in the second it has degree 2.*

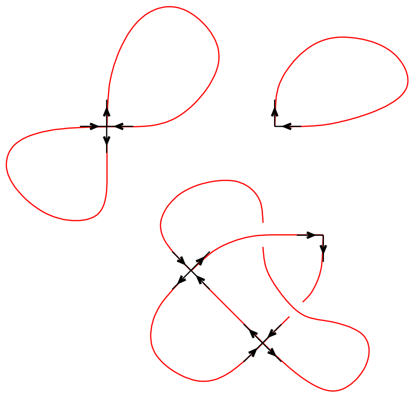
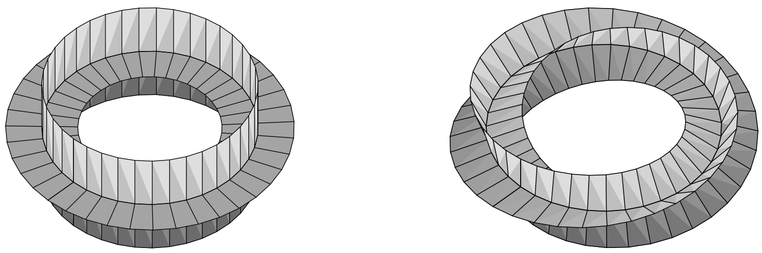


Figure 2: The vertices of G are of degree 2 or 4.

Proof. The 2-manifold $\tilde{W}^+(\gamma_i) = W^+(\gamma_i) \cup \gamma_i$ is a bundle on γ_i whose fibre is an interval. There are exactly 2 possibilities up to diffeomorphism: the trivial and the Moebius bundle. In the first case, removing γ_i gives 2 cylinders, while in the second it gives only 1 cylinder. Both bundle are isomorphic because the sum of their tangent bundle on γ_i is a $\mathbb{R}^2$−bundle that is orientable as a symplectic bundle. $\square$

Definition 1.2 *In the first case, γ_i is called* direct*, in the second case, it is called* reverse.



a) direct case b) reverse case

Figure 3: The neighbourhood of a critical circle

1.3 The classical commutant

Lemma 1.4 *For any smooth function K commuting with H_1 and H_2, the vector field $\mathcal{X}_K$ can be uniquely written (in a neighbourhood of Λ_0)*

$$\mathcal{X}_K = a\mathcal{X}_1 + b\mathcal{X}_2,$$

for smooth functions a, b commuting with H_1 and H_2.

Proof. Near any nonsingular point of F, apply the Darboux-Carathéodory theorem which makes $H_1 = \xi$ and $H_2 = \eta$ in a local symplectic coordinate chart (x, y, ξ, η). Then $K = K(\xi, \eta)$, and the result follows.

Near a critical point in some γ_i, we use Theorem 1.5 below that reduces the situation to $H_1 = \xi + a_o$ and $H_2 = \Phi(\xi, y\eta)$, and hence to $(H_1 = \xi,\ H_2 = y\eta)$, for which we use the result of [25, Lemme 2.6] ([26, Lemme 2.2.7]). □

Theorem 1.5 *Around any point $m \in \gamma_i$, there exists a canonical chart (x, y, ξ, η) in which*

$$\begin{cases} H_1 - a_o &= \xi \\ H_2 &= \Phi(\xi, y\eta), \end{cases}$$

for some smooth function Φ defined near the origin, with

$$\partial_2\Phi(0, 0) > 0. \tag{1}$$

Proof. First construct a Darboux-Carathéodory chart for H_1, i.e. a system of symplectic coordinates (x, y, ξ, η), with canonical form ω_0, in which $\xi = H_1 - a_o$. In these coordinates, the plane $\{0\} \times \{0\} \times \mathbb{R}^2$ is a Poincaré section for H_1, hence by hypothesis $(y, \eta) \to H_2(\xi, y, \eta)$ has, for each small ξ, a non-degenerate saddle point. The application of the Isochore Morse Lemma [11] with parameter ξ yields a local symplectomorphism $\phi_\xi(y, \eta)$ of the (y, η)-space, depending smoothly on ξ, such that

$$H_2(x, \xi, \phi_\xi(y, \eta)) = \Phi(\xi, y\eta), \tag{2}$$

for a function Φ with $\partial_2\Phi(0, 0) \neq 0$. Applying the canonical transformation $(y, \eta) \to (-\eta, y)$ if necessary, one can assume that Φ satisfies the condition (1) of the theorem. The map $\phi : (x, \xi, y, \eta) \mapsto (x, \xi, \phi_\xi(y, \eta))$ is a local diffeomorphism but need not be symplectic. A modification of x shall solve the problem.

The 2-form $\omega_1 := \phi^*(\omega_0)$ splits as follows :

$$\omega_1 = \omega_0 + d\xi \wedge \beta,$$

for some 1-form $\beta = \beta(\xi, y, \eta)$. From $d\omega_j = 0$, one gets $d\xi \wedge d\beta = 0$, which means $d_{(y,\eta)}\beta = 0$. Hence there exists a smooth function $f(\xi, y, \eta)$ such that $d_{(y,\eta)} f = \beta$. Let ϕ_1 be the diffeomorphism :

$$\phi_1(x, \xi, y, \eta) := (x - f(\xi, y, \eta), \xi, y, \eta), \tag{3}$$

so that

$$\phi_1^*\omega_1 = \phi_1^*\omega_0 + \phi_1^*(d\xi \wedge \beta) = \phi_1^*\omega_0 + d\xi \wedge \beta \tag{4}$$

$$= \omega_0 - d\xi \wedge df + d\xi \wedge \beta = \omega_0. \tag{5}$$

Thus $\phi \circ \phi_1$ is symplectic, and because it does not change $H_1 = \xi - a_o$, it answers the question. □

1.4 A periodic Hamiltonian H_p in Ω

Theorem 1.6 ([23]) *There exists a unique (up to additive constant) Hamiltonian H_p in Ω that fulfils the following conditions:*

1. *H_p Poisson commutes with H_1 and H_2.*
2. *The flow of H_p is 2π periodic with minimal period 2π outside Γ.*
3. *On Γ_i,*

$$\mathcal{X}_p = \alpha_i(H_1)\mathcal{X}_1, \tag{6}$$

with α_i a positive function.

Then the flow of H_p is 2π-periodic on Γ_i if the vertex $\{i\}$ is direct, and π-periodic in the reverse case.

Lemma 1.7 *Near each γ_i, there exists a unique (up to some additive constant if $a_o \neq 0$) Hamiltonian $H_{2,i} = H_2 - \lambda_i(H_1)H_1$ which is critical on Γ_i.*

Proof. If $dH_2 = f_i(H_1)dH_1$ on Γ_i, one gets the differential equation

$$t\lambda_i' + \lambda_i = f_i(t)$$

which always admits a local solution. □

Definition 1.3 *A class of path $z \to [\gamma_z]$ (where $[\,]$ means a homotopy or homology class) is called* smooth *if there exists locally a representative that smoothly depends on z.*

Lemma 1.8 *Let us denote by L_z the orbit of z by the $\mathbb{R}^2-$action. There exists a unique mapping $z \to [\gamma_z]$ that is a smooth map from Ω into $H_1(L_z, \mathbb{Z})$ and such that, if $z \in \gamma_i$, $[\gamma_z] = \nu[\gamma_i]$ where γ_i is oriented according $\mathcal{X}_1$ and $\nu = 1$ (resp. 2) if γ_i is direct (resp. reverse).*

Proof. a) The main point for this proof is to construct such a smooth family $[\gamma_z]$ near γ_i. By the Morse-Bott lemma (see Appendix 1.7) applied to the Hamiltonian $H_{2,i}$ of Lemma 1.7, there are a coordinate x_1 on the circle $\gamma_i = (\mathbb{R}/\mathbb{Z})$ and a fibre bundle $F_\pm$ of dimension 2 on γ_i defined as quotient of the trivial bundle on $\mathbb{R}$ by identifying $(x_1, w) \in \gamma_i \times \mathbb{R}^2$ with $(x_1 + 1, \pm w)$ such that $H_{2,i} = x_3 x_4$. We can assume that $\mathcal{X}_1$ has the same orientation as $\partial/\partial x_1$ on γ_i.

The above Morse-Bott lemma can be applied with the parameter H_1. Therefore, one gets coordinates (x_1, x_2, x_3, x_4) on a full neighbourhood of γ_i in M by letting $x_2 = H_1$.

We choose then γ_z to be the path given in these coordinates by $t \to (t, x_2, x_3, x_4)$ with $t \in \mathbb{R}/\mathbb{Z}$ (direct case) and $t \in \mathbb{R}/2\mathbb{Z}$ (reverse case). These path are drawn on the leaf $(H_1, H_{2,i}) = const$, and hence on a leaf of $F = (H_1, H_2)$. The last assumption of lemma 1.8 follows from proposition 1.3.

b) Far from γ_i, we construct γ_z on Λ_o by taking the cylinder equator with the orientation which in the affine structure of $\Lambda^k_{\{i,j\}}$ is given by projecting $\mathcal{X}_1$. Then we extend to the nearby Lagrangian leaves by local triviality of the foliation by orbits. Because these γ_z are homotopic to the ones constructed in a) and lying on the same Lagrangian leaf, it is easy to realise this homotopy as an isotopy, thus yielding a smooth family of loops in Ω. □

c) It remains now to use these γ_z in order to define the action variable.

Lemma 1.9 *The symplectic form ω is exact in Ω, i.e. there exists α, 1-form in Ω with $d\alpha = \omega$.*

Proof. The set Λ_o is Lagrangian and any 2-cycle can be deformed inside Λ_o. □

Proof *(of Theorem 1.6)*. Put $H_p(z) = (1/2\pi)\int_{\gamma_z} \alpha$. H_p is smooth and commutes with H_1 and H_2 (because it is constant on $\mathbb{R}^2-$orbits). Moreover on $\mathbb{R}^2-$orbits that are tori the orbits of $\mathcal{X}_p$ are $2\pi-$periodic with orbits homotopic to γ_z (by usual action-angle coordinates). On the γ_i's, the period is 2π in the direct case and π in the reverse case.

The affine structure on the Lagrangian cylinders $\Lambda^k_{\{i,j\}}$ and the condition (3.) implies the uniqueness of $\mathcal{X}_p$ on Λ_o. Now suppose H'_p is another Hamiltonian with the same properties, and let z be a point in $\Lambda_o \setminus \gamma$. Since the orbits under $\mathcal{X}_p$ and $\mathcal{X}'_p$ of z are equal, the orbits of points near z in a same level set of F (different from Λ_o) are homotopic. But these level sets are Liouville tori for which we know that $\mathcal{X}_p$ and $\mathcal{X}'_p$ must be equal. □

Remark 1.1. Step a) of the proof does not use the nature of $\Lambda^k_{\{i,j\}}$. Therefore, a)+c) gives a Hamiltonian H_p verifying the conditions of Theorem 1.6 but in a neighbourhood of γ_i only. This suffices to prove that the $\mathbb{R}^2$-action has non-trivial stabilisers, whence $\Lambda^k_{\{i,j\}}$ must be cylinders, thus finishing the proof of proposition 1.2. △

1.5 S^1 reduction

The flow of $\mathcal{X}_p$ yields a locally free Hamiltonian action of S^1 on Ω, which is free outside Γ. Let $c_o = H_p(m) = H_p(\Lambda_o)$. We denote by W the reduced space

$$W = H_p^{-1}(c_o) \cap \Omega / S^1.$$

W is a symplectic orbifold (see eg. [3]). It is a smooth manifold if and only if the action is free, that is if and only if no vertex of reverse type are present in the graph G. Otherwise, it has singularities at the critical orbits γ_i. Since these critical orbits come in families depending on the value of H_1, yielding local orbit cylinders, and because $dH_p(m) = \lambda dH_1(m)$, $\lambda \neq 0$, only one orbit of each local cylinder meets $H_p^{-1}(c_o)$, ensuring that the critical orbits give isolated singularities in W.

Let

$$H_q = -b(H_1 - a_o) + a(H_2 - b_o), \tag{7}$$

where $a > 0$ and b are the real constants such that, on Λ_o, $\mathcal{X}_p = a\mathcal{X}_1 + b\mathcal{X}_2$ (cf. Lemma 1.4). Then $\Lambda_o = H_p^{-1}(c_o) \cap H_q^{-1}(0)$. (This still holds of course for a generic choice of (a, b).) Since H_q is S^1-invariant, it defines a smooth Hamiltonian function $\tilde{H}_q$ on W. The graph G can be viewed as the quotient of Λ_o by S^1, and thus is identified to the level set $\tilde{H}_q^{-1}(0)$.

Proposition 1.10 *If the S^1-action is free (*i.e.* all vertices of G of degree 4), then a neighbourhood Ω of Λ_o in M is diffeomorphic to the direct product $S^1 \times \mathbb{R} \times W$ (hence Λ_o is diffeomorphic to the direct product $S^1 \times G$ – these diffeomorphisms are equivariant with respect to the natural action of S^1 on itself).*

Remark 1.2. In this case, W can be regarded as a “global” Poincaré section for $\mathcal{X}_p$. △

Proof. We choose now Ω to be of the form $\Omega_o \times I$ where I is some open interval around 0, Ω_o is a small invariant neighbourhood of Λ_o in $H_p^{-1}(c_o)$, and $H_p(\Omega_o \times \{\xi\}) - c_o = \xi$. If the action is free, Ω_o is a principal S^1-bundle over W. It is topologically classified by its holonomy class in $H^1(W, S^1_\mathfrak{d})$, where $S^1_\mathfrak{d}$ is the sheaf of germs of smooth functions on W with values in S^1 (see [19]). Using the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$, and the fact that the sheaf $\mathbb{R}_\mathfrak{d}$ is fine, one gets an isomorphism

$$H^1(W, S^1_\mathfrak{d}) \simeq H^2(W, \mathbb{Z}),$$

yielding the so-called Chern class of the bundle. But W retracts onto G and G is 1-dimensional, so $H^2(W, \mathbb{Z}) = 0$, and Ω_o is a trivial bundle. □

Theorem 1.11 ([23]) *If the S^1-action is free, then (Ω, ω) is symplectomorphic to a neighbourhood of $S^1 \times \{0\} \times W$ in*

$$(T^*S^1 \times W, \quad d\xi \wedge dx + \pi^*\omega_W),$$

with $H_p - c_o = \xi$. Here π is the projection onto W and ω_W is the symplectic form of W.

Proof. First apply proposition 1.10 and let $\xi = H_p - c_o$ be a coordinate for the $\mathbb{R}$ factor. Then choose the conjugate angle variable x (pick up some coordinate θ in S^1, an origin θ_0, and let $x(\theta)$ be the time required to go from θ_0 to θ under the Hamiltonian action of ξ), so that $\{\xi, x\} = 1$. Because ω is S^1-invariant, it does not depend on x; using the equivariant Darboux-Weinstein theorem [28], one can assume that $\omega = \omega_{\restriction \xi=0}$, and so does not depend on ξ either.

Because for any ξ, $\mathcal{X}_p$ is ω-orthogonal to $S^1 \times \{\xi\} \times W$, one easily checks by taking local coordinates on W that

$$\omega = d\xi \wedge dx + d\xi \wedge \pi^*\beta + \pi^*\omega_W,$$

where β is a one-form on W, and ω_W a non-degenerate 2-form on W. The closedness of ω (and its independence on ξ) implies $d\omega_W = 0$ and $d\beta = 0$, the latter yielding $d(\xi\pi^*\beta) = d\xi \wedge \pi^*\beta$. Let us now apply Moser's path method to get rid of this term. We let

$$\omega_t := d\xi \wedge dx + \pi^*\omega_W + td(\xi\pi^*\beta),$$

and wish to construct an isotopy φ_t of diffeomorphisms of Ω such that $\varphi_t^*\omega_t = \omega_0$. φ_t is then given as the flow of the vector field X_t defined by $i_{X_t}\omega_t + \xi\pi^*\beta = 0$. It is easy to check that ω_t is non-degenerate for all t so that X_t is uniquely defined. Moreover, because of its defining equation, X_t is of the form $\xi\iota_* Y_t$ (ι is the inclusion $W \hookrightarrow T^*S^1 \times W$), where Y_t is a vector field on W satisfying $i_{Y_t}(\omega_t)_{\restriction W} + \beta = 0$. Therefore φ_t is of the form

$$(x, \xi, w) \to (x, \xi, \phi_{\xi t}(w)),$$

(where ϕ_t is the flow of Y_t) and hence preserves x and ξ. If Y_t can be integrated up to the time $t_0 > 0$, then X_t can be integrated up to the time 1 for $\xi \leqslant t_0$. The diffeomorphism φ_1 then answers the question. □

Remark 1.3. The formula $\omega = d\xi \wedge dx + \pi^*\omega_W$ ensures that ω_W is the natural symplectic form on W obtained by the reduction process. △

Theorem 1.12 *In the general case, there exists a smooth double covering Ω^* of Ω in which the action is free. The reduced manifold W^* is a covering of W that is ramified of degree 2 at the critical orbits γ_i.*

Proof. Choose Ω to be a relatively compact invariant neighbourhood of Λ_o in M. Then $H_p^{-1}(c_o) \cap \Omega$ has a smooth non-empty invariant boundary, and because this boundary does not meet any critical orbit, the closure $\overline{W} = \overline{H_p^{-1}(c_o) \cap \Omega / S^1}$ is a relatively compact surface with a non-empty smooth boundary. Let $p_i, i = 1 \ldots \ell$ be the images under reduction of the critical circles γ_i and let $\check{W}$ be the surface $\overline{W}$ after removal of small disks D_i around each p_i. It is still a smooth surface with boundary, whose fundamental group is free [1], and generated by some $\mu_1, \ldots, \mu_k, \delta_1, \ldots, \delta_\ell$, with $\delta_i = \partial D_i$. Let $\mathcal{D} \subset \pi_1(\check{W})$ be the free subgroup generated by $\mu_1, \ldots, \mu_k, \delta_1^2, \ldots, \delta_\ell^2$, and let $\check{W}^*$ be the corresponding smooth covering of $\check{W}$. Gluing back the disks D_i defines a covering space W^* of W that is ramified of degree two at each p_i. Define now $\Omega^* \subset \Omega \times W^*$ such that the following diagram commutes:

$$\begin{array}{ccc} \Omega^* & \longrightarrow & W^* \\ \downarrow & & \downarrow \\ \Omega & \longrightarrow & W \end{array}$$

Since the local structure near γ_i (see Theorem 1.13 in the next section) gives a model for the covering $\Omega^* \to \Omega$, Ω^* is naturally endowed with a smooth structure compatible with that of Ω. The lifts of γ_i are critical circles in Ω^* that become of direct type. As a result, all critical circles in Ω^* are of direct type, which means that the lifted S^1 action is free. $\square$

1.6 Normal forms near γ_i

Choose any Hamiltonian function H_q near γ_i that commutes with H_1 and H_2 and such that, *for any* o in the local curve of critical values of F,

- $H_q(\Lambda_o) = 0$;
- on $\Lambda_o \setminus \gamma_i$, $\mathcal{X}_p$ and $\mathcal{X}_q$ are linearly independent, and the automorphism: $(\mathcal{X}_1, \mathcal{X}_2) \to (\mathcal{X}_p, \mathcal{X}_q)$ is orientation preserving.

For instance, the previously defined H_q (eq. (7)) is a good choice, which is independent on i, but any generic linear combination $H_q = \alpha(H_1 - a_o) + \beta(H_2 - b_o)$, $\beta > 0$ would also do.

Then the following theorem states a simple normal form near γ_i for the new system (H_p, H_q). It will be the main tool for the semi-classical analysis, for it reduces the situation to the case of a cotangent bundle.

Theorem 1.13 *There exists coordinates (x, y) on $R = \mathbb{R}\times]A, B[$ that give local coordinates on $\tilde{W}^+(\gamma_i)$ by taking the quotient by*

$$(x, y) \to (x + 2\pi, y)$$

in the direct case and

$$(x, y) \to (x + \pi, -y)$$

in the reverse case and a canonical diffeomorphism of a neighbourhood of γ_i into a neighbourhood of the "$\xi = c_o$-section" of $T^*(\tilde{W}^+(\gamma_i))$ (recall that $c_o = H_p(\Lambda_o)$) such that with respect to canonical coordinates we get

$$\left\{ \begin{array}{lcl} H_p & = & \xi \\ H_q & = & \Phi(\xi, y\eta), \end{array} \right.$$

for some smooth function Φ defined near the origin, with $\partial_2\Phi(0,0) > 0$.

Proof. 1. We first wish to prove that the restriction of H_q to the locally reduced manifold $W := H_p^{-1}(c_o)/\exp t\mathcal{X}_p$ has a non-degenerate saddle point. Let $m \in \gamma_i$ and let (s, σ, u, v) be local coordinates near m such that $\sigma = H_1 - a_o$ and the flow of $\mathcal{X}_1$ is just translation on the s variable. Then because $dH_p(m) = \lambda dH_1(m)$ with $\lambda \neq 0$, the map $(s, \sigma, u, v) \to (s, H_p - c_o, u, v)$ is a local diffeomorphism of $(\mathbb{R}^4, 0)$ that sends $(s, 0, 0, 0)$ to $(s, 0, 0, 0)$. Therefore one can take (u, v) as local coordinates for W, and we wish to prove that $(H_q)_{\restriction W}(u, v)$ has a non-degenerate saddle point at $(0, 0)$.

Let $\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of smooth functions such that

$$(dH_1, dH_2) = \mathcal{M} \cdot (dH_p, dH_q).$$

Since H_2 is of the form $H_2 = F(H_1, u, v)$, one has

$$dH_2 = K.dH_1 + A,$$

where $K = \partial_1 F$ and $A = A(H_1, u, v)$ is a one-form on $\{0\} \times \{0\} \times \mathbb{R}^2$ depending smoothly on H_1, that vanishes at m and whose differential at m is a non-degenerate quadratic form of hyperbolic type. Using $\mathcal{M}$ one gets

$$(c - Ka)dH_p = (Kb - d)dH_q + A.$$

The claim is that $(Kb - d)(m) \neq 0$. Indeed, $(c - Ka)$ and $(Kb - d)$ cannot simultaneously vanish because $\mathcal{M}$ is invertible. It suffices then to see that $A(m) = 0$ whereas by hypothesis $dH_p(m) \neq 0$.

Therefore, we have, on $TW \subset \ker dH_p$,

$$d(H_q)_{\restriction TW} = -\frac{1}{Kb - d}A,$$

which implies that $d(H_q)_{\restriction TW}$ possesses, as A does, a non-degenerate differential of hyperbolic type.

2. Now, we consider a neighbourhood of the whole critical circle γ_i and use Weinstein's theorem with S^1 action to reduce to the cotangent space of $\tilde{W}^+(\gamma_i)$ with $(H_p - c_o = \xi, H_q = H_q(\xi, y, \eta))$. In these coordinates (we consider first the direct case), W is naturally identified with $\{0\} \times \{0\} \times \mathbb{R}^2$, so the previous point shows that $(y, \eta) \to H_q(\xi, y, \eta)$ has, for each small ξ, a non-degenerate saddle point. Then the proof goes exactly as that of Theorem 1.5. Equations (2) through (5) are valid if H_1 and H_2 are replaced by H_p and H_q, and ω_0 is the canonical 2-form of $T^*(\tilde{W}^+(\gamma_i))$.

In the reverse case, the proof is the same but we need the Isochore Morse lemma for functions that are invariant by the involution $\sigma(x) = -x$: it it then possible to choose the diffeomorphism F commuting with σ. This fact follows easily from the proof given in [11]: in the *lemme principal* p. 283, we choose η such that $\sigma^\star(\eta) = -\eta$. It implies that X_t commutes with the involution. □

Remark 1.4. We decide to give to the graph G the orientation of the flow of H_q defined by (7). Near a vertex γ_j, it is also given by the flow of the normal form $y\eta$. △

1.7 Appendix: Morse-Bott lemma

Definition 1.4 *Let $f : X \to \mathbb{R}$ be a smooth function. A submanifold W of X is called a* Morse-Bott critical manifold *if every point $w \in W$ is a critical point of f and if the restriction of $f"(w)$ to the normal bundle T_wX/T_wW is non degenerate.*

Morse-Bott critical manifold arises in many situations especially when f is invariant by a Lie group action. An extension of the Morse lemma is available in that case. In some situations, there is global topological problem with the subbundles $N_\pm$ of the normal bundle generated by eigenspaces of $f"$ associated with > 0 (resp. < 0) eigenvalues.

Lemma 1.14 (Morse-Bott lemma) *Assume we have a Morse-Bott connected critical manifold W for a function $f : X \to \mathbb{R}$. Let N be the normal bundle of W and F the Hessian of f which is a non degenerate quadratic form on N. Then there exists a diffeomorphism of a neighbourhood of W on a neighbourhood of the 0-section in N which conjugates f to $F + c$.*

If W is connected, a complete set of invariant of f, up to smooth conjugacy near W, is given by the pair (N_+, N_-) of bundles on W up to isomorphism.

2 Semi-classics

The aim of this section is to express the singular Bohr-Sommerfeld quantisation rules for quantum integrable systems whose classical counterpart fulfils the hypothesis of the previous section.

Let X be a 2-dimensional differential manifold, and let $\hat{H}_1(h)$, $\hat{H}_2(h)$ be commuting h-pseudo-differential operators, with real principal symbols H_1, H_2. Assume that the momentum map $F = (H_1, H_2)$ satisfies the hypothesis of section 1. In all of this section, the 1-form α of Lemma 1.9 is taken to be the canonical Liouville 1-form of the cotangent bundle T^*X. Then H_p (Theorem 1.6) is uniquely defined as the action integral with respect to α. For any E in the image of F, the sub-principal form κ_E is the closed differentiable 1-form on $\Lambda_E := F^{-1}(E)$ defined at its regular points by $\kappa_E(\mathcal{X}_j) = -r_j$, where r_j is the sub-principal symbol of $\hat{H}_j$.

2.1 The microlocal normal form

We will prove here a semi-classical analogue of Theorem 1.13, which was particularly fit for this purpose since it reduced the situation to that of a cotangent bundle, for which the usual pseudo-differential quantisation can be used. A semi-classical analogue of Theorem 1.11 should also be interesting, but would involve symplectically reduced cotangent bundles, for which Toeplitz quantisation is needed, a theory that we don't want to enter here.

In this section a critical circle $\gamma_j \subset \Lambda_o$ is fixed. Theorem 1.13 identifies, via a symplectomorphism ψ, a neighbourhood of γ_j in T^*X with a neighbourhood of the zero section of a cotangent bundle of the form $T^*(\mathbb{R}^2/\sigma)$, where $\sigma(x,y) = (x+2\pi, y)$ in the direct case, and $\sigma(x,y) = (x+\pi, -y)$ in the reverse case.

It is easy to check that Weyl quantisation satisfies, for a symbol $a \in C_0^\infty(\mathbb{R}^2)$:

$$\sigma^* Op_h^W(a) = Op_h^W(a \circ T^*\sigma)\sigma^*,$$

where σ^* is the adjoint operator $u \mapsto u \circ \sigma$, and $T^*\sigma$ is the cotangent lift of σ. Therefore, if $a = a \circ T^*\sigma$, then $Op_h^W(a)$ acts on the space of functions u that are invariant under σ : $u \circ \sigma = u$, which is the space of functions defined on a cylinder in the direct case, and on the Mœbius strip in the reverse case. In particular, $Q_1(h) = Op_h^W(\xi)$ and $Q_2(h) = Op_h^W(y\eta)$ are well-defined differential operators on $\mathbb{R}^2/\sigma$:

$$Q_1(h) = \frac{h}{i}\frac{\partial}{\partial x}, \quad Q_2(h) = \frac{h}{i}\left(y\frac{\partial}{\partial y} + \frac{1}{2}\right). \tag{8}$$

Let Ψ^0 be the algebra of operators of the form $Op_h^W(a(h))$ for classical symbols $a(h)$ on $T^*(\mathbb{R}^2/\sigma)$, modulo those whose symbol is $O(h^\infty)$. For more details about the microlocal calculus and $O(h^\infty)$ remainders, see eg. [8], [26]. Before stating the result of this section, we introduce the following spaces :

Definition 2.1 *The classical and semi-classical commutants $\mathfrak{C}_{cl}(\gamma_j)$ and $\mathfrak{C}_h(\gamma_j)$ are defined as follows :*

$$\mathfrak{C}_{cl}(\gamma_j) = \{f \in C^\infty(T^*(\mathbb{R}^2/\sigma)), \quad \{f,\xi\} = \{f, y\eta\} = 0 \text{ near } \gamma_j\};$$

$$\mathfrak{C}_h(\gamma_j) = \{P(h) \in \Psi^0, \quad [P,Q_1] \text{ and } [P,Q_2] \text{ are } O(h^\infty) \text{ near } \gamma_j\};$$

(Recall that in our coordinates, $\gamma_j = ((\mathbb{R} \times \{0\})/\sigma) \times \{c_o\} \times \{0\}$, where $c_o = H_p(\Lambda_o)$.) Because the symbols of Q_1 and Q_2 are polynomials of degree $\leqslant 2$, the operators in $\mathfrak{C}_h(\gamma_j)$ are exactly the Weyl quantisations of symbols of the form $\sum h^k a_k$ with $a_k \in \mathfrak{C}_{cl}(\gamma_j)$.

Theorem 2.1 (Microlocal normal form) *There exists an elliptic Fourier integral operator $U(h)$ associated to the canonical transformation ψ of Theorem 1.13, an invertible 2×2 matrix $\mathcal{M}(h)$ of pseudo-differential operators in $\mathfrak{C}_h(\gamma_j)$, and complex-valued functions of h : $\epsilon_1(h)$ and $\epsilon_2(h)$ admitting an asymptotic expansion in $\mathbb{C}[\![h]\!]$:*

$$\epsilon_1(h) \sim \sum_{\ell=0}^{\infty} \epsilon_1^{(\ell)} \qquad \epsilon_2(h) \sim \sum_{\ell=0}^{\infty} \epsilon_2^{(\ell)}$$

such that, microlocally near γ_j :

$$U^{-1}(\hat{H}_1 - a_o, \hat{H}_2 - b_o)U = \mathcal{M}.(Q_1 - \epsilon_1, Q_2 - h\epsilon_2) + O(h^\infty). \tag{9}$$

If $\hat{H}_1$ and $\hat{H}_2$ are formally self-adjoint, then $U(h)$ can be chosen to be microlocally unitary, and the functions ϵ_j are real-valued.

- *The first terms of $\epsilon_1(h)$ (of order respectively h^0 and h^1) are given by the formulæ*

$$\epsilon_1^{(0)} = c_o = \frac{1}{2\pi}\int_\delta \alpha; \tag{10}$$

$$\epsilon_1^{(1)} = \frac{1}{2\pi}\int_\delta \kappa_o + \mu(\delta)/4, \tag{11}$$

where μ is the Maslov index of any regular part of Λ_o, and δ is any cycle associated to an S^1-orbit on $\Lambda_o \setminus \gamma$ (and recall that κ_o is the sub-principal form of the system).

- *The first term of $\epsilon_2(h)$ is given by the formula:*

$$\epsilon_2^{(0)} = \left(\frac{\lambda r_1 - r_2}{|\mathcal{H}_\Sigma(H_2)|^{1/2}}\right)_{\restriction \gamma_j}, \tag{12}$$

where λ is defined in Lemma 1.7 (recall that r_i is the sub-principal symbol of $\hat{H}_i$ and $\mathcal{H}_\Sigma(H_2)$ is the transversal Hessian of H_2). Note that $\mathcal{H}_\Sigma(H_2)$ is also equal to the (y,η)-Hessian of $H_{2,j}$ (the latter was defined along with λ in Lemma 1.7).

Remark 2.1. Recall that there is a choice of sign in the canonical chart ψ of Theorem 1.13. If the other sign is chosen, then ϵ_2 becomes $-\epsilon_2$. △

Proof. Consider the direct case first. The proof goes along the same lines as in [25], so we skip the details and indicate only the crucial points. First take U as any Fourier integral operator associated to ψ (note that by construction this symplectomorphism is exact in the sense that it preserve the action integral). Since H_1 and H_2 commute with H_p and H_q, Theorem 1.13 implies that the principal symbols of $U^{-1}\hat{H}_1 U$ and $U^{-1}\hat{H}_2 U$ are in the classical commutant $\mathfrak{C}_{cl}(\gamma_j)$. The following division lemma is easily proved as in [25] :

Lemma 2.2 *Any function $K \in \mathfrak{C}_{cl}(\gamma_j)$ that vanish on γ_j can be written (in a neighbourhood of γ_j) :*

$$K(x,\xi,y,\eta) = K(\xi,y,\eta) = a(\xi - c_o) + by\eta,$$

for some smooth functions a and b in $\mathfrak{C}_{cl}(\gamma_j)$.

Applying this lemma to $H_1 \circ \psi$ and $H_2 \circ \psi$ solves the principal part of equation (9).

The next steps are obtained by conjugating U by elliptic pseudo-differential operators, yielding transport equations of the form :

Lemma 2.3 *Given any functions (r_1, r_2) such that*

$$\{r_1, y\eta\} = \{r_2, \xi\},$$

there exists K_1, $K_2 \in \mathfrak{C}_{cl}(\gamma_j)$ and a function f such that :

$$\{\xi, f\} = K_1 - r_1 \quad \text{and} \quad \{y\eta, f\} = K_2 - r_2.$$

Proof. Let

$$K_1(\xi,y,\eta) = \frac{1}{2\pi}\int_0^{2\pi} r_1(x,\xi,y,\eta)dx.$$

Of course, $\{\xi, K_1\} = 0$, and using the hypothesis of the lemma, one has

$$\{K_1, y\eta\} = \frac{1}{2\pi}\int_0^{2\pi} \{r_2,\xi\}dx =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} -\frac{\partial r_2}{\partial x}dx = 0.$$

Now, let

$$f_1(x,\xi,y,\eta) = \int_0^x K_1(\xi,y,\eta) - r_1(x',\xi,y,\eta)dx'.$$

Then f_1 is a smooth function on $T^*(\mathbb{R}^2/\sigma)$ that satisfies – using the same kind of calculation as above :

$$\{f_1, y\eta\}(x,\xi,y,\eta) = r_2(x,\xi,y,\eta) - r_2(0,\xi,y,\eta).$$

Then the wanted function f is sought under the form $f = f_1 + f_2$, which leads to the system

$$\{\xi, f_2\} = 0, \quad \text{and}$$

$$\{y\eta, f_2\} = K_2 - r_2(0,\xi,y,\eta).$$

It suffices to see ξ as a parameter and apply a known lemma in the (y,η)-variables (see eg. [18, Theorem 2]). □

In the reverse case, the proof of the theorem is the same provided we deal with functions that are invariant under $T^*\sigma$. But if $K(\xi,-y,-\eta)=K(\xi,y,\eta)$, lemma 2.2 still applies, yielding functions a and b with the same properties. The same is true for the transport equation. Then each step of the proof can be quantised via Weyl's formula to yield well-defined pseudo-differential operators on $\mathbb{R}^2/\sigma$. Thus the result still holds for the reverse case.

The proofs for formulæ (10), (11) and (12) are given in section 2.4, but the formula for ϵ_1 is apparent in the proof of Theorem 2.4 below, and the formula for ϵ_2 can be directly checked using the fact that the subprincipal symbol is preserved under conjugation by an elliptic Fourier integral operator at a critical point of the principal symbol. □

2.2 Microlocal solutions

We investigate here the solutions of the system

$$(\hat{H}_1(h)-a_o)u=O(h^\infty),\quad (\hat{H}_2(h)-b_o)u=O(h^\infty),\tag{13}$$

microlocally on a neighbourhood of the critical Lagrangian Λ_o. If the operators $\hat{H}_j$ depend smoothly on some additional parameter $E\in\mathbb{R}^d$ that leaves the principal symbols intact, then all the results presented here depend smoothly and locally uniformly on E. This applies in particular to the investigation of the joint spectrum in a window of size $O(h)$ around (a_o,b_o), where $\hat{H}_j$ is to be replaced by $\hat{H}_j-hE_j$.

Theorem 2.1, applied to all critical circles γ_j, yields a finite set of semi-classical invariants $(\epsilon_{1,j}(h),\epsilon_{2,j}(h))$. We show here how these quantities are related to the solutions of (13).

Theorem 2.4 (The global quantum number)

- *The asymptotic series $\epsilon_1(h)=\epsilon_{1,j}(h)$, modulo $h\mathbb{Z}$, depend neither on j, nor on the particular way to achieve the normal form of Theorem 2.1.*
- *The system (13) admits a microlocal solution near any (and then all) S^1-orbit (including critical circles) if and only if the following condition holds :*

$$\epsilon_1(h)\in h\mathbb{Z}+O(h^\infty).\tag{14}$$

Remark 2.2. Since $\epsilon_1(h)$ is determined by $\hat{H}_1(h)$ and $\hat{H}_2(h)$, the fulfilment of equation (14) seems to impose a quantisation condition on h. While we can stick here to this interpretation, another possibility would be to recall that everything (and in particular $\epsilon_1(h)$) smoothly depends on the point o in the curve of critical values of F. Then equation (14) can be interpreted as a quantisation condition on o, which leaves h free to vary in a full neighbourhood of 0. This viewpoint is made clear in section 2.4 (cf. Corollary 2.14). △

Proof *of Theorem 2.4.* Following [26], we introduce the sheaf $(\mathfrak{L},\Lambda_o)$ of germs of microlocal solutions on Λ_o, as a sheaf of $\mathbb{C}_h$-modules, where $\mathbb{C}_h$ is the ring of all complex functions of h, $c(h)$, such that

$$|c(h)|\leqslant C.h^{-N},$$

for some constants C, N, modulo those functions that are $O(h^\infty)$. Note that the vector operator $\hat{F} = (\hat{H}_1, \hat{H}_2)$ acts on the huge sheaf over Λ_o of germs of all admissible distributions modulo microlocal equivalence, and $(\mathfrak{L}, \Lambda_o)$ can be seen as the kernel of $\hat{F}$. The question is to find out how local germs can be glued together to form a nontrivial global section of $(\mathfrak{L}, \Lambda_o)$, i.e. a solution of (13) near Λ_o.

It was shown in [26] that the restriction $(\mathfrak{L}, \Lambda_o \setminus \gamma)$ to the non-singular points of F is a locally constant sheaf, and the germs $\mathfrak{L}(p)$ at any non-singular point p form a free module of rank 1, generated by a standard WKB solution. The existence of nontrivial global sections of $(\mathfrak{L}, \Lambda_o \setminus \gamma)$ is then characterised by the nullity (mod. $O(h^\infty)$) of the associated holonomy (or "Bohr-Sommerfeld cocycle") :

$$\lambda(h) \in \mathrm{H}^1(\Lambda_o \setminus \gamma, \mathbb{R}/2\pi\mathbb{Z}).$$

Since $\Lambda_o \setminus \gamma$ is a disjoint union of cylinders $\Lambda^k_{\{i,j\}}$, whose homology $H_1(\Lambda^k_{\{i,j\}})$ is generated by the cycle represented by any oriented S^1-orbit, we get a finite set of holonomies $\lambda^k_{\{i,j\}}(h)$ characterising $(\mathfrak{L}, \Lambda_o \setminus \gamma)$.

Apply now Theorem 2.1. The system (13) is then, on a neighbourhood Ω of γ_j, equivalent to the following standard system :

$$Q_1 u = \epsilon_{1,j} u, \quad Q_2 u = h\epsilon_{2,j} u. \tag{15}$$

At any non-singular point $p \in \Omega \setminus \gamma_j$, the standard WKB u solution generating $\mathfrak{L}(p)$ is therefore of the form

$$u(p) = \mathrm{e}^{i\frac{\epsilon_{1,j}}{h}x} v(y), \quad (x, y) \sim p \in \mathbb{R}^2/\sigma.$$

This implies that

$$\frac{1}{2\pi}\int_\delta \lambda(h) \equiv \frac{\epsilon_{1,j}(h)}{h} + O(h^\infty) \quad (\mathrm{mod}\mathbb{Z}),$$

where δ is the cycle on $\Lambda_o \setminus \gamma$ associated with the orbit $S^1(p)$. This proves

1. that $\epsilon_{1,j}(h)$ does not depend on the particular way to achieve the normal form;
2. that $\int_\delta \lambda(h)$ does not depend on the choice of the Lagrangian cylinder containing p – i.e. for j fixed, all $\lambda^k_{\{i,j\}}(h)$ are equal – which in turn proves
3. that $\epsilon_1 = \epsilon_{1,j}$ does not depend on the choice of the critical circle γ_j (since Λ_o is connected).

The first part of the theorem is now proved. Moreover, the condition (14) is necessary and sufficient for the existence of a (non-trivial) solution near a regular orbit. This condition remains therefore necessary for the existence of a solution near a critical circle. We are thus left with the proof of the sufficiency of this condition for critical circles, which is achieved by the next proposition. □

Proposition 2.5 *Let $\gamma_j \subset \gamma$ be a critical circle. Let $d = 2$ or 4 be its degree in $G(\Lambda_o)$. If the condition (14) is fulfilled, then the set $\mathfrak{L}(\gamma_j)$ of germs of microlocal solutions on γ_j is a free $\mathbb{C}_h$-module of rank $\frac{d}{2}$.*

Proof. Let $n = n(h) \in \mathbb{Z}$ be such that $\epsilon_1 = hn + O(h^\infty)$, and let $p \in \gamma_j$. We know from [8, proposition 17] that the module of microlocal solutions of (15) at p is free of rank 2, generated by

$$u_\pm \stackrel{\text{def}}{=} \mathrm{e}^{inx}\left(1_{\pm y>0}\frac{1}{\sqrt{|y|}}\mathrm{e}^{i\epsilon_2 \ln|y|}\right). \tag{16}$$

If γ_j is direct, this immediately implies that the module $\mathfrak{L}(\gamma_j)$ of microlocal solutions of (13) on the whole circle γ_j is also free and of rank 2.

In the reverse case, the distribution $C_+u_+ + C_-u_-$ on $\mathbb{T}\times\mathbb{R}$ is invariant under the involution σ if and only if it has the parity of n in the variable y, which reads here

$$C_- = \mathrm{e}^{in\pi}C_+.$$

$\mathfrak{L}(\gamma_j)$ is in this case a free module of rank 1, and its generator depends on the parity of n. $\square$

2.3 The abstract Bohr-Sommerfeld rules

We assume here that the first condition (14) is fulfilled, and show that the existence of global solutions of (15) can be read on the graph $G = G(\Lambda_o)$. As before, let $n = n(h) \in \mathbb{Z}$ be such that $\epsilon_1 = n + O(h^\infty)$,

Because of Theorem 2.4, for any point $p \in \Lambda_o$, there exists a microlocal solution on a neighbourhood of the orbit $S^1(p)$. We shall use this fact to construct from the sheaf $(\mathfrak{L}, \Lambda_o)$ a new sheaf $(\bar{\mathfrak{L}}, G)$ on $G \subset W$ (recall that W is the symplectic orbifold of section 1.5) that will in some way encode whether $(\mathfrak{L}, \Lambda_o)$ has a global section. To each point $p \in G$ we associate the free module $\bar{\mathfrak{L}}(p)$ generated by the germs of **standard basis** at p, which will be of rank 1, as follows. For details of the construction when all vertices are of direct type, see [10].

Denote by $\bar{\gamma}_j$ the vertex of G corresponding to the orbit γ_j, and let $\bar{\gamma} = \bigcup \bar{\gamma}_j$.

- At a regular orbit in $\Lambda_o \setminus \gamma$, a standard basis is just any basis of the space of solutions near γ_j, so we let $\bar{\mathfrak{L}}(p) = \mathfrak{L}(S^1(p))$.
- At a vertex $\bar{\gamma}_j$ of degree 4, we use the definition of a standard basis from [10], which we recall here.

  The edges connecting $\bar{\gamma}_j$ are oriented according to the flow of $y\eta$. Moreover, near $\bar{\gamma}_j$, W is a smooth oriented surface (the orientation is given by minus the symplectic form). It is shown in [8] that Proposition 2.5 in the direct case can be restated as follows : let I_1I_2 (resp. I_3I_4) be the disjoint union of the two local edges leaving $\bar{\gamma}_j$ (resp. arriving at $\bar{\gamma}_j$) with cyclic order $(1,3,2,4)$ – with respect to the orientation of W near $\bar{\gamma}_j$. $\bar{\mathfrak{L}}(I_1I_2)$ and $\bar{\mathfrak{L}}(I_3I_4)$ are free modules of rank 2. Then there exists a linear map $T_j : \bar{\mathfrak{L}}(I_3I_4) \to \bar{\mathfrak{L}}(I_1I_2)$ such that u is a solution in a neighbourhood of γ_j if and only if its restrictions satisfy $u_{\bar{\mathfrak{L}}(I_1I_2)} = T_j u_{\bar{\mathfrak{L}}(I_3I_4)}$. In other words, if we "feed" the system with two functions on the entering edges I_3 and I_4, then these functions are propagated on the leaving edges I_1 and I_2 in a unique way (see fig.4).

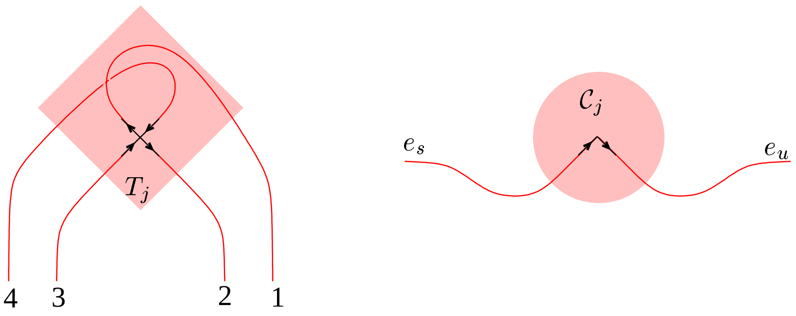


Figure 4: Propagation of solutions at vertices of degree 4 and 2

One can choose a basis element for each $\bar{\mathfrak{L}}(I_i)$, $i = 1, \ldots, 4$, and express T_j as a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (defined modulo $O(h^\infty)$). Moreover, one can show that the entries are all non-vanishing. It is then easy to check that a new choice for the basis elements does not change the cross-ratio $\rho_j = \frac{ad}{bc}$. In our situation, ρ_j can be explicitly calculated; one finds ([8]) :

$$\rho_j = \rho_j(h) = -\mathrm{e}^{2\pi\epsilon_{2,j}(h)}. \tag{17}$$

The choice of a matrix T_j fixes the basis elements up to their multiplication by a same factor. We shall call the choice of the basis elements of $\bar{\mathfrak{L}}(I_i)$, $i = 1, \ldots, 4$, a **standard basis** whenever T_j has the following expression :

$$T_j = \frac{1}{\sqrt{2\pi h}}\Gamma(\beta)\mathrm{e}^{\beta \ln h} \begin{pmatrix} \mathrm{e}^{-i\beta\frac{\pi}{2}} & \mathrm{e}^{i\beta\frac{\pi}{2}} \\ \mathrm{e}^{i\beta\frac{\pi}{2}} & \mathrm{e}^{-i\beta\frac{\pi}{2}} \end{pmatrix}, \tag{18}$$

with $\beta = \frac{1}{2} + i\epsilon_{2,j}$; or with the notations of [10] :

$$T_j = T(\epsilon_{2,j}(h)) = \mathrm{e}^{-i\frac{\pi}{4}}\mathcal{E}_j \begin{pmatrix} 1 & i\mathrm{e}^{-\epsilon_{2,j}\pi} \\ i\mathrm{e}^{-\epsilon_{2,j}\pi} & 1 \end{pmatrix}, \tag{19}$$

with

$$\mathcal{E}_j = \mathcal{E}(\epsilon_{2,j}(h)) = \frac{1}{\sqrt{2\pi}}\Gamma(\frac{1}{2} + i\epsilon_{2,j})\mathrm{e}^{\epsilon_{2,j}(\frac{\pi}{2} + i\ln h)}. \tag{20}$$

Remark 2.3. In [10] the factor $\mathrm{e}^{-i\frac{\pi}{4}}$ was absent in the definition of T_j (19). Its introduction here will greatly simplify the treatment of Maslov indices (see also [5]). $\triangle$

Remark 2.4. Equation (17) proves that $\epsilon_{2,j}$ is a semi-classical invariant (modulo $i\mathbb{Z}$) of the critical circle γ_j : it does not depend on the particular way the normal form is achieved. $\triangle$

Remark 2.5. As it is presented here, the notion of a standard basis seems to be attached to the graph G endowed with a specific labelling at vertices of degree 4. The form of the matrix T_j shows that the different possible labellings of the four hyperbolic branches I_i, $i = 1, 2, 3, 4$ yield the same set of standard basis, provided I_1 and I_2 are the local unstable manifolds (for the flow of $y\eta$ – which means for the flow of H_q), I_3 and I_4 are the local stable manifolds, and on the oriented manifold W (which is smooth at vertices of direct type) the branches appear in cyclic order $(1, 3, 2, 4)$. Furthermore, it can be easily checked using the standard basis (25) given below and the fact that $\mathcal{F}_h^2 f = \check{f}$ (where $\check{f}(y) = f(-y)$) that $T(\epsilon)T(-\epsilon) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, exchanging the local un/stable manifolds just amounts to changing the sign of $\epsilon_{2,j}$. △

- At a vertex $\bar{\gamma}_j$ of degree 2, the space of solutions has dimension 1, so we could just, as in the regular case, call any solution a standard basis. However, in order to isolate the "singular" components of the holonomy, we prefer the following convention which is more in accordance with the previous case (degree 4).

  Let I_u and I_s be the local unstable and stable manifolds of $\bar{\gamma}_j$. A choice of basis elements (e_u, e_s) for $\bar{\mathfrak{L}}(I_u)$ and $\bar{\mathfrak{L}}(I_s)$ will be called a standard basis if

$$e_u = \mathcal{C}_j e_s, \tag{21}$$

  with

$$\begin{aligned} \mathcal{C}_j &= \mathcal{C}(n, \epsilon_{2,j}(h)) = \mathrm{e}^{-i\frac{\pi}{4}} \mathrm{e}^{-in\frac{\pi}{2}} \mathcal{E}_j (1 + i(-1)^n \mathrm{e}^{-\epsilon_{2,j}\pi}). \\ &= \sqrt{\frac{2}{\pi h}} \Gamma(\beta) \mathrm{e}^{\beta \ln h} \cos(\frac{\pi}{2}(\beta + n)). \end{aligned} \tag{22}$$

  Notice that $\mathcal{C}_j$ depends on $n \mod 4$.

Remark 2.6. If $\epsilon_{2,j} \in \mathbb{R}$, one has

$$\mathcal{E}_j = \frac{1}{\sqrt{1 + \mathrm{e}^{-2\pi\epsilon_{2,j}}}} \mathrm{e}^{i \arg \Gamma(1/2 + i\epsilon_{2,j}) + i\epsilon_{2,j} \ln(h)}. \tag{23}$$

Therefore T_j is unitary and $|\mathcal{C}_j| = 1$. △

$(\bar{\mathfrak{L}}, G)$ is a locally flat sheaf of rank-one modules, and hence is characterised by its holonomy:

$$\mathbf{hol} : \mathrm{H}_1(G) \to \mathbb{C}_h .$$

In terms of Čech cohomology, if γ is a loop in G, and $\Omega_1, \Omega_2, \ldots, \Omega_\ell = \Omega_1$ is an ordered sequence of open sets covering the image of γ, each Ω_i being equipped with a standard basis u_i, then

$$\mathbf{hol}(\gamma) \stackrel{\text{def}}{=} x_{1,2} x_{2,3} \ldots x_{\ell-1,\ell},$$

where $x_{i,j}$ is defined in $\mathbb{C}_h$ by $u_i = x_{i,j} u_j$ on $\Omega_i \cap \Omega_j$.

Definition 2.2 (The singular invariants) *1. The "principal value" $\tilde{\kappa}_o$ of the sub-principal form κ_o is the cocycle on Λ_o defined as follows (see also [10]):*

- *if* $[A,B] \subset \Lambda_o \setminus \gamma$ *is a non singular path, then*
$$\int_{[A,B]} \tilde{\kappa_o} := \int_{[A,B]} \kappa_o.$$
- *if* $[A,B] \subset \Lambda_o$ *is a path intersecting once and transversally a unique critical circle* γ_j*, and oriented according to the flow of* H_q *(ie. A is on the local stable manifold and B is on the local unstable manifold) then*
$$\int_{[A,B]} \tilde{\kappa_o} := \lim_{a,b\to m} \left(\int_{[A,a]} \kappa_o + \int_{[b,B]} \kappa_o + \epsilon_{2,j}^{(0)} \ln \left| \int_{R_{a,b}} \omega \right| \right),$$
where $R_{a,b}$ *is the parallelogram (defined in any coordinate system) built on the vectors* $\overrightarrow{ma}$ *and* $\overrightarrow{mb}$ *(*$m = [A,B] \cap \gamma_j$ *– see fig.5).*

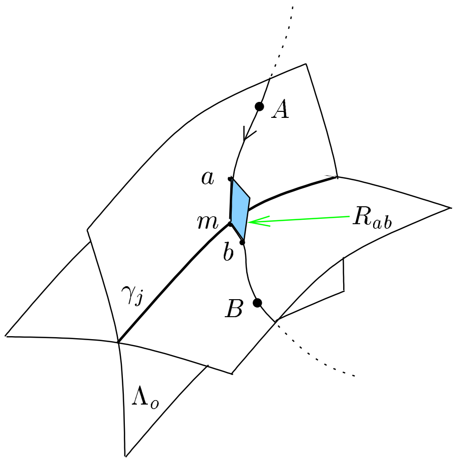


Figure 5: Regularization of κ_o

2. *The "regularized" Maslov index* $\tilde{\mu}$ *on* Λ_o *is defined as follows:*
   - *The contribution from a regular path in* $\Lambda_o \setminus \gamma$ *is the usual Maslov index of the path.*
   - *Let* $\delta = [A,B] \subset \Lambda_o$ *is a small path intersecting once and transversally a unique critical circle* γ_j*, such that A belongs to one of the hyperbolic branches* $(1,3,2,4)$ *and B to an adjacent branch (ie.* δ *makes a turn of angle* $\pm\frac{\pi}{2}$*).* δ *can be continuously deformed into a path* δ_t *drawn on a regular leaf of F. Then the usual Maslov index for this path is constant for t small enough (*$\delta_0 = \delta$*), and we define*
$$\tilde{\mu}(\delta) := \mu(\delta_t) \pm \left(\frac{1}{2} + \chi_{\{d_j=2\}} n \right), \tag{24}$$
where $\pm =$ *"+" if* δ *turns in the direct sense (with respect to the cyclic order of the branches) and "−" otherwise, and* $\chi_{\{d_j=2\}} = 1$ *if* γ_j *is of degree 2 and* $\chi_{\{d_j=2\}} = 0$ *otherwise.*

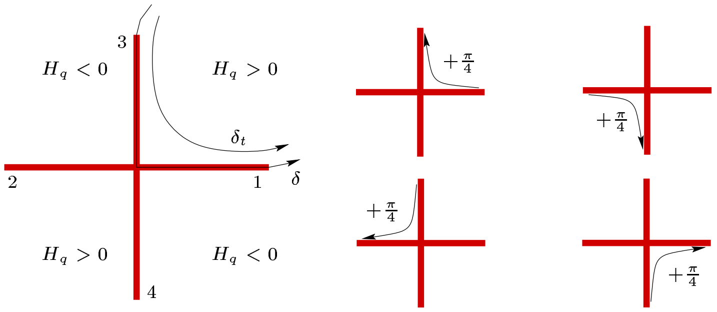


Figure 6: The local Maslov correction (for a vertex of degree 2, replace $\frac{\pi}{4}$ by $\frac{\pi}{4} + n\frac{\pi}{2}$)

Remark 2.7. The sign in (24) is negative if δ is oriented according to the flow of H_q and δ_t belongs to a region in phase space where $H_q > 0$, and changes whenever one of these two conditions changes. Of course the rule of Fig. 6 is simpler to use, but this correspondence will be used for the proofs of section 2.4. $\triangle$

Proposition 2.6 *1. The holonomy of the sheaf $(\bar{\mathfrak{L}}, G)$ has the form* **hol** $=$ $\mathrm{e}^{i[\theta(h)]/h}$, *where $[\pi^*\theta(h)] \in \check{\mathrm{H}}^1(\Lambda_o, (\mathbb{C}, +))$ admits an asymptotic expansion in non-negative powers of h. ($\pi : \Lambda_o \to G$ is the projection associated to the S^1-reduction.)*

2. Let $\sum_{\ell \geqslant 0} [\tilde{\theta}_\ell] h^\ell$ be the asymptotic expansion of $[\tilde{\theta}(h)] := \pi^*[\theta(h)]$. Then the first two terms are given by the following formulæ :

- $[\tilde{\theta}_0] = [\alpha]$ *(the Liouville 1-form on Λ_o);*
- $[\tilde{\theta}_1] = [\tilde{\kappa}_o] + \tilde{\mu}\frac{\pi}{2}$.

Proof. We just prove here the existence of the claimed asymptotic expansion. The formulæ for $\tilde{\theta}$ will follow from our refined analysis in Section 2.4 (Corollary 2.21).

For this purpose, it suffices to show that one can choose local sections u_α of $(\bar{\mathfrak{L}}, G)$ for which the transition constants $c_{\alpha,\beta}$ have the required form. On the edges of G, this follows from the regular theory of WKB solutions (see eg.[27]). At a critical circle, we apply the normal form (Theorem 2.1), and choose the following standard basis ($u_\pm^{\epsilon_2}$ is defined in Eq.(16)):

- in the direct case,

$$\left\{ \begin{array}{lcl} e_1 & = & u_+^{\epsilon_2} \\ e_2 & = & u_-^{\epsilon_2} \\ e_3 & = & \mathcal{F}_h^{-1}(u_+^{-\epsilon_2}) \\ e_4 & = & \mathcal{F}_h^{-1}(u_-^{-\epsilon_2}); \end{array} \right. \tag{25}$$

- in the reverse case,

$$\left\{ \begin{array}{lcl} e_u & = & e_1 + (-1)^n e_2 \\ e_s & = & i^n e_3 + (-i)^n e_4. \end{array} \right.$$

We see then that the restrictions of these solutions to any edge are standard WKB solutions, whose phases admit an asymptotic expansion in powers of h. □

The dimension $b_1 = \dim \mathrm{H}_1(G)$ is given by Euler-Poincaré formula :

$$b_1 = \#\{\text{edges of } G\} - N + 1$$

(recall that N is the number of vertices of G). Moreover, if we write $N = p + q$ with p the number of tetravalent vertices and q the number of divalent vertices, then it is easy to see that

$$\#\{\text{edges of } G\} = 2p + q,$$

so that $b_1 = p + 1$.

We can now cut b_1 edges of G, each one corresponding to a cycle δ_i in a basis $(\delta_1, \ldots, \delta_{b_1})$ of $\mathrm{H}_1(G)$, in such a way that the remaining graph is a tree T ($\mathrm{H}_1(T) = 0$). Then the sheaf $(\bar{\mathfrak{L}}, T)$ has a nontrivial global section, i.e. there exists a standard basis u_α on each edge e_α such that they extend to a standard basis at each vertex.

Theorem 2.7 *$(\mathfrak{L}, \Lambda_o)$ has a nontrivial global section if and only if the following linear system of $3p + q + 1$ equations with the $3p + q + 1$ unknowns $(x_\alpha \in \mathbb{C}_h)_{\alpha \in \{\text{edges of } T\}}$ has a nontrivial solution :*

1. *if the edges $(\alpha_1, \alpha_3, \alpha_2, \alpha_4)$ connect at a tetravalent vertex γ_j (with the prescribed orientation), then*
$$(x_{\alpha_3}, x_{\alpha_4}) = T_j(x_{\alpha_1}, x_{\alpha_2});$$
2. *if the edges (α_u, α_s) connect at a divalent vertex γ_j (with the prescribed orientation), then*
$$x_{\alpha_s} = \mathcal{C}_j x_{\alpha_u};$$
3. *if α_0 and α_1 are the extremities of a cut cycle δ_i, then*
$$x_{\alpha_0} = \mathbf{hol}(\delta_i) x_{\alpha_1}.$$
Here we assume the following orientation: δ_i can be represented by a closed path starting on the edge α_0 and ending on α_1.

Remark 2.8. When solving the system, it is immediate (if $p \neq 0$) to replace the equations of type 2. and 3. into those of type 1., in order to finally obtain a linear system of size $(2p) \times (2p)$. If $p = 0$ then equations of type 2. and 3. combine together to yield a unique equation in one variable. △

Proof . Any global section u of $(\mathfrak{L}, \Lambda_o)$ can be characterised by the set of constants $x_\alpha \in \mathbb{C}_h$ such that $u_{\restriction \bar{\mathfrak{L}}(e_\alpha)} = x_\alpha u_\alpha$. By definition of the standard basis (u_α), conditions (1.) and (2.) are necessary and sufficient for $(x_\alpha u_\alpha)$ to extend to a solution near every critical circle γ_j. It remains to check that the solutions at the extremities α_0, α_1 of a cut cycle δ_i can be consistently glued back together. Since (u_α) is a global section of $(\bar{\mathfrak{L}}, T)$, u_{α_1} is the parallel transport of u_{α_0} along δ_i, which means that, as local sections of $(\bar{\mathfrak{L}}, G)$ (or $(\mathfrak{L}, \Lambda_o)$), they satisfy $u_{\alpha_1} = \mathbf{hol}(\delta_i) u_{\alpha_0}$. Therefore the solutions $x_{\alpha_0} u_{\alpha_0}$ and $x_{\alpha_1} u_{\alpha_1}$ can be glued back if and only if condition (3.) holds. □

2.4 The spectral problem

The goal of this section is to investigate uniform estimates for our system when it depends on spectral parameters. Specifically, we look now at the system

$$(\hat{H}_1(h) - E_1)u = O(h^\infty), \quad (\hat{H}_2(h) - E_2)u = O(h^\infty), \tag{26}$$

where E_1 and E_2 are real numbers. Here we shall assume that $\hat{H}_1$ and $\hat{H}_2$ are formally self-adjoint. If we are only interested in studying the spectrum in a window of size $O(h)$ around the origin, we can let $E_i = h\varepsilon_i$ and there is nothing more to be done: Theorem 2.1 holds uniformly for $(\varepsilon_1, \varepsilon_2)$ varying in a compact set of $\mathbb{R}^2$, so that all the results of the previous sections apply. However, Theorem 2.1 does *not* apply to the system (26) with uniform estimates for $E = (E_1, E_2)$ in a compact. Indeed, it would imply that H_p has a unique value on the local level set $(H_1, H_2) = (E_1, E_2)$ near γ_j; if E is a *regular* value of F, this level set may fail to be connected and it is easy to construct a situation where H_p has different values on each component. Actually, H_p by definition is a function on the set of leaves of the Lagrangian foliation defined by F; and the following diagram is in general non-commutative:

$$\begin{tikzcd} \Omega \arrow[r, "F"] \arrow[dr, bend right, "H_p"] & U \arrow[d, dotted, "??"] \\ & \mathbb{R} \end{tikzcd} \tag{27}$$

Instead, we need to work with the space of leaves $\bar{\Omega}$, equipped with the momentum map $\bar{F}$.

For any $E \in U \subset (\mathbb{R}^2, 0)$, let $\Lambda_E = F^{-1}(E)$. If U is a sufficiently small ball around some critical value, the curve $\mathcal{C}_c \subset U$ of critical values of the momentum map F separates the set of regular values in U into two simply connected open sets U^+ and U^-. Using H_q defined in section 1.6, we take the following convention : $U^\pm := \{\pm H_q > 0\}$. Let $D^\pm := U^\pm \cup \mathcal{C}_c$. Let N^+ and N^- be the sets of connected components of the open sets $F^{-1}(U^+)$ and $F^{-1}(U^-)$ respectively. In each of U^+ and U^-, the levels sets of F have a unique topological type, namely they are unions of a finite number $\tilde{N}^\pm$ of Liouville tori: for any $E \in U^\pm$, $F^{-1}(E) =: \bigsqcup_{k^\pm \in N^\pm} T_{k^\pm}(E)$. Of course, $\tilde{N}^\pm = |N^\pm|$.

Proposition 2.8 *A smooth function K commuting with H_1 and H_2 in Ω is characterised by the data of $|N^+|$ functions $f_{k^+} \in C^\infty(D^+)$ and $|N^-|$ functions $f_{k^-} \in C^\infty(D^-)$ such that*

1. *For all $k^\pm \in N^\pm$, $K_{\restriction k^\pm} = f_{k^\pm} \circ F_{\restriction k^\pm}$;*
2. *For all $k^+ \in N^+$ and $k^- \in N^-$, the function equal to f_{k^+} in D^+ and to f_{k^-} in D^- is smooth on U.*

Definition 2.3 *The space of smooth functions in Ω commuting with H_1 and H_2 will be denoted by $C^{\infty F}(\Omega)$. The space of leaves together with the smooth structure described in the above proposition will be called the* ***Reeb graph*** *of F.*

Proof . a) Given K, the condition 1.) uniquely defines the functions $f_{k^\pm}$. Their smoothness is given by the Darboux-Carathéodory theorem that states

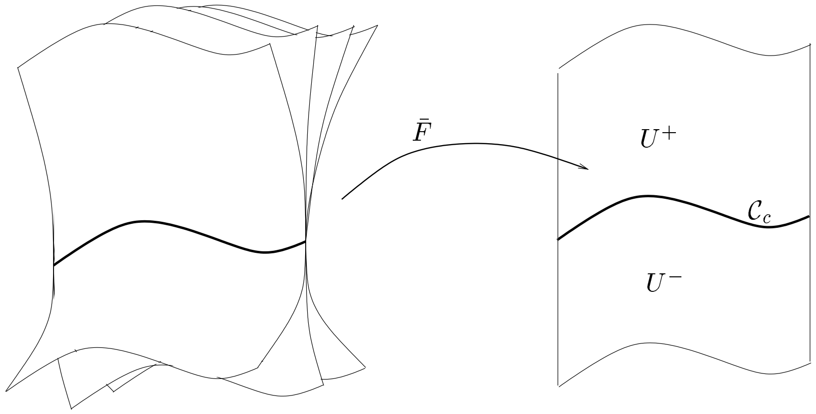


Figure 7: The Reeb graph of F

that near every non-singular point of Λ_E, there is a canonical chart (x, y, ξ, η) in which $H_1 = \xi$ and $H_2 = \eta$. The fact that such non-singular point exist even on a singular leaf Λ_{E_c}, $E_c \in \mathcal{C}_c$, shows that the smoothness extends to $D^\pm$. The same argument shows that the condition 2.) holds whenever $\overline{k}^+ \cap \overline{k}^- \neq \emptyset$. Then condition 2.) without this restriction holds because Λ_{E_c} is connected.

b) Conversely, the data of all the functions $f_{k^\pm}$ defines a unique function K. The smoothness of K outside of the critical points comes from the same argument as above; its smoothness at critical points comes from Theorem 1.5 and the following lemma 2.9 (proved as in [25]). $\square$

Lemma 2.9 *Each function K commuting with ξ and $y\eta$ near $(x, 0, 0, 0)$ is characterised by two functions $f_+, f_- \in C^\infty(\mathbb{R}^2, 0)$ satisfying*

$$f_-(\xi, t) - f_+(\xi, t) = O(t^\infty), \tag{28}$$

locally uniformly in ξ, such that

$$K(x, y, \xi, \eta) = \begin{cases} f_+(\xi, y\eta) & \text{if } y \geqslant 0 \\ f_-(\xi, y\eta) & \text{if } y < 0 \end{cases}$$

Remark 2.9. The distinction between the functions f_+ and f_- is of course irrelevant when all the data is analytic. Neither has it any impact for *semi-excited* regions, ie. for $E = O(h^\gamma)$, $0 < \gamma < 1$ (see [26, chapter 5]). In these cases, no further modification of the results of the previous sections are required (the diagram (27) is always commutative), and the following section 2.4.1 becomes rather straightforward. We have laid down all the details to cope with the C^∞ case, which makes the statements and proofs more technical. On the other hand, the statements in section 2.4.2 are non-trivial even in the analytic case, and represent some of the most crucial results of this article. $\triangle$

2.4.1 The "global" quantum number

In this section, the issue is to generalise the "global" quantum number $\epsilon_1(h)$ of Theorem 2.4. In the smooth, non-analytic category, this leads to a subtle repartition property for the semi-classical spectrum.

For any $E \in U \setminus \mathcal{C}_c$, denote by $(\mathfrak{L}, \Lambda_E)$ the sheaf of germs of microlocal solutions of (26) on Λ_E. We know from the regular theory (see [26]) that $(\mathfrak{L}, \Lambda_E)$ is just a flat bundle characterised by its holonomy, which was called in [26] the Bohr-Sommerfeld cocycle, and denoted by $\lambda_h \in \mathrm{H}^1(\Lambda_E, \mathbb{R}/2\pi\mathbb{Z})$. When E is restricted to any compact subset $K \subset U^\pm$, and λ_h is restricted to some connected component $k^\pm$, $h\lambda_h$ has a uniform asymptotic expansion in $C^\infty(K)[\![h]\!]$. This is *no* longer true on $D^\pm$. However, the following statement holds :

Theorem 2.10 *The function that assigns to a leaf $T_{k^\pm}(E)$ the integral*

$$\frac{h}{2\pi}\int_\delta \lambda_h,$$

where δ is any S^1-orbit in $T_{k^\pm}(E)$, defines an element $\mathfrak{e}_1(h) \in C^{\infty F}(\Omega)$ that admits an asymptotic expansion of the form :

$$\mathfrak{e}_1(h) = \sum_{\ell=0}^{\infty} h^\ell \mathfrak{e}_1^{(\ell)} \qquad \in C^{\infty F}(\Omega)[\![h]\!],$$

with $\mathfrak{e}_1^{(0)} = H_p$, and $\mathfrak{e}_1^{(1)} = -\frac{a}{2\pi}\int_\delta r_1 - \frac{b}{2\pi}\int_\delta r_2 + \mu(\delta)/4$, where r_j is the sub-principal symbol of $\hat{H}_j$, and a, b in $C^{\infty F}(\Omega)$ are the functions defined by $\mathcal{X}_p = a\mathcal{X}_1 + b\mathcal{X}_2$, and μ is the Maslov cocycle of $T_{k^\pm}(E)$.

Remark 2.10. Fix $k^\pm \in N^\pm$ and realise $(\mathfrak{e}_1)_{\restriction k^\pm}$ as a smooth function $E \to \mathfrak{e}_1(E;h)$ in $C^\infty(D^\pm)$. Now let $E = (a_o, b_o) + h(e_1, e_2)$, where e_j vary in a compact (recall that (a_o, b_o) is a critical value of F); then $\mathfrak{e}_1(E;h)$ admits an asymptotic expansion in powers of h whose coefficients are smooth functions of (e_1, e_2). But these coefficients are made out of the Taylor series of $E \to \mathfrak{e}_1(E;h)$, and therefore, in view of the definition of $C^{\infty F}$, they do not depend on the component $k^\pm$. We obtain this way an element $\epsilon_1(h) \in C^\infty[\![h]\!]$ that is nothing else but the global quantum number of Theorem 2.4.

Notice that, in order to compute the principal term in $\epsilon_1(h)$, it is not interesting to use the formula $\epsilon_1(h) = \mathfrak{e}_1(E;h)$, since it would involve the derivative of $\mathfrak{e}_1^{(0)}$. Instead, apply the formula of the theorem above to a system whose principal symbol is independent of E, and whose sub-principal symbol is $(r_1 - e_1, r_2 - e_2)$. We obtain this way the claim (10)-(11) in Theorem 2.1. $\triangle$

Proof *of the theorem.* The fact that $\mathfrak{e}_1(h) \in C^{\infty F}(\Omega)$ is obvious form the construction. To prove the existence of the claimed asymptotic expansion, it suffices to microlocalize near a critical circle γ_j.

Using Lemma 2.9 in the coordinates of Theorem 1.13, one sees that the functions f_+ and f_- of Lemma 2.9 are the same if the degree $d = 2$. In this case, Theorem 2.1 generalises to

$$U^{-1}(\hat{H}_1 - E_1, \hat{H}_2 - E_2)U = \mathcal{M}.(Q_1 - \mathfrak{e}_1, Q_2 - \mathfrak{e}_2) + O(h^\infty),$$

where $\mathcal{M}$, $\mathfrak{e}_1$, and $\mathfrak{e}_2$ depend smoothly on E, which gives the result.

The case $d = 4$ is more intricate, and follows from Proposition 2.12 below. $\square$

We shall need the following slightly weaker version of Theorem 2.1 :

Proposition 2.11 *There exists an elliptic Fourier integral operator $U(h)$ associated to the canonical transformation ψ of Theorem 1.13 such that, microlocally near γ_j :*

$$U^{-1}\hat{H}_j U = \hat{K}_j,$$

where $\hat{K}_j \in \mathfrak{C}_h(\gamma_j)$.

Proof. The same proof scheme as that of Theorem 2.1 applies, using Lemma 2.9 instead of Lemma 2.2. □

Proposition 2.12 *Let γ_j be a critical circle of degree 4. For each E close to zero, the set of microlocal solutions of (26) on a small neighbourhood of any point of γ_j is a free $\mathbb{C}_h$-module of rank 2. In the coordinates of Theorem 1.13, it has a basis of the form*

$$u_E^{\pm} = \mathrm{e}^{i\mathfrak{c}_1^{\pm} x/h}\left(1_{\pm y>0}\frac{1}{\sqrt{|y|}}\mathrm{e}^{i\mathfrak{c}_2^{\pm}\ln|y|/h}\right),$$

where $\mathfrak{c}_j^{\pm} = \mathfrak{c}_j^{\pm}(E;h)$ admits an asymptotic expansion in non-negative powers of h whose coefficients are smooth functions of E. For this basis, the system (26) is solved locally uniformly with respect to (E_1, E_2) near $(0,0)$. Moreover, the functions $(E_1, E_2) \to \mathfrak{c}_j^{+} - \mathfrak{c}_j^{-}$ are flat on the set $\mathcal{C}_c$ of critical points of F.

The proof of this proposition relies on the following lemma :

Lemma 2.13 *Let $p \in C_0^{\infty}(T^*S^1 \times T^*\mathbb{R})$ be a Hamiltonian satisfying*

$$p(x,\xi,y,\eta) = 0 \text{ for } y \geqslant 0$$

(i.e. $p = 1_{y\leqslant 0}p$). Then

$$Op_h^W(p) = Op_h^W(p)\circ 1_{y\leqslant 0} = 1_{y\leqslant 0}.Op_h^W(p) \quad (\text{mod } O(h^{\infty})).$$

Of course the symmetric result (with respect to $y=0$) holds.
Proof. Recall that Weyl quantisation of p is defined by :

$$v(x,y) = Op_h^W(p)u(x,y) =$$

$$\frac{1}{(2\pi h)^2}\int \mathrm{e}^{\frac{i}{h}((x-x')\xi+(y-y')\eta)} p(\frac{x+x'}{2},\xi,\frac{y+y'}{2},\eta)u(x',y')dx'dy'd\xi d\eta.$$

We prove the first estimate by showing that $\|v\| = O(h^{\infty})$ whenever $u = 1_{y\geqslant 0}u$. For this, we consider the two regions $|y| \geqslant h^{\delta_1}$ and $|y| \leqslant h^{\delta_2}$, with $0 < \delta_2 \leqslant \delta_1 < 1$. If $|y| \leqslant h^{\delta_2}$, then only the domain $|\frac{y+y'}{2}| \leqslant h^{\delta_2}$ contributes to the integral; and the result follows from the flatness of p with respect to its third variable : for all $N \in \mathbb{N}, |v| = O(h^{N\delta_2})$. Let us now look at the region $|y| \geqslant h^{\delta_1}$. Since $v(\cdot,\cdot,y\geqslant 0,\cdot) = 0$, one can assume that $y \leqslant -h^{\delta_1}$, which implies $|y-y'| \geqslant h^{\delta_1}$. Now the usual trick applies : a repeated integration by parts with respect to the operator $\frac{h}{i(y-y')}\frac{\partial}{\partial\eta}$ (or standard estimates for the Fourier transform) gives $|v| = O(h^{N(1-\delta_1)})$ for any integer N.

The same method can be applied to show that $\|1_{y\geqslant 0}v\| = O(h^\infty)$ whenever $u = 1_{y\leqslant 0}u$, thus proving the second estimate. □

Proof *of Proposition 2.12.* The fact that the set of solutions is a free module of rank 2 is due, for $E \notin \mathcal{C}_c$, to the regular theory (the local Lagrangian manifold has two connected components, on each of which the set of solutions is a free module of rank 1), and, for $E \in \mathcal{C}_c$, to Proposition 2.5.

We prove the rest of the proposition for u_E^+; the same argument applies to u_E^-. First apply proposition 2.11 to assume in what follows that $\hat{H}_j \in \mathfrak{C}_h(\gamma_j)$. Since $y\eta$ is a quadratic function, every element of $\hat{K} \in \mathfrak{C}_h(\gamma_j)$ can be written $\hat{K} = Op_h^W(K_h)$, with $K_h \sim \sum_{\ell\geqslant 0} h^\ell K^{(\ell)}$ and $K^{(\ell)} \in \mathfrak{C}_{cl}(\gamma_j)$. Because of Lemma 2.9, each $K^{(\ell)}$ is defined by two functions $f_\pm^{(\ell)}$. Let

$$F_h(x,\xi,y,\eta) \sim \sum_{\ell\geqslant 0} h^\ell f_+^{(\ell)}(\xi, y\eta),$$

and $R_h = K_h - F_h$. Let us prove now that there is a unique symbol $g_h(\xi,t) \sim \sum_{\ell\geqslant 0} h^\ell g^{(\ell)}(\xi,t)$ such that

$$Op_h^W(F_h) = g_h(Q_1,Q_2) + O(h^\infty),$$

(Q_j is defined in Eq.(8)). Indeed, $g^{(0)}$ is necessarily equal to $f_+^{(0)}$; therefore,

$$Op_h^W(F_h) = g^{(0)}(Q_1,Q_2) + h\hat{S},$$

where $\hat{S} \in \mathfrak{C}_h(\gamma_j)$ and is of order 0. Then $\hat{S}$ is similarly decomposed – and the claim is proved by induction – provided we show that its Weyl symbol is, as for F_h, a function of $(\xi, y\eta)$. This is achieved by applying Lemma 2.9 and remarking that $Op_h^W(F_h)$, as well as $g^{(0)}(Q_1,Q_2)$, commute with the involution $y \to -y$ (and thus their Weyl symbols are invariant under $(y,\eta) \to (-y,-\eta)$). Summing up, we have proved so far that any operator $\hat{K} \in \mathfrak{C}_h(\gamma_j)$ can be written :

$$\hat{K} = g_h(Q_1,Q_2) + Op_h^W(R_h),$$

where all the coefficients in the expansion of R_h are smooth functions verifying the hypothesis of Lemma 2.13.

Applying this to $\hat{H}_j$, we obtain the existence of two symbols $g_{1,h}$ and $g_{2,h}$ such that

$$\hat{H}_j u_E^+ = g_{j,h}(\mathfrak{e}_1^+, \mathfrak{e}_2^+) u_E^+ + O(h^\infty).$$

The independence of H_1 and H_2 implies that the principal term :

$$(\xi,t) \to (g_1^{(0)}(\xi,t), g_2^{(0)}(\xi,t))$$

is local diffeomorphism; therefore the symbol $(g_{1,h}, g_{2,h})$ is invertible, and the proposition is proved with

$$(\mathfrak{e}_1^+, \mathfrak{e}_2^+) \sim (g_{1,h}, g_{2,h})^{-1}(E_1,E_2).$$

□

Corollary 2.14 *Fix $k^\pm \in N^\pm$ and realise $(\mathfrak{e}_1)_{\restriction k^\pm}$ as a smooth function $E \to \mathfrak{e}_1(E;h)$ in $C^\infty(D^\pm)$. Then the condition*

$$\mathfrak{e}_1(E;h) \in h\mathbb{Z} + O(h^\infty) \tag{29}$$

is necessary and sufficient for the existence of a uniform solution of (26) microlocalised in a neighbourhood (in $\overline{k}^\pm$) of any S^1-orbit in $\overline{k}^\pm$.

Definition 2.4 *For any natural integers ℓ^+ and ℓ^-, we call an (ℓ^+,ℓ^-)-**curve** the union of ℓ^+ smooth curves in D^+ and ℓ^- smooth curves in D^- that are transversal to $\mathcal{C}_c$ and infinitely tangent to each other on $\mathcal{C}_c$.*

These curves are just the image by F of a level set of a smooth function K on the Reeb graph of F, if $\mathcal{X}_K \neq 0$ on the critical leaves Λ_{E_c} (this is a consequence of Proposition 2.8). This holds for instance for $K = H_p$ (Fig. 8).

Corollary 2.15 *There exists a fixed neighbourhood U in $\mathbb{R}^2 = \{(E_1,E_2)\}$ of any critical value of F in which the joint spectrum of $\hat{H}_1$ and $\hat{H}_2$ is distributed (modulo $O(h^\infty)$) on the union of $(|N^+|,|N^-|)$-curves $L_n(h)$ ($n \in \mathbb{Z}$) defined as the image by F of the level sets $\mathfrak{e}_1(h) = hn$. The principal part of these curves is thus given by the level sets of H_p.*

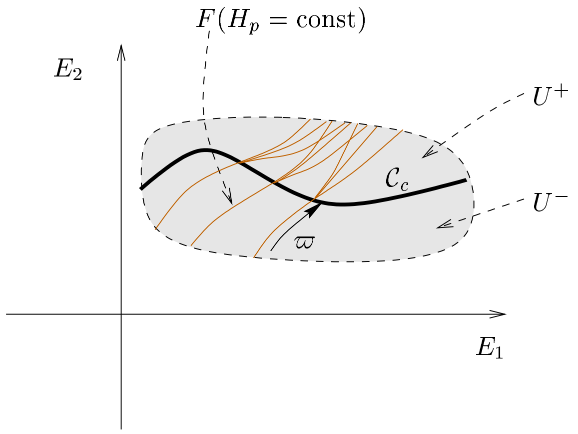


Figure 8: Level sets of H_p

Proof. Fix $k^\pm \in N^\pm$. Then Proposition 2.8 says that the restriction $(\mathfrak{e}_1(h))_{\restriction \overline{k}^\pm}$ is equal to $f_h \circ F_{\restriction \overline{k}^\pm}$ for some smooth function $f_h \sim \sum_{\ell=0}^\infty f_\ell$ admitting an asymptotic expansion in $C^\infty(\mathbb{R}^2,0)[\![h]\!]$. Since $f_0 \circ F_{\restriction \overline{k}^\pm} = (H_p)_{\restriction \overline{k}^\pm}$, the hypothesis (6) of Theorem 1.6 implies that the foliation $f_h = \text{const}$ is transversal to $\mathcal{C}_c$, and we can define the projection $\varpi_h : \mathbb{R}^2 \to \mathcal{C}_c$ such that $\varpi_h(f = \text{const})$ is a point. The pre-image of $\{f_h = \text{const}\} \cap D^\pm$ by $F_{\restriction \overline{k}^\pm}$ is a leaf of the foliation $\{\mathfrak{e}_1(h) = \text{const}\}$ in $\overline{k}^\pm$. The value of H_p on $\mathcal{C}_c$ can be taken as a coordinate on $\mathcal{C}_c$, and via this identification, it is natural to view $\mathfrak{e}_1(h)$ as a function with values in $\mathcal{C}_c$. By Corollary 2.14, any microlocal eigenfunction of $(\hat{H}_1,\hat{H}_2)$ microlocalised in $\overline{k}^\pm$

defines a joint eigenvalue that belongs to $\varpi_h^{-1}(h\mathbb{Z} + O(h^\infty))$.

$$\begin{tikzcd} \overline{k}^{\pm} \arrow[r, "F"] \arrow[drr, bend right, "H_p"'] & U \arrow[r, "\varpi_0"] & \mathcal{C}_c \arrow[d, "\simeq"] \\ & & \mathbb{R} \end{tikzcd} \tag{30}$$

□

2.4.2 Regularization of $\lambda_h(E)$

This section contains some of the most central results of this article (Theorems 2.19 and 2.20). They give a new interpretation of the holonomy of section 2.3 and provide for the proofs of the various formulæ claimed before.

To each $k^\pm \in N^\pm$ and $E_c \in \mathcal{C}_c$ we associate the subset $T_{k^\pm}(E_c) = \overline{k}^\pm \cap \Lambda_{E_c}$ which is the "limit" of the torus $T_{k^\pm}(E)$ as $E \to E_c$. Suppose we are given a continuous family of piecewise differentiable loops $(\delta_E)_{E\in D^\pm}$ on $T_{k^\pm}(E)$ that are everywhere transversal to the S^1-orbits (such a family can be constructed using for instance the normal form of Theorem 1.13), and assume that they are oriented by the flow of H_q. For non-singular values of E, δ_E together with an S^1-orbit form a basis of $\mathrm{H}_1(T_{k^\pm}(E))$, and $E \to \int_{\delta_E}\lambda_h$ is a smooth function. To complete the result of Theorem 2.10, it is natural to investigate here the behaviour of that function as E approaches a critical value.

To each $E_c \in \mathcal{C}_c$ corresponds a real number x via the diffeomorphism (30); $T_{k^\pm}(E_c)$ is an S^1-invariant subset of $H_p^{-1}(x)$, and hence can be reduced to a cycle $G_{k^\pm}(E_c)$ of the graph $G(E_c) := G(\Lambda_{E_c})$ in the reduced manifold $W(E_c)$. The second goal of this section is to show how the asymptotic behaviour of the function $E \to \int_{\delta_E}\lambda_h$ is related to the *holonomy* **hol** of section 2.3. Actually we shall prove that $\int_{\delta_E}\lambda_h$ diverges as E approaches a critical value; but there is a universal way of *regularizing* this divergence. The regularized value is precisely $\mathbf{hol}(\bar\delta_E)$ – modulo some Maslov corrections in presence of vertices of degree 2 –, where $\bar\delta_E$ is the projection of δ_E onto the reduced manifold and is actually equal to $G_{k^\pm}(E_c)$.

Unfortunately, the set of cycles $G_{k^\pm}(E_c)$ doesn't necessarily generate the group $\mathrm{H}_1(G(E_c),\mathbb{Z})$ – see eg. Fig. 9 – (but it does indeed if $G(E_c)$ is planar). So there's a little bit more to it than just taking the limit of regular cycles.

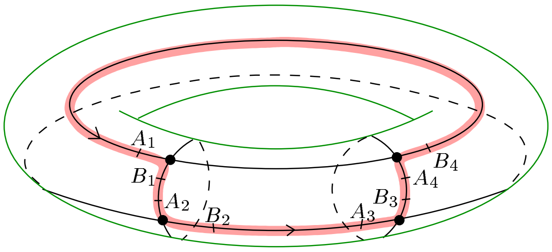


Figure 9: A graph on a torus. The cycle in gray cannot be obtained as a combination of boundary faces of the graph.

To compute **hol** of all possible cycles of $G(E_c)$, we have to replace the "natural" object δ_E with a local path near a critical circle. In order to give some sense to the expression $\int_{\delta_E} \lambda_h$, where δ_E is not closed, we could abstractly choose a smooth family of closed 1-forms on Λ_E, $E \in U^\pm$, whose cohomology class is $[\lambda_h(E)]$ – which is always possible as $\mathrm{H}^1(\Lambda_E) \to U^\pm$ is a trivial bundle if U is small enough. However, this does not allow us to have a local control of the divergence of the holonomy. Instead, we follow [27] and interpret λ_h as the phase of the multiplicative Čech holonomy of the sheaf $(\mathfrak{L}, \Lambda_E)$, as follows:

Definition 2.5 *Near each Γ_j, we let $A_j(E)$ and $B_j(E)$ in $C^\infty(D^\pm, \bar{k}^\pm \setminus \Gamma_j)$ be families of points such that for a critical value E_c, $A(E_c)$ and $B(E_c)$ lie respectively in the local stable or unstable manifold. We endow a small neighbourhood of A_j (or B_j) with a standard WKB microlocal solution u_{A_j} (resp. u_{B_j}) whose phase admits an asymptotic expansion in $h^{-1}C^\infty(D^\pm)[\![h]\!]$.*

Then the integrals $\int_{A_j}^{B_{j'}} \lambda_h$ are defined as the phase of the Čech holonomy of $(\mathfrak{L}, \Lambda_E)$ for paths joining A_j and $B_{j'}$ with the sections u_{A_j} and $u_{B_{j'}}$ fixed.

In other words, if the path δ between A_j and $B_{j'}$ is covered by open sets $\Omega_0, \ldots, \Omega_\ell$, each of which being endowed with a microlocal solution u_α with $u_0 = u_{A_j}$ and $u_\ell = u_{B_{j'}}$, then

$$\int_\delta \lambda_h = -i \ln (c_{0,1} c_{1,2} \cdots c_{\ell-1,\ell}),$$

where $c_{i,j}$ is the transition constant $u_i = c_{i,j} u_j$ on $\Omega_i \cap \Omega_j$.

Note that if another admissible choice for the local sections $u_{A/B}$ is made, then the holonomy is modified by an additive term admitting an asymptotic expansion in $h^{-1}C^\infty(D^\pm)[\![h]\!]$. Therefore the singular behaviour of the holonomy at a critical value is fully preserved. Note also that this additive term is necessary a Čech *coboundary*, and hence has no influence on the value of the holonomy along a *closed* loop.

Definition 2.6 *In what follow, $(\delta_E)_{E \in D^\pm}$ designates a continuous family of paths in $T_{k^\pm}(E)$ such that:*

- *for each $E \in U^\pm$, δ_E is smooth;*
- *either for all $E_c \in \mathcal{C}_c$, δ_{E_c} does not meet the critical set Γ (then (δ_E) is called **regular**) or for each E_c, δ_{E_c} meets uniquely a unique critical circle γ_j (in which case (δ_E) is called **local**);*
- *the end points $A_j(E)$ and $B_{j'}(E)$ are one of those defined in Definition 2.5, and we will write $\delta_E = [A_j(E), B_{j'}(E)]$;*
- *δ_E is always transversal to the S^1-orbits.*

Here again, the normal form of Theorem 1.13 proves the existence of such a family of paths near any critical circle γ_j.

The goal of this section is finally to investigate the behaviour of the function $E \to \int_{\delta_E} \lambda_h$ as E tends to a critical value, and to relate it to the holonomy of the sheaf $(\bar{\mathfrak{L}}, G(E_c))$. The previous case where δ_E was a loop can always be recovered by composing paths of the type of Definition 2.6. Moreover, the regular theory implies that the so-called "local" paths can indeed be restricted to paths that are local in small neighbourhoods of the critical circles, since the following proposition holds:

Proposition 2.16 *If $\delta_E = [B_j(E), A_{j'}(E)]$ is a* regular *family of paths (in the sense of Definition 2.6), then $E \to \int_{\delta_E} \lambda_h$ is smooth in $D^\pm$ and admits an asymptotic expansion in $h^{-1}C^\infty(D^\pm)[\![h]\!]$. The first terms of this expansion are the following:*

$$\int_{\delta_E} \lambda_h = \Phi_{B_j}(B_j) - \Phi_{A_{j'}}(A_{j'}) + \frac{1}{h}\int_{\delta_E} \alpha + \int_{\delta_E} \kappa + \mu(\delta_E)\frac{\pi}{2} + O(h), \qquad (31)$$

where Φ_{B_j} (resp. $\Phi_{A_{j'}}$) is the phase of the principal symbol (viewed as a section of the Keller-Maslov bundle over the Lagrangian manifold $\Lambda_E \setminus \Gamma_j$ – see eg. [13] or [4]) of the fixed solution u_{B_j} (resp. $u_{A_{j'}}$).

To study the neighbourhood of a critical circle, we shall use Theorem 1.5 to generalise the semi-classical invariant $\epsilon_{2,j}(h)$ of equation (17) in a better way than Proposition 2.12 would do. That theorem still holds if H_1 and H_2 are replaced by $H_1 - E_1$ and $H_2 - E_2$, for a parameter $E = (E_1, E_2)$ varying near (a_o, b_o). If we fix a critical circle γ_j and $m \in \gamma_j$, the theorem yields a canonical change of coordinates (x, y, ξ, η) near m, depending smoothly on E, and a function $\Phi_E \in C^\infty(\mathbb{R}^2, 0)$ depending smoothly on E, such that

$$H_1 - E_1 = \xi, \quad H_2 - E_2 = \Phi_E(\xi, y\eta). \qquad (32)$$

(We have still $\partial_2\Phi_E(0,0) > 0$.) This leads to yet another semi-classical normal form :

Proposition 2.17 *Let γ_j be a critical circle and $m \in \gamma_j$. There exists a microlocally unitary Fourier integral operator $U(h)$ associated to the canonical coordinates (x, y, ξ, η), elliptic pseudo-differential operators $M_1(h)$, $M_2(h)$ commuting (modulo $O(h^\infty)$) with Q_1 and Q_2, and a real-valued function of h (independent on m, U, M_i): $\mathfrak{e}_2 = \mathfrak{e}_{2,j}(E; h) \sim \sum_{\ell=0}^{\infty} \mathfrak{e}_{2,j}^{(\ell)}(E) h^\ell$, such that, microlocally near m :*

$$\begin{aligned} U^{-1}(\hat{H}_1 - E_1)U &= Q_1 + O(h^\infty) \\ U^{-1}(\hat{H}_2 - E_2)U &= M_1 Q_1 + M_2.(Q_2 - \mathfrak{e}_2) + O(h^\infty). \end{aligned}$$

M_i, U and $\mathfrak{e}_2$ depend smoothly on E.

- *$\mathfrak{e}_{2,j}^{(0)}(E)$ is equal to the value of $y\eta$ on Λ_E. In particular, $\mathfrak{e}_{2,j}^{(0)} > 0$ if $E \in U^+$ and $\mathfrak{e}_{2,j}^{(0)} < 0$ if $E \in U^-$;*
- *If $E = E_c \in \mathcal{C}_c$,*

$$\mathfrak{e}_{2,j}^{(1)}(E) = \left(\frac{\lambda r_1 - r_2}{|\mathcal{H}_\Sigma(H_2)|^{1/2}}\right)_{\restriction \Gamma_j \cap \Lambda_E},$$

where r_i is the sub-principal symbol of $\hat{H}_i$, $\lambda = \lambda(E_c)$ is the unique real number such that $H_2 - \lambda H_1$ is critical on $\Gamma_j \cap \Lambda_E$ (see Lemma 1.7), and $|\mathcal{H}_\Sigma(H_2)|$ is the absolute value of the determinant of the transversal Hessian of H_2. Note also that this denominator is equal to $\partial_2\Phi_E(0,0)$, and $\lambda = \partial_1\Phi_E(0,0)$.

Remark 2.11. (See Remark 2.10). If E is restricted to a domain of the form $E = (a_o, b_o) + h(e_1, e_2)$, where $(a_o, b_o) \in \mathcal{C}_c$, then Theorem 2.1 applies with e_1 and e_2 as parameters, and yields an invariant $\epsilon_{2,j}$, which can be recovered from $\mathfrak{e}_{2,j}$ by the following formula :

$$\epsilon_{2,j}(e_1, e_2) = \frac{1}{h}\mathfrak{e}_{2,j}((a_o, b_o) + h(e_1, e_2)) + O(h^\infty),$$

or merely by viewing $-(e_1, e_2)$ as a correction of the subprincipal symbols and applying the formulæ of the Proposition. This proves the claim (12) of Theorem 2.1. △

Using this proposition, we let $\beta = \frac{1}{2} + i\mathfrak{e}_2/h$ and $\zeta_j^\pm$ be the h-dependent functions in $C^\infty(U)$ defined by

$$\zeta_j^+ := \frac{1}{\sqrt{2\pi h}}\Gamma(\beta)\mathrm{e}^{\beta \ln h}\mathrm{e}^{-i\beta\frac{\pi}{2}} = \mathrm{e}^{-i\frac{\pi}{4}}\mathcal{E}_j(\mathfrak{e}_{2,j}/h); \tag{33}$$

$$\zeta_j^- := \frac{1}{\sqrt{2\pi h}}\Gamma(\beta)\mathrm{e}^{\beta \ln h}\mathrm{e}^{i\beta\frac{\pi}{2}} = \mathrm{e}^{i\frac{\pi}{4}}\mathrm{e}^{-\mathfrak{e}_{2,j}\pi/h}\mathcal{E}_j(\mathfrak{e}_{2,j}/h). \tag{34}$$

($\mathcal{E}_j$ was defined in (20).) Next lemma, which directly follows from Stirling's formula, will be crucial for our analysis.

Lemma 2.18 *For any $E \in U^\pm$,*

$$\frac{1}{i}\ln\zeta_j^\pm = \frac{1}{h}(\mathfrak{e}_{2,j}^{(0)}\ln|\mathfrak{e}_{2,j}^{(0)}| - \mathfrak{e}_{2,j}^{(0)}) + \mathfrak{e}_{2,j}^{(1)}\ln|\mathfrak{e}_{2,j}^{(0)}| \mp \frac{\pi}{4} + O_E(h).$$

Theorem 2.19 *Fix a component $k^\pm$, and let $\delta_E = [A_j(E), B_j(E)]$ be a local path near the critical component Γ_j (see Definition 2.6). Assume moreover that δ_E is oriented according to the flow of H_q (otherwise just take the opposite of the holonomy !). Then there exists an h-dependent $\mathbb{R}/2\pi\mathbb{Z}$-valued function $g_\delta(h) : E \to g_{\delta_E}(h) \in C^\infty(D^\pm)$ admitting a uniform asymptotic expansion of the form*

$$g_\delta(E; h) \sim \sum_{\ell=-1}^{\infty} g_\delta^{(\ell)}(E)h^\ell, \quad g_\delta^{(\ell)} \in C^\infty(D^\pm),$$

such that

$$\forall E \in U^\pm, \quad g_\delta(E; h) = \int_{\delta_E} \lambda_h - i\ln(\zeta_j^\pm(E)) \pmod{2\pi\mathbb{Z}}. \tag{35}$$

The principal terms of $g_\delta(h)$ are given by the following formulæ, for $E \in U^\pm$:

$$g_\delta^{(-1)}(E) = \int_{\delta_E} \alpha + \left(\mathfrak{e}_{2,j}^{(0)}\ln|\mathfrak{e}_{2,j}^{(0)}| - \mathfrak{e}_{2,j}^{(0)}\right) + \Phi_{A_j}^{(-1)}(A_j) - \Phi_{B_j}^{(-1)}(B_j); \tag{36}$$

$$g_\delta^{(0)}(E) = \int_{\delta_E} \kappa_E + \mu(\delta_E)\frac{\pi}{2} + \left(\mp\frac{\pi}{4} + \mathfrak{e}_{2,j}^{(1)}\ln|\mathfrak{e}_{2,j}^{(0)}|\right) + \Phi_{A_j}^{(0)}(A_j) - \Phi_{B_j}^{(0)}(B_j), \tag{37}$$

where $\Phi_{A_j} = \frac{1}{h}\Phi_{A_j}^{(-1)} + \Phi_{A_j}^{(0)}$ is the phase of the principal symbol of the fixed solution u_{A_j} (and similarly for u_{B_j}).

Theorem 2.20 *Let $E_c = (a_o, b_o) \in \mathcal{C}_c$. Let $\tilde{\delta}_{E_c}$ be a loop in Λ_{E_c} oriented according to the flow of H_q and of the form*

$$\tilde{\delta}_{E_c} = \delta_1^{\mathrm{loc}} \cdot \delta_1^{\mathrm{reg}} \cdot \delta_2^{\mathrm{loc}} \cdot \delta_2^{\mathrm{reg}} \cdots \delta_q^{\mathrm{loc}} \cdot \delta_q^{\mathrm{reg}},$$

where $\delta_\ell^{\mathrm{loc}}$ and $\delta_\ell^{\mathrm{reg}}$ are respectively "local" and "regular" paths in the sense of Definition 2.6. (The components $k^\pm$ used for these paths may vary. – see eg. Fig 9). Let

$$g(E_c; h) \sim \sum_{\ell=-1}^{\infty} g^{(\ell)}(E_c) h^\ell, \quad g^{(\ell)} \in C^\infty(\mathcal{C}_c)$$

be defined as the sum

$$g(E_c; h) := \left(g_{\delta_1^{\mathrm{loc}}} + g_{\delta_1^{\mathrm{reg}}} + \cdots + g_{\delta_q^{\mathrm{loc}}} + g_{\delta_q^{\mathrm{reg}}} \right)_{\restriction E = E_c},$$

where $g_{\delta_k^{\mathrm{loc}}}$ is given by Theorem 2.19 and $g_{\delta_k^{\mathrm{reg}}} \stackrel{\mathrm{def}}{=} \int_{\delta_k^{\mathrm{reg}}} \lambda_h$ (see Proposition 2.16). Then, under the hypothesis of section 2.3,

$$\mathbf{hol}(\bar{\delta}_{E_c}) := \mathrm{e}^{ig(E_c;h)} \mathrm{e}^{i\frac{\pi}{2} n (N_2^- - N_2^+)} + O(h^\infty), \tag{38}$$

where $\bar{\delta}_{E_c}$ is the projection of $\tilde{\delta}_{E_c}$ onto the graph $G(E_c)$ in the reduced orbifold $W(E_c)$, n is the "global quantum number" of Theorem 2.4, and $N_2^\pm$ is the number of local paths through a vertex of degree 2 and defined by a component $k \in N^\pm$.

Proof *of Theorem 2.19.* Fix a critical value $E_c \in \mathcal{C}_c$. Let m be the intersection of the cycle δ_{E_c} with γ_j, and Ω_m an open set in which Proposition 2.17 applies. We can assume that the paths δ_E, $E \in D^\pm$ all entirely lie in Ω_m (using Proposition 2.16, this will only modify $\int_{\delta_E} \lambda_h$ by an additive term entering in $g_\delta(E; h)$). As before, label the local un/stable manifolds with cyclic order I_1, I_3, I_2 and I_4. δ_{E_c} enters Ω_m on a local stable manifold I_s, $s = 3, 4$, and leaves it on a local unstable manifold I_u, $u = 1, 2$ (u is the index *preceding* s in the cycle $(1, 3, 2, 4)$ if $k^\pm = k^+ \in N^+$ and the index *following* s if $k^\pm = k^- \in N^-$). As before, we endow a neighbourhood of each I_α with the distribution e_α:

$$\begin{aligned}
e_1 &:= u_+^{\mathfrak{e}_{2,j}/h} := 1_{\pm y > 0} \frac{1}{\sqrt{|y|}} \mathrm{e}^{i \mathfrak{e}_{2,j} \ln |y| / h}, \\
e_2 &:= u_-^{\mathfrak{e}_{2,j}/h} := 1_{\pm y < 0} \frac{1}{\sqrt{|y|}} \mathrm{e}^{i \mathfrak{e}_{2,j} \ln |y| / h}, \\
e_3 &:= \mathcal{F}_h^{-1}(u_+^{-\mathfrak{e}_{2,j}/h}), \\
e_4 &:= \mathcal{F}_h^{-1}(u_-^{-\mathfrak{e}_{2,j}/h}).
\end{aligned}$$

These distributions are classical Lagrangian distributions whose phases admit an asymptotic expansion in $\frac{1}{h} C^\infty(D^\pm)[\![h]\!]$. Moreover, they are microlocal solutions of (26) in Ω_m uniformly for $E \in D^\pm$, and hence constitute an admissible choice in view of Definition 2.5. Note that this choice possibly implies another additive term entering in g_{δ_E}. In a small ball around $A_j(E_c)$, the space of solutions has

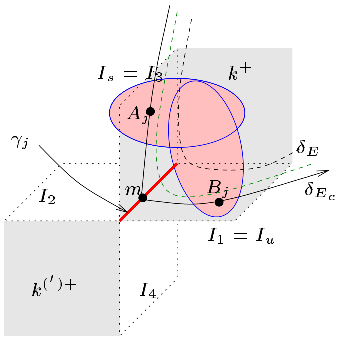


Figure 10: The local holonomy

dimension 1 and we must have a constant $C^{\pm}(E,h)$ such that $e_u \sim C^{\pm}(E,h)e_s$. Now Definition 2.5 says that, with respect to the fixed solutions e_u and e_s,

$$C^{\pm}(E,h) = \mathrm{e}^{-i\int_{\delta_E}\lambda_h} \qquad (\mathrm{mod}\ O(h^\infty)).$$

The expression of the Fourier transform involved in e_s shows – as in [8] – that

$$C^{\pm}(E,h) = \zeta_j^{\pm} \qquad (\mathrm{mod}\ O(h^\infty)). \tag{39}$$

This proves the existence of the claimed g_δ satisfying Eq.(35). The formulæ (36) and (37) now follow from Stirling's formula (Lemma 2.18) and from the fact that the asymptotic expansion for g_δ is uniform in $D^\pm$, and hence is given for any fixed $E \in U^\pm$ by the difference of the (non-uniform) asymptotic expansions of $\int_{\delta_E}\lambda_h$ and $i\ln(\zeta_j^{\pm}(E))$. We have also used the formula (31) which holds only for non-critical E, and which comes from the definition of the bundle of principal symbols.

Note that we don't see here any difference between critical circles of direct/reverse type, for the statement is purely local near a point of γ_j. □

Proof *of Theorem 2.20.* To each $\delta_\ell^{\mathrm{loc}}$ is associated a unique component k^{s_ℓ}, $s_\ell = \pm$, a unique critical circle γ_{j_ℓ}, and a (non-unique) local family of paths $\delta_\ell^{\mathrm{loc}}(E)$ (in the sense of Definition 2.6) such that $\delta_\ell^{\mathrm{loc}} = \delta_\ell^{\mathrm{loc}}(E_c)$. Let $\mathcal{I}_h^{s_\ell}$ be the set of $E \in D^{s_\ell}$ such that $(\mathfrak{e}_1)_{\restriction k^{s_\ell}} \in h\mathbb{Z} + O(h^\infty)$ (ie. $\mathcal{I}_h^{s_\ell} = \varpi_h^{-1}[h\mathbb{Z} + O(h^\infty)]$ in the notation of the proof of Corollary 2.14). Then the (assumed) hypothesis of section 2.3 "$\epsilon_1 = n$" says that $E_c \in \mathcal{I}_h^{s_\ell}$. Using Proposition 2.12 one can construct smooth families $(u_E)^{1,2,3,4}_{E\in\mathcal{I}_h^{s_\ell}}$ of solutions on a neighbourhood of γ_{j_ℓ} in $\overline{k^{s_\ell}}$ such that

$$(u_E^1, u_E^2, u_E^3, u_E^4) \tag{40}$$

– in the direct case – or

$$\left\{ \begin{array}{ccc} u_E^1 & + & (-1)^n u_E^2 \\ \mathrm{e}^{i\frac{\pi}{2}n}u_E^3 & + & \mathrm{e}^{-i\frac{\pi}{2}n}u_E^4 \end{array} \right. \tag{41}$$

– in the reverse case – form at $E = E_c$ a *standard basis* for the graph $G(E_c)$ at the vertex g_{j_ℓ}.

Since these solutions are smooth WKB solutions and hence admissible in the sense of Definition 2.5, we shall use them to define the local holonomies $\int_{\delta_\ell^{\mathrm{loc/reg}}(E)} \lambda_h$; since they are standard basis at $E = E_c$, we shall in the same way use them to define the local "reduced" holonomies $\mathbf{hol}(\overline{\delta}_\ell^{\mathrm{loc/reg}})$. But then by definition of the sheaf $(\bar{\mathfrak{L}}, G(E_c))$ we have

$$\begin{cases} \mathbf{hol}(\overline{\delta}_\ell^{\mathrm{loc}}) = 1 \qquad \text{and} \\ \mathbf{hol}(\overline{\delta}_\ell^{\mathrm{reg}}) = \exp\left(i \int_{\delta_\ell^{\mathrm{reg}}(E_c)} \lambda_h\right) = \exp\left(i g_{\delta_\ell^{\mathrm{reg}}}(E_c)\right). \end{cases} \tag{42}$$

On the other hand, we know from the proof of Theorem 2.19 that for such a choice of microlocal solutions, we have (modulo 2π)

$$\begin{cases} g_{\delta_\ell^{\mathrm{loc}}}(E) = 0 & \text{if } \gamma_{j_\ell} \text{ is of degree 4;} \\ g_{\delta_\ell^{\mathrm{loc}}}(E) = s_\ell \frac{\pi}{2} n & \text{if } \gamma_{j_\ell} \text{ is of degree 2.} \end{cases} \tag{43}$$

Therefore, if we decompose

$$\mathbf{hol}(\overline{\delta}_{E_c}) = \prod_{\ell/\deg \gamma_{j_\ell}=2} \mathbf{hol}(\overline{\delta}_\ell^{\mathrm{loc}}) \times \prod_{\ell/\deg \gamma_{j_\ell}=4} \mathbf{hol}(\overline{\delta}_\ell^{\mathrm{loc}}) \times \prod_\ell \mathbf{hol}(\overline{\delta}_\ell^{\mathrm{reg}}),$$

we obtain by (42) and (43):

$$\mathbf{hol}(\overline{\delta}_{E_c}) = \exp\left(i \sum_{\ell/\deg \gamma_{j_\ell}=2} \left(g_{\delta_\ell^{\mathrm{loc}}}(E_c) - s_\ell \frac{\pi}{2} n \right) + \right.$$

$$\left. + \sum_{\ell/\deg \gamma_{j_\ell}=4} g_{\delta_\ell^{\mathrm{loc}}}(E_c) + \sum_\ell g_{\delta_\ell^{\mathrm{reg}}}(E_c) \right),$$

which proves the theorem. □

Corollary 2.21 *Theorem 2.20 together with formulæ (36) and (37) finally prove the second point of Proposition 2.6.*

Proof.

1. The principal action – Since $\mathfrak{e}_{2,j}^{(0)}(E_c) = 0$, it is clear from (36) that for any $\ell = 1, \ldots, q$,

$$g_{\delta_\ell^{\mathrm{loc}}}^{(-1)}(E_c) = \int_{\delta_\ell^{\mathrm{loc}}} \alpha + \Phi_{A_{j_\ell}}^{(-1)}(A_{j_\ell}(E_c)) - \Phi_{B_{j_\ell}}^{(-1)}(B_{j_\ell}(E_c))$$

and from (31) that for any $\ell = 1, \ldots, q$ (identifying $\ell = q+1$ with $\ell = 1$),

$$g_{\delta_\ell^{\mathrm{reg}}}^{(-1)}(E_c) = \int_{\delta_\ell^{\mathrm{reg}}} \alpha + \Phi_{B_{j_\ell}}^{(-1)}(B_{j_\ell}(E_c)) - \Phi_{A_{j_{\ell+1}}}^{(-1)}(B_{j_{\ell+1}}(E_c)).$$

Therefore,

$$g^{(-1)}(E_c) = \int_{\tilde{\delta}_{E_c}} \alpha.$$

2. The sub-principal action and the Maslov index – Fix $\ell = 1, \ldots, q$ and let $\gamma_j = \gamma_{j_\ell}$. Let $m = m(E_c)$ be the point where $\delta_\ell^{\mathrm{loc}}$ meets γ_j. As in Proposition 2.17, we shall use the local canonical coordinates at m given by Theorem 1.5.

Recalling the notation of Proposition 2.17, (32) implies that

$$(\mathcal{X}_1, \mathcal{X}_2) = \begin{pmatrix} 1 & 0 \\ \partial_1 \Phi_E & \partial_2 \Phi_E \end{pmatrix} \cdot (\mathcal{X}_\xi, \mathcal{X}_{y\eta}).$$

Since $\partial_2 \Phi_E \neq 0$, there exist a smooth function $\rho_2 = \frac{-\partial_1 \Phi_E r_1 + r_2}{\partial_2 \Phi_E}$ (depending also smoothly on E) such that the sub-principal form κ_E is given by

$$\kappa_E.(\mathcal{X}_\xi, \mathcal{X}_{y\eta}) = -(r_1, \rho_2).$$

Note that for a critical value $E = E_c$, $(\rho_2)_{\restriction y=\eta=0} = -\mathfrak{e}_{2,j}^{(1)}$. The closedness of κ_E on each Λ_E implies that

$$\{r_1, y\eta\} = \{\rho_2, \xi\}.$$

Using a local analogue of Lemma 2.3, we can decompose (r_1, ρ_2) in the following way:

$$(r_1, \rho_2) = (0, K) - (\{\mathcal{X}_\xi, \tilde{f}\}, \{\mathcal{X}_{y\eta}, \tilde{f}\}) \tag{44}$$

for some smooth functions $\tilde{f}$, K where K commutes with $y\eta$ and ξ. Therefore the function

$$f := \tilde{f} - K \ln |y| \qquad (\text{or } \tilde{f} + K \ln |\eta| \text{ where } y = 0),$$

restricted to Λ_E, satisfies $d_{\Lambda_E} \tilde{f} = \kappa_E$. We can now compute

$$\int_{\delta_\ell^{\mathrm{loc}}} \kappa_E = f(B_j) - f(A_j) = \tilde{f}(B_j) - \tilde{f}(A_j) - K \ln |y_{B_j} \eta_{A_j}| + K \ln |y_{A_j} \eta_{A_j}|$$

Since $A_j(E)$ and $B_j(E)$ are in Λ_E, $y_{A_j} \eta_{A_j} = \mathfrak{e}_{2,j}^{(0)}$ and we have

$$\int_{\delta_\ell^{\mathrm{loc}}} \kappa_E + \mathfrak{e}_{2,j}^{(1)} \ln |\mathfrak{e}_{2,j}^{(0)}| = \tilde{f}(B_j) - \tilde{f}(A_j) - K \ln |y_{B_j} \eta_{A_j}| + (\mathfrak{e}_{2,j}^{(1)} + K) \ln |\mathfrak{e}_{2,j}^{(0)}|.$$

Because of (44), $\rho_2 - K$ vanish at $y = \eta = 0$, hence $K = -\mathfrak{e}_{2,j}^{(1)} + O(y\eta)$ and $(\mathfrak{e}_{2,j}^{(1)} + K) \ln |\mathfrak{e}_{2,j}^{(0)}|$ tends to zero as E tends to a critical value E_c. Since $g_{\delta_\ell^{\mathrm{loc}}}^{(0)}$ is smooth at E_c, the formula (37) implies that $\mu(\delta_\ell^{\mathrm{loc}}(E))$ is continuous at E_c and hence constant; let us denote it by $\mu(\delta_\ell^{\mathrm{loc}}(E_c))$.

Suppose now that $E = E_c$ and let a, b be points on $\delta_\ell^{\mathrm{loc}}(E_c)$ located respectively in $[A_j, m]$ and $[m, B_j]$. Then

$$\tilde{f}(B_j) - \tilde{f}(A_j) - K \ln |y_{B_j} \eta_{A_j}| =$$

$$= \lim_{a,b \to m} \left(\tilde{f}(a) - \tilde{f}(A_j) + \tilde{f}(B_j) - \tilde{f}(b) - K \ln |y_{B_j} \eta_{A_j}| \right).$$

The term in the limit is equal to

$$\int_{[A_j,a]} \kappa_{E_c} + \int_{[b,B_j]} \kappa_{E_c} + \mathfrak{e}_{2,j}^{(1)} \ln |y_b \eta_a|$$

Therefore, by Definition 2.2, the limit is equal to $\int_{\delta_\ell^{\mathrm{loc}}} \tilde{\kappa}_{E_c}$, and (37) yields:

$$g^{(0)}_{\delta_\ell^{\mathrm{loc}}}(E_c) = \int_{\delta_\ell^{\mathrm{loc}}} \tilde{\kappa}_{E_c} + \mu(\delta_\ell^{\mathrm{loc}}(E_c))\frac{\pi}{2} - s_\ell \frac{\pi}{4} \quad +$$

$$+ \quad \Phi^{(0)}_{A_j}(A_j(E_c)) - \Phi^{(0)}_{B_j}(B_j(E_c)).$$

Then as before, if we sum up all the contributions from regular and local paths, we obtain

$$g^{(0)}(E_c) = \int_{\tilde{\delta}_{E_c}} \tilde{\kappa}_{E_c} + \mu(\tilde{\delta}_{E_c})\frac{\pi}{2} + \sum_\ell -s_\ell \frac{\pi}{4}.$$

Using the definition 2.2 of the regularized Maslov cocycle, we finally obtain

$$g^{(0)}(E_c) + \frac{\pi}{2}(N_2^- - N_2^+) = \int_{\tilde{\delta}_{E_c}} \tilde{\kappa}_{E_c} + \tilde{\mu}(\tilde{\delta}_{E_c})\frac{\pi}{2},$$

and equation (38) concludes the proof. $\square$

3 Examples

We propose in this section several examples for which our theory applies. Many other could probably be found; the ones presented here are interesting by their simplicity and yet by their rich structure and behaviour.

The recipe

Let us recall here briefly the recipe for obtaining the semi-classical quantisation rules. The first thing to do is to locate the critical value of transversal hyperbolic type in the image of the momentum map $F = (H_1, H_2)$. Then choose one of these points o and describe the singular level set $\Lambda_o = F^{-1}(o)$, in order to have: a) the graph G, b) a formula for the vector fields $\mathcal{X}_1$ and $\mathcal{X}_2$ on Λ_o – and if it is not one of these, for the periodic vector field $\mathcal{X}_p$.

Compute the semi-classical invariants (action integral, Maslov index) for a periodic cycle – this implies only regular tools – in order to derive the first quantisation condition of Theorem 2.4 up to $O(h)$, and fix the quantum number n.

From the graph, apply Theorem 2.7 to obtain the second quantisation rule in the form of a determinantal equation. It remains to compute the holonomy **hol** up to $O(h)$, which involves the singular semi-classical invariants of Definition 2.2. The fulfilment of these quantisations rules determine the spectrum up to an error of order $O(h^2)$ in a window of size $O(h)$ (in fact, it is easy to determine the smooth dependence of the semi-classical invariants in o, and the spectral window can be extended to a rectangular domain of size $O(1)$ along the curve of critical values, and of size $O(h)$ in the transversal direction).

Notation– The reader must be warned that the symbol e (italic) is used as a subprincipal spectral parameter (as in "$E = he$"), while the exponential is denoted by $\mathrm{e}^a = \exp(a)$.

3.1 Laplacians on Ellipsoids

Let us consider the ellipsoid in the Euclidian space $\mathbb{R}^3$ defined by :

$$E = \{\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1\}$$

with $0 < a_1 < a_2 < a_3$. The geodesic flow on E has been discovered to be integrable by Jacobi in 1838 using Abelian integrals. For a recent presentation, one can read [21], [2], [24] or [22].

3.1.1 Classics

Let us denote by $P, Q, P' = -P, Q' = -Q$ the four umbilics of E which are located on the ellipse $\{x_2 = 0\}$. If

$$X_1 = \sqrt{a_1(a_2 - a_1)(a_3 - a_1)} \text{ and } X_3 = \sqrt{a_3(a_3 - a_2)(a_3 - a_1)}\ ,$$

we have

$$P = (X_1, 0, X_3),\ Q = (-X_1, 0, X_3)\ .$$

We will consider the (unique up to global dilatation) conformal representation Φ of $E_+ = E \cap \{x_2 > 0\}$ on a rectangle $R =]0, T_1[\times]0, T_2[$ such that the four umbilics are going on the four vertices of R, according to figure 14. Using such coordinates $(x, y) \in R$, we get (see [12] (vol. 2 p. 308 and vol. 3 p.13) or [20]) the following expression for the metric of E :

$$ds^2 = \left(a^2(x) + b^2(y)\right)(dx^2 + dy^2) \tag{45}$$

where a, b are given in terms of hyperelliptic integrals and extends to smooth functions on $\mathbb{R}$ which satisfy: a is > 0 on $]0, T_1[$, vanishes exactly at the points $kT_1,\ k \in \mathbb{Z}$, and is odd with respect to $T_1\mathbb{Z}$ and b satisfies the same properties with respect to T_2. Moreover $a'(0) = b'(0) > 0$. Let us denote by Γ the lattice $T_1\mathbb{Z} \oplus T_2\mathbb{Z}$. Then ds^2 extends into a smooth metric on $\mathbb{R}^2 \setminus \Gamma$, which is Γ-periodic. Let us consider the torus $T = \mathbb{R}^2/2\Gamma$. Then the map $\sigma : T \to T$ defined by $T(z) = -z$ defines an isometric involution of T with four fixed points and we get a natural identification of E with T/σ as a 2-sheeted branched covering Π of T over E with automorphism σ. The metric ds^2 admits conical singular points of total angle 4π at the umbilics which makes the metric on E smooth. More precisely, it follows from the formulae of [20] that there exists an analytic function G defined near 0, with $G(0) = 0, G'(0) > 0$ such that near $(0, 0)$ we have $ds^2 = (G(x^2) - G(-y^2))(dx^2 + dy^2)$. It is rather easy to check that there exists an analytic function $A(u, v)$ with $A(0, 0) > 0$ such that $G(x^2) - G(-y^2) = (x^2 + y^2)A(x^2 - y^2, 2xy)$ and, if locally $Z = \Pi(z) = z^2$, we have : $ds^2 = \Pi^\star\left(4A(X, Y)(dX^2 + dY^2)\right)$.

We will use the fundamental domain $D = [0, 2T_1] \times [0, T_2]$. E can be recovered from this rectangle by gluing edges as indicated on the figure 14.

If a, b were non vanishing, ds^2 would be called a *Liouville metric* on T. Our case corresponds to a degenerate Liouville metric on the sphere. It is well known that Liouville metrics are integrable. Let us denote by

$$H_1 = \frac{\xi^2 + \eta^2}{a^2(x) + b^2(y)}$$

the geodesic flow and by

$$H_2 = \frac{b^2(y)\xi^2 - a^2(x)\eta^2}{a^2(x) + b^2(y)}\ .$$

The manifold $L_{E,F} = \{H_1 = E,\ H_2 = F\}$ is given by:

$$L_{E,F} = \{\xi^2 = F + a^2(x)E,\ \eta^2 = b^2(y)E - F\}$$

which is obviously Lagrangian.

We are interested in the singular value $o = (E = 1, F = 0)$ of the moment map (H_1, H_2) and the corresponding Λ_o. Geodesics passing through P (resp. Q) contain also P' (resp. Q') and vice-versa. Λ_o is the set of unit covectors corresponding to geodesics passing through P or Q. We have $\Lambda_o = \cup L_{\pm,\pm}$ where $L_{\pm,\pm} = \{\xi = \pm a(x),\ \eta = \pm b(y)\}$. In particular $L_P = L_{+,+} \cup L_{-,-}$ and $L_Q = L_{+,-} \cup L_{-,+}$ are smooth Lagrangian tori whose intersection is $\gamma_+ \cup \gamma_-$. Here γ_+ (resp. γ_-) is the lift of the ellipse $x_2 = 0$ with orientation (P, Q, P', Q') (resp. opposite). L_P (resp. L_Q) is the set of unit covectors of geodesics of E

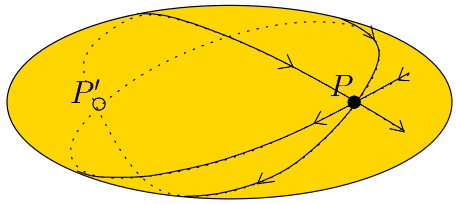


Figure 11: a geodesic passing through P and P'

containing P and P' (resp. Q and Q'). L_P (resp. L_Q) is the stable manifold of γ_- (resp. γ_+) and the unstable manifold of γ_+ (resp. γ_-).

The associated graph G is the union of 2 circles corresponding to L_P and L_Q intersecting at 2 points corresponding to $\gamma_\pm$.

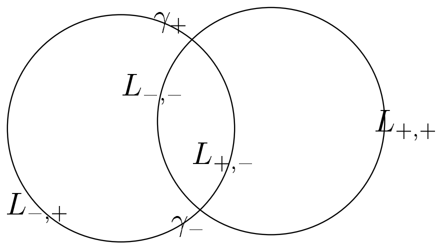


Figure 12: associated graph for E

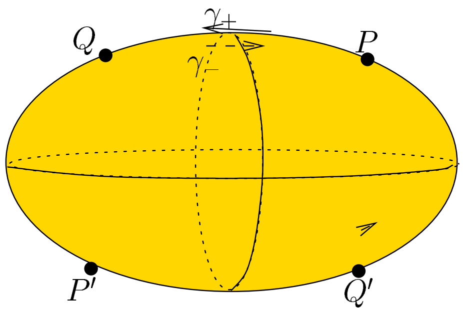


Figure 13: the ellipsoid

3.1.2 Quantum

We now introduce the quantum Hamiltonian $\hat{H}_1 = h^2 \Delta_E$ which is given in the coordinates (x, y) by:

$$\hat{H}_1 = -\frac{h^2}{a^2 + b^2}(\partial_x^2 + \partial_y^2)$$

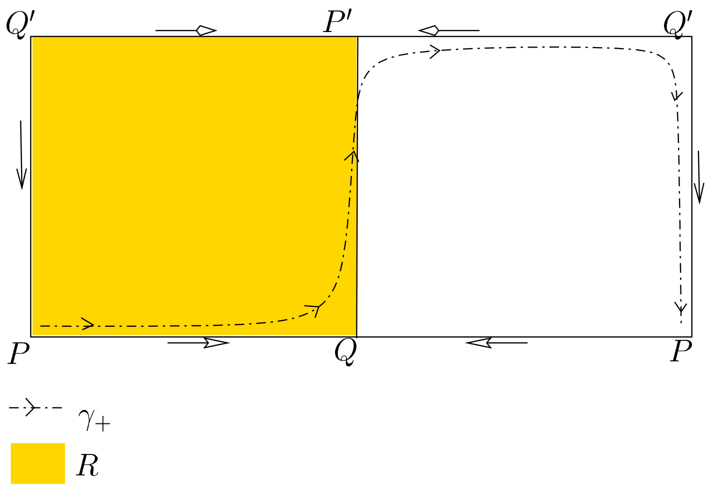


Figure 14: fundamental domain

and the operator

$$\hat{H}_2 = -\frac{h^2}{a^2+b^2}(b^2\partial_x^2 - a^2\partial_y^2) \ .$$

It is possible to check directly that $[\hat{H}_1, \hat{H}_2] = 0$ outside of the lattice Γ which is the set of singular points. We need a stronger form of commutation, namely if

$$\hat{H}_1\varphi = \lambda\varphi \ , \tag{46}$$

we need to prove that $\hat{H}_2\varphi$ is in the domain of $\hat{H}_1$ and $\hat{H}_1(\hat{H}_2\varphi) = \lambda\hat{H}_2\varphi$, so that $\hat{H}_1$ and $\hat{H}_2$ have a commun eigenbasis.

For that, we need a caracterisation of the functions $\tilde{\varphi} = \varphi \circ \Pi$ where φ is an eigenfunction of the Riemannian Laplacian Δ_E on E.

Lemma 3.1 *$\tilde{\varphi} : T \to \mathbb{C}$ is of the form $\tilde{\varphi} = \varphi \circ \Pi$ with $\Delta_E\varphi = \lambda\varphi$ if and only if $\tilde{\varphi}$ is σ invariant and satifies*

$$-\tilde{\varphi}''_{x^2} - \tilde{\varphi}''_{y^2} = \lambda(a^2+b^2)\tilde{\varphi} \ . \tag{47}$$

Proof. Starting from φ an eigenfunction of Δ_E and using smoothness of Π gives the trivial direction. For the other, using the fact that $\tilde{\varphi} \circ \sigma = \tilde{\varphi}$, we get a bounded φ with $\tilde{\varphi} = \varphi \circ \Pi$ and $(\Delta_E - \lambda)\varphi = T$ where T is supported insides the (finite) set of umbilics. We deduce that T is 0 or that φ (and hence $\tilde{\varphi}$) is unbounded. □

It is easy to check that $\hat{H}_2\tilde{\varphi}$ is smooth using equation (46). Then $\hat{H}_2\tilde{\varphi}$ satisfies equation (47) outside Γ and hence everywhere on T and we get a commun eigenbasis for $\hat{H}_1$ and $\hat{H}_2$.

We prefer to rewrite the eigenvectors equations

$$\hat{H}_1\tilde{\varphi} = \lambda\tilde{\varphi}, \ \ \hat{H}_2\tilde{\varphi} = \mu\tilde{\varphi}$$

in the following simpler way:

$$\hat{P}\tilde{\varphi} := h^2 \frac{\partial^2 \tilde{\varphi}}{\partial x^2} + (a^2(x)\lambda + \mu)\tilde{\varphi} = 0$$

$$\hat{Q}\tilde{\varphi} := h^2 \frac{\partial^2 \tilde{\varphi}}{\partial y^2} + (b^2(y)\lambda - \mu)\tilde{\varphi} = 0 \ .$$

We are interested in solutions of this system which are σ invariant. If we denote by $\sigma_1(x, y) = (-x, y)$ and $\sigma_2(x, y) = (x, -y)$, we get $\sigma = \sigma_1 \circ \sigma_2$ and because σ_j commutes with $\hat{P}$ and $\hat{Q}$ we are reduced to find solutions of the form $\tilde{\varphi}(x, y) = f(x)g(y)$ with f a $2T_1$-periodic solution of $\hat{P}f = 0$ and g a $2T_2$-periodic solution of $\hat{Q}f = 0$. We ask moreover that f and g are both even or both odd. We assume $\lambda = 1$ which corresponds to quantize h and $\mu = \varepsilon h$. The associated fiber of the momentum map is then Λ_o.

This way we are reduced to 2 one dimensional problems and because $\hat{P}$ and $\hat{Q}$ are semi-classical stationnary Schrödinger operators with potentials $-a^2$ and $-b^2$, we are reduced to the computations of [10, p. 489-490] for periodic double wells.

3.2 1 : 2-resonance

3.2.1 Birkhoff normal forms

Consider a Hamiltonian $H : T^*\mathbb{R}^2 \to \mathbb{R}$ with a non-degenerate minimum at the origin. We can assume using a symplectic linear change that $H(z_1, z_2) = K_2(z) + O(|z|^3)$ with

$$K_2(z) = \omega_1|z_1|^2 + \omega_2|z_2|^2 \ ,$$

and $\omega_j > 0$. Here $z_j = x_j + i\xi_j$, where (x_1, x_2, ξ_1, ξ_2) are canonical coordinates for $T^*\mathbb{R}^2$. We will say that the quadratic part is resonant if ω_1/ω_2 is a rational number. It is possible to derive a Birkhoff normal form H of the following form

$$H = K_2 + R + O(|z|)^\infty$$

with $R = O(|z|^3)$ and $\{K_2, R\} = 0$. The same result is true on the quantum level (see chapter 5 of [26]) with commuting operators $\hat{K}_2$ and $\hat{R}$. If we are able to analyse the joint spectrum of the operators $\hat{K}_2, \hat{R}$ we can deduce some sharp results for eigenstates in the energy domain $E = O(h^\alpha)$ with $\alpha > 0$. In the case of the $1 : 1$ resonance – ie. $\omega_1 = \omega_2$ – the flow of K_2 induces a free circle action on the energy hypersurface $K_2 = \text{const}$ and the reduced space is smooth. Then, via the use of Toeplitz operators, the problem is fully reduced to a 1-dimensional one. This is no longer the case for the $1 : 2$ resonance, where the reduced phase space has a conical singularity. For this simple example, we will show that our analysis applies. Another application would be the *near* $1 : 2$ resonance with $R = \varepsilon|z_1|^2 + R'$.

3.2.2 1 : 2 resonance

We consider the following Poisson commuting Hamiltonians on $T^\star\mathbb{R}^2$:

$$H_1 = \frac{1}{2}|z_1|^2 + |z_2|^2, \quad H_2 = (x_1^2 - \xi_1^2)x_2 + 2x_1\xi_1\xi_2 = \Re(z_1^2\bar{z}_2) \qquad (48)$$

with $z_j = x_j + i\xi_j,\ j = 1, 2$. The image of the momentum map $F = (H_1, H_2)$ is

$$F(T^*\mathbb{R}^2) = \{(X, Y)| \quad 16X^3 \geqslant 27Y^2\}.$$

The singular values consist of the boundary (which corresponds to transversal elliptic points, except for the origin which is degenerate) and the half line $\mathcal{C}_c = \{(X, 0),\ X > 0\}$, whose points are transversally hyperbolic (see Fig.15). Here we shall be interested in the critical values on $\mathcal{C}_c$. Because of the homogeneity of H_j, it is sufficient to consider the point $o = (1, 0)$.

The corresponding commuting quantum Hamiltonians are:

$$\hat{H}_1 = \frac{1}{2}(-h^2\frac{\partial^2}{\partial x_1^2} + x_1^2) + (-h^2\frac{\partial^2}{\partial x_2^2} + x_2^2), \qquad (49)$$

$$\hat{H}_2 = x_2(h^2\frac{\partial^2}{\partial x_1^2} + x_1^2) - h^2\frac{\partial}{\partial x_2}(2x_1\frac{\partial}{\partial x_1} + 1). \qquad (50)$$

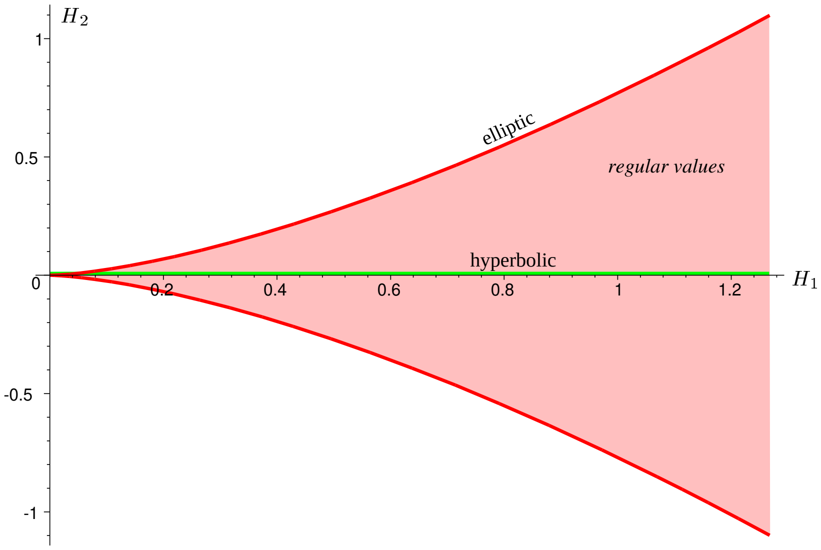


Figure 15: Image of the momentum map for the 1 : 2 resonance

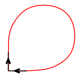
3.2.3 Classical description

Here we are interested in the singular Lagrangian leaf Λ_o defined by $H_1 = 1, H_2 = 0$. The singular part of Λ_o is the closed trajectory $\gamma_0 = \{z_1 = 0\} \cap \{|z_2| = 1\}$. From its defining equations, it is easy to find a parameterisation that shows that Λ_o is a Lagrangian immersion of a Klein bottle $\mathbb{K}$ with γ_0 as a double loop:

$$\Phi : \mathbb{K} \ni (\theta, \varphi) \mapsto (\sqrt{2}e^{i\theta} \sin \varphi, -ie^{2i\theta} \cos \varphi) \in \Lambda_o, \tag{51}$$

where $\mathbb{K}$ is the quotient of $\mathbb{T}^2_{(\theta,\varphi)} = \mathbb{R}^2/\mathbb{Z}^2$ by the equivalence relation

$$(\theta + \pi, -\varphi) \sim (\theta, \varphi)$$

A fundamental domain D is given by $D = \{(\theta, \varphi) \mid \quad 0 \leqslant \theta \leqslant \pi, \quad -\pi \leqslant \varphi \leqslant \pi\}$. The singular line γ_0 corresponds to $\{\varphi = 0\} \cup \{\varphi = \pm\pi\}$ and we have there the identifications $\Phi(\theta, 0) = \Phi(\theta + \pi/2, \pm\pi)$. The graph $G =$  corresponding to Λ_o has just one vertex γ_0 and one edge.

Remark 3.1. Although we don't really need it, it can be helpful to have a representation of the reduced phase space $W = H_1^{-1}(1)/S^1$ – where the S^1-action is the flow of the harmonic oscillator H_1. Using a priori argument, one can show that W is a 2-sphere with a conical singular point; however, one can find an explicit equation for W. The algebra of S^1-invariant polynomials – that is, those that commute with H_1 – is generated by

$$\pi_1 = |z_1|^2, \quad \pi_2 = |z_2|^2, \quad \pi_3 = \Re(z_1^2 \bar{z}_2), \quad \pi_4 = \Im(z_1^2 \bar{z}_2),$$

which are subject to the relation $\pi_3^2 + \pi_4^2 = \pi_1^2 \pi_2$. One can show (see the book [3]) that this relation together with $\pi_1 \geqslant 0$ and $\pi_2 \geqslant 0$ define the orbit

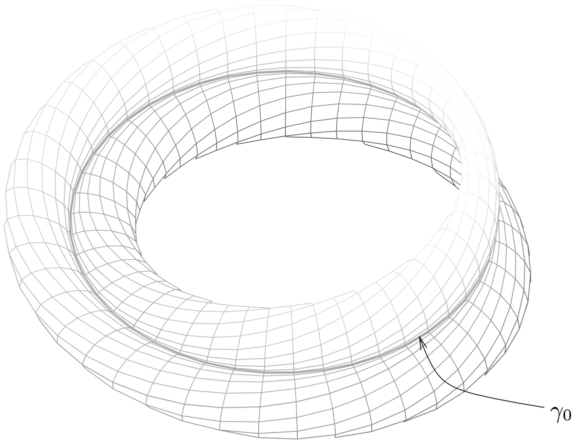


Figure 16: The manifold Λ_o

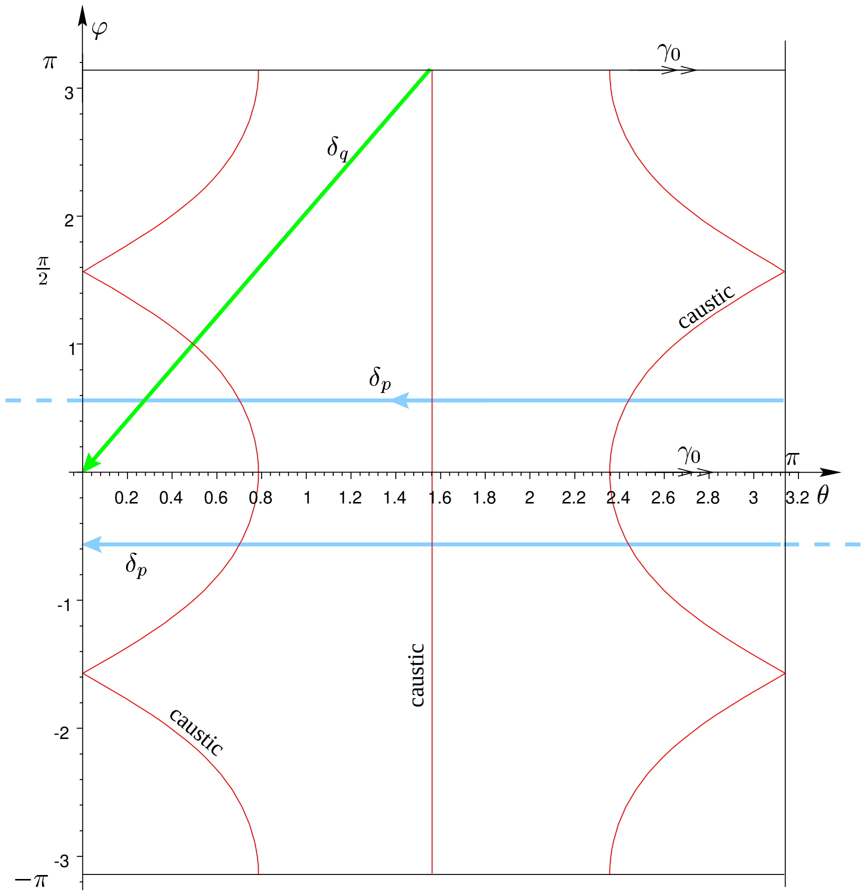


Figure 17: Parameters set for Λ_o

space $T^*\mathbb{R}^2/S^1$ in terms of the variables π_j. The energy level set is the section $\{\pi_1 \geqslant 0\} \cap \{\pi_2 \geqslant 0\}$ of the 3-dimensional hyperplane $\pi_1 + 2\pi_2 = 1$. Therefore, W is defined in the space $\mathbb{R}^3 = (\pi_1, \pi_4, \pi_3)$ by the equation

$$\pi_3^2 = \pi_1^2(1-\pi_1)/2 - \pi_4^2, \qquad \text{with } \pi_1 \in [0,1].$$

W is a surface of revolution around the π_1-axis, homeomorphic to a 2-sphere, with a conical singularity at the origin. Note that $H_2 = \pi_3$ so that the restriction $(H_2)_{\upharpoonright W}$ is just the height function (see Fig. 18) and is a Morse function on $W \setminus \{0\}$. The manifold Λ_o reduces to the singular equator $\pi_3 = 0$. $\triangle$

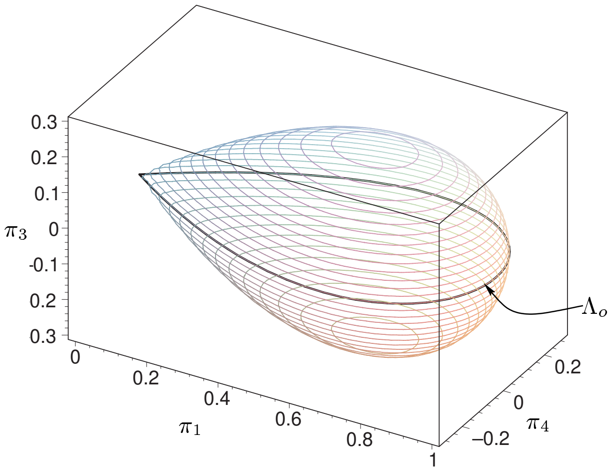


Figure 18: The singular reduced phase space W. Here the function $(H_2)_{\upharpoonright W}$ is equal to the π_3 coordinate.

3.2.4 Semi-classical computations

We consider the solutions of the system

$$(\hat{H}_1 - 1)u = 0, \quad (\hat{H}_2 - eh)u = 0, \tag{52}$$

for bounded e. The microsupport of the solutions is Λ_o.

If we denote by $\mathcal{X}_j$ the Hamiltonian vector fields of H_j we get on Λ_o:

$$\mathcal{X}_1 = -\frac{\partial}{\partial\theta}, \ \mathcal{X}_2 = -2\sin\varphi\frac{\partial}{\partial\varphi}\ , \tag{53}$$

which leads to the following sub-principal form κ for $\hat{F} = (\hat{H}_1, \hat{H}_2 - eh)$:

$$\kappa = \frac{-e}{2\sin\varphi}d\varphi\ . \tag{54}$$

Note also that, since the flow of $\mathcal{X}_1$ is 2π-periodic outside γ_0, we have $\mathcal{X}_p = \mathcal{X}_1$. The canonical 1-form $\alpha = \xi_1 dx_1 + \xi_2 dx_2$ is given by:

$$\alpha = -2(\sin^2\theta\sin^2\varphi + \cos^2 2\theta\cos^2\varphi)d\theta + \frac{1}{2}(\sin 2\theta(1+\cos 2\theta)\sin 2\varphi)d\varphi\ .$$

Finally, the caustic set C of Λ_o is given by

$$C = \{\cos\theta = 0\} \cup \{\tan^2\theta = \cos^2\varphi\}.$$

In order to compute the quantisation rules, let us introduce the following loops on Λ_o:

$$\delta_p(\pi - s) = \Phi(-s, \pm\varphi_0), \quad s \in [0,\pi] \qquad \text{for some } \varphi_0 \neq 0(\pi), \tag{55}$$

$$\delta_q(\frac{\pi}{2} - s) = \Phi(-s, -2s), \quad s \in [0, \pi/2] \quad . \tag{56}$$

δ_p is an oriented S^1-orbit, and δ_q is a loop which is everywhere transversal to the S^1-action and oriented according to the flow of $\mathcal{X}_2$.

The first quantisation condition:

$$\frac{1}{2\pi}\int_{\delta_p} \alpha + h\mu(\delta_p)/4 \in h\mathbb{Z} + O(h^2)$$

is actually exact since $\hat{H}_1$ is a harmonic oscillator, and therefore reads:

$$1 - \frac{6}{4}h = hn \qquad \text{or} \qquad h = \frac{1}{n+3/2}. \tag{57}$$

Because of the homogeneity property, we choose here to view this condition as a discrete quantisation of h (see remark 2.2).

Assuming now that (57) holds, we can compute the semi-classical invariants associated to δ_q:

The action integral is easily computed to be

$$\int_{\delta_q} \alpha = \frac{\pi}{2},$$

and the invariant ε given by (12) is equal to $e/2$. The sub-principal action I_{δ_q} of Definition 2.2 is given by

$$I_{\delta_q} := \int_{\delta_q} \tilde{\kappa} = 3\varepsilon \ln 2.$$

Finally, we can show by slightly shifting δ_q to the right (in the θ direction) that its Maslov index is -2. Moreover δ_q turns around γ_0 in the direct sense, hence the regularised Maslov index is $-2 + (\frac{1}{2} + n)$.

We can now write down the second quantisation condition: $\mathcal{C}_0 = \mathbf{hol}(\delta_q)$, which reads:

$$\mathrm{e}^{-i\frac{\pi}{4} - in\frac{\pi}{2}} \left(1 + i(-1)^n \mathrm{e}^{-\varepsilon\pi}\right) \Gamma\left(\frac{1}{2} + i\varepsilon\right) \mathrm{e}^{\varepsilon\left(\frac{\pi}{2} + i\ln h\right)} = \tag{58}$$

$$= \mathrm{e}^{i\left(\frac{\pi}{2h} + 3\varepsilon\ln 2 + \frac{\pi}{2}(-2 + (\frac{1}{2}+n)) + O(h)\right)}. \tag{59}$$

Using (57) and $\varepsilon = e/2$, we obtain the equation in e and n:

$$\left(1 + i(-1)^n \mathrm{e}^{-\frac{e\pi}{2}}\right) \Gamma\left(\frac{1}{2} + \frac{ie}{2}\right) \mathrm{e}^{\frac{e}{2}\left(\frac{\pi}{2} - i\ln(n+3/2)\right)} = \mathrm{e}^{i\left(\frac{3}{2}e\ln 2 - \frac{\pi}{2}n + \frac{\pi}{4} + O(h)\right)}. \tag{60}$$

Remark 3.2. The semi-classical invariants were computed explicitly; this is related to the fact that Λ_o admits a parameterisation as a "rational" variety. Somewhat paradoxically, it would be more technical to compute the WKB invariants attached to *regular* tori, since no rational parameterisation of these tori exists. On the other hand, the regular invariants can be asymptotically recovered from the singular ones using Stirling's formula (35). △

Remark 3.3. The obtained formula (60) yields easily (exactly as in [27]) the fact that the level spacings for the eigenvalues in a region of bounded e are of order $O(1/n \ln n) = O(h/\ln h)$ – while they are of order $O(h)$ in a regular region. Moreover, the precise shape of the spacing function is readily derived and involves the log-derivative of the Gamma function. △

3.2.5 Matrix form for $\hat{H}_2$

The goal here is to study the restriction of $\hat{H}_2$ to the eigenspace $\mathfrak{E}_n$ of $\hat{H}_1$ corresponding to the quantum number n (ie. to the eigenvalue $h(n+3/2)$).

In analogy with formulæ (52), the operators (49) and (50) can be written

$$\hat{H}_1 = h\left(a_1(h)b_1(h) + 2a_2(h)b_2(h) - \frac{3}{2}\right), \tag{61}$$

$$\hat{H}_2 = \sqrt{2}h^{\frac{3}{2}}\left(a_2(h)b_1(h)^2 + a_1(h)^2 b_2(h)\right), \tag{62}$$

with $a_1(h) = (2h)^{-1/2}(h\frac{\partial}{\partial x} + x)$ and $b_1(h) = a_1(h)^* = (2h)^{-1/2}(-h\frac{\partial}{\partial x} + x)$ (and similarly for $a_2(h)$ and $b_2(h)$ with the variable y). Using the unitary transform in $L^2(\mathbb{R}^2) : f(x) \to \sqrt{h}f(\sqrt{h}x)$, the operators $\hat{H}_1$ and $\hat{H}_2$ are transformed into those given by the equations (61) and (62) with $a_j(h)$ and $b_j(h)$ replaced by $a_j := a_j(1)$ and $b_j := b_j(1)$. Note that this shows that the homogeneity argument used for the classical analysis has an analogue in the quantum setting: if we know the spectrum for some value of $h > 0$, then the spectrum for any other value of h immediately follows.

Now, using the Bargmann representation, we identify a_j (respectively b_j) with the operator $\frac{\partial}{\partial z_j}$ (resp. z_j), and let them act on the monomials $\frac{z_1^k z_2^\ell}{k!\ell!}$ which form a Hilbert basis of eigenvectors of $\hat{H}_1$ (corresponding to the eigenvalues $E_1 = h(k + 2\ell + 3/2)$). Then it is easy to find the matrix of $\hat{H}_2$ in this basis of $\mathfrak{E}_n$ $(n = k + 2\ell)$:

$$\hat{H}_{2\restriction\mathfrak{E}_n} = \sqrt{2}h^{\frac{3}{2}}\begin{pmatrix} 0 & A_{n,1} & & & \\ A_{n,1} & 0 & A_{n,2} & & 0 \\ & A_{n,2} & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & 0 \end{pmatrix}, \tag{63}$$

with $A_{n,\ell} = \sqrt{\ell(n-2\ell+1)(n-2\ell+2)}, \qquad \ell = 1, 2, \ldots E[\frac{n}{2}].$

3.2.6 Numerical computations

Since h is of order $1/n$, one sees that the coefficients of (63) are bounded as $n \to \infty$. Moreover, since no coefficient $A_{n,\ell}$ vanishes, the spectrum is simple. For these reasons, it is reasonable to expect a good accuracy of numerically computed eigenvalues. The resulting spectrum will be called the "quantum" spectrum.

On the other hand, numerically solving equation (60) in the variable e – assuming that he remains in the bounds of the momentum map – yields the so-called "semi-classical" spectrum for $E_1 = 1$ and $h = (n + 3/2)^{-1}$. If we wish now to fix h and compute the rest of the joint spectrum, the same formulæ (58) and (59) can be used if one lets $\varepsilon = \frac{e}{2\sqrt{E_1}}$ and replaces h by $\tilde{h} = h/E_1$.

The results are displayed in the following figures. In Fig. 19 we have superposed the quantum and the semi-classical joint spectra. The differences are hardly noticeable (they are theoretically of order h^1 – which means h^2 for the unscaled spectrum – and experimentally much better at the critical value $H_2 = 0$), even for a very large h and very small E_1's – both of these conditions are supposed to reach the limitations of our analysis.

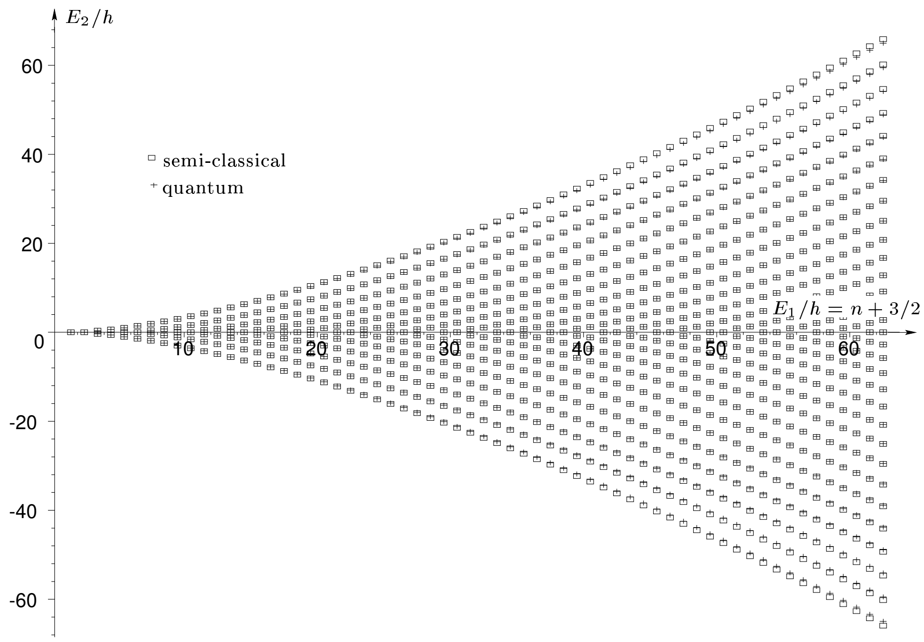


Figure 19: A comparison between semi-classical and quantum results. Here $h = 2/63$ (so that $E_1 = 1$ corresponds to $n = 30$)

In the other figures we focus on one spectrum (here at $E_1 = 1$) around the critical value $E_2 = 0$ – which is the most interesting feature.

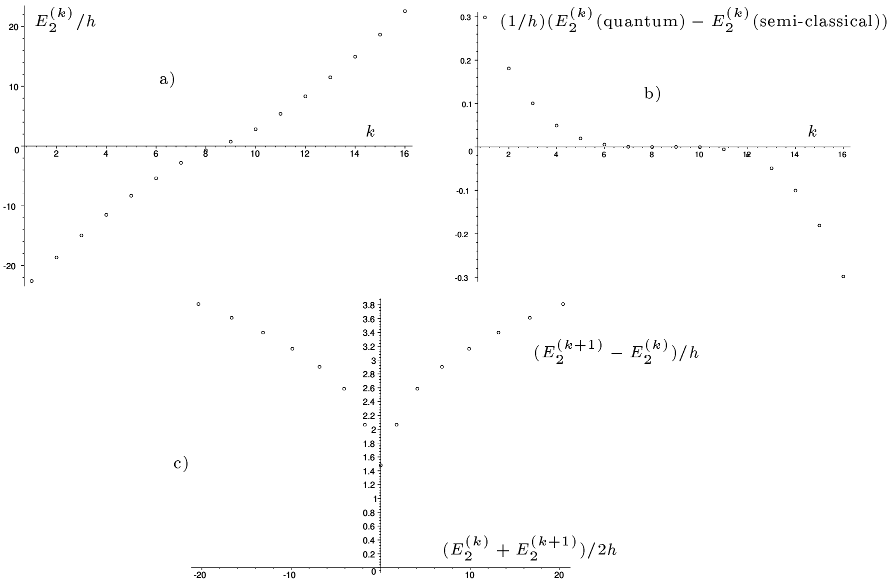


Figure 20: The spectrum at $E_1 = 1$, $n = 30$ (quantum and semi-classical are indistinguishable). a) The spectrum sorted in increasing order and displayed versus the eigenvalue number. b) The difference "quantum-semi-classical". c) The energy spacings.

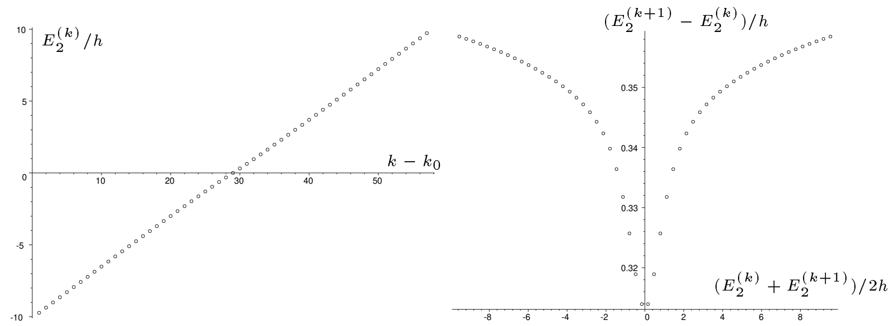


Figure 21: Contrary to the numerical diagonalisation of the matrix (63), the semi-classical formula allows very small values of h – and the results are supposedly even more accurate. Here is displayed a window of size $[-10h, 10h]$ for the spectrum and the corresponding eigenvalue spacings, where we have let $E_1 = 1$ and $n = 10^{15}$ ($h \simeq 10^{-15}$!! of course we haven't tried the matrix diagonalisation !)

3.3 Schrödinger Operators on S^2

3.3.1 Setting of the problem

We consider now the operator $\hat{H} = \Delta + V$ where Δ is the canonical Laplacian on $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ whose spectral theory is given by the spherical harmonics and $V : S^2 \to \mathbb{R}$ is a smooth potential. We introduce the pseudo-differential operator $\hat{H}_2$ on S^2 which is obtained by averaging V using the 2π-periodic quantum unitary flow $U(t) = \exp(it\sqrt{\Delta + 1/4})$:

Definition 3.1

$$\hat{H}_2 = \frac{1}{2\pi} \int_0^{2\pi} U(-t) V U(t) dt \ .$$

The following results have been obtained by Weinstein and Guillemin (see [29], [15], [17], [16] and also [6] [7]):

Theorem 3.2 • *$\hat{H}_2$ commute with Δ.*

- *$\hat{H}_2$ is a PDO of order 0 whose principal symbol is the Radon transform of V:*

$$H_2(z) = \frac{1}{2\pi} \int_0^{2\pi} V(\varphi_t(z)) dt \ ,$$

  where φ_t is the geodesic flow with unit speed. The sub-principal symbol of $\hat{H}_2$ vanishes.

- *There exists an unitary FIO Ω such that*

$$\Omega^{-1} \hat{H} \Omega = \Delta + \hat{H}_2 + R$$

  where R is a PDO of order -2.

- $$\hat{H}_2 = \oplus_{l=0}^{\infty} \Pi_l V \Pi_l$$

  where the Π_l's are the orthogonal projections on the spaces $\mathcal{H}_l$ of spherical harmonics of degree l.

Proofs can be found in [15] and [17].

In such a way, we get a quantum integrable system $\hat{H}_1 = h^2 \Delta, \ \hat{H}_2$. The spectrum of $\hat{H}$ is related to the joint spectrum

$$(h^2 l(l+1), \mu_{l,m}), \ l = 0, \cdots, \infty, \ \ -l \leqslant m \leqslant l$$

of $(\hat{H}_1, \hat{H}_2)$ by $\lambda_{l,m} = l(l+1) + \mu_{l,m} + O(l^{-2})$ and high energy asymptotics $(l \to \infty)$ corresponds in the usual way to semi-classical asymptotics $h^2.l(l+1) = 1, \ h \to 0$. We will study the system :

$$\hat{H}_1 \varphi = h^2 l(l+1) \varphi = \varphi, \ \hat{H}_2 \varphi = eh\varphi \ ,$$

assuming that 0 is a critical value of saddle type of H_2.

The Radon transform H_2 of V is a function on the manifold *Geod* of oriented closed geodesics of S^2. *Geod* is a global Poincaré section for H_1 and can be identified with $S^2 \subset \mathbb{R}^3_{X,Y,Z}$ by associating to the circle $t \to \gamma(t) = u \cos t + v \sin t$ the unit vector $u \wedge v$. Then reversing the orientation of γ corresponds to antipodal symmetry σ on $Geod = S^2$ and H_2 is even with respect to that symmetry. We can then interpret H_2 as a function on the projective plane. This fact implies that, if H_2 is a Morse-Bott function, it cannot have only local maxima and minima: it has always saddle points for which our analysis is needed.

We will from now assume that we are in the simplest situation where $H_2 : Geod \to \mathbb{R}$ has only 2 maxima, 2 minima and 2 nondegenerate saddle points. The singular manifold Λ_0 is then the union of 2 tori which intersect along 2 circles. The projection of Λ_0 on $Geod = S^2$, i.e. the reduction of Λ_0, is the union of 2 circles which are invariant by σ and which intersect at 2 antipodal points.

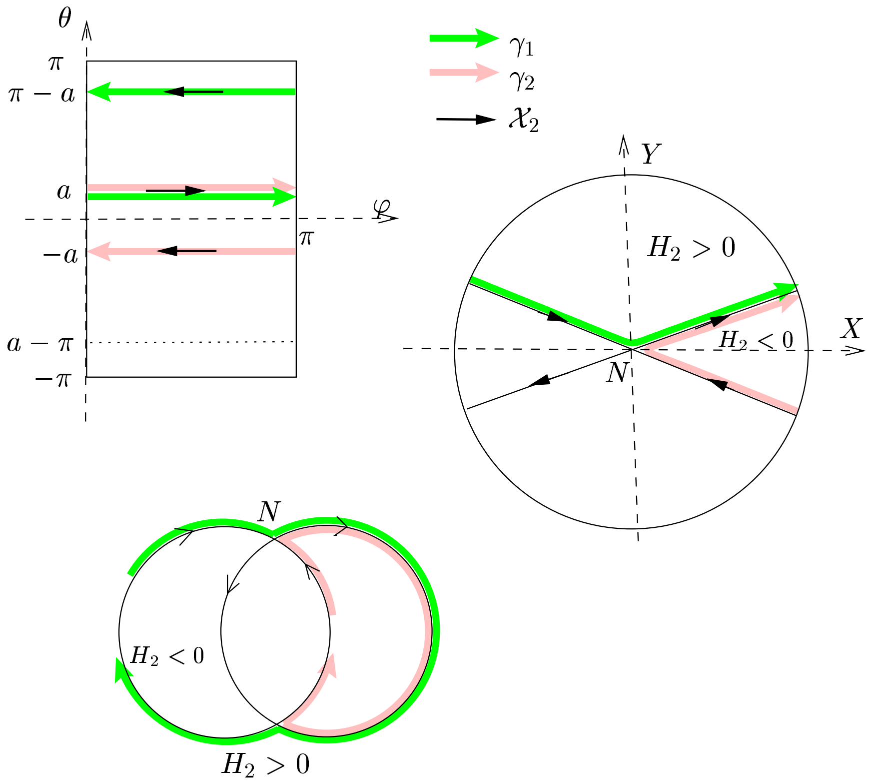


Figure 22: G in *Geod*

3.3.2 Semi-classical computations

We will complete the computations in the simplest case where V itself is a generic harmonic polynomial of degree 2. Using $SO(3)$ invariance, we have only to consider the 2-parameter family given by

$$V_{a,b,c}(x, y, z) = 2(ax^2 + by^2 + cz^2) \ ,$$

with $a + b + c = 0$ and $a < c < b$.

Because the Radon transform commutes with the $SO(3)$ action, by Schur's lemma, H_2 itself is an harmonic polynomial of degree 2 on *Geod* which is given by

$$H_2 = aX^2 + bY^2 + cZ^2 \ .$$

The critical values of H_2 are $a < c < b$ and $H_2 - c = B^2Y^2 - A^2X^2$ with $A = \sqrt{c-a}$ and $B = \sqrt{b-c}$. It is clear that the singular value $(1, c)$ of (H_1, H_2) is of hyperbolic type, so that we can apply the previous tools.

Let us denote by G the projection of Λ_0 on *Geod* by the map π which associates to a point $m \in U^\star S^2$ the geodesic to which m belongs; G is the graph introduced in the general situation and is the union of the two circles $C_\tau = \{AX = \tau BY,\ \tau = \pm 1\}$. We can compute the projections on G of the vector field $\mathcal{X}_2$ and of the sub-principal form κ on Λ_0 (because $\kappa(\mathcal{X}_1) = 0$). We will use spherical coordinates $(\theta, \varphi), 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \varphi \leqslant \pi$, on *Geod*:

$$X = \sin\varphi \cos\theta,\ Y = \sin\varphi\sin\theta,\ Z = \cos\varphi \ .$$

The symplectic form on *Geod* is $SO(3)$ invariant and of total area 4π. We will assume that:

$$\omega = \sin\varphi \, d\theta \wedge d\varphi \ .$$

By direct computation we get on C_τ:

$$\mathcal{X}_2 = 2\tau AB \sin\varphi \frac{\partial}{\partial\varphi}$$

and

$$\kappa = \frac{\tau e}{2AB\sin\varphi} d\varphi \ .$$

Let us denote by $\gamma_j,\ j = 1, \cdots, 4$, the four cycles of G oriented by $\mathcal{X}_2$, consisting of the union of one arc of C_1 and one arc of C_{-1} and bounding a topological disk in *Geod*. γ_1 and γ_2 are defined on figure 22 and $\gamma_3 = \sigma(\gamma_1),\ \gamma_4 = \sigma(\gamma_2)$. It is then easy to check using the explicit formula for the integral $\int d\varphi/\sin\varphi$ that

$$\forall j = 1, \cdots, 4,\ I_{\gamma_j} = \frac{e}{AB} \log \frac{8AB}{A^2 + B^2} \ . \tag{64}$$

Moreover, we find easily, using formula (12), that

$$\varepsilon_2 = \frac{e}{2AB} \ . \tag{65}$$

We put $\alpha = \mathrm{atan}\frac{A}{B}$ and $\beta = \mathrm{atan}\frac{B}{A} = \frac{\pi}{2} - \alpha$. Action integrals are $A_1 = A_3 = -4\beta$ and $A_2 = A_4 = 4\alpha$.

Let $\tau_j = \mathbf{hol}(\gamma_j)$. We observe first, using the fact that $h(l + \frac{1}{2}) = 1$, the relations $\tau_1 = \tau_3$ and $\tau_2 = \tau_4$ (modulo $O(h)$). We get

$$\tau_1 = \mathrm{e}^{i(-(4l+2)\beta + I + \pi + O(h))} \ . \tag{66}$$

and

$$\tau_2 = \mathrm{e}^{i((4l+2)\alpha + I + O(h))} \ . \tag{67}$$

It follows that $H := \tau_1 = \tau_2 = \tau_3 = \tau_4$. It would be nice to prove that this relation holds mod $O(h^\infty)$.

Using the computations of [10] p. 493 and putting $T = T(\varepsilon)$ with $\varepsilon = \varepsilon_2 + O(h)$ we get the following quantisation rule:

$$\det\left(\mathrm{Id} - T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T \begin{pmatrix} 0 & H^{-1} \\ H^{-1} & 0 \end{pmatrix}\right) = 0 \ . \tag{68}$$

Putting

$$T = \mathcal{E}\begin{pmatrix} 1 & \omega \\ \omega & 1 \end{pmatrix},$$

we get:

$$H = (1 \pm \omega)^2 \mathcal{E}^2$$

and the quantisation rule:

$$\mathrm{e}^{i\left((4l+2)\alpha + I + O(h)\right)} = \frac{1}{2\pi}\Gamma(\frac{1}{2} + i\varepsilon)^2 \mathrm{e}^{\varepsilon\left(\pi - 2i\log(l+\frac{1}{2})\right)}(1 \pm i\mathrm{e}^{-\varepsilon\pi})^2 \ , \tag{69}$$

which has to be solved in e where e enters in I and in ε.

Remark 3.4. Because of the $\pm$ sign in (60), the spectrum can be separated in two spectra. For each of these, the spacings of eigenvalues are easily computed as in remark 3.3. Moreover, formula (60) shows that far from the critical value (ie. $e \to \pm\infty$), the "+" and "−" eigenvalues associate in doublets, and that there is a universal transition happening when crossing the critical value ($e = 0$), where a doublet "++" becomes a doublet "−+". The details for these formulæ are similar to [9] and left to the reader. △

3.3.3 Matrix form for $\hat{H}_2$

Since V is a harmonic polynomial of degree 2, the Toeplitz operator $\hat{H}_2$ is given by

$$\hat{H}_2 = \Pi_2 V \Pi_2,$$

where V here is just the multiplication by V and Π_2 is the orthogonal projector on the space $\mathcal{H}_2$ of spherical harmonics of degree 2. We shall first determine the explicit formula for $\hat{H}_2$ with a generic $V \in \mathcal{H}_2$ and then apply it to the specific form $V = 2ax^2 + 2by^2 + 2cz^2$.

The spaces $\mathcal{H}_l$ are seen as the spaces of the irreducible representation of $\mathfrak{so}(3)$ on $L^2(S^2)$, acting via the differential operators

$$\begin{array}{rcl} L_x & = & y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \\ L_y & = & z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\ L_z & = & z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \end{array}$$

which are subject to the relation

$$[L_x, L_y] = L_z \quad (\text{ and cyclic permutations of } (x, y, z)\ ).$$

These operators commute with Δ and thus preserve $\mathcal{H}_l$. As usual, (see eg. [30]), we use the coordinates $\zeta = x + iy$ and z, and let

$$\Omega_\pm = iL_x \pm L_y,$$

so that $\Omega_+ = -\bar{\zeta}\frac{\partial}{\partial z} + 2z\frac{\partial}{\partial \zeta}$ and $\Omega_- = \Omega_+^* = \zeta\frac{\partial}{\partial z} - 2z\frac{\partial}{\partial \bar{\zeta}}$. A natural basis of $\mathcal{H}_l$ is then the following:

$$\mathfrak{B}_l = (\zeta^l, \Omega_+\zeta^l, \Omega_+^2\zeta^l, \ldots, \Omega_+^m\zeta^l, \ldots, \Omega_+^{2l}\zeta^l). \tag{70}$$

We shall use the convenient equivalent representation given by the action of $\mathfrak{su}(2)$ on the spaces $\mathcal{P}_{2l}$ of homogeneous polynomials of degree $2l$ in $\mathbb{C}^2 = (\xi, \eta)$. Using the following identification

$$\begin{aligned} L_x &= \frac{1}{2i}\left(\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}\right) \\ L_y &= \frac{1}{2}\left(-\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}\right) \\ L_z &= \frac{1}{2i}\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right), \end{aligned}$$

we get $\Omega_+ = \xi\frac{\partial}{\partial\eta}$, and a natural basis of $\mathcal{P}_{2l}$ is the following:

$$(\eta^{2l}, \Omega_+\eta^{2l}, \Omega_+^2\eta^{2l}, \ldots, \Omega_+^m\eta^{2l}, \ldots, \Omega_+^{2l}\eta^{2l}).$$

In the rest of the argument, this basis together with the basis (70) will be used to identify $\mathcal{H}_l$ and $\mathcal{P}_{2l}$. With this identification, V assumes the form:

$$V = \frac{a-b}{2}(\eta^4 + \xi^4) - 3(a+b)\eta^2\xi^2.$$

Up to a multiplicative constant (depending on l), there exists, for each l, a unique equivariant morphism $\mathbf{\Pi} : \mathcal{H}_2 \otimes \mathcal{H}_l \to \mathcal{H}_l$. Hence $\mathbf{\Pi}$ is a multiple of $\mathcal{D}^2$, where

$$\mathcal{D} = \frac{\partial}{\partial\eta} \otimes \frac{\partial}{\partial\xi} - \frac{\partial}{\partial\xi} \otimes \frac{\partial}{\partial\eta}.$$

(A subsequent composition by the multiplication $f \otimes g \to fg$ is always assumed.)

On the other hand, using the fact that every element of the form fg, $f \in \mathcal{H}_2$ and $g \in \mathcal{H}_l$ splits into

$$fg = f_{l+2} + r^2 f_l + r^4 f_{l-2},$$

where r is the radial distance and $f_j \in \mathcal{H}_j$, one easily computes

$$f_l = \mathbf{\Pi}(f \otimes g) = \frac{1}{6+4l}\left(\Delta_{\mathbb{R}^3}(fg) - \frac{r^2}{2(2l-1)}\Delta^2_{\mathbb{R}^3}(fg)\right).$$

Testing this formula and $\mathcal{D}^2$ with, for instance, $f = x^2 - z^2$ and $g = (x+iy)^l$, one gets

$$\mathbf{\Pi} = -\frac{1}{6(2l-1)(3+2l)}\mathcal{D}^2$$

$$= -\frac{1}{6(2l-1)(3+2l)}\left(\frac{\partial^2}{\partial\xi^2} \otimes \frac{\partial^2}{\partial\eta^2} - 2\frac{\partial^2}{\partial\xi\partial\eta} \otimes \frac{\partial^2}{\partial\xi\partial\eta} + \frac{\partial^2}{\partial\eta^2} \otimes \frac{\partial^2}{\partial\xi^2}\right).$$

Now it is easy to let this operator act on $V\mathcal{H}_l$, and one finally obtains the matrix representation of $\hat{H}_2$ in the basis $\mathfrak{B}_l$

$$\hat{H}_{2\restriction\mathcal{H}_l} = -\frac{1}{3(2l-1)(3+2l)} \begin{pmatrix} A_0 & 0 & B_0 & & \mathbf{0} \\ 0 & \ddots & \ddots & \ddots & \\ B_{2l-2} & \ddots & \ddots & \ddots & B_{2l-2} \\ & \ddots & \ddots & \ddots & 0 \\ \mathbf{0} & & B_0 & 0 & A_{2l} \end{pmatrix}, \qquad (71)$$

with

$$\begin{cases} A_m &= 6(a+b)(3m(2l-m) - l(2l-1)) \\ B_m &= 3(a-b)(m+1)(m+2) \end{cases}$$

Notice the symmetry of the matrix: $A_m = A_{2l-m}$.

3.3.4 Numerical computations

As it is, the matrix (63) is very badly prepared for being numerically diagonalised. Indeed, the spectrum exhibits near degeneracies – as is expected from tunneling effects – and usual algorithms will rapidly fail as l increases. Fortunately, there is an easy way to cope with this, for two commuting transformations can split the matrix: the projection onto the subspace spanned by the vectors from the basis (70) having even index, and the central symmetry ($m \to 2l-m$) of the matrix. We arrive at the following 4-blocks decomposition for the matrix $-3(2l-1)(3+2l)\hat{H}_{2\restriction\mathcal{H}_l}$, each block being a tridiagonal matrix:

- if $l = 2k+1$ is odd:

$$\begin{pmatrix} A_0 & B_0 & & & \\ B_{2l-2} & A_2 & B_2 & & 0 \\ & B_{2l-4} & \ddots & \ddots & \\ & 0 & \ddots & A_{2k-2} & B_{2k-2} \\ & & & B_{2k+2} & (A_{2k}+B_{2k}) \end{pmatrix} \oplus \begin{pmatrix} A_0 & B_0 & & & \\ B_{2l-2} & A_2 & B_2 & & 0 \\ & B_{2l-4} & \ddots & \ddots & \\ & 0 & \ddots & A_{2k-2} & B_{2k-2} \\ & & & B_{2k+2} & (A_{2k}-B_{2k}) \end{pmatrix}$$

$$\oplus \begin{pmatrix} A_1 & B_1 & & & \\ B_{2l-3} & A_3 & B_3 & & 0 \\ & B_{2l-5} & \ddots & \ddots & \\ & & \ddots & \ddots & B_{2k-3} \\ & 0 & & A_{2k-1} & 2B_{2k-1} \\ & & & B_{2k+1} & A_{2k+1} \end{pmatrix} \oplus \begin{pmatrix} A_1 & B_1 & & & \\ B_{2l-3} & A_3 & B_3 & & 0 \\ & B_{2l-5} & \ddots & \ddots & \\ & 0 & \ddots & \ddots & B_{2k-3} \\ & & & B_{2k+3} & A_{2k-1} \end{pmatrix}.$$

- if $l = 2k$ is even:

$$\begin{pmatrix} A_0 & B_0 & & & \\ B_{2l-2} & A_2 & B_2 & & 0 \\ & B_{2l-4} & \ddots & \ddots & \\ & & \ddots & \ddots & B_{2k-4} \\ & 0 & & A_{2k-2} & 2B_{2k-2} \\ & & & B_{2k} & A_{2k} \end{pmatrix} \oplus \begin{pmatrix} A_0 & B_0 & & & \\ B_{2l-2} & A_2 & B_2 & & 0 \\ & B_{2l-4} & \ddots & \ddots & \\ & 0 & \ddots & \ddots & B_{2k-4} \\ & & & B_{2k+2} & A_{2k-2} \end{pmatrix}$$

$$\oplus \begin{pmatrix} A_1 & B_1 & & & \\ B_{2l-3} & A_3 & B_3 & & 0 \\ & B_{2l-5} & \ddots & \ddots & \\ & 0 & \ddots & A_{2k-3} & B_{2k-3} \\ & & & B_{2k+1} & (A_{2k-1}+B_{2k-1}) \end{pmatrix} \oplus \begin{pmatrix} A_1 & B_1 & & & \\ B_{2l-3} & A_3 & B_3 & & 0 \\ & B_{2l-5} & \ddots & \ddots & \\ & 0 & \ddots & A_{2k-3} & B_{2k-3} \\ & & & B_{2k+1} & (A_{2k-1}-B_{2k-1}) \end{pmatrix}.$$

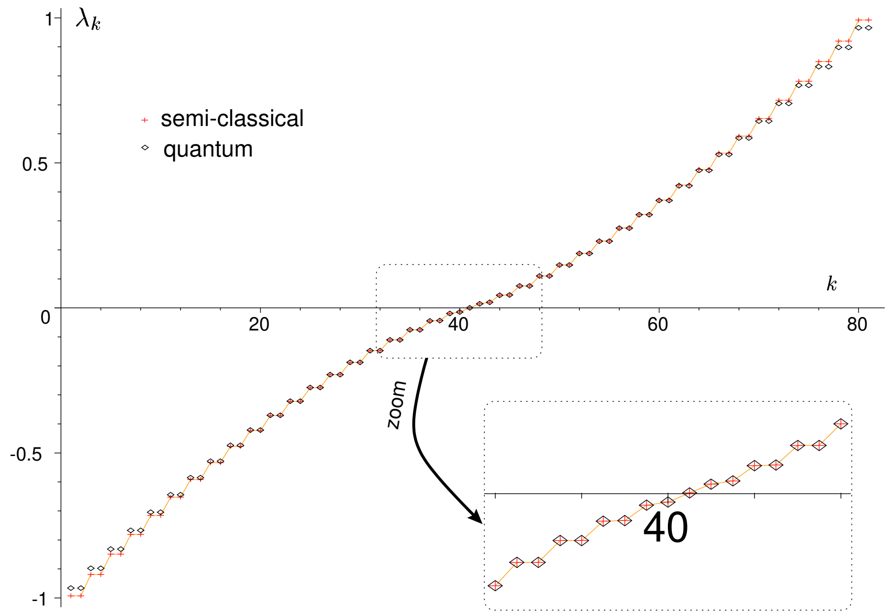


Figure 23: A comparison between semi-classical and quantum computations. Here is displayed the spectrum of $\hat{H}_{2\restriction\mathcal{H}_l}$ in increasing order versus the eigenvalue number, for $l = 40$ and the potential V defined by $a = -1$ and $b = 1$. The light crosses linked by line segments are the semi-classical computations while the quantum eigenvalues are the dark diamonds. We observe a very good accuracy (of mean order $O(h) = O(1/l)$ predicted by the theory, but much better near the critical value). Notice also how the eigenvalue doublets reassociate when passing through the critical value.

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