Non equilibrium in statistical and fluid mechanics.
Ensembles and their equivalence. Entropy driven intermittency.

Giovanni Gallavotti

Fisica, Università di Roma 1
P.le Moro 2, 00185 Roma, Italia

Abstract: *We present a review of the chaotic hypothesis and discuss its applications to intermittency in statistical mechanics and fluid mechanics proposing a quantitative definition. Entropy creation rate is interpreted in terms of certain intermittency phenomena. An attempt to a theory of the experiment of Ciliberto–Laroche on the fluctuation law is presented.*

§1. Introduction.

A general theory of non equilibrium stationary phenomena extending classical thermodynamics to stationary non equilibria is, perhaps surprisingly, still a major open problem more than a century past the work of Boltzmann (and Maxwell, Gibbs,...) which made the breakthrough towards an understanding of properties of matter based on microscopic Newton's equations and the atomic model.

In the last thirty years, or so, some progress appears to have been achieved since the recognition that non equilibrium statistical mechanics and stationary turbulence in fluids are closely related problems and, in a sense, in spite of the apparently very different nature of the equations describing them they are essentially the same.

The unifying principle, originally proposed for turbulent motions by Ruelle, [Ru78], in the early 970's, has been extended to statistical mechanics and eventually called the "*chaotic hypothesis*", [GC95]:

Chaotic hypothesis: Asymptotic motions of a chaotic system, be it a multi particle system of microscopic particles or a turbulent macroscopic fluid, can be regarded as a transitive Anosov system for the purposes of computing time averages in stationary states.

It may be useful to make a few comments on how this is supposed to be interpreted. The conclusions that we draw here from the chaotic hypothesis are summarized in §13 which might be consulted at this point. For a review on the subject seen from a different perspective see [Ru99a]

§2. Meaning of the chaotic hypothesis.

Anosov systems are well understood dynamical systems: they play a paradigmatic role with respect to chaotic systems parallel to the one harmonic oscillators play with respect to orderly motions. They are so simple, and yet very chaotic, that their properties are likely to be the ones everybody develops in thinking about chaos, even without having any familiarity with Anosov systems which certainly are not (yet) part of the background of most contemporary physicists.[1]

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[1] Informally a map $x \to Sx$ is a Anosov map if at every point x of the bounded phase space M one can set up a local system of coordinates with origin at x, continuously dependent on x and covariant under the action of S and such that in this comoving system of coordinates the point x appears as a hyperbolic fixed point for S. The corresponding continuous time motion, when the evolution is $x \to S_t x, t \in R$, requires that the local system of coordinates contains the phase space velocity $\dot{x}$ as one of the coordinate axes and that the motion transversal to it sees x as a hyperbolic fixed point: note that a motion in continuous time cannot possibly be hyperbolic in all directions and it has to be neutral in the direction of $\dot{x}$ because the velocity has to be bounded if M is bounded, while hyperbolicity would imply exponential growth as either $t \to +\infty$ or $t \to -\infty$. Furthermore there should be no equilibrium

In general an Anosov system has asymptotic motions which approach one out of finitely many invariant closed sets $C_1, \ldots, C_q$ each of which contains a dense orbit,:one says that the systems (C_j, S_t) are "transitive". One of them, at least, must be an attractive set.
To say that "the asymptotic motions form a transitive Anosov system" means that

(1) each of the sets C_j which is attractive is a smooth surface in phase space and
(2) only one of them is attractive:

The last "transitivity" assumption is meant to exclude the trivial case in which there are more than one attractive sets and the system *de facto* consists of several independent systems.
The smoothness of C_j is a *strong assumption* that means that one does not regard possible lack of smoothness, *i.e.* fractality, as a really relevant property in systems with large number of degrees of freedom. In any event one could consider (if necessary) replacing "Anosov systems" with some slightly weaker property like "axiom A" systems which could permit more general asymptotic motions. Here we adhere strictly to the chaotic hypothesis in the stated original form, [GC95].

§3. Basic implications of the chaotic hypothesis and relation with the ergodic hypothesis.

The chaotic hypothesis boldly extends to non equilibrium the *ergodic hypothesis*: applied to equilibrium systems, *i.e.* to systems described by Hamiltonian equations, it implies the latter, [Ga98]. This means that if a Hamiltonian system at a given energy is assumed to verify the chaotic hypothesis, *i.e.* to be a transitive Anosov system, then for all observables F (*i.e.* for all smooth functions F defined on phase space)

3.1
$$T^{-1}\int_0^T F(S_t x)\,dt \xrightarrow[T\to\infty]{} \int_M F(y)\,\mu_L(dy) \tag{3.1}$$

where μ_L is the Liouville distribution on the constant energy surface M, and (3.1) holds for almost all points $x \in M$, *i.e.* for x outside a set $\mathcal{N}$ of zero Liouville volume on M.
Being very general one cannot expect that the chaotic hypothesis will solve any special problem typical of non equilibrium physics, like "proving" the Fourier's law of heat conduction, the Ohm's law of electric conduction or the K41 theory of homogeneous turbulence.
Nevertheless, like the ergodic hypothesis in equilibrium, the chaotic hypothesis accomplishes the remarkable task of giving us the "statistics" of motions. If M is the phase space, which we suppose a smooth bounded surface, and $t \to S_t x$ is the motion starting at $x \in M$, the time average:

3.2
$$T^{-1}\int_0^T F(S_t x)\,dt \xrightarrow[T\to\infty]{} \int_M F(y)\,\mu_{SRB}(dy) \tag{3.2}$$

of the observable F exists for x outside a set $\mathcal{N}$ of *zero phase space volume* and it is x–independent, thus defining the probability distribution μ_{SRB} via (3.2).
Note, in fact, that the probability distribution μ_{SRB} defined by the l.h.s. of (3.2) is uniquely determined (provided it exists): it is usually called the "*statistics of the motion*" or the "*SRB distribution*" associated with the dynamics of the system.
To appreciate the above property (existence and uniqueness of the statistics) the following considerations seem appropriate.

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points and the periodic points should be dense in phase space. When the system has one or more (the so called "*hysteresis phenomenon*") attracting sets which do not occupy the whole phase space the chaotic hypothesis can be interpreted as saying that each attracting set is a smooth surface on which the time evolution flow (or map) acts as an Anosov flow (map).

An essential feature, and the main novelty, with respect to equilibrium systems is that *non conservative forces* may act on the system: this is in fact the very definition of "non equilibrium system".

Since non conservative forces perform work it is necessary that on the system act also other forces that take energy out of it, at least if we wish that the system reaches a stationary state, showing a well defined statistics.

As a consequence *any model* of the system must contain, besides non conservative forces which keep it out of equilibrium by establishing "*flows*" on it (like a heat flow, a matter flow, ...), also dissipative forces preventing the energy to increase indefinitely and forcing the motion to visit only a finite region of phase space.

The dissipation forces, also called "thermostatting forces", will in general be such that the volume in phase space is *no longer* invariant under time evolution. Mathematically this means that the divergence $-\sigma(x)$ of the equations of motion will be not zero and its time average $\int_M \sigma(y)\mu_{SRB}(dy) \stackrel{def}{=} \sigma_+$ will be positive or zero as it cannot be negative ("because phase space is supposed bounded": see [Ru96]).

One calls a system "*dissipative*" if $\sigma_+ > 0$ and we expect this to be the case as soon as there are non conservative forces acting on it.

We see that if a system is dissipative then its statistics μ_{SRB} *must* be concentrated on a set of *zero volume* in M: this means that μ_{SRB} cannot be very simple, and in fact it is somewhat hard to imagine it.

If the acting forces depend on a parameter E, "strength of the non conservative forces", and for $E = 0$ the system is Hamiltonian we have a rather unexpected situation. At $E = 0$ the chaotic hypothesis and the weaker ergodic hypothesis imply that the statistics μ_{SRB} is equal to the Liouville distribution μ_L; but if $E \neq 0$, no matter how small, it*will not* be possible to express μ_{SRB} via some density $\rho_E(y)$ in the form $\mu_{SRB}(dy) = \rho_E(y)\mu_L(dy)$, because μ_{SRB} attributes probability 1 to a set $\mathcal{N}$ with zero volume in phase space (*i.e.* $\mu_L(\mathcal{N}) = 0$). Nevertheless *natura non facit saltus* (no discontinuities appear in natural phenomena) so that sets that have probability 1 with respect to μ_{SRB} may be all still dense in phase space, at least for E small. In fact this is a "*structural stability*" property for systems which verify the chaotic hypothesis (see [Ga96c])

The above observations show one of the main difficulties of non equilibrium physics: the unknown μ_{SRB} is intrinsically more complex than a function $\rho_E(y)$ and we cannot hope to proceed in the familiar way we might have perhaps expected from previous experiences: namely to just set up some differential equations for the unknown $\rho_E(y)$.

Hence it is important that the chaotic hypothesis not only guarantees us the existence of the statistics μ_{SRB} but *also* that it does so in a "constructive way" *giving at the same time formal expressions for the distribution* μ_{SRB} which should possibly play the same role as the familiar formal expressions used in equilibrium statistical mechanics in writing expectations of observables with respect to the microcanonical distribution μ_L.

For completeness we write a popular expression for μ_{SRB}. If γ is a periodic orbit in phase space, x_γ a point on γ, $T(\gamma)$ the period of γ then

3.3
$$\int F(y)\mu_{SRB}(dy) = \lim_{T\to\infty} \frac{\sum_{\gamma:\, T(\gamma)\le T} e^{-\int_0^{T(\gamma)} \sigma(S_t x_\gamma)\, dt} \int_0^{T(\gamma)} F(S_t x_\gamma)\, dt}{\sum_{\gamma:\, T(\gamma)\le T} e^{-\int_0^{T(\gamma)} \sigma(S_t x_\gamma)\, dt}\, T(\gamma)} \tag{3.3}$$

This is simple in the sense that it does not require, to be formulated, an even slight understanding of any of the properties of Anosov or hyperbolic dynamical systems. But in many respects *it is not* a natural formula: as one can grasp from the fact that it is far from clear that in the equilibrium cases (3.3) is an alternative definition of the microcanonical ensemble (*i.e.* of the Liouville distribution μ_L), in spite of the fact that in this case $\sigma \equiv 0$ and (3.3) becomes slightly simpler.

To prove (3.3) one first derives alternative and much more useful expressions for μ_{SRB} which, however, require a longer discussion to be formulated, see [Ga99a], [Ga86c]: the original work is due to Sinai and in cases more general than Anosov systems, to Ruelle and Bowen.

§4. What can one expect from the chaotic hypothesis?

In equilibrium statistical mechanics we know the statistics of the motions, if the ergodic hypothesis is taken for granted. However this hardly solves the problems of equilibrium physics simply because evaluating the averages is a difficult task which is also model dependent. Nevertheless there are a few general consequences that can be drawn from the ergodic hypothesis: the simplest (and first) is embodied in the "*heat theorem*" of Boltzmann.

Imagine a system of N particles in a box of volume V subject to pair interactions and to external forces with potential energy W_V, due to the walls and providing the confinement of the particles to the box. Define

4.1
$$\begin{aligned} T =& \text{average kinetic energy}\\ U =& \text{total energy}\\ p =& \text{average of } \ \partial_V W_V \end{aligned} \tag{4.1}$$

where the averages are taken with respect to the Liouville distribution on the surface of energy U.

Imagine varying the parameters on which the system depends (*e.g.* the energy U and the volume V) so that dU, dV are the corresponding variations of U, V, then

4.2
$$(dU + p\,dV)/T = \ \text{exact} \tag{4.2}$$

expresses the heat theorem of Boltzmann.

It is a consequence of the ergodicity assumption, but it is *not* equivalent to it as it only involves a relation between a few averages (U, p, V, T), see [Bo66], [Bo84], [Ga99]. Not only it gives us a relation which is a very familiar property of macroscopic systems, but it also suggests us that even if the ergodic hypothesis is not strictly valid some of its consequences might, still, be regarded as correct.

The proposal is to regard the chaotic hypothesis in the same way: it is possible to imagine that mathematically speaking the hypothesis is not strictly valid and that, nevertheless, it yields results which are physically correct for the few macroscopic observables in which one is really interested in.

The ergodic hypothesis implies the heat theorem as a general ("somewhat trivial") *mechanical identity* valid for systems of N particles with $N = 1, 2, \ldots, 10^{23}, \ldots$. For small N it might perhaps be regarded as a curiosity: such it must have been considered by most readers of the key paper [Bo84] who were possibly misled by several examples with $N = 1$ given by Boltzmann in this and other previous papers. Like the example of the system consisting of one "averaged" Saturn ring, *i.e.* one homogeneous ring of mass rotating around Saturn with energy U, kinetic energy T and "volume" V (improbably identified with the strength of the gravitational attraction!). But for $N = 10^{23}$ it is no longer a curiosity and it is a fundamental law of thermodynamics in equilibrium: which, therefore, can be regarded on the same footing of a symmetry being a direct consequence of the structure of the equations of motion, [Ga99] appendices to Ch.1 and Ch.9. It reflects in macroscopic terms a simple microscopic assumption (*i.e.* Newton's equations for atomic motions, in this case).

No new consequences of even remotely comparable importance are known to follow from the chaotic hypothesis besides the fact that it implies the validity of the ergodic hypothesis itself (hence of all its consequences, first of them classical equilibrium statistical mechanics).

Nevertheless the chaotic hypothesis *does* have some rather general consequences. We mention here the *fluctuation theorem.* Let $\sigma(x)$ be the phase space contraction rate and σ_+ be its SRB average (*i.e.* $\sigma_+ = \int \sigma(x)\,\mu_{SRB}(dx)$), let $\tau > 0$ and define

4.3
$$p(x) = \tau^{-1} \int_{-\tau/2}^{\tau/2} \frac{\sigma(S_t x)}{\sigma_+}\, dx \tag{4.3}$$

and study the fluctuations of the observable $p(x)$ in the stationary state μ_{SRB}. We write $\pi_\tau(p)\,dp$ the probability that, in the distribution μ_{SRB}, the quantity $p(x)$ has actually value between p and $p + dp$ as

4.4
$$\pi_\tau(p)\,dp \;=\; \text{const}\; e^{\zeta_\tau(p)\,\tau} dp \tag{4.4}$$

Them $\lim_{\tau\to\infty} \zeta_\tau(p) = \zeta(p)$ exists and is convex in p; and

Theorem: *(fluctuation theorem) Assume the chaotic hypothesis and suppose that the dynamics is reversible, i.e. that there is an isometry I of phase space such that*

4.5
$$I S_t = S_{-t} I, \qquad I^2 = 1 \tag{4.5}$$

and that the attracting set is the full phase space.[2] *Then*

4.6
$$\zeta(-p) = \zeta(p) - \sigma_+\, p, \qquad \text{for all } p \tag{4.6}$$

where $\sigma_+ = \mu_{SRB}(\sigma)$.

It should be pointed out that the above relation was first discovered in an experiment, see [ECM93], where also some theoretical ideas were presented, correctly linking the result to the SRB distributions theory and to time reversal symmetry. Although such hints were not followed by what can be considered a proof, [CG99], still the discovery has plaid a major role and greatly stimulated further research.

The interest of (4.6) is that, in general, it is a relation without free parameters. The above theorem, proved in [GC95] for discrete evolutions (maps) and in [Ge97] for continuous time systems (flows), is one among the few general consequences of the chaotic hypothesis, see [Ga96a], [Ga96b], [Ga99b] for others.

§5. Non equilibrium ensembles. Thermodynamic limits. Equivalence.

The chaotic hypothesis gives us, unambiguously, the probability distribution μ_{SRB} which has to be employed to compute averages of observables in stationary states.

For each value of the parameters on which the system depends we have, therefore a well defined probability distribution μ_{SRB}. Calling $\underline{\alpha} = (\alpha_1, \ldots, \alpha_p)$ the parameters and $\mu_{\underline{\alpha}}$ the corresponding SRB distribution we consider the collection $\mathcal{E}$ of probability

[2] It is perhaps important to stress that we distinguish between *attracting set* and *attractor*: the first is a closed set such that the motions that start close enough to it approach it ever closer; an attractor is a subset of an attracting set that
(1) has probability 1 with respect to the statistics μ of the motions that are attracted by the attracting set (a notion which makes sense when such statistics exists, but for a zero volume set of initial data, and is unique) and that
(2) has the smallest Hausdorff dimension among such probability 1 sets. Hence density of an attracting set in phase space does not mean that the corresponding attractor has dimension equal to that of the phase space: it could be substantially lower, see [GC95].

distributions $\mu_{\underline{\alpha}}$ obtained by letting the parameters $\underline{\alpha}$ vary. We call such a collection an "*ensemble*".

For instance $\underline{\alpha}$ could be the average energy U of the system, the average kinetic energy T, the volume V, the intensity E of the acting non conservative forces, *etc*

Non equilibrium thermodynamics can be defined as the set of relations that the variations of the parameters $\underline{\alpha}$ and of other average quantities are constrained to obey as some of them are varied. In equilibrium the heat theorem is an example of such relations. In reversible non equilibria the fluctuation theorem (4.6) is an example.

In non equilibrium systems the equations of motion play a much more prominent role than in equilibrium: in fact one of the main properties of equilibrium statistical mechanics is that dynamics enters only marginally in the definition of the statistical distributions of the equilibrium states.

The necessity of a reversibility assumption in the fluctuation theorem already hints at the usefulness of considering the equations of motion themselves as "parameters" for the ensembles describing non equilibrium stationary states: we are used to irreversible equations in describing non equilibrium phenomena (like the heat equation, the Navier Stokes equation, *etc*) and unless we are able to connect our experiments with reversible dynamical models we shall be unable to make use of the fluctuation theorem.

Furthermore it is quite clear that once a system is not in equilibrium and thermostatting forces act on it, the exact nature of such forces might be irrelevant within large equivalence classes: *i.e.* it might be irrelevant which particular "cooling device" we use to take heat out of the system. Hence one would like to have a frame into which to set up a more precise analysis of such arbitrariness. Therefore we shall set

Definition 1: *A stationary ensemble $\mathcal{E}$ for a system of particles or for a fluid is the collection of SRB distributions, for given equations of motion, obtained by varying the parameters entering into the equations.*

It can happen that for the *same system* one can imagine *different models*. In this case we would like that the models give the same results, *i.e.* the same averages to the same observables, at least in some relevant limit. Like in the limit of infinite size in which the number N, the volume V and the energy U tend to infinity but N/V and U/V stay constant. Or in the limit in which the Reynolds number R tends to infinity in the case of fluids.

This gives the possibility of giving a precise meaning to the equivalence of different thermostatting mechanisms. We shall declare

Definition 2, *(equivalence of ensembles): Two thermostatting mechanisms are equivalent "in the thermodynamic limit" if one can establish a one to one correspondence between the elements of the ensembles $\mathcal{E}$ and $\mathcal{E}'$ of SRB distributions associated with the two models in such a way that the same observables, in a certain class $\mathcal{L}$ of observables, have the same averages in corresponding distributions, at least when some of the parameters of the system are sent to suitable limiting values to which we assign the generic name of "thermodynamic limit".*

In the following sections we illustrate possible applications of this concept.

§6. Drude–Lorentz' electric conduction models.

Understanding of electric conduction is in a very unsatisfactory state. It is usually based on linear response theory and very seldom a fundamental approach is attempted. Of course this is so for a good reason, because a fundamental approach would require imposing an electric field E on the system and, at the same time, a thermostatting force to keep the system from blowing up and to let it approach a steady state with a current J_E flowing in it, and then taking the ratio J_E/E (with or without taking also the limit as $E \to 0$).

However, as repeatedly mentioned, it is an open problem to study steady states out of equilibrium. Hence most theories have recourse to linear response where the problem of studying stationary non equilibria does not even arise.

The reason why this is unsatisfactory is that as long as we are *in principle* unable to study stationary non equilibria we are also *in principle* unable to estimate the size of the approximation and errors of linear response.

In spite of many attempts the old theory of Drude, see [Be59], [Se87], seems to be among the few conduction theories which try to establish a conductivity theory based on the study of electric current at non zero fields.

We imagine a set of obstacles distributed randomly or periodically and among them conduction electrons move, roughly with density of one per obstacle.

The (screened) interactions between the electrons are, at a first approximation, ignored. The collisions between electrons and obstacles ("nuclei") will take place in the average after the electrons have traveled a distance $\lambda = (\rho a^2)^{-1}$ if ρ is the nuclei density and a is their radius.

Between collisions the electrons, with electric charge e, accelerate in the direction of an imposed field $\underline{E}$ incrementing, in that direction, velocity by

6.1 $$\delta v = \frac{eE\lambda}{mv} = \frac{eE(\rho a^2)^{-1}}{m\sqrt{k_B T/m}} \tag{6.1}$$

where k_B is Boltzmann's constant. At collision they are "thermalized": an event that is modeled by giving them a new velocity of size $v = \sqrt{k_B T/m}$ and a random direction.

The latter is the "thermostatting mechanism" which is a, somewhat rough, description of the energy transfer from electrons to lattice which physically corresponds to electrons losing energy in favor of lattice phonons, which *in turn* are kept at constant temperature by some other thermostatting mechanism which prevents the wire melting. All things considered the total current that flows will be

6.2 $$J_E = \frac{e^2}{\rho a^2\sqrt{m k_B T}}\, E \stackrel{def}{=} \chi E \tag{6.2}$$

obtaining Ohm's law.

To the same conclusion we arrive by a different thermostat model. We imagine that the electrons move exchanging energy with lattice phonons but keeping their *total energy* constant and equal to $N k_B T$: *i.e.* $2^{-1}\sum_{j=1}^N m\,\underline{\dot x}_j^{\,2} = 3Nk_BT/2$, where k_B is Boltzmann's constant. There are several forces that can achieve this result

we select the "Gaussian minimal constraint" force.[3] This is the force that is required to keep $\sum m\,\underline{\dot x}_j^{\,2}$ strictly constant and that is determined by "Gauss least effort" principle, see [Ga99], ch. 9, appendix 4, for instance: as is well known this is, on the i–th particle, a force

6.3 $$-\alpha\,\underline{\dot x}_i \stackrel{def}{=} -\frac{e\underline{E}\cdot\sum_j \underline{\dot x}_j}{\sum_j \underline{\dot x}_j^{\,2}}\,\underline{\dot x}_i \equiv -\frac{m\underline{E}\cdot N\underline{J}}{3Nk_BT}\,\underline{\dot x}_i \tag{6.3}$$

If there are N particles and N is large it follows that $\underline{J} = N^{-1}e\sum_j \underline{\dot x}_j$ is essentially constant, see [Ru99b], and each particle evolves, almost independently of the others, according to an equation:

[3] Not because it plays any fundamental role but because it has been studied by many authors and because it represents a mechanism very close to that proposed by Drude.We recall, for copleteness, that the *effort* of a constraint reaction on a motion on which the active force is $\underline{f}$ (with $3N$ components) and $\underline{a}$ is the acceleration of the particles (with $3N$ components) and m is the mass is $\mathcal{E}(\underline{a}) = (\underline{f} - m\,\underline{a})^2/m$; then Gauss' principle is that the effort is minimal if $\underline{a}$ is given the actual value of the acceleration, at fixed space positions and velocities.

6.4
$$m\,\ddot{\underline{x}}_i = e\,\underline{E} - \nu\,\dot{\underline{x}}_i \tag{6.4}$$

between collisions, with a suitably fixed constant ν. If we imagine that the velocity of the particles between collisions changes only by a small quantity compared to the average velocity the "friction term" which in the average will be of order E^2 will be negligible *except* for the fact that its "only" effect will be of insuring that the *total* kinetic energy stays constant and the speeds of the particles are constantly renormalized. In other words this is the same as having continuously collisions between electrons and phonons even when there is no collision between electrons and obstacles. Hence the resulting current is the same (if N is large) as in (6.2).

§7. Ensemble equivalence: the example of electric conduction theories.

We have derived three models for the conduction problem, namely

(1) the classical model of Drude, [Se87], in which at *every collision* the electron velocity is reset to the average velocity at the given temperature, with a random direction, *c.f.r.* (6.1) and (6.2).

(2) the Gaussian model in which the total kinetic energy is kept constant by a thermostat force

7.1
$$m\,\ddot{\underline{x}}_i = \underline{E} - \frac{m\,\underline{E}\cdot\,\underline{J}}{3k_BT}\,\dot{\underline{x}}_i + \text{“collisional forces''} \tag{7.1}$$

where $3Nk_BT$ is the total kinetic energy (a constant of motion in this model). The model has been widely studied and it was introduced by Hoover and Evans (see for instance [HHP87] and [EM90]).

(3) a "friction model" in which particles independently experience a constant friction

7.2
$$m\,\ddot{\underline{x}}_i = \underline{E} - \nu\,\dot{\underline{x}}_i + \text{“collisional forces''} \tag{7.2}$$

where ν is a constant tuned so that the *average kinetic energy* is $eNk_BT/2$. This model was considered in the perspective of the conjectures of ensemble equivalence in [Ga95], [Ga96b].

The first model is a "stochastic model" while the second and third are deterministic: the third is "irreversible" while the second is reversible because the involution $I(\underline{x}_i, \underline{v}_i) = (\underline{x}_i, -\underline{v}_i)$ anticommutes with the time evolution flow S_t defined by the equation (7.1): $IS_t = S_{-t}I$ (as the "friction term" is *odd* under I).

Let $\mu_{\delta,T}$ be the SRB distribution for (7.1) for the stationary state that is reached starting from initial data with energy $3Nk_BT/2$. The collection of the distributions $\mu_{\delta,T}$ as the kinetic energy T and the density $\delta = N/V$ vary, define a "statistical ensemble" $\mathcal{E}$ of stationary distributions associated with the equation (7.1).

Likewise we call $\tilde{\mu}_{\delta,\nu}$ the class of SRB distributions associated with (7.2) which forms an "ensemble" $\tilde{\mathcal{E}}$.

We establish a correspondence between distributions of the ensembles $\mathcal{E}$ and $\tilde{\mathcal{E}}$: we say that $\mu_{\delta,T}$ and $\tilde{\mu}_{\delta',\nu}$ are "corresponding elements" if

7.3
$$\delta = \delta', \qquad T = \int \frac{1}{2}\Big(\sum_j m\,\dot{\underline{x}}_j^{\,2}\Big)\,\tilde{\mu}_{\delta,\nu}(d\,\underline{x}\;d\,\dot{\underline{x}}) \tag{7.3}$$

Then the following conjecture was proposed in [Ga96b].

Conjecture 1: *(equivalence conjecture) Let F be a "local observable", i.e. an observable depending solely on the microscopic state of the electrons whose positions is inside a fixed box V_0. Then, if $\mathcal{L}$ denotes the local smooth observables*

7.4 $$\lim_{N\to\infty,N/V=\delta}\tilde\mu_{\delta,\nu}(F)=\lim_{N\to\infty,N/V=\delta}\mu_{\delta,T}(F)\qquad F\in\mathcal{L}\tag{7.4}$$

if T and ν are related by (7.3).

This conjecture has been discussed in [Ga95], sec. 5, and [Ga96a], see sec. 2 and 5: and in [Ru99b] arguments in favor of it have been developed.

Clearly the conjecture is very similar to the equivalence in equilibrium between canonical and microcanonical ensembles: here the friction ν plays the role of the canonical inverse temperature and the kinetic energy that of the microcanonical energy.

It is remarkable that the above equivalence suggests equivalence between a "reversible statistical ensemble", *i.e.* the collection $\mathcal{E}$ of the SRB distributions associated with (7.1) and a "irreversible statistical ensemble", *i.e.* the collection $\tilde{\mathcal{E}}$ of SRB distributions associated with (7.2).

Furthermore it is natural to consider also the collection $\mathcal{E}'$ of stationary distributions for the original stochastic model (1) of Drude, whose elements $\mu'_{\nu,T}$ can be parameterized by the quantities T, temperature (such that $\frac12\sum_j m\,\underline{\dot x}_j^{\,2}=\frac32 Nk_BT$, and $N/V=\delta$). This is an ensemble $\mathcal{E}'$ whose elements can be put into one to one correspondence with the elements of, say, the ensemble $\mathcal{E}$ associated with model (2), *i.e.* with (7.1): an element $\mu'_{\nu,T}\in\mathcal{E}'$ corresponds to $\mu_{\delta,\nu}\in\mathcal{E}$ if T verifies (7.3). Then

Conjecture 2: *If $\mu_{\delta,T}\in\mathcal{E}$ and $\mu'_{\delta,\nu}\in\mathcal{E}'$ are corresponding elements (i.e. (7.3) holds) then*

7.5 $$\lim_{N\to\infty,N/V=\delta}\mu_{\delta,T}(F)=\lim_{N\to\infty,N/V=\delta}\mu'_{\delta,T}(F)\qquad F\in\mathcal{L}\tag{7.5}$$

for all local observables $F\in\mathcal{L}$.

Hence we see that there can be statistical equivalence between a viscous irreversible dissipation model and either a stochastic dissipation model or a reversible dissipation model, at least as far as the averages of special observables are concerned.

The argument in [Ru99b] in favor of conjecture 1 is that the coefficient α in (6.3) is essentially the average J of the current over the *whole* box containing the system of particles, $J=N^{-1}\,e\sum_j\underline{\dot x}_i$: hence J should be constant with probability 1, at least if the stationary SRB distributions can be reasonably supposed to have some property of ergodicity with respect to *space translations.*

§8. Entropy driven intermittency in reversible dissipation.

A further argument for the equivalence conjectures in the above electric conduction models can be related to the fluctuation theorem: the quantity $\alpha(x)$ is also proportional to the phase space contraction rate $\sigma(x)=(3N-1)\alpha(x)$. Therefore, denoting in general with a subscript $+$ the SRB average (or the time average) of an observable, the probability that $\sigma(x)$ deviates from its average $\sigma_+=(3N-1)\,\alpha_+$ can be studied as follows.

If the number N of particles is large the time scale τ_0 over which $\sigma(S_tx)$ evolves will be large compared to the microscopic evolution rates, because $\sigma_t(x)$ is the *sum* of the $\sim 6N$ rates of expansion and contraction of the $\sim 6N$ phase space directions out of x (sometimes called the "*local Lyapunov exponents*").[4]

[4] The exact number of exponents depends on how many constants of motion the system has: for instance in the case of the conduction model (1) in §6 above the number of exponents is $6N-1$ because the kinetic energy is conserved and the system has no other (obvious) first integrals. Furthermore one of such exponents is 0 since every dynamical system in continuous time has one zero exponent (corresponding to the direction $\dot x$ of the flow).

Consider a large number m of time intervals $I_1, I_2, \ldots, I_m$ of size τ_0 and let σ_j be the (average) value of $\sigma(S_t x)$ for $t \in I_j$. Then the fraction of the j's such that $\sigma_j - \sigma_+ \simeq \sigma_+ p$ will be proportional to

8.1
$$\pi_{\tau_0}(p) \simeq e^{\tau_0 \zeta(p)} \tag{8.1}$$

and $\zeta(p) < \zeta(1)$ if $p \neq 1$. Since we can expect that$\zeta(p)$ is proportional to N we see that the fraction of time intervals I_j in which $\sigma_j \neq \sigma_+$ will be exponentially small with N.

For instance the fraction of time intervals in which $\sigma_j \simeq -\sigma_+$ will be, by the fluctuation theorem

8.2
$$e^{-(3N-1)\alpha_+ \tau_0} \tag{8.2}$$

In order that the above argument holds it is essential that N is large to the point that we can think that the time scale τ_0 over which $\sigma(S_t x)$ varies is much larger than the microscopic scales: so that we can regard τ_0 large enough for the fluctuation theorem to apply. In this respect this is not really different from the previously quoted argument in [Ru99b]. However the change of perspective gives further information.

In fact we get the following picture: N is large and for most of the time the (stationary) evolution uneventfully proceeds as if $\sigma(S_t x) \equiv \sigma_+$ (thus justifying conjecture 1). Very rarely, however, it proceeds as if $\sigma(S_t x) \neq \sigma_+$, for instance with $\sigma(S_t x) = -\sigma_+$: such "*bursts of anomalous behavior*" occur very rarely. But when they occur "everything else goes the wrong way" because, as discussed in detail in [Ga99c], while the phase space contraction is opposite to what it "should be" (in the average) then it also happens that *all observables evolve following paths that are the time reversal of the expected paths*, This is the content, see [Ga99c], of the following theorem which is quite close (particularly if one examines its derivation) to the Machlup–Onsager theory of fluctuation patterns (note that, however, it does not require closeness to equilibrium)

Theorem*(conditional reversibility theorem): If F is an observable with even (or odd), for simplicity, time reversal parity and if τ is large then the evolution or "fluctuation pattern" $\varphi(t)$ and its time reversal $I\varphi(t) \equiv \varphi(-t)$, $t \in [-\tau_0/2, \tau_0/2]$, will be followed with equal likelihood if the first is conditioned to an entropy creation rate p and the second to the opposite $-p$.*

In other words systems with reversible dynamics can be equivalent to systems with irreversible dynamics but they show "*intermittent behavior*" with intermittency lapses that become extremely rare very quickly as $N \to \infty$. Sometimes they can be really dramatic, as in the cases in which $\sigma = -\sigma_+$: alas they are unobservable just for this reason and one can wonder (see §9 below) whether this is really of any interest.

§9. Local fluctuations and observable intermittency.

As a final comment upon the analysis of the equivalence of ensembles attempted above we consider a very large system with volume V and a small subsystem of volume V_0 which is large but not yet really macroscopic so that the number of particles in V_0 is not too large, a nobler way to express the same notion is to say that we consider a "mesocopic" subsystem of our macroscopic system.

Here it is quite important to specify the system because we want to make use of aspects of the equivalence conjectures that are model dependent. Therefore we consider the conduction models (2) or (3) of §5: these are models in which dissipation occurs "homogeneously" throughout the system. In this case we can imagine to look at the part of the system in the box V_0: if $j_1, \ldots, j_{N_0}$ are the particles which at a certain instant are inside V_0 and $\underline{\dot{x}}_j = \underline{f}_j(\underline{x})$ are the equations of motion we can define

9.1

$$\sigma_{V_0}(x) = \sum_{i=1}^{N_0} \partial_{x_{j_k}} f_{j_k}(x) \tag{9.1}$$

which is (by definition) the part of phase space contraction due to the particles in V_0.

Since the part of the system inside the microscopically large but macroscopically small V_0 can be regarded as a new dynamical system whose properties should not be different from the ones of the full system enclosed in the full volume V we may expect that the subsystem inside V_0 is in a stationary state and the quantity σ_{V_0} has the same fluctuation properties as σ_V, *i.e.*

9.2

$$\begin{array}{lll} (1) & \langle\sigma_{V_0}\rangle_+ = V_0\,\overline{\sigma}_+, & \langle\sigma_V\rangle_+ = V\,\overline{\sigma}_+ \\ (2) & \pi_\tau^{V_0}(p) = e^{\overline{\zeta}(p)\,\tau\,V_0}, & \pi_\tau^{V}(p) = e^{\overline{\zeta}(p)\,\tau\,V} \end{array} \tag{9.2}$$

where $\overline{\zeta},\overline{\sigma}_+$ are *the same for* V, V_0 and $p = \tau^{-1}\int_{-\tau/2}^{\tau/2}\sigma_{V_0}(S_t x)/\langle\sigma_{V_0}\rangle\,dt$ or respectively $p = \tau^{-1}\int_{-\tau/2}^{\tau/2}\sigma_V(S_t x)/\langle\sigma_V\rangle\,dt$. Here σ_{V_0} is naively defined as the contribution to σ coming from the particles in V_0.

In other words in large stationary systems with homogeneous reversible dissipation phase space contractions fluctuate in an extensive way, i.e. they are regulated by the same deviation function $\overline{\zeta}(p)$ (volume independent).

This is very similar to the well known property of equilibrium density fluctuations in a gas of density ρ: if $V \supset V_0$ are a very large volume V in a yet larger container and V_0 is a small but microscopically large (*i.e.* mesoscopic) volume V_0 then the total numbers of particles in V and V_0 will be N and N_0 and the average numbers will be ρV and ρV_0 respectively. Then setting

9.3

$$p = (N - \rho\,V)/\rho V, \qquad or, \qquad p = (N_0 - \rho\,V_0)/\rho V_0 \tag{9.3}$$

the probability that the variable p has a given value will be proportional to

9.4

$$\pi^V(p) = e^{\overline{\zeta}(p)V}, \qquad \pi^{V_0}(p) = e^{\overline{\zeta}(p)V_0} \tag{9.4}$$

again with the same function $\overline{\zeta}(p)$.

This means that we *can observe* $\overline{\zeta}(p)$ *by performing fluctuations experiments in small boxes*, ideally carved out of the large container, where the density fluctuations are not too rare. A "local fluctuation law" should hold more generally in cases of models in which dissipation occurs homogeneously across the system, like the above considered conduction models.

The intuitive picture for the above "local fluctuation relation" inspired (and was substantiated) a mathematical model in which a local fluctuation relation can be proved as a theorem: it has een discussed in [Ga99c], see also below.

Going back to the conduction model we see that the intermittency phenomena discussed above can be actually observed by looking at the fluctuations of the contribution to phase space contraction due to a small subsystem.

And such "*entropy driven*" intermittency will be model independent for models which are equivalent in the sense of the previous sections provided the models used are equivalent and one of them is reversible.

An extreme case is provided by models (1)%(3), §7, for electric conduction (conjectured to be equivalent, see §7). In fact at first the model (3), the viscous thermostat, might look uninteresting as, obviously, in this case

9.5

$$\sigma^V(x) = 3N\nu, \qquad \sigma^{V_0}(x) = 3\,N_0\,\nu \tag{9.5}$$

and σ^V/V has no fluctuations.

However the equivalence conjecture makes a statement about expectation values of the *same observable*: hence we should consider the quantity $\hat\sigma^{V_0}(x)=\underline{E}\cdot\underline{J}_{V_0}/\sum_j\underline{\dot x}_j^2$ and we should expect that its statistics with respect to an element of the ensemble $\mathcal{E}'$ is the same as that of the same quantity with respect to the corresponding elements of the ensembles $\mathcal{E},\tilde{\mathcal{E}}$. Hence in particular the functions $\overline{\zeta}(p)$ which control the large fluctuations of $\sigma^V(p)$ will verify

9.6 $$\overline{\zeta}(-p)=\overline{\zeta}(p)-p\langle\hat\sigma^{V_0}\rangle_+/V_0=\overline{\zeta}(p)-3\rho\nu\,p=\overline{\zeta}(p)-\frac{eEm\,J_+}{k_BT}\,p \tag{9.6}$$

where the first equality expresses the validity of a fluctuation theorem type of relation due to the fact that the small system, by the equivalence conjecture, should behave as a closed system; the second equality expresses a consequence of the equivalence conjecture between models (2) and (3) while the third is obtained by expressing the current via Drude's theory (again assuming the conjectures of equivalence 1,2 of §7).

§10. Fluids.

The chaotic hypothesis was originally formulated to understand developed turbulence, [Ru78]: it is therefore interesting to revisit fluid motions theory.

The incompressible Navier Stokes equation for a velocity field $\underline{u}$ in a periodic container V of side L can be considered as an equation for the evolution in time of its Fourier coefficients $\underline{u}_{\underline{k}}$ where the "*mode*" $\underline{k}$ has the form $2\pi L^{-1}\underline{n}$ with $\underline{n}\ne\underline{0}$ and $\underline{n}$ an integer components vector.[5] Furthermore $\underline{u}_{\underline{k}}=\overline{\underline{u}}_{-\underline{k}}$ and $\underline{k}\cdot\underline{u}_{\underline{k}}\equiv0$. If p is the pressure field and $\underline{f}$ a simple forcing we shall fix the ideas by considering $\underline{f}(\underline{x})=f\,\underline{e}\,\sin\underline{k}_f\cdot\underline{x}$ where $\underline{k}_f$ is some prefixed mode and $\underline{e}$ is a unit vector orthogonal to $\underline{k}_f$.

The Navier Stokes equation is then

10.1 $$\underline{\dot u}+\underline{u}\cdot\underline{\partial}\,\underline{u}=-\underline{\partial}p+\underline{f}+\nu\Delta\underline{u} \tag{10.1}$$

and it is convenient to use *dimensionless variables* $\underline{u}_0,p_0,\underline{\varphi}_0,\underline{\xi},\tau$: so we define them as

10.2 $$\begin{aligned}\underline{u}(\underline{x},t)=&fL^2\nu^{-1}\,\underline{u}_0(L^{-1}\underline{x},L^{-2}\nu t), &\qquad \underline{\xi}=L^{-1}\underline{x},\ \tau=L^{-2}\nu t\\ p(\underline{x},t)=&fLp_0(L^{-1}\underline{x},L^{-2}\nu t), & R\stackrel{def}{=}f\,L^3\,\nu^{-2}\\ \underline{f}(\underline{x},t)=&f\,\underline{\varphi}_0(L^{-1}\underline{x})\end{aligned} \tag{10.2}$$

with $\max|\underline{\varphi}_0|=1$. The result, *dropping the label* 0 and calling again $\underline{x},t$ the new variables $\underline{\xi},\tau$, is that the Navier Stokes equations become an equation for a divergenceless field $\underline{u}$ defined on $V=[0,1]^3$, with periodic boundary conditions and equations

10.3 $$\underline{\dot u}+R\,\underline{u}\cdot\underline{\partial}\,\underline{u}=-\underline{\partial}p+\underline{\varphi}+\Delta\underline{u},\qquad\underline{\partial}\cdot\underline{u}=0 \tag{10.3}$$

with $\max|\underline{\varphi}|=1$.

Equation (10.3) is our model of fluid motion, where R plays the role of "forcing intensity" and the term $\Delta\underline{u}$ represents the "thermostatting force". As R varies the stationary distributions μ_R which describe the SRB statistics of the motions (10.3) define a set $\mathcal{E}$ of probability distributions which forms an "ensemble".

The mathematical theory of the Navier Stokes equations is far from being understood: however phenomenology establishes quite clearly a few key points. The main property

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[5] The value $\underline{n}=\underline{0}$ is excluded because, having periodic boundary conditions, it is not restrictive to suppose that the space average of $\underline{u}$ vanishes (galilean invariance). The convention for the Fourier transform that we use is $\underline{u}(\underline{x})=\sum_{\underline{k}}e^{i\,\underline{k}\cdot\underline{x}}\,\underline{u}_{\underline{k}}$.

is that if (10.3) is written as an equation for the Fourier components of $\underline{u}$ then one can assume that $\underline{u}_{\underline{k}} \equiv \underline{0}$ for $|\underline{k}| > K(R)$, for some finite $K(R)$.
Therefore the equation (10.3) should be thought of as a "truncated equation" in momentum space by identifying it with the equation obtained by projecting also $\underline{u} \cdot \underline{\partial}\, \underline{u}$ on the same function space.

Should one develop anxiety about the mathematical aspects of the Navier Stokes equation one should therefore think that an equally good model for a fluid is the mentioned truncation *provided* $K(R)$ is chose large enough.
The idea is that for $K(R) = R^\kappa$, with κ larger than a suitable κ_0 the results of the theory, *i.e.* the statistical properties of μ_R become κ–independent for R large.
The simplest evaluation of κ_0 gives $\kappa_0 = 9/4$ as a consequence of the so called K41 theory of homogeneous turbulence, see [LL71].
If (10.3) is a good model for a fluid when L is large then it provides us with an "ensemble" $\mathcal{E}$ of SRB distributions (on the space of the velocity fields components $\underline{u}_{\underline{k}}$ of dimension $\sim 8\pi K(R)^3/3$).[6]

We should expect, following the discussion of the statistical mechanics cases, that there can be other "ensembles" $\tilde{\mathcal{E}}$ which are equivalent to $\mathcal{E}$.
Here R plays the role of the volume in non equilibrium statistical mechanics, so that $R \to \infty$ will play the role of the thermodynamic limit, a limit in which the effective number of degrees of freedom, $\sim 4\pi R^{3\kappa}/3$, becomes infinite. The role of the local observables will be plaid by the (smooth) functions $F(\underline{u})$ of the velocity fields $\underline{u}$ which depend on $\underline{u}$ only via its Fourier components that have mode $\underline{k}$ with $|\underline{k}| < B$ for some B: $F(\underline{u}) = F(\{\underline{u}_{\underline{k}}\}_{|\underline{k}|\le B})$.
We shall call $\mathcal{L}$ the space of such observables: examples can be obtained by setting $F(\underline{u}) = |\int e^{i\underline{k}\cdot\underline{x}}\, \underline{u}(\underline{x})\, d\underline{x}|^2$ or $F(\underline{u}) = \int \underline{\psi}(\underline{x}) \cdot \underline{u}(\underline{x})\, d\underline{x}$ where the function has only a finte number of harmonics, $\underline{\psi}(\underline{x}) = \int \sum_{|\underline{k}|<B} e^{i\underline{k}\cdot\underline{x}}\, \underline{u}(\underline{x})\, d\underline{x}$, *etc.*

As in non equilibrium statistical mechanics we can expect that the equations of motion themselves become part of the definition of the ensembles. For instance one can imagine defining the ensemble $\tilde{\mathcal{E}}$ of the SRB distributions $\tilde{\mu}_V$ for the equations

10.4
$$\dot{\underline{u}} + R\, \underline{u} \cdot \underline{\partial}\, \underline{u} = -\underline{\partial} p + \underline{\psi} + \nu(\underline{u})\Delta\, \underline{u} \tag{10.4}$$

called GNS equations in [Ga97a], or "gaussian Navier Stokes" equations, where $\nu(\underline{u})$ is so defined that

10.5
$$\Xi = \int_V (\underline{\partial}\, \underline{u})^2\, d\underline{x}/(2\pi)^3 = \sum_{\underline{k}} \underline{k}^2\, |\underline{u}_{\underline{k}}|^2 \tag{10.5}$$

is exactly constant and equal to Ξ. The equations (10.4) are interpreted as above with the same momentum cut off $K(R) = R^\kappa$.
An element $\tilde{\mu}_\Xi$ of $\tilde{\mathcal{E}}$ and one μ_R of $\mathcal{E}$, SRB distributions for the two different dynamics (10.3) and (10.4), "correspond to each other" if

10.6
$$\Xi = \int \mu_R(d\underline{u})\, \Big(\int_V (\underline{\partial}\, \underline{u})^2\, d\underline{x}/(2\pi)^3\Big) \stackrel{def}{=} \Xi_R \tag{10.6}$$

where $\mu_R \in \mathcal{E}$ is the SRB distribution at Reynolds number R for the previous viscous Navier Stokes equation, (10.3), and we naturally conjecture

[6] There are about $4\pi K(R)^3/3$ vectors with integer components inside a sphere of radius $K(R)$, thus the number of complex Fourier components with mode label $|\underline{k}| < K(R)$ would be 3 times as much, but the divergenceless condition leaves only 2 complex components for $\underline{u}_{\underline{k}}$ along the two unit vectors orthogonal to $\underline{k}$ and the reality condition further divides by 2 the number of "free" components.

Conjecture 3 *(equivalence GNS–NS): If $R\to\infty$ then for all local observables $F\in\mathcal{L}$ it is $\mu_R(F)=\tilde\mu_{\Xi_R}(F)$ if (10.6) holds.*

It is easy to check that the GNS model "viscosity" $\nu(\underline{u})$, having to be such that the quantity Ξ in (10.5) is exactly constant must be

10.7
$$\nu(\underline{u})=\frac{\int_V\big(\underline{\varphi}\cdot\Delta\underline{u}-R\Delta\underline{u}\cdot(\underline{u}\cdot\underline{\partial}\,\underline{u})\big)\,d\underline{x}}{\int_V(\Delta\underline{u})^2\,d\underline{x}}\tag{10.7}$$

and we see that while (10.3) is an *irreversible* equation the (10.4) is *reversible*, with time reversal symmetry given by

10.8
$$I\,\underline{u}(\underline{x},t)=-\,\underline{u}(\underline{x},t)\tag{10.8}$$

as one can check.

More generally one may wish to leave the "Kolmogorov parameter" κ as a free parameter: in this case the SRB distributions will form an ensemble whose elements can be parameterized by R,κ and the equivalence conjecture can be extended to this case yielding equivalence between $\mu_{R,\kappa}$ and $\tilde\mu_{\Xi,\kappa}$. This is of interest, particularly if one has numerical experiments in mind.

If $\kappa>\kappa_0$ then the value of κ *should be irrelevant*: but if $\kappa<\kappa_0$ the phenomenology will be different from the one of the Navier Stokes equation and equivalence might still hold but one cannot expect either equation to have the properties that we expect for the usual Navier Stokes equations (*i.e.* in this situation one would have to be careful in making statements based on common experience).

If we take κ to be exactly equal to the value $\kappa_0=9/4$ (*i.e.* if we take the ultraviolet cut–off to be such that, according to the K41 theory, for larger values it is *needlessly large* and for lower values it is *incorrectly low* and shows a phenomenology which will depend on its actual value) then we may speculate that the "attracting set" is the full phase space (available compatibly with the constraint $\Xi=\Xi_R$). Therefore the divergence of the equations of motion, which is given by a rather involved expression in which only the first term seems to dominate at large R, namely

10.9
$$\begin{aligned}\sigma(\underline{u})=\Big(&\sum_{|\underline{k}|<K(R)}\underline{k}^2\Big)\,\nu(\underline{u})-\Big(\int_V\Delta\underline{\varphi}\cdot\Delta\underline{u}\,d\underline{x}\Big)\Big(\int_V[(\Delta\underline{u})^2-\\&-R\,\Delta\underline{u}\cdot(\Delta(\underline{u}\cdot\underline{\partial}\,\underline{u}))-R(\Delta\underline{u})\cdot(\Delta\underline{u})\cdot(\underline{\partial}\,\underline{u})-R\Delta\underline{u}\cdot(\Delta\underline{\partial}\,\underline{u})\underline{u}+\\&+\nu(\underline{u})\Delta\underline{u}\cdot\Delta^2\underline{u}]d\underline{x}\Big)\;/\int_V(\Delta\underline{u})^2\,d\underline{x}\end{aligned}\tag{10.9}$$

will verify the fluctuation theorem, *i.e.* the rate function $\zeta(p)$ for the average phase space contraction $p=\tau^{-1}\int_{-\tau/2}^{\tau/2}\sigma(S_t\,\underline{u})dt/\tau\sigma_+$ will be such that $\zeta(-p)=\zeta(p)-p\sigma_+$.

If the chaotic hypothesis is valid together with the equivalence conjecture the validity of the fluctuation relation can be taken as a criterion for determining κ: it would be the last κ before which the fluctuation relation between $\zeta(p)$ and $\zeta(-p)$ holds. *However this conclusion can only be drawn if the attracting set in phase space is the full ellipsoid* $\Xi=\Xi_R$ at least for $K(R)=R^{\kappa_0}$.

The latter property might not be realized: and in such case the fluctuation theorem does not apply directly, although the equivalence conjectures still hold. In fact one can try to extend the fluctuation theorem to cover reversible cases in which the attracting set is smaller than the full phase space left available by the constraints. In such cases *under* *suitable geometric assumptions*, [BG97] and the earlier work [BGG97], one can derive a relation like

10.10 $$\zeta(-p) = \zeta(p) - p\sigma_+ \vartheta, \qquad 0 \le \vartheta \le 1 \tag{10.10}$$

where ϑ is a coefficient that can be related to the Lyapunov spectrum of the system, *c.f.r.* [BG97], [Ga97a]. In fact numerical work to check the theory proposed in [Ga97a] is currently being performed (private communication by Rondoni and Segre) with not too promising results which, optimistically, can be attributed to the fact that the ultraviolet cut off is too small due to numerical litmitations: clearly there is more work to do here. The preliminary numerical results give, so far, the *somewhat surprising linearity* in p but with a slope that, although of the correct order of magnitude, seems to have a value that does not match the theory within the error bounds.

Coming back to the Navier Stokes equation we mention that we may imagine to write it as (10.3) *but* with the different constraint

10.11 $$U = \int_V \underline{u}^{\,2} \, d\underline{x} \; = \; \text{const} \tag{10.11}$$

rather than (10.5).
This case has been considered in [RS99] and the multiplier $\nu(\,\underline{u}\,)$ is in this case

10.12 $$\nu(\,\underline{u}\,) = \frac{\int_V \underline{\varphi} \cdot \underline{u} \, d\underline{x}}{\int_V \underline{u}^{\,2} \, d\underline{x}}, \qquad \sigma(\,\underline{u}\,) = (3 \sum_{|\,\underline{k}\,| < K(R)} |\,\underline{k}\,|^2 - 1)\,\nu(\,\underline{u}\,) \tag{10.12}$$

and we can (almost) repeat the above considerations and equivalence conjectures. This constraint is a gaussian constraint that U is constant obtained by imposing its constancy on the Euler evolution via Gauss' principle with a suitable definition of the notion of "constraint effort" (this notion is not unique, see [Ga97a] for another definition) and we do not discuss it here to avoid overlapping with §12 below.

Thd intuitive motivation for the equivalence conjectures is that for large R the phase space contraction $\sigma(\,\underline{u}\,)$ and the coefficient $\nu(\,\underline{u}\,)$ [7] are "global quantities" and depend on the global properties of the system (*e.g.* $\sigma(\,\underline{u}\,)$ is the sum of all the local Lyapunov exponents of the system whose number is $O(K(R)^3)$): they will "therefore" vary over time more slowly than any time scale of the system and can be considered constant.
The argument is not very convincing in the case of the equations with the constraint (10.11) because the $\sigma(\,\underline{u}\,)$ in (10.12) is proportional to $\int_V \underline{\varphi} \cdot \underline{u} \, d\underline{x}$ which clearly depends *only* on harmonics of $\underline{u}$ with $\underline{k}$ small, *i.e.* it is a "local observable". Note that this does not apply to the GNS equations with the constrained vorticity Ξ, (10.6) where the "main" contribution to $\sigma(\,\underline{u}\,)$, see (10.7), comes from the term proportional to R which contains all harmonics. Therefore the result in [RS99] about the equivalence between the GNS equations, (10.4) with the constraint (10.5), and the equations with constraint (10.11) is interesting and puzzling: it might be an artifact of the smallness of the cut off that one has to impose in order to have numerically feasible simulations.

Finally $\sigma_+(\,\underline{u}\,)/\sigma_+$, *i.e.* essentially $\nu(\,\underline{u}\,)/\nu_+$ will fluctuate taking values sensibly different from their average value 1, at very rare intervals of time: but when such fluctuations will occurr one shall see "bursts" of anomalous behavior: *i.e.* the motion will be "*intermittent*" as in the case discussed in non equilibrium statistical mechanics.

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[7] Which in the case (10.9) are simply proportional and in the case of (10.4) they are related in a more involved way, see (10.8),(10.9), but which are still probably proportional to leading order as $R \to \infty$.

11. Entropy creation rate and entropy driven intermittency.

Of course if R is large the number of degrees of freedom is large and intermittency on the scale of the fluid container will not be observable due to its extreme unlikelyhood (expected and quantitatively predicted by the fluctuation theorem).

Therefore we look also here, in fluid motions, for a *local fluctuation relation*. Fluids seem particularly suitable for verifying such local fluctuations relations because dissipation occurs *homogeneously*, *i.e.* friction strength is translation invariant.

This implies that we can regard a very small volume V_0 of the fluid as a system in itself (as always done in the derivation of the basic fluid equations, *e.g.* see [Ga97b]) and we can expect that the phase space contraction due to such volume elements is simply $\sigma(\underline{u})$, given by (10.9) or (10.12) ("equivalently" because of our equivalence conjectures) with the integrals in the numerator and denominator being extended to the volume V_0 rather than to the whole box, and expressing (essentially by definition) the "local phase space contraction" $\sigma_{V_0}(\underline{u})$.

Then $p = \tau^{-1}\int_{-\tau/2}^{\tau/2}\sigma_{V_0}(S_t\,\underline{u})/\langle\sigma_{V_0}\rangle_+$ will have a rate function $\zeta(p)$ which will verify, under the same assumptions as in (10.10), a large deviation relation as

11.1 $$\zeta(-p) = \zeta(p) - p\langle\sigma_{V_0}\rangle_+\,\vartheta \tag{11.1}$$

for some ϑ: as mentioned the theoretical value of this slope ϑ seems currently inaccessible to theory (as the theory proposed in [BG67], [Ga97a] may need substantial modifications, *c.f.r.* comment following (10.10)). The $\langle\sigma_{V_0}\rangle$ and $\zeta(p)$ will be proportional to V_0: $\zeta(p) = V_0\,\overline{\zeta}(p)$ with a V_0–independent $\overline{\zeta}(p)$. Note that $\zeta(p)$ depends also on R.

The small volume element of the fluid will therefore be subject to rather frequent variations: in spite of $\zeta(p)$ being proportional to V_0, because now V_0 is not large. The consequent intermittency phenomena can therefore be observed. And as in §9 once the phase space contraction is intermittent all properties of the system show the same behavior.

And in fact intermittency in observations averaged ove a time span τ will appear with a time frequency of the form $e^{V_0\,(\overline{\zeta}(p)-\overline{\zeta}(1))\,\tau}$: the quantity p can be interpreted as a measure of the "*strength of intermittency*" observable in easurements averaged over a time τ because as noted in §9 and in [Ga99b] the size of p controls the statistical properties of "most" other observables. *Therefore the function $\overline{\zeta}(p)$ (hence $\zeta(p)$) might be directly measurable and it should be rather directly related to the quantities that one actually observes in intermittency experiments.* And the difference $\zeta(p)-\zeta(-p)$ can be tested for linearity in p as predicted by the analysis above.

Note that in an extended system the volume V is much larger than V_0 and we shall see "for sure" intermittency (for observables averaged over a time τ) of strength p in a region of volume V_0 *somewhere* within a volume W such that

11.2 $$\frac{W}{V_0}\,e^{V_0\,(\overline{\zeta}(p)-\overline{\zeta}(1))\tau} \simeq 1 \tag{11.2}$$

At this point it seems relevant to recall that it is rather heatedly being debated whether the name of "*entropy creation rate*" that some authors (including the present one) give to the phase space contraction rate is justified or not, see [An82]. The above properties not only propose the physical meaning of the quantity p and bring up the possibility of measuring its rate function $\zeta(p)$ in actual experiments but also provide a further justification of the name given to σ as "entropy creation rate" and fuel for the debate that *inevitably* the word entropy generates at each and every occurrence.

§12. Benard convection, intermittency and the Ciliberto–Laroche experiment.

A very interesting attempt at checking some of the above ideas has been made recently by Cilberto and Laroche in an experiment on real fluids which has been performed with the aim of testing the relation (11.1) locally in a small volume element, [CL98]. By "real" we mean here *non numerical*: a distinction that, however, has faded away together with the XX–th century but that some still cherish: the system is physically macroscopic (water in a container of a size of the order of a liter).

This being a real experiment one has to stretch quite a bit the very primitive theory developed so far in order to interpret it and one has to add to the chaotic hypothesis other assumptions that have been discussed in [BG97], [Ga97a] in order to obtain the fluctuation relation (10.12) and its local couterpart (11.1).

The experiment attempts at measuring a quantity that is eventually interpreted as the difference $\zeta(p)-\zeta(-p)$, by observing the fluctuations of the product ϑu^z where ϑ is the *deviation* of the temperature from the average temperature in a *small volume element* Δ of water at a fixed position in a Couette flow and u^z is the velocity in the z direction of the water in the same volume element.

The result of the experiment is in a way quite unexpected: it is found that the function $\zeta(p)$ is rather irregular and lacking symmetry around $p=1$: *nevertheless the function $\zeta(p)-\zeta(-p)$ seems to be strikingly linear.* As discussed in [Ga97a], predicting the slope of the entropy creation rate would be difficult but if the equivalence conjecture considered above and discussed more in detail in [Ga97a] is correct then we should expect linearity of $\zeta(p)-\zeta(-p)$.

In the experiment of [CL98] the quantity ϑu_z did not appear to be the divergence of the phase space volume simply because there was no model proposed for a theory of the experiment. Nevertheless Ciliberto–Laroche select the quantity $\int_\Delta \vartheta\, u^z\, d\underline{x}$ on the basis of considerations on entropy and dissipation so that there is hope that in a model of the flow this quantity can be related to the entropy creation rate discussed in §10,§11.

Here we propose that a model for the fluid, that can be reasonably used, is Rayleigh's model of convection, [Lo63], [LL71] and [Ga97b] sec. 5. An attempt for a theory of the experiment could be the following.

One supposes that the equations of motion of the system in the *whole container* (of linear size of the order of $30\,cm$) are written for the quantities t,x,z,ϑ, $\underline{u}$ in terms of the height H of the container (assumed to be a horizontal infinite layer), of the temperature difference between top and bottom δT and in terms of the phenomenological "*friction constants*" ν,χ of viscosity, dynamical thermal conductivity and of the thermodynamic dilatation coefficient α. We suppose that the fluid is 3–dimensional but stratified, so that velocity and temperature fields do not depend on the coordinate y, and gravity is directed along the z–axis: $\underline{g}=g\,\underline{e}$, $\underline{e}=(0,0,-1)$. The temperature deviation ϑ is defined as the difference betwen the temperature $T(x,y,z)$ and the temperature that the fluid would have at height z in absence of convection, *i.e.* $T_0-z\,\delta T/H$ if T_0 is the bottom temperature.

In such conditions the equations, including the boundary conditions (of fixed temperature at top and bottom and zero normal velocity at top and bottom), the convection equations in the Rayleigh model, see [Lo63] eq. (17), (18) where they are called the *Saltzman equations*, and [Ga97b] §1.5, become

12.1

$$\begin{aligned}
&\underline{\partial}\cdot\underline{u}=0, &&\int u_x d\underline{x}=\int u_y d\underline{x}=0\\
&\dot{\underline{u}}+\underset{\sim}{u}\cdot\underset{\sim}{\partial}\,\underline{u}=\nu\Delta\underline{u}-\alpha\vartheta\,\underline{g}-\underline{\partial}\,p', && \qquad (12.1)\\
&\dot\vartheta+\underset{\sim}{u}\cdot\underset{\sim}{\partial}\vartheta=\chi\Delta\vartheta+\frac{\delta T}{H}u_z \qquad \vartheta(0)=0=\vartheta(H),\quad u_z(0)=0=u_z(H), &&
\end{aligned}$$

The function p' is related to the pressure p: within the approximations it is $p = p_0 - \rho_0 g z + p'$. We shall impose for simplicity horizontal periodic boundary conditions in x, y so that the fluid can be considered in a finite container V of side a for some $a > 0$ prefixed (which in theoriginal variable would correspond to a container of horizntal size aH).

It is useful to define the following adimensional quantities

12.2
$$\begin{aligned} &\tau = t\nu H^{-2},\ \xi = xH^{-1},\ \eta = yH^{-1},\ \zeta = zH^{-1},\\ &\vartheta^0 = \frac{\alpha\vartheta}{\alpha\,\delta T},\ \underline{u}^{\,0} = (\sqrt{gH\alpha\,\delta T})^{-1}\ \underline{u}\\ &R^2 = \frac{gH^3\alpha\,\delta T}{\nu^2}, \qquad R_{Pr} = \frac{\nu}{\kappa} \end{aligned} \tag{12.2}$$

and one checks that the Rayleigh equations take the form

12.3
$$\begin{aligned} &\dot{\underline{u}} + R\underset{\sim}{\underline{u}}\cdot\underset{\sim}{\underline{\partial}}\ \underline{u} = \Delta\,\underline{u} - R\vartheta\,\underline{e} - \underline{\partial}\,p,\\ &\dot\vartheta + R\underset{\sim}{\underline{u}}\cdot\underset{\sim}{\underline{\partial}}\,\vartheta = R_{Pr}^{-1}\Delta\vartheta + Ru_z,\\ &\underline{\partial}\cdot\underline{u} = 0\\ &u_z(0) = u_z(1) = 0, \qquad \vartheta(0) = \vartheta(1) = 0,\\ &\int_V u_x d\,\underline{x} = \int_V u_y d\,\underline{x} = 0 \end{aligned} \tag{12.3}$$

where we again call $t, x, y, z, \underline{u}, \vartheta$ the adimensional coordinates $\tau, \xi, \eta, \zeta, \underline{u}^{\,0}, \vartheta^0$ in (12.2). The numbers R, R_{Pr} are respectively called the *Reynolds and Prandtl numbers* of the problem: $R_{Pr} =\sim 6.7$ for water while R is a parameter that we can adjust, to some extent, from 0 up to a rather large value.

According to the principle of equivalence stated in [Ga97a] here one could impose the constraints

12.4
$$\int_V \big(\,\underline{u}^{\,2} + \frac{1}{R_{Pr}}\vartheta^2\,\big)\,d\,\underline{x} = C \tag{12.4}$$

on the "*frictionless equations*" (*i.e.* the ones without the terms with the laplacians) and determine the necessary forces via Gauss' principle of minimal effort, see footnote [3] and [Ga96a], [Ga97a]. We use as *effort functional* of an acceleration field $\underline{a}$ and of a temperature variation field s the quantity

12.5
$$\begin{aligned} &\mathcal{E}(\,\underline{a}\,, s) \stackrel{def}{=} \big((\,\underline{a} + \underline{\partial}p - \underline{f}\,), (-\Delta)^{-1}(\,\underline{a} + \underline{\partial}\,p - \underline{f}\,)\big) +\\ &+ \big((s-\varphi), (-\Delta)^{-1}(s-\varphi)\big) \qquad\qquad \text{with}\\ &\underline{f} \stackrel{def}{=} -R\vartheta\,\underline{e}\,, \qquad \varphi \stackrel{def}{=} Ru_z \end{aligned} \tag{12.5}$$

and require it to be minimal over the variations $\underline{\delta}\,(\,\underline{x}\,)$ of $\underline{a} = \frac{d\,\underline{u}}{dt}$ and $\tau(\,\underline{x}\,)$ of $s = \frac{d\vartheta}{dt}$ with the constraints that for all $\underline{x}$ it is $\underline{\partial}\cdot\underline{\delta} = 0$, besides those due to the boundary conditions. The result is

12.6
$$\begin{aligned} &\underline{\partial}\cdot\underline{u} = 0\\ &\dot{\underline{u}} + R\,\underset{\sim}{\underline{u}}\cdot\underset{\sim}{\underline{\partial}}\ \underline{u} = R\vartheta\ \underline{e} - \underline{\partial}\,p' + \underline{\tau}_{\,th}\\ &\dot\vartheta + R\underset{\sim}{\underline{u}}\cdot\underset{\sim}{\underline{\partial}}\vartheta = Ru^z + \lambda_{th}\\ &\vartheta(0) = 0 = \vartheta(H), \quad \int_V u_x d\,\underline{x} = \int_V u_y d\,\underline{x} = 0 \end{aligned} \tag{12.6}$$

where the frictionless equations are modified by *the thermostats forces* $\underline{\tau}_{th},\lambda_{th}$: the latter impose the nonholonomic constraint in (12.4) with the effort functional defined by (12.5). Looking only at the bulk terms we see that the equations obtained by imposing the con1straints via Gauss' principle become the (12.3) with coefficients in front of the Laplace operators equal to $\nu_G, \nu_G R_{Pr}^{-1}$, respectively, with the "gaussian multiplier" ν_G being an *odd* functions of $\underline{u}$, see [Ga97a]: setting $\tilde C_V(\underline{u},\vartheta)=\int_V\big((\underline{\partial}\,\underline{u})^2+R_{Pr}^{-1}(\underline{\partial}\,\vartheta)^2\big)d\,\underline{x}$ one finds

12.7
$$\nu_G=\tilde C_V(\underline{u},\vartheta)^{-1}R(1+R_{Pr}^{-1})\int_V u^z\vartheta\;d\,\underline{x} \tag{12.7}$$

And the equations become, finally

12.8
$$\begin{aligned}
&\underline{\partial}\cdot\underline{u}=0\\
&\dot{\underline{u}}+R\,\underline{u}\cdot\underline{\partial}\,\underline{u}=R\vartheta\,\underline{e}-\underline{\partial}\,p'+\nu_G\,\Delta\,\underline{u}\\
&\dot\vartheta+R\underline{u}\cdot\underline{\partial}\vartheta=Ru^z+\nu_G\frac{1}{R_{Pr}}\Delta\vartheta\\
&\vartheta(0)=0=\vartheta(H),\qquad\int u_x d\,\underline{x}=\int u_y d\,\underline{x}=0
\end{aligned}\tag{12.8}$$

If one wants the equivalence between the ensembles of SRB distributions for the equation (12.8) and for (12.3) one has to tune, [Ga97a], the value of the constant C in (12.4) so that the time average value $\langle\nu_G\rangle_+$ of ν_G is precisely the physical one: namely $\langle\nu_G\rangle=1$ by (12.3). This is (again) the same, in spirit, as fixing the temperature in the canonical ensemble so that it agrees with the microcanonical temperature thus implying that the two ensembles give the same averages to the local observables.

The equations (12.8) are time reversible (unlike the (12.3)) under the time reversal map:

12.9
$$(\underline{u},\vartheta)=(-\underline{u},\vartheta) \tag{12.9}$$

and they should be supposed, by the arguments in [Ga97a] and §10,11: "equivalent" to the irreversible ones (12.3),

The (12.8) should therefore have a "divergence" $\sigma(\underline{u},\vartheta)$ whose fluctuation function $\zeta(p)$ verifies a linear fluctuation relation, *i.e.* $\zeta(p)-\zeta(-p)$ should be linear in p. Note that the divergence of the above equations is proportional to ν_G if one supposes that the high momenta modes with $|\underline{k}|>K(R)=R^\kappa$ with κ suitable can be set equal to 0 so that the equation (12.8) becomes a system of finite differential equations for the Fourier components of $\underline{u},\vartheta$.

For instance the Lorenz' equations, [Lo63] see also §17 of [Ga97b], reduced the number of Fourier components necessary to describe (12.3) to just three components, thus turning it into a system of three differential equations.

Proceeding in this way the divergence of the equations of motion can be computed as a sum of two integrals one of which proportional to ν_G in (12.7). If instead of integrating over the whole sample we integrate over a small region Δ, like in the experiment of [CL98], we can expect to see a fluctuation relation for the entropy creation rate if the fluctuation theorem *holds locally*, *i.e.* for the entropy creation in a small region.

As for the cases in §11 this is certainly not implied by the proof in [GC95]: however *when the dissipation is homogeneous through the system*, as it is the case in the Rayleigh model there is hope that the fluctuation relation holds locally because "a small subsystem should be equivalent to a large one". As noted in §9 the actual possibility of a local fluctuation theorem in systems with homogeneous dissipation has been shown in [Ga99c], after having been found through numerical simulations in [GP99], and this example was relevant because it gave us some justification to imagine that it might apply to the present situation as well.

The entropy creation is due to the term $R\int_\Delta u_z\vartheta\, d\underline{x}\,/\tilde C_\Delta(\,\underline{u}\,,\vartheta)$, where Δ is the region where the measurements of [CL98] are performed, hence we have a proposal for the explanation of the remarkable experimental result. Unfortunately in the experiment [CL98] the contributions not explicitly proportional to R to the entropy creation rates have not been measured nor has been the $\hat C$ in (12.7) which also fluctuates (or might fluctuate). In any event they might be measurable by improving the same apparatus, so that one can check whether the above attempt to an explanation of the experiment is correct, or try to find out more about the theory in case it is not right. If correct the above "theory" the experiment in [CL98] would be quite important for the status of the chaotic hypothesis.

§13. Conclusions.

The chaotic hypothesis promises a point of view on non equilibrium that has proved so far of some interest. Here we have exposed the basic ideas and attempted at drawing some consequences: admittedly the most interesting rely on rather phenomenological and heuristic grounds. They are summarized below.

(1) The definition of nonequilibrium ensembles with the proposal that out of equilibrium also the equation of motion should be considered as part of the definition of ensemble. This is take into account that while in equilibrium the system is uniquely defined by its microscopic forces and constituents *in non equilibrium it is not so.* Systems must be put in contact with thermostats if we want them to become stationary after a transient time. And (for large systems) there may be several equivalent ways of taking heat out of a system, *i.e.* several thermostats, without affecting the properties of stationary state that is eventually reached by the system itself.
(2) Equivalence of ensembles has the most striking aspect that systems which evolve with equations that are very different may exhibit the same statistical properties. In particular reversible evolutions might be equivalent to non reversible ones, thus making it possible to apply results that require reversibility, in particular the fluctuation relations, to cases in which it is not valid.
(3) An interpretation of the quantity p that intervenes in the fluctuation theorems in terms of an intermittency phenomenon and as a further quantitative measure of it.
(4) The possibility of applying the theory to strongly turbulent motions was the origin of the Ruelle's principle that evolved into the chaotic hypothesis: therefore not surprisingly the ideas can be applied to fluid dynamics. We have discussed a possible approach. The approach leads again to a proposal for the theory of certain intermittency phenomena which appear quantitatively related to entropy creation fluctuations.
(5) The possibility of measurement of the rate $\zeta(p)$ leads to a a possible prediction of the spatial frequency of internittent events of strength p or, as I prefer, with entropy creation rate p (see (4) above, (11.2) and §12). This seems testable in concrete experiments (both real and numerical).
(6) We have used the results in (2)%(8) to hint at an interpretation of the experiment by Ciliberto and Laroche on Benard convection in water.

Although the theory is still at its beginning and it migh turn out to be not really of interest it seems that at this moment it is worth trying to test it both in its safest, *c.f.r.* §2%§8, and in its most daring, *c.f.r.* §9%§12, predictions.

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