SUB-STABLE AND WEAK-STABLE MANIFOLDS ASSOCIATED WITH FINITELY NON-RESONANT SPECTRAL SUBSPACES

MOHAMED S. ELBIALY

ABSTRACT. In this work we study $C^{k,\delta}, 2 \leq k \leq \infty, 0 \leq \delta \leq 1$, maps of a Banach space near a fixed point. We show the existence and uniqueness of a class of $C^{k,\delta}$ local invariant sub-manifolds of the stable manifold which correspond to a spectral subspace satisfying a finite non-resonance condition of order $L \leq k$ and an overriding condition of order $L \leq k$ (condition (3) of Theorem 1). We study the dependence of these invariant manifolds on a parameter that lies in a Banach space. We also show that a $C^{k,\delta}$ local weak-stable manifold that satisfies these two conditions is unique in the class of $C^{k,\delta}$ maps. The uniqueness is due to the fact that our method does not require a cut-off function. An infinite dimensional Banach space does not always admit smooth cut-off functions.

1. Introduction

Invariant manifolds at a fixed point of a map have been studied extensively since the works of Perron and Hadamard. See the partial list of references we give and references therein. Let Φ be a map of a Banach space Z to itself with a fixed point, say 0. Let $\sigma(D\Phi(0))$ be the spectrum of $D\Phi(0)$. Assume that Z has a direct sum decomposition $Z = X \oplus Y$ which is invariant under $D\Phi(0)$. Let $\sigma_1 = \sigma(D\Phi(0) \mid X)$ and $\sigma_2 = \sigma(D\Phi(0) \mid Y)$. Thus $\sigma(D\Phi(0)) = \sigma_1 \cup \sigma_2$. An invariant manifold associated with σ_1 (respectively σ_2) is a sub-manifold through the fixed point 0, which is invariant under Φ and tangent to X (respectively Y) at the fixed point. The traditional invariant manifolds (stable, pseudo-stable, weak-stable, centre-stable, and strong-stable manifolds and their unstable cousins) correspond to an annular gap between σ_1 and σ_2. The smoothness of any invariant manifold from this traditional group depends on the so called gap condition. More precisely, assume that $|\sigma_1| < a < b < |\sigma_2|$ where $|\sigma_i| = \{|\lambda| \mid \lambda \in \sigma_i\}$. In this case the spectral decomposition theorem guarantees the existence of a decomposition $Z = X \oplus Y$ which is invariant under $D\Phi(0)$. The gap condition can be stated as follows: *If $a^r < b$, then there is a C^r invariant manifold associated with σ_1. On the other hand if $a < b^r$, then there is a C^r invariant manifold associated with σ_2.* In either case, if the map is $C^k, k > r$, one cannot guarantee that an invariant manifold will be C^k. The reason is that all proofs reduce the problem to a transformation on an appropriately constructed Banach space, which will be a contraction if $a^r < b$ ($a < b^r$), a condition that does not imply that $a^{r+1} < b$ ($a < b^{r+1}$). It is obvious that the gap condition is trivially satisfied for the stable, unstable, centre-stable, centre-unstable, strong-stable, and strong-unstable manifolds, but not for the rest of the group mentioned above.

Key words and phrases. Invariant manifolds, weak-stable manifold, linearization, resonance.

In this work we study a new class of invariant manifolds, which was introduced previously only in [16]. Our theorems are proved for infinite dimensional Banach spaces. We start with an example in a finite dimensional space which captures most of the characteristics of the new object we are studying. We call this object a *sub-stable manifold* since it is a sub-manifold of the stable manifold. A certain type of *local weak-stable manifolds* belong to this new class of invariant manifolds. A *local weak-stable manifold* which is guaranteed by our theorem is unique. Now for the example: Suppose the eigenvalues of $D\Phi(0)$ are

$$0 = \alpha_1 < \beta_1 = \beta_2 < \alpha_2 < \alpha_3 = \beta_3 < \alpha_4 = \alpha_5 < \beta_4 < \beta_5 = 1 < \beta_6$$

Let the α-generalized eigenvectors be $< u_1, \cdots, u_5 >$ and the β-ones be $< v_1, \cdots, v_6 >$ with matching subscripts. We assume that the eigenvalue $\alpha_3 = \beta_3$ is a multiple eigenvalue with two linearly independent eigenvectors u_3 and v_3. Notice also that $|\alpha_i| < 1, i = 1, \cdots, 5$. Let $Z = \mathbb{R}^{11} = X \oplus Y$ where $X = span\{u_1, \cdots, u_5\}$ and $Y = span\{v_1, \cdots, v_6\}$. Write $z = (x, y), D\Phi(0) = blockdiag(A, B)$, and $\Phi = (A + f, B + g)$.

In terms of this example our two main Theorems 1 and 2 can be stated as follows. Assume the following:

1. There is a constant $0 < a < 1$ such that $0 \leq |\alpha| < a < 1, \alpha \in \sigma(A)$.
2. There is a constant $0 < b$ such that $|\beta| > b > 0, \beta \in \sigma(B)$.
3. $a^L < b$ for some $L \geq 2$.
4. $\beta_j \neq \alpha_1^{m_1} \cdots \alpha_5^{m_5}, 2 \leq m_1 + \cdots + m_5 \leq L, j = 1, \cdots, 6$.
5. The map Φ is $C^{k,\delta}, k = L, L+1, L+2, \cdots, \infty, 0 \leq \delta \leq 1$.

Then the following are true:

a: In a sufficiently small neighbourhood $\mathcal{U}$ of the fixed point $(0, 0)$, there is an invariant manifold of the form $y = w(x), w \in C^{k,\delta}(\mathcal{U}, Y)$, whose graph is tangent to $X \oplus \{0\}$ at the fixed point. Moreover, $w(x)$ is unique in $C^L(\mathcal{U}, Y)$.

b: If Φ_λ varies continuously with a parameter λ in the $C^{k,\delta}$ topology, then w_λ varies continuously with λ in the $C^{k,\gamma}$ topology for all $0 \leq \gamma < \delta \leq 1$. If $\delta = 0$, then w varies continuously with λ in the $C^{k-1,\gamma}$ topology for all $0 \leq \gamma < 1$. The parameter λ is allowed to be in a Banach space.

A family $\{h_\lambda\}$ varies continuously with λ in the $C^{k,\beta}$ topology iff $\|h_\lambda - h_\mu\|_{C^{k,\beta}} \to 0$ as $\lambda \to \mu$.

Remarks 1.1. 1. If we apply our theorem to the question of the existence of a *local weak-stable manifold*, in both finite and infinite dimensional spaces, we obtain the following for a local weak-stable manifold $\mathcal{W}$ which satisfies the hypotheses of our theorem:

- $\mathcal{W}$ is unique in the class of C^L functions. Thus, any other *local weak-stable manifold* will be less smooth than C^L.
- If $\mathcal{W}$ is C^L, then it is $C^{k,\delta}$ for all $L \leq k \leq \infty$ as long as the map is $C^{k,\delta}$.

In [16], one can find examples with several invariant manifolds with less smoothness.

A *local weak-stable manifold* obtained by the gap-condition method is not unique because one has to use a cut-off function. For each cut-off function one obtains a different local weak-stable manifold. Our method does not require a cut-off function.

The situation with the gap-condition method is worse in infinite dimensional Banach spaces because such a space does not necessarily admit a C^1 cut-off function. In [15], it is shown that the space of continuous functions on an interval does not admit a C^1 cut-off function. In [19], it is shown that a separable Banach space admits a Fréchet differentiable cut-off function iff its dual space is separable. We will comment on the usage of cut-off functions for any invariant manifold in more details in Section 3.3.

2. In general, the spectrum of $D\Phi(0)$ does not have to consist of only real numbers as in the example.
3. Hypothesis (3) is not a gap condition. In fact the two parts of the spectrum do not have to be disjoint. In this case, if Z is infinite dimensional, the spectral decomposition theorem does not guarantee a decomposition of the Banach space Z. In Theorem 1, it will be assumed that such a decomposition exists. We use adapted norms on X and Y in which $\|A\| < a$ and $\|B\| > b$. Thus $\|A\|^L < \|B\|$. When hypothesis (3) is satisfied we will say that A *overrides* B *in* L *steps* or A L*-overrides* B.
4. Hypothesis (4) is a finite nonresonance condition of order L similar to the one we need in studying finitely smooth linearization near a fixed point. In terms of the proof, our current problem is closer to the linearization problem than to the invariant-manifold problem. The nonresonance condition (4) allows us to eliminate the j-monomials in x from $g(x, y)$ for $j = 2, 3, \cdots, L$.
5. However, the degree of smoothness of our invariant manifold is not bounded above by L in the non-resonance condition (L in (4)) as it is in linearization theorems. This is because the L-overriding condition (3) allows us to eliminate the j-terms of degree $j = L + 1, \cdots, k$. In this way we can show that if the invariant manifold is $C^{L,\delta}$, then it is $C^{k,\delta}$ for any $L \le k \le \infty$ as long as the map is $C^{k,\delta}$. In other words, it is enough to verify the assumptions of the theorem up to order L.
6. When we consider a family of maps $\{\Phi_\lambda\}$, we only require that the map $\lambda \to \Phi_\lambda$ is continuous, when the family is equipped with the $C^{k,\delta}$ topology.
7. Article [16] is the only place known to the author where this class of invariant manifolds was introduced. However, in [16] there is a loss of differentiability when $\delta = 0$; that is, if the map is C^k then the invariant manifold obtained in [16] is only $C^{k-1,1}$. In this work we show how to overcome this problem. There is no loss of differentiability in the invariant manifolds that we obtain.
8. We also note that in [16] $L < k$ while in our theorem $L \le k$.
9. The only places in the literature, known to the author, where the possible loss of differentiability of an invariant manifold when the map is $C^k, k \in \mathbb{N}$ is studied are [2], [9], and [18].
10. In [16] it is shown how the class of invariant manifolds studied in this work and in [16] is relevant to some finite dimensional applications of renormalization groups. Also in [16] the reader will find counter examples when the nonresonance condition fails as well as some examples of the lack of smoothness of invariant circles with periodic orbits which appear after a Hopf bifurcation. Moreover, [16] contains examples that admits several invariant manifolds but these additional invariant manifolds are less smooth than $C^{L,\delta}$.

Acknowledgment. The author would like to thank Rafael de la Llave for suggesting this problem and for fruitful conversations. Most of this work was done while the author was on sabbatical at The Department of Mathematics, The University of Texas at Austin. The author would like to thank its members for their great hospitality.

2. Main Results

2.1. General setup.

1. Let Z be a Banach space which has a direct sum decomposition $Z = X \oplus Y$ where X and Y are closed subspaces of Z and the projections $\pi^X : Z \to X$ and $\pi^Y : Z \to Y$ are bounded.
2. Accordingly, we write $z = (x, y), z \in Z, x \in X, y \in Y$. We use the box norm $|z| = \max\{|x|, |y|\}$.
3. Let $\mathcal{L}(E, F)$ be the Banach space of linear maps from a Banach space E to a Banach space F.
4. Let $\Phi : Z \to Z$ be a $C^{0,1}$ map which takes the form $\Phi(x, y) = (x^*, y^*)$ where

$$\begin{aligned} x^* &= Ax + f(x, y), \\ y^* &= By + g(x, y), \end{aligned} \tag{2.1}$$

where $A \in \mathcal{L}(X, X), B \in \mathcal{L}(Y, Y), F = (f, g)$ and $F(x, y) = o(|x| + |y|)$.

Definition 2.2. 1. Let $\sigma(A)$ and $\sigma(B)$ be the spectra of A and B respectively. We say that A is L-nonresonant relative to B and denote this by $NR(A^L, B)$ iff

$$\beta \neq \alpha_1\alpha_2 \cdots \alpha_j, \qquad \beta \in \sigma(B), \alpha_i \in \sigma(A), \qquad 1 \leq i \leq j, \qquad 2 \leq j \leq L. \tag{2.2}$$

In other words,

$$\sigma(A)^j \cap \sigma(B) = \emptyset, \qquad 2 \leq j \leq L,$$

where

$$\sigma(A)^j = \{\alpha_1\alpha_2 \cdots \alpha_j \mid \alpha_i \in \sigma(A), 1 \leq i \leq j\}.$$

2. If A and B satisfy $NR(A^j, B)$ for all $j \geq 2$ we say that A is infinitely nonresonant relative to B.
3. If there is $0 < a < 1$ and $b > 0$ such that $|\alpha| < a, \alpha \in \sigma(A), |\beta| > b, \beta \in \sigma(B)$, and $a^L < b$, we say that A overrides B in L steps or A L-overrides B. Notice that in this case we can use adapted norms so that $\|A\| < a$ and $\|B\| > b$ and hence $\|A\|^L < \|B\|$.
4. For $\gamma > 0$, let $B_\gamma(0) \subset E$ be the closed ball of radius γ centred at $0 \in X$ in a banach space E.

A $C^{k,\delta}$ function $h : U \to Y$ is a C^k function whose k^{th} derivative is uniformly δ-Hölder continuous. That is

$$H_\delta(D^k h) = \sup\left\{\frac{\|D^k h(x) - D^k h(x')\|}{|x - x'|^\delta} \mid x \neq x', x, x' \in U\right\} < \infty.$$

In this case, the $C^{k,\delta}$ norm is given by

$$\|h\|_{C^{k,\delta}} = \max\{\|h\|, \|Dh\|, \cdots, \|D^k h\|, H_\delta(h)\}$$

Notice that when $\delta = 1, H_1 = Lip$. Now we state our three main results.

Theorem 1 (Existence and uniqueness)**.** *Consider the map Φ given by(2.1) in the setup of 2.1. Let $2 \le L \le k$. Assume the following in a neighbourhood of the origin:*

1. *There is $0 < a < 1$ such that $|\alpha| < a$ for all $\alpha \in \sigma(A)$.*
2. *There is $b > 0$ such that $|\beta| > b$ for all $\beta \in \sigma(B)$.*
3. *$a^L < b$.*
4. *A is L-nonresonant relative to B. That is A and B satisfy the condition $NR(A^L, B)$.*
5. *$F(x, y)$ is $C^{k,\delta}$, $L \le k \le \infty$, $0 \le \delta \le 1$.*

Let $s_o = \cdots = s_{L-1} = 1$. When $k = L$, let $s_L = \delta$. When $k > L$, let $s_L = 1$.

Then the following are true:

1. *For sufficiently small $\gamma > 0$, the map (2.1) has an invariant manifold of the form $y = w(x), w \in C^{k,\delta}(B_\gamma(0), Y)$, where $B_\gamma(0) \subset X$ is the closed ball of radius γ centred at $0 \in X$.*
2. *$w = w^{(L)} + p_2 + \cdots + p_L$, where p_j is unique in the Banach space $\mathcal{S}(X^j, Y)$ of symmetric j-multilinear functions from X^j to Y, $2 \le j \le L$. Since $w(0) = 0$ and $Dw(0) = 0$, the tangent space of the graph of w at $(0, 0)$ is $X \oplus \{0\}$.*
3. *Let $0 \le \beta \le s_L$. Then $w^{(L)}$ is unique in the class of functions $v : B_\gamma(0) \to Y$ which satisfy the following:*

$$\sup\left\{\frac{|v(x)|}{|x|^{L+\beta}} \mid 0 < |x| \le \gamma\right\} \le \frac{1}{\gamma^{L+\beta-1}}, \tag{2.3}$$

$$\sup\left\{\frac{|v(x) - v(x')|}{|x - x'|(|x| + |x'|)^{L+\beta-1}} \mid x \ne x', |x| \le \gamma, |x'| \le \gamma\right\} \le \frac{1}{\gamma^{L+\beta-1}}. \tag{2.4}$$

   Notice that a function that satisfies (2.4) is uniformly Lipschitz continuous.

4. *Moreover, $w^{(L)}$ satisfies the following for $0 \le j \le L$ and $|x| \le \gamma$:*

$$\|D^j w^{(L)}(x)\| \le \frac{1}{\gamma^{L+\beta-1}}|x|^{L+\beta-j}, \tag{2.5}$$

$$\|D^j w^{(L)}(x) - D^j w^{(L)}(x')\| \le \frac{1}{\gamma^{L+\beta-1}}|x - x'|^{s_j}(|x| + |x'|)^{L+\beta-j-s_j}. \tag{2.6}$$

   When $s_L = 0$ (i.e. when $L = k$ and $\delta = 0$) omit (2.6) for $j = L$.

5. *$w^{(L)}$ is unique in the class of $C^{L,\beta}(B_\gamma(0), Y)$ functions which satisfy the following:*

$$\sup\left\{\frac{|v(x)|}{|x|^{L+\beta}} \mid 0 < |x| \le \gamma\right\} < \infty, \tag{2.7}$$

$$\sup\left\{\frac{|v(x) - v(x')|}{|x - x'|(|x| + |x'|)^{L+\beta-1}} \mid x \ne x', |x| \le \gamma, |x'| \le \gamma\right\} < \infty. \tag{2.8}$$

6. *The invariant manifold $y = w(x)$ is unique in $C^{L,\beta}(B_\gamma(0), Y)$. It follows that the smoothness of any other invariant manifold is less than $C^{L,\beta}$. Moreover, if $w \in C^{L,\beta}(B_\gamma(0), Y)$, then $w \in C^{k,\delta}(B_\gamma(0), Y), L \le k \le \infty$ as long as the map is $C^{k,\delta}$.*

Remarks 2.3. 1. Notice that we do not require that $\sigma(A) \cap \sigma(B) = \emptyset$. All we require is the existence of a direct-sum decomposition. This might occur in a finite dimensional space if a multiple eigenvalue has, say, two linearly independent eigenvectors one of them lying in X and the other in Y.

2. Notice also that A is not expected to be invertible.
3. The first step in our proof is to take advantage of hypotheses (3) and (4) of Theorem 1 and eliminate certain nonresonant terms of order 2 to k. In fact it is essential to do so since we only assume that $a^L < b$. It is possible that $a \not< b$. If $a < b$ we have the previously known strong-stable manifold. We will accomplish this elimination in Lemma 2.5, Lemma 2.7 and Lemma 2.8.

Theorem 2 (Dependence on parameters). *Let Λ be an open subset of a Banach space. Let $\{\Phi_\lambda : \lambda \in \Lambda\}$ be a family of maps each of which satisfies the hypotheses of Theorem 1 with constants $(a_\lambda, b_\lambda, L, k, \delta)$. Assume that $\{\Phi_\lambda : \lambda \in \Lambda\}$ depends continuously on λ in the $C^{k,\delta}$ topology. That is, for all $\mu, \lambda \in \Lambda$*

$$\|\Phi_\lambda - \Phi_\mu\|_{C^{k,\delta}} \to 0, \qquad \text{as } \lambda \to \mu.$$

Let $y = w_\lambda(x), w_\lambda \in C^{k,\delta}(B_{\gamma_\lambda}(0), Y)$, be the unique invariant manifold of Φ_λ guaranteed by Theorem 1. Let $\mu \in \Lambda$ be fixed but arbitrary. Let $\mathcal{U} = B_c^o(\mu) \subset \Lambda$ be the open ball of radius c centered at μ. Then the following are true for sufficiently small $c > 0$:

1. *We may assume that $a_\lambda = a := a_\mu$ and $b_\lambda = b := b_\mu$ for $\lambda \in \mathcal{U}$. Using adapted norms, we may assume that $\|A_\lambda\| < a$ and $\|B_\lambda^{-1}\| < b^{-1}$, for $\lambda \in \mathcal{U}$.*
2. *The neighbourhood $B_{\gamma_\lambda}(0)$ can be chosen uniformly in $\lambda \in \mathcal{U}$. That is, we can assume that $\gamma_\lambda = \gamma$ for $\lambda \in \mathcal{U}$.*
3. *For sufficiently small $\gamma > 0, w_\lambda$ depends continuously on λ in the $C^{k,\beta}(B_\gamma(0), Y)$ topology, for all $0 \le \beta < \delta$. Let $\alpha = (k+\beta)/(k+\delta)$. Then for all $k \ge 2, 2/3 \le \alpha < 1$, amd*
$$\|D^k w_\lambda - D^k w_\mu\| = O(\|\Phi_\lambda - \Phi_\mu\|^{\beta_k(1-\alpha)}),$$
$$\beta_k = \frac{\beta}{(k+\beta)(k!)}.$$
4. *If $\delta = 0$, w_λ depends continuously on λ as above but in the $C^{k-1,\beta}(B_\gamma(0), Y)$ for all $0 \le \beta < 1$.*

Proof of assertion 1. Assertion 1 follows from the fact that $\sigma(D\Phi_\lambda(0))$ depends continuously on $D\Phi_\lambda(0)$ which, by assumption, depends continuously on λ. □

Remark 2.4. Notice that in theorem 2 the map $\lambda \mapsto \Phi_\lambda$ is required to be only continuous in the $C^{k,\delta}$ topology.

If we apply our theorems to *local weak-stable manifolds* we obtain some new results. The following theorem follows immediately from Theorem 1 and Theorem 2.

Theorem 3 (Local weak-stable manifolds). *A $C^{k,\delta}$ local weak-stable manifold obtained from Theorem 1 is unique in $C^{L,\beta}(B_\gamma(0), Y)$. Moreover, the smoothness of any other weak-stable manifold is necessarily less than $C^{L,\beta}$. More precisely, consider the map (2.1). In addition to the assumptions of Theorem 1, assume that there are two additional numbers $0 < b_1 < a_1 < 1$ such that*

1. *$0 < b < b_1 < a_1 < a$.*
2. *$|\beta| < b_1, \beta \in \sigma(B)$.*
3. *$|\alpha| > a_1, \alpha \in \sigma(A)$.*

Then,

1. *The $C^{k,\delta}$ invariant manifold guaranteed by Theorem 1 is a local weak-stable manifold $\mathcal{W}$.*
2. *$\mathcal{W}$ is unique in $C^{L,\beta}(B_\gamma(0), Y)$ for $0 \leq \beta \leq s_L$.*
3. *Moreover, if $\mathcal{W}$ is $C^{L,\beta}$, then it is $C^{k,\delta}$ for all $L \leq k \leq \infty$ as long as the map is $C^{k,\delta}$.*
4. *The smoothness of any other local weak-stable manifold is less than $C^{L,\beta}$.*
5. *If we are dealing with a family of maps $\{\Phi_\lambda\}$, the dependence of $\mathcal{W}_\lambda$ on λ is as in Theorem 1.*

In sections 2.5-2.10 we reduce Theorem 1 to Theorem 4 below by eliminating all nonresonant symmetric j-multilinear terms in $x, 2 \leq j \leq k$, from $g(x,y)$ and by creating a smallness assumptions on $F(x,y)$ via a linear transformation $\Phi \to \gamma^{-1}\Phi \circ (\gamma id_X)$.

Lemma 2.5. *The linear operator A is infinitely nonresonant relative to B, i.e. (A,B) satisfy $NR(A^j, B)$ for $j = 2, 3, \cdots$.*

Proof. By assumption 4, A and B satisfy $NR(A^L, B)$. For $j > L$, let $\beta \in \sigma(B)$, and $\alpha_i \in \sigma(A), 1 \leq i \leq j$. Since $a < 1$ and $a^j < a^L < b$, it follows that

$$|\beta| > b > a^L > a^j \geq |\alpha_1| \cdots |\alpha_j|$$

Thus $\beta \neq \alpha_1 \cdots \alpha_j$ for $j \geq L$. $\square$

Definition 2.6. For $j \geq 2$, let $\mathcal{L}(X^j, Y)$ be the Banach space of j-multilinear functions from X^j to Y, and let $\mathcal{S}(X^j, Y)$ be the Banach space of symmetric j-multilinear functions from X^j to Y. Although X and Y are infinite dimensional Banach spaces, we will continue to call elements in $\mathcal{S}(X^j, Y)$ j-monomials or monomials of degree j.

Lemma 2.7. *Let $A \in \mathcal{L}(X, X)$ and $B \in \mathcal{L}(Y, Y)$. For $j \geq 2$, define the two linear operators*

$$\begin{aligned} K_{A,B} &: \mathcal{S}(X^j, Y) \to \mathcal{S}(X^j, Y), & K_{A,B}p_j &= B^{-1}p_j \circ A\,, \\ L_{A,B} &: \mathcal{S}(X^j, Y) \to \mathcal{S}(X^j, Y)\,, & L_{A,B} &= I - K_{A,B}\,. \end{aligned}$$

Then for each fixed but arbitrary $j \geq 2$, the following are true:

1. *The spectrum of $K_{A,B}$ satisfies*

$$\sigma(K_{A,B}) \subseteq \{\beta^{-1}\alpha_1 \cdots \alpha_j \mid \beta \in \sigma(B), \alpha_i \in \sigma(A), 1 \leq i \leq j\}. \tag{2.9}$$

2. *Assume also that (A,B) satisfy $NR(A^j, B)$. Then $1 \notin \sigma(K_{A,B})$.*
3. *Thus, $L_{A,B}$ is a bijection.*
4. *$L_{A,B}^{-1}$ is continuous (equivalently bounded).*
5. *Consider a family $\{(A_\lambda, B_\lambda) \mid \lambda \in \Lambda\}$ satisfying the assumptions of Theorem 2. Let $\mu \in \Lambda$ be fixed but arbitrary. To simplify notation let $L_\lambda = L_{A_\lambda, B_\lambda}, K_\lambda = K_{A_\lambda, B_\lambda}, \Delta L = L_\lambda - L_\mu$, and $\Delta B = B_\lambda - B_\mu, \ldots$. Then for λ sufficiently close to μ*

$$B_\lambda^{-1} = [I + B_\mu^{-1}\Delta B]^{-1} B_\mu^{-1}, \tag{2.10}$$

$$L_\lambda^{-1} = [I + L_\mu^{-1}\Delta L]^{-1} L_\mu^{-1} = [I + L_\mu^{-1}\Delta K]^{-1} L_\mu^{-1}. \tag{2.11}$$

Thus B_λ^{-1} and L_λ^{-1} are continuous in λ. In fact,

$$\|L_\lambda^{-1}\| \le \frac{\|L_\mu^{-1}\|}{1-\|L_\mu^{-1}\|\|\Delta L\|}, \tag{2.12}$$

$$\|B_\lambda^{-1}\| \le \frac{\|B_\mu^{-1}\|}{1-\|B_\mu^{-1}\|\|\Delta B\|}.$$

Thus L_λ^{-1} and B_λ^{-1} are uniformly bounded for λ sufficiently close to μ. (We know from assertion (1) of Theorem 2 that $\|B_\lambda^{-1}\| < b^{-1}, \lambda \in B_c^o(\mu)$ when $c > 0$ is sufficiently small).

Proof. Assertion (1) is shown in [17]. In Appendix 9 we present the proof in our notation for completeness.

Assertion (2) follows from (2.9) and the nonresonance condition. Assertion (3) follows from assertion (2) and the concept of a spectrum. Assertion (4) follows from the Inverse Mapping Theorem (which is an immediate corollary to the Open Mapping Theorem in the present case).

To prove assertion (5) we write

$$B_\lambda = B_\mu[I + B_\mu^{-1}\Delta B], \tag{2.13}$$

$$L_\lambda = L_\mu[I + L_\mu^{-1}\Delta L] = L_\mu[I - L_\mu^{-1}\Delta K]. \tag{2.14}$$

By the assumptions of Theorem 2, B_λ is continuous in λ and $\|B_\mu^{-1}\| < b^{-1}$. Thus we obtain (2.10) for λ sufficiently close to μ. It follows from (2.10) that B_λ^{-1} is continuous in λ.

Moreover, K_λ, and hence L_λ, are continuous in λ since B_λ^{-1} is continuous in λ and since for all $p \in \mathcal{S}(X^j, Y)$

$$\|p \circ A_\lambda - p \circ A_\mu\| \le Lip(p)\|\Delta A\| \le \|p\|_{C^{0,1}}\|\Delta A\|.$$

Also notice that L_μ^{-1} is bounded by assertion (4) above. Thus, for λ sufficiently close to μ we can invert (2.14) and obtain (2.11). It follows also that L_λ^{-1} is continuous in λ. □

Now we are in position to eliminate all j-monomials in x of degree $2 \le j \le k$ from $g(x, y)$.

Lemma 2.8. *The following are true under the assumptions of Theorem 1:*

1. *There is a change of variables of the form*
$$(x, y) \mapsto (x, \hat{y}), \qquad \hat{y} = y - p(x),$$
which eliminates all j-monomials in $x, 2 \le j \le k$, from $g(x, y)$.
2. *Moreover, $p(x)$ is a sum*
$$p(x) = p_2(x) + \cdots + p_k(x),$$
where each $p_j(x)$ is unique in $\mathcal{S}(X^j, Y)$. Thus $p(x)$ is unique in $\bigcup_{j=2}^k \mathcal{S}(X^j, Y)$.
3. *The new map takes the form $(x, \hat{y}) \mapsto (x^*, \hat{y}^*)$, where*
$$x^* = Ax + f(x, \hat{y} + p(x)), \tag{2.15}$$
$$\hat{y}^* = B\hat{y} + h_{k+1}(x, \hat{y}).$$

and

$$|h_{k+1}(x,\hat{y})| = O(|x|^{k+\delta}) + O(|\hat{y}|[|x|+|\hat{y}|]). \tag{2.16}$$

If $\delta = 0$, replace $O(|x|^{k+\delta})$ by $o(|x|^k)$.

4. *Let $v \in C^{k,\delta}(B_\gamma(0), Y)$ and write*

$$v(x) = v_1(x) + v_2(x) + \cdots + v_k(x) + v^{(k)}(x),$$

where $v_1 \in \mathcal{L}(X,Y)$, $v_j \in \mathcal{S}(X^j,Y)$ for $2 \le j \le k$, $|v^{(k)}(x)| = O(|x|^{k+\delta})$ when $0 < \delta \le 1$, and $|v^{(k)}(x)| = o(|x|^k)$ when $\delta = 0$.

Then the manifold $y = v(x), |x| \le \gamma$, is invariant under the original map of Theorem 1 iff $v_1(x) \equiv 0$, $v_j = p_j, 2 \le j \le k$, i.e.

$$v = v^{(k)} + p_2 + \cdots + p_k. \tag{2.17}$$

and the manifold $\hat{y} = v^{(k)}(x), |x| \le \gamma$ is invariant under the new map (2.15). In this case, the tangent space to the graph of v at the origin is $X \oplus \{0\}$.

5. *Consider the family of maps in Theorem 2. Let $\mu \in \Lambda$ be fixed but arbitrary. Then, $\|p_{\lambda,j} - p_{\mu,j}\| \to 0$, and, since $p_{\lambda,j}$ and $p_{\mu,j}$ are monomials, $\|p_{\lambda,j} - p_{\mu,j}\|_{C^{k,\delta}} \to 0$, as $\lambda \to \mu$.*

Proof. Assume that we have already eliminated symmetric i-monomials in x for $2 \le i \le j-1$. In order not to complicate notation we will continue to denote the new nonlinear y-component by $g(x,y)$.

Write the new $g(x,y)$ as $g(x,y) = g_j(x) + \bar{h}_{j+1}(x,y)$, where $g_j(x) \in \mathcal{S}(X^j,Y)$ and

$$|\bar{h}_{j+1}(x,y)| = O(|x|^{j+s}) + O(|y|[|x|+|y|]),$$

where $s = 1$ for $2 \le j \le k-1$ and $s = \delta$ for $j = k$. If $\delta = 0$, replace $O(|x|^{k+\delta})$ by $o(|x|^k)$.

To eliminate $g_j(x)$ we need to find $p_j \in \mathcal{S}(X^j,Y)$ such that

$$p_j(x) = B^{-1}p_j(Ax) - B^{-1}g_j(x), \tag{2.18}$$

In other words we need to solve the homological equation

$$L_{A,B}p_j = (I - K_{A,B})p_j = -B^{-1}g_j \tag{2.19}$$

By (3) and (4) of Lemma 2.7,

$$L_{A,B} : \mathcal{S}(X^j,Y) \to \mathcal{S}(X^j,Y)$$

is a bijection and $L_{A,B}^{-1}$ is continuous (equivalently bounded). It follows that we can solve (2.19) uniquely and find the desired

$$p_j = -L_{A,B}^{-1}B^{-1}g_j.$$

Let $\bar{y} = y - p_j(x)$. Since $f(0,0) = 0$, it follows that for some $0 \le c \le 1$

$$\begin{aligned}|p_j(x^*) - p_j(Ax)| &\le \|Dp_j(Ax + cf(x,y))\|\|f(x,y)| \\ &\le O([|x|^{j-1} + |y|][|x|^2 + |y|^2]) \le O(|x|^{j+1} + |y|[|x|+|y|]).\end{aligned}$$

Thus, after eliminating $g_j(x)$ the map takes the following form in the $(x, \bar{y})$ variables:

$$
\begin{aligned}
x^* &= Ax + f(x, \bar{y} + p_j(x)), \\
\bar{y}^* &= y^* - p_j(x^*) = B\bar{y} + h_{j+1}(x, \bar{y}), \\
h_{j+1}(x, \bar{y}) &= \bar{h}_{j+1}(x, \bar{y} + p_j(x)) - p_j(x^*) + p_j(Ax).
\end{aligned} \tag{2.20}
$$

But $|p_j(x)| = O(|x|^j)$. Thus,

$$|h_{j+1}(x, \bar{y})| = |\bar{h}_{j+1}(x, \bar{y} + p_j(x))| = O(|x|^{j+s}) + O(|\bar{y}|(|x| + |\bar{y}|)).$$

When $j = k$ and $\delta = 0$, replace $O(|x|^{k+s})$ by $o(|x|^k)$.

Applying this procedure successively for $j = 2, 3, \cdots, k$, we obtain the desired $p(x) = p_2(x) + \cdots + p_k(x)$. Since each $p_j(x)$ is unique in $\mathcal{S}(X^j, Y)$, it follows that $p(x)$ is unique in $\bigcup_{j=2}^k \mathcal{S}(X^j, Y)$. This proves assertions (2) and (3).

Assertion (4) follows from the same calculations that led to (2.18).

To prove assertion (5) notice that

$$
\begin{aligned}
p_{\lambda,j} =& - L_\lambda^{-1} B_\lambda^{-1} g_{\lambda,j}, \\
\|p_{\lambda,j} - p_{\mu,j}\| \leq& \, b^{-1} \|g_{\lambda,j}\| \, \|L_\lambda^{-1} - L_\mu^{-1}\| \\
&+ \|L_\mu^{-1}\| \, \|g_{\lambda,j}\| \, \|B_\lambda^{-1} - B_\mu^{-1}\| \\
&+ b^{-1} \|L_\mu^{-1}\| \, \|g_{\lambda,j} - g_{\mu,j}\|.
\end{aligned} \tag{2.21}
$$

By Lemma 2.7, L_λ^{-1} and B_λ^{-1} are continuous near $\lambda = \mu$ and the family $\{L_\lambda^{-1}\}$ is uniformly bounded near $\lambda = \mu$. It follows that $\|p_{\lambda,j} - p_{\mu,j}\| \to 0$ as $\lambda \to \mu$. □

The following lemma is obvious.

Lemma 2.9. *Consider a C^1 map $\Psi(z) = D\Psi(0)z + G(z), G(0) = 0, DG(0) = 0$. For $0 < \gamma < 1$ define*

$$\Psi^{(\gamma)}(z) := \frac{1}{\gamma}\Psi(\gamma z) = D\Psi(0)z + G^{(\gamma)}(z) = D\Psi(0)z + \frac{1}{\gamma}G(\gamma z), \quad |z| \leq 1. \tag{2.22}$$

Then the graph of

$$y = w_{(\gamma)}(x), \qquad |x| \leq 1,$$

is invariant under $\Psi^{(\gamma)}$ iff the graph of

$$w(x) := \gamma w_{(\gamma)}(x/\gamma), \qquad |x| \leq \gamma, \tag{2.23}$$

is invariant under Ψ.

Notice that the subscript in $w_{(\gamma)}$ is in accordance with the superscript in $\Psi^{(\gamma)}$ and not with the neighbourhood $|x| \leq 1$ in which $w_{(\gamma)}$ is defined.

Corollary 2.10. *Let $k \geq 1, r = k + \delta$ and $0 \leq \delta \leq 1$. In view of Lemma 2.8 and Lemma 2.9, for $0 < \gamma < 1$, $F^{(\gamma)}(x, y)$ satisfies the following estimates for $(x, y) \in B_1(0, 0)$:*
For $0 \leq j \leq k$,

$$\|D^j F^{(\gamma)}\| \leq O(\gamma), \tag{1}$$

(2) $$H_\delta(D^k F^{(\gamma)}) \le O(\gamma),$$

(3) $$|g^{(\gamma)}(x,y)| \le O(\gamma)(|x|^r + |y|),$$

(4) $$\|D_x^j g^{(\gamma)}(x,y)\| \le O(\gamma)(|x|^{r-j} + |y|),$$

For $0 \le j \le k-1$,

(5) $$\|D_x^j g^{(\gamma)}(x,y) - D_x^j g^{(\gamma)}(x',y)\| \le O(\gamma)|x-x'|[(|x|+|x'|)^{r-j-1} + |y|],$$

(6) $$\|D_x^j g^{(\gamma)}(x,y) - D_x^j g^{(\gamma)}(x,y')\| \le O(\gamma)|y-y'|,$$

(7) $$\|D_x^j g^{(\gamma)}(x,y) - D_x^j g^{(\gamma)}(x,y')\| \le O(\gamma)|y-y'|^\delta[(|y|+|y'|)^{1-\delta} + |x|^{r-j-\delta}].$$

Moreover, for all $\epsilon > 0$, for sufficiently small $\gamma > 0$ all the factors $O(\gamma) < \epsilon$.

Proof. The $O(\gamma)$ factors in all these estimates follow from Lemma 2.9.

Estimates (3) and (4) follow from Lemma 2.8.

When $0 \le j \le k-1$, $D_x^j g^{(\gamma)}(x,y)$ is $C^{k-j,\delta}$. Thus, for some $0 \le c \le 1$

(*) $$\|D_x^j g^{(\gamma)}(x,y) - D_x^j g^{(\gamma)}(x',y)\| \le \|D_x^{j+1} g^{(\gamma)}(x + c(x'-x), y)\||x-x'|.$$

Now, (5) follows from (*) and (4).

Estimate (6) follows from the fact that $D^j g^{(\gamma)}$ is uniformly Lipschitz continuous for $0 \le j \le k-1$.

Estimate (7) follows from (6) if we notice that $|y-y'| \le |y-y'|^\delta(|y|+|y'|)^{1-\delta}$. □

Remarks 2.11. 1. Notice that small ϵ for $F^{(\gamma)}|_{B_1(0,0)}$ in Lemma 2.9 corresponds to small γ for $F|_{B_{(\gamma)}(0,0)}$.
2. Corollary 2.10 and Lemma 2.9 will allow us to reduce Theorem 1 to Theorem 4 below where we study $\Phi^{(\gamma)}$ on $B_1(0,0)$ under the additional assumptions (H5)-(H9).
3. To simplify notation we will drop the superscript γ. However, we will bring it back when we prove Theorem 1 in Section 7.
4. From now on, we will use adapted norms on X and Y in which $\|A\| < a$ and $\|B^{-1}\| < b^{-1}$.

Theorem 4. *Consider the map (2.1) in the setup of section 2.1. Let $0 \le n \le k$. When $n = 0$, let $\delta = 1$. When $1 \le n \lneqq k$, let $0 \lneqq \delta \le 1$. When $n = k$, let $0 \le \delta \le 1$. Let $r = k + \delta$. Assume the following in $B_1(0,0)$ for some $\epsilon > 0$:*

- **(H1):** *There is $0 < a < 1$ such that $\|A\| < a$.*
- **(H2):** *There is $b > 0$ such that $\|B^{-1}\| < b^{-1}$.*
- **(H3):** *$a^r < b$.*
- **(H4):** *$F(x,y)$ is $C^{n,\delta}$.*
- **(H5):** *$|F(x,y)| \le \epsilon o(|x|+|y|)$ and $\|D^j F\| < \epsilon, 0 \le j \le n$.*
- **(H6):** *$H_\delta(D^n F) < \epsilon$. (Notice that when $n = 0, \delta = 1$, and hence $H_\delta = Lip$).*
- **(H7):** *$\|D_x^j g(x,y)| \le \epsilon(|x|^{r-j} + |y|), 0 \le j \le n$.*
- **(H8):** *Let $s_o = s_1 = \cdots = s_{n-1} = 1$ and $s_n = \delta$. Assume that for $0 \le j \le n$*
$$\|D_x^j g(x,y) - D_x^j g(x',y)\| \le \epsilon|x-x'|^{s_j}[(|x|+|x'|)^{r-j-s_j} + |y|].$$
- **(H9):** *Let $l_j = 1 - s_j, 0 \le j \le n-1$. If $n < k$, let $l_n = 1 - s_n$. If $n = k$, let $l_n = 0$. Assume*
$$\|D_x^j g(x,y) - D_x^j g(x,y')\| \le \epsilon|y-y'|^{s_j}[(|y|+|y'|)^{l_j} + |x|^{r-j-s_j}], \qquad 0 \le j \le n.$$

Then for sufficiently small $\epsilon > 0$ the following is true:

1. ***Existence.*** *The map (2.1) has an invariant manifold of the form $y = w(x), |x| \leq 1$, and $w \in C^{n,\delta}(B_1(0), Y)$.*
2. ***Uniqueness.*** *Moreover, $w(x)$ is unique in the class of functions that satisfy*

$$|w(x)| \leq |x|^r, \tag{2.24}$$

$$|w(x) - w(x')| \leq |x - x'|(|x|^{r-1} + |x'|^{r-1}). \tag{2.25}$$

In addition, for $0 \leq j \leq n$ and $|x| \leq 1$:

$$\|D^j w(x)\| \leq |x|^{r-j} \tag{2.26}$$

$$\|D^j w(x) - D^j w(x')\| \leq |x - x'|^{s_j}(|x|^{r-j-s_j} + |x'|^{r-j-s_j}) \tag{2.27}$$

When $n = k$ and $\delta = 0$ omit (2.27) for $j = k$.

Remarks 2.12.

1. We will prove Theorem 4 by induction on $1 \leq n \leq k$. We will use the inequality $a^{k+\delta} < b$. If we tried induction on k we would need to use the inequalities $a^{j+\delta} < b$ for $0 \leq j \leq L - 1$. But these inequalities are not true.
2. The hypotheses of Theorem 4 are kept to the minimum needed to prove the conclusions. This way we can see which hypotheses of Theorem 1 are needed at each step. Notice that in Theorem 4 we do not assume the non-resonance condition nor the high degree of $C^{k,\delta}$ smoothness. These two requirements are needed in Theorem 1 in order to reduce $g(x, y)$ to a form that satisfies (H7)-(H9). What is really needed in the proof of Theorem 4 is to have the same exponent r in both (H3) and (H7)-(H9) and that $F(x, y)$ is $C^{n,\delta}$. Also notice that $s_n = \delta$ although for the original map if $n < k$ we can take $s_n = 1$. See also the remark about hypothesis (H9) below.
3. Hypothesis (H9) needs some explanation. Here also, we tried to keep the assumptions of Theorem 4 to a minimum: The map of Theorem 1 is $C^{k,\delta}$ and hence $C^{n,1}$ for $0 \leq n \leq k-1$. We do not need all this smoothness in Theorem 4. But assuming that the map is only $C^{n,\delta}$, which leads to

$$\|D_x^n g(x, y) - D_x^n g(x, y')\| \leq \epsilon|y - y'|^\delta, \tag{2.28}$$

   is not enough: In estimate (4.5) in the proof of part (6) of Lemma 4.4, we will need $D_x g$ to satisfy an estimate of the form (H9) and not (2.28) to cover the case $n = 1$ and $s_1 = \delta \lneqq 1$. If $s_1 = 1$, we can use (2.28). Also, in (4) of Lemma 5.4 we will need $D_x^m g$ to satisfy an estimate of the form (H9) to cover the case $n = m$ and $s_m = \delta \lneqq 1$.

   It is obvious that (H9) is weaker than assuming that $g(x, y)$ is $C^{n,1}$ but stronger than (2.28).
4. Theorem 1 will follow from Theorem 4. Briefly, let k and L be as in Theorem 1 and let $n = k$ and $r = n + \delta = k + \delta$. Thus $a^r < b$ which is hypothesis (H3). By Lemma 2.8, Corollary 2.10, and Lemma 2.9 we can assume that F and g satisfy assumptions (H5)-(H9).
5. In Section 3 we show the existence of a $C^{0,1}$ invariant manifold for the map in Theorem 4. In this case, $\delta = 1$. We will need to use norms which are appropriate for the class of functions satisfying (2.26) and (2.27). In particular we cannot use the usual Lipschitz constant $Lip(u)$ since $Lip(B^{-1}u \circ A - B^{-1}v \circ A) \leq b^{-1}aLip(u - v)$. But $b^{-1}a \not< 1$.

However, $b^{-1}a^{k+\delta} < 1$. Instead of $Lip(u)$ we use $L_{r,o}(u)$ which is given in Definition 3.1 below.

6. In section 4 we prove Theorem 4 when $n = 1$ and $0 \lneqq \delta \leq 1$ using the Fiber Contraction Theorem [12, page 136]. It is here that the problem of a possible loss of differentiability arises. It is straight forward to show that we have a fiber contraction. The difficulty is to show that *the fiber contraction is continuous in the base points*. (The base points are the functions $u(x)$). We will comment on the $\delta = 0$ case later in this set of remarks.
7. In Section 5 we use the Fiber Contraction Theorem again to prove Theorem 4 for $2 \leq n \leq k$ and $0 \lneqq \delta \leq 1$.
8. When $0 \lneqq \delta \leq 1$ we have some form of uniform Hölder continuity which allows us to show that the fiber contraction is continuous in the base points and conclude that the manifold is $C^{k,\delta}$ in Section 5. If $\delta = 0$ and we are in a finite dimensional space (it is enough that X is finite dimensional) then we can use the fact that a continuous map on a closed bounded ball is uniformly continuous. This statement is not true in an infinite dimensional Banach space. In Section 6 we introduce a new method to deal with the case $\delta = 0$ in an infinite dimensional space.
9. When $\delta = 0$ we have, from Section 5, a $C^{k-1,1}$ invariant manifold. We devote Section 6 to showing that this manifold is in fact C^k. The blue print for the proof is as follows: Let the $C^{k-1,1}$ invariant manifold be given by $y = w(x)$. Although the fiber contraction we have is not necessarily continuous in the base points, it still has an invariant fiber $\mathcal{F}_{\Delta^{k-1}w}$ above $\Delta^{k-1}w$ since $\Delta^{k-1}w$ is a unique attractive fixed point in the base. Thus, the fiber $\mathcal{F}_{\Delta^{k-1}w}$ has a unique attractive fixed point $\tau \in \mathcal{F}_{\Delta^{k-1}w}$. Although τ is attractive only in the invariant fiber $\mathcal{F}_{\Delta^{k-1}w}$, it is our candidate for $D^k w$. We show that $D^k w = \tau$ by showing that given any fixed but arbitrary x and h

$$D^{k-1}w(x+h) - D^{k-1}w(x) = \int_0^1 \tau(x+th)h dt.$$

In order to do that we pick any C^k function u_o and let $u_n = \Gamma u_{n-1}, n \geq 1$, where Γ is the transformation (3.2) given below. Thus, u_n is C^k and $\Gamma u_n \to w$ as $n \to \infty$. Then, we restrict our fiber contraction to the sequence $\{\Delta^{k-1}u_n|_J\}_{n=0}^{n=\infty}$, where $J = \{\xi = x + th \mid 0 \leq t \leq 1\}$ and $u_\infty := w$. This way we obtain a fiber contraction on the compact line segment $J = \xi([0,1])$ instead of the whole unit ball. We will need to show that the following three sets are compact:

$$\mathcal{U} = \bigcup_{n=0}^{n=\infty} \{u_n(\xi) \mid \xi \in J\},$$
$$\mathcal{V} = \bigcup_{n=0}^{n=\infty} \{u_n(z_n) \mid \xi \in J\},$$
$$\mathcal{Z} = \bigcup_{n=0}^{n=\infty} \{H_{u_n}(\xi) \mid \xi \in J\},$$
$$z_n = H_{u_n}(\xi) := A\xi + f(\xi, u_n(\xi)).$$

This is not particularly hard. Now, a continuous function on a compact set is uniformly continuous. This allows us to show that the restricted fiber contraction is continuous in the base points.

10. For the rest of this article the main statements will be given as propositions. Each proposition will be preceded by one or more lemmas in which we prove the estimates needed for the proposition.

3. The $\mathbf{C^{0,1}}$ Manifold For Theorem 4

In this section we prove Theorem 4 for $n = 0$. In this case $\delta = 1$. So, we show the existence of a unique $C^{0,1}$ invariant manifold. Also, when $n \geq 1$, the map is automatically $C^{0,1}$ regardless of the value of δ. Therefore, the conclusions of this section apply to the following sections in particular the uniqueness of the invariant manifold in the class of functions that satisfy (2.24) and (2.25).

Definition 3.1. Let $C^0(B_1(0), Y)$ be the space of continuous functions from $B_1(0)$ to Y. Let $s_o = 1, q_o = r - 1$, and $p_o = r$. Also define the following:

$$\begin{aligned}
N_{r,o}(u) &= \sup\left\{\frac{|u(x)|}{|x|^{p_o}} \mid 0 < |x| \leq 1\right\}, \\
L_{r,o}(u) &= \sup\left\{\frac{|u(x) - u(x')|}{|x - x'|^{s_o}(|x| + |x'|)^{q_o}} \mid x \neq x'; x, x' \in B_1(0)\right\}, \\
\mathfrak{Z}_{r,o} &= \{u \in C^0(B_1(0), Y) \mid N_{r,o}(u) < \infty\}, \\
\mathfrak{X}_{r,o} &= \{u \in \mathfrak{Z}_{r,o} \mid u(0) = 0, L_{r,o}(u) \leq 1, N_{r,o}(u) \leq 1\}.
\end{aligned}$$

Since convergence in the $N_{r,o}$ norm implies uniform convergence on bounded sets, it follows that $(\mathfrak{Z}_{r,o}, N_{r,o})$ is a Banach space and that $\mathfrak{X}_{r,o}$ is a closed subset.

3.2. The Reversed Graph Transform (RGT).

1. The graph of a function $y = u(x)$ is invariant under the map (2.1) if and only if $u(x)$ satisfies

$$\begin{aligned}
u(x) &= B^{-1}u(z) - B^{-1}g(x, u(x)), \\
z &= Ax + f(x, u(x)).
\end{aligned} \tag{3.1}$$

Therefore we define a transformation $\Gamma_F : \mathfrak{X}_{r,o} \to \mathfrak{X}_{r,o}$ by

$$\begin{aligned}
\Gamma_F u(x) &= B^{-1}u(z) - B^{-1}g(x, u(x)), \\
z &= H_u(x) := Ax + f(x, u(x)).
\end{aligned} \tag{3.2}$$

We call this transformation the *reversed graph transform* (RGT).

2. We will be looking for a fixed point for Γ_F. The transformation (3.2) is not the usual graph transform. The usual graph transformation (GT) is given by

$$\begin{aligned}
T_F u(x) &= Bu(z) + g(z, u(z)), \\
z &= H_u(x) = A^{-1}x - A^{-1}f(z, u(z)).
\end{aligned} \tag{3.3}$$

To use (3.3) we need to invert A which is not necessarily invertible. Even if A were invertible, we would need to replace the condition $a^r < b$ by the gap condition $\|B\|\|A^{-1}\|^r < 1$. But this condition would give us just the usual pseudo-unstable manifold theorem and not the new theorem we are trying to prove.

Many of the graph transform proofs use (3.2). We will comment on this issue in Section 3.3 below.

3. Now, u is a fixed point for Γ_F if and only if it satisfies (3.1). But u satisfies (3.1) if and only if it is a fixed point of T_F.
4. For the rest of this work we will use the convention that $t = t(\epsilon) \to 0$ as $\epsilon \to 0$, where $t(\epsilon)$ is continuous and may vary from one estimate to another but depends always on F and not on $u \in \mathfrak{X}_{r,o}$.
5. Since $b^{-1}a^r < 1$ we have the following: For sufficiently small $\epsilon > 0$, $\theta = b^{-1}a^r(1+t) < 1$. So, for the rest of this work θ will stand for such a quantity that can be made less than one by taking $\epsilon > 0$ small independent of $u \in \mathfrak{X}_{r,o}$.

3.3. On cut-off functions, GT and RGT. As we mentioned in Remarks 1.1 using a cut-off function is not possible in infinite dimensional Banach spaces and hence any smallness assumptions on the map have to be global assumptions. In finite dimensional spaces we have cut-off functions but using them leads to non uniqueness of the invariant manifold we are seeking. Therefore, the question of when we need a cut-off function is important.

Another question is whether to use the GT or the RGT.

1. In general, whether to use the GT or the RGT is determined by which linear map we can invert, A or B. If both are invertible, then we have a choice. However, each choice leads to different manifold as we mentioned in Section 3.2(2).
2. Notice that in our problem we do not need to use a cut-off function because in the RGT (3.2) $\|A\| < 1$ and hence $H_u : B_1(0) \to B_1(0)$ for sufficiently small $\epsilon > 0$. However, in the GT (3.3) $\|A^{-1}\| > 1$ and hence we have to deal with $H_u : X \to X$. Therefore, we have to use a cut-off function to show the existence of a pseudo-unstable manifold under a gap condition.
3. In general, whether to use a cut-off function is determined by whether the linear part of H_u is a contraction or not. That is, if $\|A\| < 1$ in the GT or $\|A^{-1}\| < 1$ in the RGT then we do not need a cut-off function: We use the method of Lemma 2.9 to obtain the smallness assumptions we need. Thus, $H_u : B_1(0) \to B_1(0)$.

   If the converse is true, then we have to live with $H_u : X \to X$. In this case, if we are working in a finite dimensional space we use a cut of function to obtain the smallness assumptions we need. In an infinite dimensional space which does not admit cut-off functions we have to impose global smallness assumptions.
4. In [6] we give a unified proof for the existence of all the standard invariant manifolds for $C^{k,\delta}$ (not necessarily invertible) maps of Banach space under the appropriate gap conditions, some of which are trivially satisfied. We group them in two groups. The first are called S-manifolds. These are the stable, unstable, strong-stable and strong-unstable manifolds. For S-manifold, $H_u : B_1(0) \to B_1(0)$ and hence, we do not need cut-off functions. The second group are called W-manifolds: the weak-stable, weak-unstable, centre-stable, centre-unstable, pseudo-stable and pseudo-unstable manifolds.

For W-manifold, $H_u : X \to X$ and hence, we have to use cut-off functions when the space X admits them or impose global smoothness assumptions.

Lemma 3.4. *Let $v : B_1(0) \to Y$ and assume that $N_{r,o}(v) < \infty$. Then,*

$$\begin{aligned} |H_v(x)| &\leq \|A\||x| + |f(x, u(x))| \\ &\leq a|x| + \epsilon[1 + N_{r,o}(v)]|x| \\ &\leq [a + \epsilon + \epsilon N_{r,o}(v)]|x| \end{aligned} \tag{3.4}$$

Let $u \in \mathfrak{X}_{r,o}$. Let $x \neq x'$. Then the following are true for sufficiently small $\epsilon > 0$:

$$|z| = |H_u(x)| \leq \ a(1 + t)|x| \leq 1, \tag{3.5}$$

$$H_u(0) = 0, \tag{3.6}$$

$$|H_u(x) - H_u(x')| \leq a(1 + t)|x - x'|. \tag{3.7}$$

Proof. The assertions follow immediately from the facts that $f(0, 0) = 0$, $Lip(f) < \epsilon$, $N_{r,o}(u) \leq 1$ and $|x| \leq 1$. □

Proposition 3.5. *For sufficiently small $\epsilon > 0$, there is $\theta = b^{-1}a^r(1+t) \in (0, 1)$ such that the following are true for all $u \in \mathfrak{X}_{r,o}$:*

1. $N_{r,o}(\mathit{\Gamma}_F u) < \theta N_{r,o}(u) + t < 1$.
2. $L_{r,o}(\mathit{\Gamma}_F u) < \theta L_{r,o}(u) + t < 1$. *Moreover, $t/(1 - \theta) < 1/2$ uniformly in $u \in \mathfrak{X}_{r,o}$.*
3. *Thus $\mathit{\Gamma}_F$ indeed maps $\mathfrak{X}_{r,o}$ to itself.*

Proof. (1) Let $u \in \mathfrak{X}_{r,o}$. By (H7) with $j = 0$ and (3.5)we obtain

$$\begin{aligned} |\mathit{\Gamma}_F u(x)| &\leq b^{-1}N_{r,o}(u)|z|^r + \epsilon b^{-1}(1 + N_{r,o}(u))|x|^r \\ &\leq [b^{-1}(a + t)^r N_{r,o}(u) + 2\epsilon b^{-1}]|x|^r \\ &\leq (\theta N_{r,o}(u) + t)|x|^{p_o} \leq (\theta + t)|x|^{p_o}, \\ \theta &= b^{-1}a^r(1 + t), \end{aligned}$$

for sufficiently small $\epsilon > 0, 0 < \theta + t < 1$. It follows that $N_{r,o}(\mathit{\Gamma}_F u) < \theta N_{r,o}(u) + t < 1$.
(2) Let $x \neq x'$, $z = H_u(x)$, and $z' = H_u(x')$. Recall that $s_o + q_o = p_o = r$. By Lemma 3.4 and (H8) we obtain

$$\begin{aligned} |\mathit{\Gamma}_F u(x) - \mathit{\Gamma}_F u(x')| &\leq b^{-1}L_{r,o}(u)|z - z'|^{s_o}(|z| + |z'|)^{q_o} \\ &\quad + \epsilon|x - x'|^{s_o}(|x| + |x'|)^{q_o} \\ &\leq [\theta L_{r,o}(u) + t]|x - x'|^{s_o}(|x| + |x'|)^{q_o}. \end{aligned} \tag{3.8}$$

Thus, for sufficiently small ϵ, $L_{r,o}(\mathit{\Gamma}_F u) \leq \theta L_{r,o}(u) + t < 1$.

Notice that for sufficiently small $\epsilon > 0, 1 - \theta > 0$ is bounded away from 0 uniformly in $u \in \mathfrak{X}_{r,o}$. Thus for possibly smaller $\epsilon > 0$, we have $t/(1 - \theta) < 1/2$ uniformly in $u \in \mathfrak{X}_{r,o}$.
(3) Also, $u(0) = 0$ implies that $\mathit{\Gamma}_F u(0) = 0$. Then by (1) and (2) $\mathit{\Gamma}_F u \in \mathfrak{X}_{r,o}$. □

Remarks 3.6. 1. The second statement of assertion (2) is needed in Proposition 3.9 in which we show that the unique attractive fixed point of $\mathit{\Gamma}_F : \mathfrak{X}_{k+\delta,o} \to \mathfrak{X}_{k+\delta,o}, 0 < \delta \leq 1$, (which we are still looking for at this point) is also the unique attractive fixed point of $\mathit{\Gamma}_F : \mathfrak{X}_{k+\beta,o} \to \mathfrak{X}_{k+\beta,o}$ for all $0 \leq \beta < \delta \leq 1$.

2. Estimate (3.4) shows that the assumption that $N_{r,o}(v) < \infty$ does not guarantee that $z = H_v(x) \leq 1$ when $|x| \leq 1$. This estimate has important ramifications when we consider the uniqueness issue. For even if $z = H_v(x) \leq 1$ when $|x| \leq 1$, calculations similar to the ones used to prove assertion (1) of Proposition 3.5 show that

$$N_{r,o}(\Gamma_F) \leq b^{-1}a^r[1 + a^{-1} + \epsilon a^{-1}N_{r,o}(v)]^r N_{r,o}(v) + \epsilon b^{-1}(1 + N_{r,o}(v)) \tag{3.9}$$

If v is a fixed point of Γ_F which satisfies only $N_{r,o}(v) < \infty$, estimate (3.9) does not allow us to deduce that $N_{r,o}(v) < t/(1-\theta) < 1$, and then conclude that $v = w$ where $w \in \mathfrak{X}_{r,o}$ is the unique fixed point of Γ_F which we are going to find in this section. The more general uniqueness issue will be dealt with in Proposition 7.6.
3. The second estimate of Proposition 3.5 above and the second estimate of the following Lemma 3.7 below are the first two places where it is essential to use $L_{r,o}(u)$ rather than the standard $Lip(u)$.

Lemma 3.7. *Let u and v be in $\mathfrak{X}_{r,o}$. Let $z = H_u(x)$ and $z^* = H_v(x)$. Then*

$$|z - z^*| \leq \epsilon N_{r,o}(u-v)|x|^{p_o}, \tag{3.10}$$

$$|u(z) - v(z^*)| \leq a^r[1 + t]N_{r,o}(u-v)|x|^{p_o}. \tag{3.11}$$

Proof. Since $F \in C^{0,1}(B_1(0), Y)$, we have

$$\begin{aligned} |z - z^*| &\leq |f(x, u(x)) - f(x, v(x))| \\ &\leq \epsilon|u(x) - v(x)| \leq \epsilon N_{r,o}(u-v)|x|^{p_o}. \end{aligned}$$

By (3.5) and(3.10),

$$\begin{aligned} |u(z) - v(z^*)| &\leq |u(z) - v(z)| + |v(z) - v(z^*)| \\ &\leq N_{r,o}(u-v)(a+t)^{p_o}|x|^{p_o} + L_{r,o}(v)|z - z^*|^{s_o}(|z| + |z^*|)^{q_o} \\ &\leq N_{r,o}(u-v)a^r[1+t]|x|^{p_o}. \end{aligned}$$

□

Proposition 3.8. *For sufficiently small $\epsilon > 0$, $\Gamma_F u$ is uniformly Lipschitz continuous in u in the norm $N_{r,o}$. More precisely, there is $\theta \in (0,1)$ such that for all $u, v \in \mathfrak{X}_{r,o}$*

$$N_{r,o}(\Gamma_F u - \Gamma_F v) < \theta N_{r,o}(u-v).$$

Thus $\Gamma_F : \mathfrak{X}_{r,o} \to \mathfrak{X}_{r,o}$ is a contraction, and hence, has a unique attractive fixed point $w \in \mathfrak{X}_{r,o}$.

Proof. Let u and v be in $\mathfrak{X}_{r,o}$. Let $z = H_u(x)$ and $z^* = H_v(x)$. By Lemma 3.4 and Lemma 3.7 we have

$$\begin{aligned} |\Gamma_F u(x) - \Gamma_F v(x)| &\leq b^{-1}[|u(z) - u(z^*)| + |g(x, u(x)) - g(x, v(x))|] \\ &\leq b^{-1}(a^r + t)N_{r,o}(u-v)|x|^r + b^{-1}\epsilon|u(x) - v(x)| \\ &\leq b^{-1}a^r(1+t)N_{r,o}(u-v)|x|^r \\ &= \theta N_{r,o}(u-v)|x|^r, \end{aligned}$$

and $\theta \in (0,1)$, for sufficiently small $\epsilon > 0$. Thus $N_{r,o}(\Gamma_F u - \Gamma_F v) < \theta N_{r,o}(u-v)$. □

Proposition 3.9. *Let $w \in \mathfrak{X}_{k+\delta}$ be the unique attractive fixed point of $\Gamma_F : \mathfrak{X}_{k+\delta,o} \to \mathfrak{X}_{k+\delta,o}$. Let $0 \le \beta < \delta \le 1$. Then, w is the unique attractive fixed point of $\Gamma_F : \mathfrak{X}_{k+\beta,o} \to \mathfrak{X}_{k+\beta,o}$.*

Proof. We know that $\Gamma_F : \mathfrak{X}_{k+\beta,o} \to \mathfrak{X}_{k+\beta,o}$ has a unique attractive fixed point $\hat{w} \in \mathfrak{X}_{k+\beta,o}$. To show that $w = \hat{w}$ we need to show that $N_{k+\beta,o}(w) \le 1$ and $L_{k+\beta,o}(w) \le 1$ which implies that $w \in \mathfrak{X}_{k+\beta,o}$.

Since $|x| \le 1$ it follows that for all $u \in \mathfrak{X}_{k+\delta,o}$,

$$\begin{aligned} |u(x)| &\le N_{k+\delta,o}(u)|x|^{k+\beta}|x|^{\delta-\beta} \le N_{k+\delta,o}(u)|x|^{k+\beta}, \\ N_{k+\beta,o}(u) &\le N_{k+\delta,o}(u) \le 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} |u(x) - u(x')| &\le L_{k+\delta,o}(u)|x - x'|(|x| + |x'|)^{k+\beta-1}(|x| + |x'|)^{\delta-\beta}, \\ L_{k+\beta,o}(u) &\le 2^{\delta-\beta} L_{k+\delta,o}(u) \le 2^{\delta-\beta}, \end{aligned}$$

which does not allow us to conclude that $L_{k+\beta,o}(u) \le 1$. But by assertion (2) of Proposition 3.5 we have $L_{k+\delta,o}(w) \le t/(1-\theta) < 1/2$. Thus

$$L_{k+\beta,o}(w) \le 2^{\delta-\beta} L_{k+\delta,o}(w) \le \frac{1}{2^{1-\delta+\beta}} < 1, \qquad 0 \le \beta < \delta \le 1.$$

It follows that w is the unique attractive fixed point of $\Gamma_F : \mathfrak{X}_{k+\beta,o} \to \mathfrak{X}_{k+\beta,o}$. ∎

4. The $\mathbf{C^{1,\delta}}$ Manifold, $\mathbf{0 \lneqq \delta \le 1}$, For Theorem 4

Let $w \in \mathfrak{X}_{r,o}$ be the unique fixed point obtained in Section 3. If w is C^1 it satisfies

$$\begin{aligned} Dw(x) = {} & B^{-1}Dw(z)[A + Df(x, w(x))(I, Dw(x))] \\ & - B^{-1}Dg(x, w(x))(I, Dw(x)) \end{aligned} \tag{4.1}$$

It is obvious that Dw will be a fixed point for a transformation over an appropriately constructed function-space which we define presently.

Definition 4.1. When $1 \le n + \delta \le r = k + \delta$ define the following:

1. Extend (s_o, q_o, p_o) as defined in Definition 3.1 to

$$\begin{aligned} s_j &= 1, & 0 \le j \le n-1, \\ s_j &= \delta, & j = n, \\ q_j &= r - j - s_j, & 0 \le j \le n, \\ p_j &= r - j, & 0 \le j \le n. \end{aligned}$$

2. Let $X^j = X \times \cdots \times X$, j times. Let $\mathcal{S}(X^j, Y)$ be the Banach space of symmetric continuous multilinear maps from X^j to Y. Define the following for $\sigma : B^1(0) \to \mathcal{S}(X^j, Y), j \geq 2$:

$$\|\sigma\| = \sup_{|x| \leq 1} \|\sigma(x)\|,$$
$$N_{r,j}(\sigma) = \sup_{0<|x|\leq 1} \frac{\|\sigma(x)\|}{|x|^{p_j}},$$
$$L_{r,j}(\sigma) = \sup \left\{ \frac{\|\sigma(x) - \sigma(x')\|}{|x - x'|^{s_j} (|x| + |x'|)^{q_j}} \mid x \neq x', x, x' \in B_1(0) \right\}.$$

3. Also define the following spaces for $1 \leq j \leq k$:

$$\mathfrak{Z}_{r,j} = \{\sigma \in C^0(B_1(0), \mathcal{S}(X^j, Y)) \mid N_{r,j}(\sigma) < \infty\},$$
$$\mathfrak{X}_{r,j} = \{\sigma \in \mathfrak{Z}_{r,j} \mid N_{r,j}(\sigma) \leq 1, L_{r,j}(\sigma) \leq 1, \sigma(0) = 0\}.$$

4. If h is C^1 define

$$\Delta h(x, \xi) = (h(x), Dh(x)\xi), \qquad x \in B_1(0), \xi \in X.$$

5. Define the following transformation on $\mathfrak{X}_{r,o} \times \mathfrak{X}_{r,1}$:

$$\mathit{\Gamma}_{\Delta F}(u, \sigma) = (\mathit{\Gamma}_F u, \mathit{\Gamma}^u \sigma), \tag{4.2}$$
$$\mathit{\Gamma}^u \sigma(x) = B^{-1} \sigma(z) \mathcal{H}^u \sigma(x) + \mathcal{G}^u \sigma(x), \tag{4.3}$$
$$\mathcal{H}^u \sigma(x) = A + Df(x, u(x))(I, \sigma(x)),$$
$$\mathcal{G}^u \sigma(x) = -B^{-1} Dg(x, u(x))(I, \sigma(x)),$$
$$z = H_u(x) = Ax + f(x, u(x)),$$

where $\mathit{\Gamma}_F(u)$ is given by (3.2).

Since convergence in the $N_{r,j}$ norm implies uniform convergence on bounded sets, it follows that $(\mathfrak{Z}_{r,j}, N_{r,j})$ is a Banach space and that $\mathfrak{X}_{r,j}$ is a closed subset.

Remark 4.2. If we added $L_{r,j}(u) < \infty$ to the definition of $\mathfrak{Z}_{r,j}$ and used the norm

$$\|u\|_* = \max\{N_{r,j}(u), L_{r,j}(u)\},$$

then $\mathfrak{Z}_{r,j}$ would be a Banach space still, and $\mathfrak{X}_{r,j}$ would be a closed subset. However, we would be able to prove that $\mathit{\Gamma}^u \sigma$ is continuous in u only if we replace $r = k + \delta$ by $r = k + \gamma, 0 \leq \gamma < \delta$ in the definition of $N_{r,j}$ and $L_{r,j}$ which would lead to loss of differentiability.

In order to study $\mathit{\Gamma}_{\Delta F}$ we need the fiber contraction theorem [12, page 136].

4.3. The Fiber Contraction Theorem. *Let $\mathfrak{B}$ be a metric space and $\phi : \mathfrak{B} \to \mathfrak{B}$ a map having an attractive fixed point p (i.e. $\forall u \in \mathfrak{B} : \phi^n(u) \to p$ as $n \to \infty$.) Let $\mathfrak{Y}$ be a metric space and $\{\psi^u \mid u \in \mathfrak{B}\}$ a family of maps $\psi^u : \mathfrak{Y} \to \mathfrak{Y}$ such that the formula $\mathit{\Gamma}(u, \sigma) = (\phi(u), \psi^u(\sigma))$ defines a continuous map $\mathit{\Gamma} : \mathfrak{B} \times \mathfrak{Y} \to \mathfrak{B} \times \mathfrak{Y}$. Let $q \in \mathfrak{Y}$ be a fixed point for ψ^p. Then, (p, q) is an attractive fixed point for $\mathit{\Gamma}$ provided that*

$$\limsup_{n\to\infty} Lip(\psi^{\phi^n(u)}) < 1, \qquad \forall u \in \mathfrak{B}. \tag{*}$$

We will apply the Fiber Contraction Theorem to $\Gamma_{\Delta F}(u,\sigma)$. We will show that there is $\theta \in (0,1)$ such that $Lip(\Gamma^u) < \theta < 1, \forall u \in \mathfrak{X}_{r,o}$, which is stronger than (*). In order to apply the Fiber Contraction Theorem we need a few estimates that are given in the following few lemmas.

Lemma 4.4. *Let $u \in \mathfrak{X}_{r,o}$, $v \in \mathfrak{X}_{r,o}$ and $\sigma \in \mathfrak{X}_{r,1}$. Then,*

1. $\|\mathcal{H}^u\sigma(x)\| \le \|A\| + \epsilon \le a + \epsilon$.
2. $\|\mathcal{G}^u\sigma(x)\| \le t|x|^{p_1}$.
3. $\|\mathcal{H}^u\sigma(x) - \mathcal{H}^v\sigma(x)\| \le \epsilon[N_{r,o}(u-v)]^{s_1}|x|^{rs_1}$.
4. $\|\mathcal{G}^u\sigma(x) - \mathcal{G}^v\sigma(x)\| \le t[N_{r,o}(u-v)]^{s_1}|x|^{p_1}$.
5. $\|\mathcal{H}^u\sigma(x) - \mathcal{H}^u\sigma(x')\| \le t|x-x'|^{s_1}$.
6. $\|D_xg(x,u(x)) - D_xg(x',u(x'))\| \le t|x-x'|^{s_1}(|x|+|x'|)^{q_1}$
7. $\|\mathcal{G}^u\sigma(x) - \mathcal{G}^u\sigma(x')\| \le t|x-x'|^{s_1}(|x|+|x'|)^{q_1}$.
8. *Let $z = H_u(x)$ and $z^* = H_v(x)$. Then,*

$$\begin{aligned}\|\sigma(z) - \sigma(z^*)\| &\le t[N_{r,o}(u-v)]^{s_1}\|x\|^{q_1+rs_1}\\ &\le t[N_{r,o}(u-v)]^{s_1}\|x\|^{p_1}.\end{aligned}$$

Proof. Since $N_{r,1}(\sigma) \le 1$ and $\|x\| \le 1$, we have $\|\sigma(x)\| \le 1$.

1. Assertion (1) follows from (H5) and the fact that $\|\sigma(x)\| \le 1$.
2. Using (H5) and (H7)we obtain

$$\|\mathcal{G}^u\sigma(x)\| \le b^{-1}\|D_xg(x,u(x))\| + b^{-1}\|D_yg(x,u(x))\|\|\sigma(x)\| \le t|x|^{p_1}.$$

3. Estimate (3) follows from hypothesis (H6) on DF.
4. Notice that $(s_1 + l_1)r = r = p_1 + 1$. Using hypothesis (H9), we obtain

$$\begin{aligned}\|\mathcal{G}^u\sigma(x) - \mathcal{G}^v\sigma(x)\| &\le \|D_xg(x,u(x)) - D_xg(x,v(x))\|\\ &\quad + \|D_yg(x,u(x)) - D_yg(x,v(x))\|\,\|\sigma(x)\|\\ &\le \epsilon|u(x)-v(x)|^{s_1}[(|u(x)| + v(x)|)^{l_1} + |x|^{r-1-s_1}]\\ &\quad + \epsilon|x|^{r-1}|u(x)-v(x)|^{s_1}\\ &\le t[N_{r,o}(u-v)]^{s_1}[|x|^{r(s_1+l_1)} + |x|^{(r-1)(1+s_1)} + |x|^{r-1+rs_1}]\\ &\le t[N_{r,o}(u-v)]^{s_1}|x|^{p_1}.\end{aligned}$$

5. Since $|x|, |x'| \le 1$, $H_{s_1}(\sigma) \le 2^{q_1}L_{r,1}(\sigma)$ and $H_{s_1}(u) \le 2^{1-s_1}H_{s_o}(u) \le 2^{r-s_1}L_{r,o}(u)$. Now the fifth estimate follows from the fact that and Df, u and σ are at least uniformly s_1-Hölder continuous.
6. Notice that $p_o > q_1$ and hence, $|x|^{p_o} \le |x|^{q_1} \le (|x|+|x'|)^{q_1}$. Also, $q_o = q_1 + 1$. Using (H8) we obtain

$$\begin{aligned}\|D_xg(x,u(x)) - D_xg(x',u(x))\| &\le \epsilon|x-x'|^{s_1}[(|x|+|x'|)^{q_1} + |u(x)|]\\ &\le \epsilon|x-x'|^{s_1}[(|x|+|x'|)^{q_1} + |x|^{p_o}]\\ &\le 2\epsilon|x-x'|^{s_1}(|x|+|x'|)^{q_1}.\end{aligned} \tag{4.4}$$

Notice that $(s_1 + l_1)q_o = q_o = 1 + q_1$ and that $s_1 r \geq s_1 q_o$. Using (H9) we obtain

$$\begin{aligned}\|D_x g(x', u(x)) - D_x g(x', u(x'))\| &\leq \epsilon |u(x) - u(x')|^{s_1}[(|u(x)| + |u(x')|)^{l_1} + |x'|^{q_1}] \\ &\leq t|x - x'|^{s_1}(|x| + |x'|)^{rs_1}[(|x| + |x'|)^{l_1 q_o} + |x'|^{q_1}] \\ &\leq t|x - x'|^{s_1}(|x| + |x'|)^{q_1}.\end{aligned} \tag{4.5}$$

Estimate (6) follows from estimates (4.4) and (4.5).

7. Using estimate (6) of this Lemma, we obtain

$$\begin{aligned}\|\mathcal{G}^u\sigma(x) - \mathcal{G}^u\sigma(x')\| &\leq \|D_y g(x', u(x'))\|\|\sigma(x) - \sigma(x')\| \\ &\quad + \|D_y g(x, u(x)) - D_y g(x', u(x'))\|\|\sigma(x)\| \\ &\quad + \|D_x g(x, u(x)) - D_x g(x', u(x'))\| \\ &\leq \epsilon L_{r,1}(\sigma)|x - x'|^{s_1}(|x| + |x'|)^{q_1} \\ &\quad + t|x - x'|^{s_1} N_{r,1}(\sigma)|x|^{p_1} \\ &\quad + t|x - x'|^{s_1}(|x| + |x'|)^{q_1} \\ &\leq t|x - x'|^{s_1}(|x| + |x'|)^{q_1}.\end{aligned}$$

8. Recall that we are considering the case $s_1 = \delta \in (0, 1]$. By (3.10) and (3.5), we have

$$\begin{aligned}\|\sigma(z) - \sigma(z^*)\| &\leq L_{r,1}(\sigma)|z - z^*|^{s_1}(|z| + |z^*|)^{q_1} \\ &\leq t[N_{r,o}(u - v)]^{s_1}|x|^{q_1 + rs_1}.\end{aligned}$$

The estimate follows from the fact that $q_1 + rs_1 \geq q_1 + s_1 = p_1$.

□

Proposition 4.5. *For sufficiently small $\epsilon > 0$, the transformation $\Gamma_{\Delta F}$ maps $\mathfrak{X}_{r,o} \times \mathfrak{X}_{r,1}$ to itself. More precisely, for sufficiently small $\epsilon > 0$ there is $\theta = b^{-1}a^r(1 + t) \in (0, 1)$ such that for all $(u, \sigma) \in \mathfrak{X}_{r,o} \times \mathfrak{X}_{r,1}$, $\Gamma^u\sigma(0) = 0$ and*

$$\begin{aligned}N_{r,1}(\Gamma^u\sigma) &\leq \theta N_{r,1}(\sigma) + t < 1, \\ L_{r,1}(\Gamma^u\sigma) &\leq \theta L_{r,1}(\sigma) + t < 1.\end{aligned}$$

Proof. We already know that Γ_F maps $\mathfrak{X}_{r,o}$ to itself. Let $(u, \sigma) \in \mathfrak{X}_{r,o} \times \mathfrak{X}_{r,1}$. Since $u(0) = 0$ and $Df(0, u(0)) = 0$, it follows that if $\sigma(0) = 0$ then $\Gamma^u\sigma(0) = 0$. It remains to show that $N_{r,1}(\Gamma^u\sigma) \leq 1$ and $L_{r,1}(\Gamma^u\sigma) \leq 1$. By (1) and (2) of Lemma 4.4 we have

$$\begin{aligned}\|\Gamma^u\sigma(x)\| &\leq b^{-1}N_{r,1}(\sigma)|z|^{p_1}\|\mathcal{H}^u\sigma(x)\| + \|\mathcal{G}^u\sigma(x)\| \\ &\leq b^{-1}a^r(1 + t)[N_{r,1}(\sigma) + t]|x|^{p_1}, \\ N_{r,1}(\Gamma^u\sigma) &\leq \theta N_{r,1}(\sigma) + t \leq \theta + t < 1,\end{aligned}$$

for sufficiently small $\epsilon > 0$, since $b^{-1}a^r < 1$ and $N_{r,1}(\sigma) \leq 1$.

To prove the second estimate let $x \neq x'$, $z = H_u(x)$, and $z' = H_u(x')$. By assertions (5) and (7) of Lemma 4.4 and (3.7); and since $q_1 + s_1 = p_1$, we have

$$\begin{aligned}
\|\Gamma^u\sigma(x) - \Gamma^u\sigma(x')\| &\leq b^{-1}\|\sigma(z) - \sigma(z')\|\|\mathcal{H}^u(x)\| \\
&\quad + b^{-1}\|\sigma(z')\|\|\mathcal{H}^u\sigma(x) - \mathcal{H}^u\sigma(x')\| \\
&\quad + \|\mathcal{G}^u(x) - \mathcal{G}^u(x')\| \\
&\leq b^{-1}(a+t)L_{r,1}(\sigma)|z - z'|^{s_1}(|z| + |z'|)^{q_1} \\
&\quad + tN_{r,1}(\sigma)|x - x'|^{s_1}|x'|^{p_1} \\
&\quad + t|x - x'|^{s_1}(|x| + |x'|)^{q_1} \\
&\leq [b^{-1}a^r(1+t)L_{r,1}(\sigma) + t]|x - x'|^{s_1}(|x| + |x'|)^{q_1}, \\
L_{r,1}(\Gamma^u\sigma) &\leq \theta L_{r,1}(\sigma) + t \leq \theta + t.
\end{aligned}$$

Since $b^{-1}a^r < 1$, it follows that $\theta < 1$ for sufficiently small $\epsilon > 0$. Thus $L_{r,1}(\Gamma^u\sigma) < 1$. □

Proposition 4.6. *Let $u \in \mathfrak{X}_{r,o}$, $v \in \mathfrak{X}_{r,o}$ and $\sigma \in \mathfrak{X}_1$. Then*

$$N_{r,1}(\Gamma^u\sigma - \Gamma^v\sigma) \leq t[N_{r,o}(u - v)]^{s_1}. \tag{4.6}$$

Thus $\Gamma_{\Delta F}(u,\sigma)$ is uniformly s_1-Hölder continuous in u.

Proof. Let $z = H_u(x)$ and $z^* = H_v(x)$. By (3), (4), and (8) of Lemma 4.4 we have

$$\begin{aligned}
\|\Gamma^u\sigma(x) - \Gamma^v\sigma(x)\| &\leq b^{-1}\|\sigma(z) - \sigma(z^*)\|\|\mathcal{H}^u\sigma(x)\| \\
&\quad + b^{-1}\|\sigma(z^*)\|\|\mathcal{H}^u\sigma(x) - \mathcal{H}^v\sigma(x)\| \\
&\quad + \|\mathcal{G}^u\sigma(x) - \mathcal{G}^v\sigma(x)\| \\
&\leq t[N_{r,o}(u - v)]^{s_1}|x|^{p_1},
\end{aligned}$$

which implies (4.6). We also know from Proposition 3.8 that $\Gamma_F u$ is uniformly Lipschitz continuous in u. Thus $\Gamma_{\Delta F}(u,\sigma)$ is uniformly s_1-Hölder continuous in u since $N_{r,o}(u) \leq 1$ for $u \in \mathfrak{X}_{r,o}$. □

It remains to show that $\Gamma_{\Delta F}(u,\sigma)$ is continuous in σ and that $\Gamma^u\sigma$ is uniformly Lipschitz continuous in σ with Lipschitz constant less than one. This will be done simultaneously, but first we need the following lemma.

Lemma 4.7. *Let $u \in \mathfrak{X}_{r,o}$ and $\sigma, \rho \in \mathfrak{X}_{r,1}$. Then,*

1. $\|\mathcal{H}^u\sigma(x) - \mathcal{H}^u\rho(x)\| \leq \epsilon N_{r,1}(\sigma - \rho)|x|^{p_1}$.
2. $\|\mathcal{G}^u\sigma(x) - \mathcal{G}^u\rho(x)\| \leq \epsilon N_{r,1}(\sigma - \rho)|x|^{p_1}$.

Proof. The proof of (1) and (2) are identical and simple, so we prove (1).

$$\begin{aligned}
\|\mathcal{H}^u\sigma(x) - \mathcal{H}^u\rho(x)\| &\leq \|D_yf(x,u(x))\|\|\sigma(x) - \rho(x)\| \\
&\leq \epsilon N_{r,1}(\sigma - \rho)|x|^{p_1}.
\end{aligned}$$

□

Proposition 4.8. *For sufficiently small $\epsilon > 0$, there is $0 < \theta < 1$ such that for all u in $\mathfrak{X}_{r,o}$, and for all σ and ρ in $\mathfrak{X}_{r,1}$*

$$N_{r,1}(\Gamma^u\sigma - \Gamma^u\rho) \le \theta N_{r,1}(\sigma - \rho).$$

In fact $\theta = b^{-1}a^r(1+t) < 1$ for sufficiently small $\epsilon > 0$. It follows that $\Gamma_{\Delta F}(u,\sigma)$ is uniformly Lipschitz continuous in σ.

Proof. Recall that $|z| = |H_u(x)| \le (a+\epsilon)|x|$ and that $1+p_1 = r = p_o$. By Lemma 4.7 we have

$$\begin{aligned}\|\Gamma^u\sigma(x) - \Gamma^u\rho(x)\| &\le b^{-1}\|\sigma(z) - \rho(z)\|\|\mathcal{H}^u\sigma(x)\| \\ &\quad + b^{-1}\|\rho(z)\|\|\mathcal{H}^u\sigma(x) - \mathcal{H}^u\rho(x)\| \\ &\quad + \|\mathcal{G}^u\sigma(x) - \mathcal{G}^u\rho(x)\| \\ &\le b^{-1}N_{r,1}(\sigma-\rho)|z|^{p_1}(a+\epsilon) + 2\epsilon N_{r,1}(\sigma-\rho)|x|^{p_1} \\ &\le b^{-1}a^r(1+t)N_{r,1}(\sigma-\rho)|x|^{p_1}.\end{aligned}$$

Thus $N_{r,1}(\Gamma^u\sigma - \Gamma^u\rho) \le \theta N_{r,1}(\sigma-\rho)$ with $\theta = b^{-1}a^r(1+t) < 1$ for sufficiently small $\epsilon > 0$. □

Proposition 4.9. *For sufficiently small $\epsilon > 0$, $\Gamma_{\Delta F} : \mathfrak{X}_{r,o} \times \mathfrak{X}_{r,1} \to \mathfrak{X}_{r,o} \times \mathfrak{X}_{r,1}$ has a unique attractive fixed point (w,τ), and $Dw = \tau$. Let $G(w)$ be the graph of w. Then, $T_{(0,0)}G(w) = X \oplus \{0\}$.*

Proof. By Proposition 4.6 and Proposition 4.8, $\Gamma_{\Delta F}$ is a continuous fiber contraction. We know from Lemma 3.8 that Γ_F has a unique attractive fixed point $w \in \mathfrak{X}_{r,o}$. The fiber $\{w\} \times \mathfrak{X}_{r,1}$ is invariant under $\Gamma_{\Delta F}$. But Γ^w is a contraction. Thus Γ^w has a unique attractive fixed point $\tau \in \mathfrak{X}_{r,1}$. By the Fiber Contraction Theorem 4.3 (w,τ) is a unique attractive fixed point for $\Gamma_{\Delta F}$.

Now we would like to show that $Dw = \tau$. Recall (3) of Definition 4.1 and notice that

$$\begin{aligned}\Gamma_{\Delta F}(u, Du) &= \Delta(\Gamma_F u), \\ (\Gamma_{\Delta F})^n(u, Du) &= \Delta(\Gamma_F^n u) \qquad n = 1,2,3,\cdots.\end{aligned} \tag{*}$$

Let $u_0 = 0$ and $u_n = \Gamma_F^n u$. Then, $\lim_{n\to\infty} u_n = w$ and u_n is C^1 for $n \ge 1$. Thus , by (*), we have

$$\begin{aligned}\lim_{n\to\infty}(u_n, Du_n) &= \lim_{n\to\infty}\Delta u_n = \lim_{n\to\infty}\Delta(\Gamma_F^n u_0) \\ &= \lim_{n\to\infty}(\Gamma_{\Delta F})^n(u_0, Du_0) = (w,\tau),\end{aligned}$$

where all the limits are taken in the norms we have been using. Thus $\lim_{n\to\infty} N_{r,1}(Du_n - \tau) = 0$. Since convergence in the norm $N_{r,1}$ implies uniform convergence on bounded sets, it follows that $Dw = \tau$.

Since $Dw(0) = \tau(0) = 0$, it follows that $T_{(0,0)}G(w) = X \oplus \{0\}$. Finally,

$$\begin{aligned}N_{r,1}(Dw) &= N_{r,1}(\tau) \le 1, \\ L_{r,1}(Dw) &= L_{r,1}(\tau) \le 1.\end{aligned}$$

□

So far we have proven the first induction step in the proof of Theorem 4 for $0 < \delta \le 1$.

5. The $\mathbf{C^{n,\delta}}$ Manifold, $\mathbf{0 \lneqq \delta \leq 1, 2 \leq n \leq k}$, For Theorem 4

In Section 4 we proved Theorem 4 for $n = 1$ and $0 \lneqq \delta \leq 1$. We did not include $\delta = 0$ because we would not be able to use the standard Fiber Contraction Theorem 4.3. The reason is that $\Gamma_{\Delta F}(u, \sigma)$ would not be necessarily continuous in u when $\delta = 0$. That is, Proposition 4.6 would not be necessarily true. The points where the argument fails are assertions (3), (4), and (8) of Lemma 4.4. These are the places where $s_1 = \delta$ appears as an exponent of $N_{r,o}(u-v)$. In assertion (8), for example, the exponent appears because $\sigma(x)$ is C^{0,s_1}, $s_1 = \delta \neq 0$.

In this section we prove Theorem 4 for $2 \leq n \leq k$ and $0 \lneqq \delta \leq 1$. In Section 6 we will prove Theorem 4 for $n = k$ and $\delta = 0$. We will explain in Remark 6.9 why we have to take this path.

Remarks 5.1. 1. Let $D^{[j]}u = (D^1 u, \cdots, D^j u), j = 1, \cdots, k$. Thus $\Delta^j u = (u, D^{[j]}u)$.

2. Assume we have proved Theorem 4 for $n = m - 1 < k$ and $0 \lneqq \delta \leq 1$ and hence have a $C^{m-1,\delta}$ invariant manifold $y = w(x), |x| \leq 1$, unique in the class of functions satisfying (2.24) and (2.25). In fact, it is also unique in the class of functions that satisfy (2.26) and (2.27). If w is C^m, then $D^m w(x)$ satisfies

$$
\begin{aligned}
D^m w(x) = {} & B^{-1} D^m w(z)[A + Df(x, w(x))(I, Dw(x))]^m \\
& + B^{-1} Dw(z) D^m f(x, w(x))[(I, Dw(x))]^m \\
& + B^{-1} Dw(z) D_y f(x, w(x)) D^m w(x) \\
& - B^{-1} D_y g(x, w(x)) D^m w(x) \\
& + S_m^{\Delta^{m-1}w}(x) \\
& + T_m^{\Delta^{m-1}w}(D^m g(x, w(x)), Dw(x)) \\
& + P_{m-1}^{\Delta^{m-1}w}(D^{[m-1]}w(z), \Delta^{m-1} f(x, w(x)), D^{[m-1]}w(x)) \\
& + Q_{m-1}^{\Delta^{m-1}w}(\Delta^{m-1} g(x, w(x)), D^{[m-1]}w(x)), \\
z = {} & Ax + f(x, w(x)), \\
S_m^{\Delta^{m-1}w}(x) = {} & -B^{-1} D_x^m g(x, w(x)), \\
T_m^{\Delta^{m-1}w}(D^m g(x, u(x), Dw(x)) = {} & -B^{-1} D^m g(x, w(x))[(I, Dw(x))]^m - S_m^{\Delta^{m-1}w}(x),
\end{aligned} \tag{5.1}
$$

where $P_{m-1}^{\Delta^{m-1}w}(\gamma, \alpha, \beta)$ and $Q_{m-1}^{\Delta^{m-1}w}(\alpha, \beta)$ are multilinear in the components of α, β and γ and satisfy some vanishing properties which will be described in Definition 5.2 below.

3. It is obvious that $D^m w$ should be a fixed point for some transformation defined on an appropriately constructed function-space which we define presently.

Definition 5.2. 1. Let $\mathfrak{Z}_{r,o}, \cdots, \mathfrak{Z}_{r,m}, \mathfrak{X}_{r,o}, \cdots, \mathfrak{X}_{r,m}$ be as in Definitions 3.1 and 4.1. Let

$$
\begin{aligned}
\mathfrak{B}_{m-1} &= \mathfrak{X}_{r,o} \times \mathfrak{X}_{r,1} \times \cdots \times \mathfrak{X}_{r,m-1}, \\
U &= (u, \Sigma) = (u, \sigma_1, \cdots, \sigma_{m-1}) \in \mathfrak{B}_{m-1}, \\
\|\Sigma\|_* &= \max\{N_{r,1}(\sigma_1), \cdots, N_{r,m-1}(\sigma_{m-1})\}, \\
\|U\|_* &= \max\{N_{r,o}(u), N_{r,1}(\sigma_1), \cdots, N_{r,m-1}(\sigma_{m-1})\}.
\end{aligned}
$$

2. To simplify notation we will write

$$\varGamma_j := \varGamma_{\Delta^j F}, \qquad j = 0, 1, 2, \cdots, k.$$

3. Inductively, define $\varGamma_m : \mathfrak{B}_{m-1} \times \mathfrak{X}_{r,m} \to \mathfrak{B}_{m-1} \times \mathfrak{X}_{r,m}$ by

$$\varGamma_m(U, \rho) = (\varGamma_{m-1}(U), \varGamma_m^U \rho), \tag{5.2}$$

$$\begin{aligned} \varGamma_m^U \rho(x) &= \mathcal{K}^U \rho(x) + \mathcal{A}^U \rho(x) + \mathcal{B}^U \rho(x) \\ &\quad + C_m^U(x) + S_m^U(x) + T_m^U(x) \\ &\quad + P_{m-1}^U(x) + Q_{m-1}^U(x), \end{aligned} \tag{5.3}$$

$$z = H_u(x) = Ax + f(x, u(x)),$$

$$\mathcal{H}^u \sigma_1(x) = A + Df(x, u(x))(I, \sigma_1(x)),$$

where

$$\mathcal{K}^U \rho(x) = B^{-1} \rho(z)[\mathcal{H}^u \sigma_1(x)]^m, \tag{5.4}$$

$$\mathcal{A}^U \rho(x) = B^{-1} \sigma_1(z) D_y f(x, u(x)) \rho(x), \tag{5.5}$$

$$\begin{aligned} \mathcal{B}^U \rho(x) &= -B^{-1} D_y g(x, u(x)) \rho(x), \\ C_m^U(x) &= B^{-1} \sigma_1(z) D^m f(x, u(x))[(I, \sigma_1(x))]^m, \\ S_m^U(x) &= -B^{-1} D_x^m g(x, u(x)), \\ T_m^U(x) &= T_m^U(D^m g(x, u(x)), \sigma_1(x)) \\ &= -B^{-1} D^m g(x, u(x))[(I, \sigma_1(x))]^m - S_m^U(x) \\ &= \sum_{\substack{i+j=m \\ j \geq 1}} \theta^{i,j}(x), \\ \theta^{i,j}(x) &= D_x^i D_y^j g(x, u(x))[\sigma_1(x)]^j, \\ T_m^U(\cdot, 0) &\equiv 0, \qquad T_m^U(0, \cdot) \equiv 0, \\ P_{m-1}^U(x) &= P_{m-1}^U(\Sigma(z), \Delta^{m-1} f(x, u(x)), \Sigma(x)), \\ P_{m-1}^U(0, \cdot, \cdot) &\equiv 0, \qquad P_{m-1}^U(\cdot, 0, \cdot) \equiv 0, \end{aligned}$$

$$Q_{m-1}^U(x) = Q_{m-1}^U(\Delta^{m-1} g(x, u(x)), \Sigma(x)) = \sum \psi_{r_1 \cdots r_{m-1}}^{i,j}(x), \tag{5.6}$$

$$\psi_{r_1 \cdots r_{m-1}}^{i,j}(x) = D_x^i D_y^j g(x, u(x))[\sigma_1(x)]^{r_1} \cdots [\sigma_{m-1}(x)]^{r_{m-1}},$$

$$Q_{m-1}^U(\cdot, 0) \equiv 0, \qquad Q_{m-1}^U(0, \cdot) \equiv 0, \tag{5.7}$$

where the sum in (5.6)is taken over the finite set of indices

$$\left\{ (i, j, r_1, \cdots, r_{m-1}) \;\middle|\; i + j \leq m - 1,\ r_1 + \cdots + r_{m-1} = j \geq 1,\ i + \sum_{l=1}^{m-1} l r_l = m \right\},$$

and $P_{m-1}^U(\gamma, \alpha, \beta)$ is multilinear in the components of α, β and γ.

Since convergence in the norm $\|\cdot\|_*$ implies uniform convergence on bounded sets, it follows that $\mathfrak{B}_{m-1}$ is a closed subset of the Banach space $\mathfrak{Z}_{r,o} \times \cdots \times \mathfrak{Z}_{r,m-1}$. We do not need to write $P^U_{m-1}(x)$ in great detail as we did with $Q^U_{m-1}(x)$ and $T^U_m(x)$ because of the presence of a component of $\Sigma(z)$ in each term which causes $P^U_{m-1}(x)$ to be of order $|x|^{m-1}$.

5.3. Induction Hypothesis IH(m). Assume that Γ_{m-1} maps $\mathfrak{B}_{m-1}$ to itself and has a unique attractive fixed point $W := \Delta^{m-1}w = (w, \Omega) \in \mathfrak{B}_{m-1}$. Assume that the hypotheses of Theorem 4 are satisfied for $n = m \leq k$.

Lemma 5.4. *Let $z = H_u(x)$ and $z^* = H_v(x)$. Let $M = P, Q, R =, C, S, T$ and $\mathcal{E} = \mathcal{A}, \mathcal{B}$. Then for sufficiently small $\epsilon > 0$, the following are true for :*

$$\|M^U_{m-1}(x)\| \leq t|x|^{p_{m-1}} \leq t|x|^{p_m}, \tag{1}$$

$$\|R^U_m(x)\| \leq t|x|^{p_m} \tag{2}$$

$$\|M^U_{m-1}(x) - M^U_{m-1}(x')\| \leq t|x-x'|(|x|+|x'|)^{q_{m-1}} \leq t|x-x'|(|x|+|x'|)^{q_m}, \tag{3}$$

$$\|R^U_m(x) - R^U_m(x')\| \leq t|x-x'|^{s_m}(|x|+|x'|)^{q_m} \tag{4}$$

$$\|M^U_{m-1}(x) - M^V_{m-1}(x)\| \leq t\|U-V\|_*|x|^{p_{m-1}} \leq t\|U-V\|_*|x|^{p_m}, \tag{5}$$

$$\|R^U_m(x) - R^V_m(x)\| \leq t[\|U-V\|_*]^{s_m}|x|^{p_m}, \tag{6}$$

$$\begin{aligned}\|\rho(z) - \rho(z^*)\| &\leq L_{r,m}(\rho)|z-z^*|^{s_m}(|z|+|z^*|)^{q_m} \\ &\leq tL_{r,m}(\rho)[N_{r,o}(u-v)]^{s_m}|x|^{rs_m+q_m} \\ &\leq t[\|U-V\|_*]^{s_m}|x|^{p_m},\end{aligned} \tag{7}$$

$$\begin{aligned}\|\mathcal{K}^U\rho(x)\| &\leq b^{-1}a^{m+p_m}(1+t)N_{r,m}(\rho)|x|^{p_m} \\ &= b^{-1}a^r(1+t)N_{r,m}(\rho)|x|^{p_m},\end{aligned} \tag{8}$$

$$\begin{aligned}\|\mathcal{K}^U\rho(x) - \mathcal{K}^U\rho(x')\| &\leq [b^{-1}a^{s_m+q_m+m}(1+t)L_{r,m}(\rho)+t]|x-x'|^{s_m}(|x|+|x'|)^{q_m} \\ &= [b^{-1}a^r(1+t)L_{r,m}(\rho)+t]|x-x'|^{s_m}(|x|+|x'|)^{q_m},\end{aligned} \tag{9}$$

$$\begin{aligned}\|\mathcal{K}^U\rho(x) - \mathcal{K}^V\rho(x)\| &\leq t[N_{r,o}(u-v)]^{s_m}|x|^{p_o s_m+q_m} \\ &\leq t\|U-V\|^{s_m}_*|x|^{p_m},\end{aligned} \tag{10}$$

$$\begin{aligned}\|\mathcal{K}^U\rho(x) - \mathcal{K}^U\tau(x)\| &\leq b^{-1}a^{m+p_m}(1+t)N_{r,m}(\rho-\tau)|x|^{p_m} \\ &= b^{-1}a^r(1+t)N_{r,m}(\rho-\tau)|x|^{p_m},\end{aligned} \tag{11}$$

$$\|\mathcal{E}^U\rho(x)\| \leq t|x|^{p_m}, \tag{12}$$

$$\|\mathcal{E}^U\rho(x) - \mathcal{E}^U\rho(x')\| \leq t|x-x'|^{s_m}(|x|+|x'|)^{q_m}, \tag{13}$$

$$\|\mathcal{E}^U\rho(x) - \mathcal{E}^V\rho(x)\| \leq t\|U-V\|_*|x|^{p_m}. \tag{14}$$

Proof. The Lemma follows by straightforward calculations similar to those of Lemma 4.4 if we recall properties (5.4)-(5.7) and the multilinearity of P^U_{m-1}.

Notice that in each term in $\Gamma^U_m\rho$, the factors that need special attention are ρ and components of $D^m f$ and $D^m g$. The rest are at least uniformly Lipschitz continuous. For example, $\Delta^{m-1}f(x,y)$ and $\Delta^{m-1}g(x,y)$ are $C^{1,\delta}$ and $\Sigma(x)$ is $C^{0,1}$ since $s_o = \cdots = s_{m-1} = 1$. Notice also that since $\|U\|_* \leq 1$ and $\|V\|_* \leq 1$, we have $\|U-V\|_* \leq 2^{1-s_m}\|U-V\|^{s_m}_*$. However, in all calculations, this happens only when we have a factor of "t" which takes care of the "2^{1-s_m}" factor. It is important to recall also that $x \in B_1(0)$. $\square$

Proposition 5.5. *For sufficiently small $\epsilon > 0$, the following are true:*

1. *Γ_m maps $\mathfrak{B}_{m-1} \times \mathfrak{X}_{r,m}$ to itself.*
2. *$\Gamma_m^U \rho$ is uniformly s_m-Hölder continuous in U. In fact,*
$$\|\Gamma_m^U \rho(x) - \Gamma_m^V \rho(x)\| \leq t[\|U - V\|_*]^{s_m} |x|^{p_m}.$$
3. *$\|\Gamma_m^U \rho(x) - \Gamma_m^U \tau(x)\| \leq \theta N_{r,m}(\rho - \tau)|x|^{p_m}$ where $\theta = b^{-1}a^r(1+t) < 1$.*
4. *Thus, Γ_m^W has a unique attractive fixed point μ in the invariant fiber $\{W\} \times \mathfrak{X}_{r,m}$.*
5. *By the Fiber Contraction Theorem, it follows that (W, μ) is a unique attractive fixed point for $\Gamma_m : \mathfrak{B}_{m-1} \times \mathfrak{X}_{r,m} \to \mathfrak{B}_{m-1} \times \mathfrak{X}_{r,m}$.*
6. *$D^m w = \mu$.*

Proof. Assertions (1)-(3) follow from Lemma 5.4 by straightforward calculations Assertion (4) and (5) are obvious. Assertion (6) follows from the fact that $\Gamma_{\Delta^m F}(\Delta^m u) = \Delta^m(\Gamma_F u)$ using an argument similar to that in the proof of Proposition 4.9. $\square$

6. The $\mathbf{C^k}$ Manifold, $\delta = \mathbf{0}$, For Theorem 4

In this section we prove Theorem 4 when the map is only $C^{k,0}$. That is when $n = k = r$ and $\delta = 0$. In Remark 6.9 at the end of this section we will explain why it is necessary to consider the case $\delta = 0$ for $n = k$ and not $n < k$.

When $\delta = 0$, $\Gamma_k(U, \rho)$ of Section 5 is still a fiber contraction but not necessarily continuous in U. More precisely, $\Gamma_k^U \rho$ is not continuous in the base point U. Even at the C^1 level, $\Gamma_1^u \rho$ is not necessarily continuous in u. That is, Proposition 4.6 is not true. The points where the argument fails are assertions (3), (4), and (8) of Lemma 4.4. These are the places where $s_1 = \delta = 0$ appears as an exponent of $N_{r,o}(u - v)$. At the C^k level, assertion (2) of Proposition 5.5 is false due to the falsehood of assertions (6), (7) and (10) of Lemma 5.4 when $m = k$, $s_k = \delta = 0$. Assertions (3), (4), (9), (13) are redundant when $\delta = 0$.

In spite of this superficial setback, the argument in Section 5 still shows that Γ_k maps $\mathfrak{B}_{k-1} \times \mathfrak{X}_{k,k}$ to itself and is a fiber contraction which is not necessarily continuous in the base point U. Since W is the unique attractive fixed point for Γ_{k-1}, the fiber $\{W\} \times \mathfrak{X}_{k,k}$ is invariant under Γ_k. Thus $\Gamma_k^W : \mathfrak{X}_{k,k} \to \mathfrak{X}_{k,k}$ is a contraction and, by the standard contraction mapping theorem, has a unique attractive fixed point μ which is our candidate for $D^k w$.

Theorem 4, with $n = k = r$ and $\delta = 0$, follows from the following Proposition 6.1:

Proposition 6.1. *If x and h are in X such that both x and $x + h$ are in $B_1(0)$, then*

$$D^{k-1}w(x+h) - D^{k-1}w(x) = \int_0^1 \mu(x+th)h\,dt. \tag{6.1}$$

We prove Proposition 6.1 by taking a fixed but arbitrary sequence $\{u_n \in \mathfrak{X}_{r,o}\}_{n=0}^{\infty}$, such that $\Delta^k u_o \in \mathfrak{B}_{k-1} \times \mathfrak{X}_{k,k}$, and $u_n = (\Gamma_F)^n u_o$. Thus $\|\Delta^{k-1}u_n - \Delta^{k-1}w\|_* \to 0$. Then we reformulate the problem on the line segment $J = \{x + th \mid 0 \leq t \leq 1\}$ which is compact.

The reason for using this approach is the simple fact that a continuous map on a compact set is obviously uniformly continuous but a continuous map on a closed bounded subset of an infinite dimensional Banach space is not necessarily uniformly continuous. We will need compactness in assertions (5)-(10) of Lemma 6.6.

If X is finite dimensional, then $B_1(0)$ is compact and the argument of this section can be used on all of $B_1(0)$.

If $0 < \delta \leq 1$, a $C^{0,\delta}$ function is always uniformly continuous. This is what we used in Lemma 4.4, especially assertion (8).

Definition 6.2. Let x and h be as in Proposition 6.1 and define the following:

1. Let $I = [0,1]$ and $\xi : [0,1] \to X$ be given by $\xi(t) = x + th$. Let $J = \xi(I)$.
2. Let $u_o \in \mathfrak{X}_{r,o}$ be such that $\Delta^k u_o \in \mathfrak{B}_{k-1} \times \mathfrak{X}_{k,k}$. Let $u_n = \Gamma_o u_{n-1}, n \geq 1$. Thus, for $n \geq 0, \Delta^k u_n \in \mathfrak{B}_{k-1} \times \mathfrak{X}_{k,k}$. Let $u_\infty = w$.
3. Let $U_n = \Delta^{k-1} u_n$ and $U_\infty = W = \Delta^{k-1} w$. Thus, for $n \geq 0, U_n \in \mathfrak{B}_{k-1}, U_n = (\Gamma_{k-1})^n(U_o)$, and $\|U_n - W\|_* \to 0$ as $n \to \infty$.
4. Let $\phi_n = U_n|_J$, for $n \geq 0$, and $\phi_\infty = W|_J = U_\infty|_J$.
5. Let
$$\begin{aligned} \mathbb{V} &= \{\phi_n \mid 0 \leq n \leq \infty\}, \\ \mathbb{Y} &= \{\hat{\rho} = \rho|_J \ \mid \ \rho \in \mathfrak{X}_{k,k}\}, \\ \|\phi_n\|_o &= \|U_n|_J\|_*, \\ \|\hat{\rho}\|_1 &= \|\rho|_J\| = \sup_{\xi \in J} \|\rho(\xi)\|, \qquad \hat{\rho} = \rho|_J \in \mathbb{Y}. \end{aligned}$$
6. Let $\Lambda : \mathbb{V} \times \mathbb{Y} \to \mathbb{V} \times \mathbb{Y}$ be the obvious restriction of Γ_k to $\mathbb{V} \times \mathbb{Y}$ given by:
$$\Lambda(\phi_j, \hat{\rho}) := (\Lambda_o \phi_j \ , \ \Lambda^{\phi_j} \hat{\rho}) := [\Gamma_k(U_j, \rho)]|_J. \tag{6.2}$$
Thus, for $\xi \in J$,
$$\begin{aligned} \Lambda_o \phi_j(\xi) &:= \Lambda_o(U_j|_J)(\xi) := [\Gamma_{k-1} U_j]|_J(\xi), \\ \Lambda^{\phi_j} \hat{\rho}(\xi) &:= [\Gamma^{U_j} \rho]|_J(\xi) \\ &= \mathcal{K}^{U_j} \rho(\xi) + \mathcal{A}^{U_j} \rho(\xi) + \mathcal{B}^{U_j} \rho(\xi), \\ &\quad + C_k^{U_j}(\xi) + S_k^{U_j}(\xi) + T_k^{U_j}(\xi) \\ &\quad + P_{k-1}^{U_j}(\xi) + Q_{k-1}^{U_j}(\xi), \\ z_j &= H_{u_j}(\xi) = A\xi + f(\xi, u_j(\xi)), \\ \mathcal{H}^{u_j} \sigma_{j,1}(\xi) &= [A + Df(\xi, u_j(\xi))(I, \sigma_{j,1}(\xi))], \\ U_j &= (u_j, \Sigma_j) = (u_j, \sigma_{j,1}, \cdots, \sigma_{j,k-1}), \end{aligned} \tag{6.3}$$
where $\mathcal{K}, \mathcal{A}, \cdots, Q$ are given by (5.4)-(5.7) with $m = k$.

Remarks 6.3. 1. It is obvious that $(\mathbb{V}, \|\cdot\|_o)$ is a complete metric space since it consists of one sequence and its limit point. Also $(\mathbb{Y}, \|\cdot\|_1)$ is a metric space. Recall that the fiber contraction theorem is stated for metric spaces.

2. Recall that

$$(6.4)\qquad \begin{aligned}(\Gamma_{k-1}U_j\ ,\ \Gamma^{U_j}D^k u_j) &= \Gamma_k(U_j, D^k u_j) = \Gamma_k(\Delta^k u_j)\\ &= \Delta^k(\Gamma_o u_j) = \Delta^k u_{j+1} = (U_{j+1}, D^k u_{j+1}).\end{aligned}$$

Restricting (6.4) to J, we obtain

$$(6.5)\qquad \begin{aligned}\Lambda_o\phi_j &= [\Gamma_{k-1}U_j]|_J = U_{j+1}|_J = \phi_{j+1},\\ \Lambda^{\phi_j}(D^k u_j|_J) &= [\Gamma^{U_j}D^k u_j)]|_J = [D^k u_{j+1}]|_J.\end{aligned}$$

3. When $\delta = 0, p_k = 0$. Thus, $N_{k,k}(\cdot) = \|\cdot\|$, the standard sup-norm. Also the condition $L_{k,k}(\sigma) \leq 1$ becomes redundant.

Proposition 6.4. *The following are true for sufficiently small $\epsilon > 0$:*
1. *Λ maps $\mathbb{V}\times\mathbb{Y}$ to itself.*
2. *The fiber $\mathfrak{Y}_\infty := \{\phi_\infty\}\times\mathbb{Y}$ is invariant under Λ.*
3. *$\Lambda : \mathbb{V}\times\mathbb{Y}\to\mathbb{V}\times\mathbb{Y}$ is a fiber contraction (not necessarily continuous in U_j yet).*
4. *$\Lambda^{\phi_\infty} : \mathfrak{Y}_\infty\to\mathfrak{Y}_\infty$ has a unique attractive fixed point τ.*
5. *$\tau = \mu|_J$ where μ is the unique attractive fixed point of $\Gamma_k^W : \{W\}\times\mathfrak{X}_{k,k}\to\{W\}\times\mathfrak{X}_{k,k}$.*

Proof. All estimates of Lemma 5.4 and Proposition 5.5 which do not have an exponent $s_m = s_k = 0$ are valid here with $m = k$ and $x = \xi$. Thus assertions (1)-(4) follow from Proposition 5.5. As for (5), it is obvious that $\mu|_J$ is a fixed point for $\Lambda^{\phi_\infty} = \Gamma^w\ |_J$. But τ is the unique fixed point of Λ^{ϕ_∞}. Thus $\tau = \mu|_J$. □

In order to be able to apply the Fiber Contraction Theorem we need to show that $\Lambda(\phi_j, \hat\rho)$ is continuous in ϕ_j. We need the following elementary but important lemma.

Lemma 6.5. *The following sets are compact subsets of Y:*

$$\begin{aligned}
&\mathcal{V}_j = u_j(H_{u_j}(J)), && \mathcal{U}_j = u_j(J), && \mathcal{Z}_j = H_{u_j}(J),\\
&\mathcal{V}_\infty = w(H_w(J)), && \mathcal{U}_\infty = w(J), && \mathcal{Z}_\infty = H_w(J),\\
&\mathcal{V} = \bigcup_{j=0}^{j=\infty}\mathcal{V}_j, && \mathcal{U} = \bigcup_{j=0}^{j=\infty}\mathcal{U}_j, && \mathcal{Z} = \bigcup_{j=0}^{j=\infty}\mathcal{Z}_j,\\
&\mathcal{W} = \mathcal{U}\cup\mathcal{V}\cup\mathcal{Z}.
\end{aligned}$$

Proof. Recall that X and Y are Banach spaces. All the sets defined, except $\mathcal{Z}, \mathcal{U}, \mathcal{V}$ and $\mathcal{W}$, are continuous images of the interval I, and hence, compact. $\mathcal{W}$ is compact, being the union of three compact sets. First we will show that $\mathcal{V}$ is compact.

Since $\mathcal{V}_\infty$ is compact, for all $\eta > 0$ there is a finite set of points $\{x_1, \cdots, x_{n_\eta}\}\subset\mathcal{V}_\infty$ such that

$$\mathcal{V}_\infty\subset\mathcal{O}(\eta) = \bigcup_{i=1}^{n_\eta}\mathcal{O}(x_i,\eta), \qquad \mathcal{O}(x,\eta) = \{x'\in B_1(0)\mid |x'-x| < \eta\}.$$

By (3.11) of Lemma 3.7 we have

$$(6.6)\qquad \|u_j(H_{u_j}(\xi)) - w(H_w(\xi))\| \leq a^r(1+t)N_{r,o}(u_j - w)|\xi|^{p_o}\to 0$$

as $j \to \infty$ uniformly in $|\xi| \le 1$.

It follows that for all $\eta > 0$ there is j_o such that: If $j \ge j_0$, then $\mathcal{V}_j \subset \mathcal{O}(\eta)$. The finite union of compact sets $\mathcal{V}_0 \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{j_o - 1}$, is compact. Thus, for all $\eta > 0$, $\mathcal{V}$ has a finite cover of open balls of radius η. Since Y is also a metric space, it follows that $\mathcal{V}$ is compact.

We can show in a similar fashion that $\mathcal{U}$ and $\mathcal{Z}$ are compact. We only need to replace (6.6) by

$$|u_j(\xi) - w(\xi)\| \le N_{r,o}(u_j - w)|\xi|^{p_o} \to 0,$$
$$|H_{u_j}(\xi) - H_w(\xi)| \le tN_{r,o}(u_j - w)|\xi|^{p_o} \to 0,$$

respectively. The first inequality is obvious. The second follows from (3.10). □

Lemma 6.6. *Let $\phi = U|_J$ and $\psi = V|_J$ be in $\mathbb{V}$ and $\rho|_J$ be in $\mathbb{Y}$. Write $z = H_u(\xi)$ and $z^* = H_v(\xi)$. The following are true:*

1. $|z - z^*| < tN_{r,o}((u-v)\ |_J)|\xi|^r \le t\|\phi - \psi\|_o|\xi|^r$.
2. $|u(z) - v(z^*)| \le a^k(1+t)N_{r,o}((u-v)|_J)|\xi|^r \le a^k(1+t)\|\phi - \psi\|_o|\xi|^r$.
3. $\|M_{k-1}^U(\xi) - M_{k-1}^V(\xi)\| < t\|(U-V)|_J\|_*|\xi| = t\|\phi - \psi\|_o|\xi|, M = P, Q$.
4. $\|\mathcal{E}^U\rho(\xi) - \mathcal{E}^V\rho\| \le t\|U - V\|_*, \mathcal{E} = \mathcal{A}, \mathcal{B}$.

   Moreover, for all $\eta > 0$, there is $c > 0$ such that: if $\|\phi - \psi\|_o < c$, then the following hold for all $\xi \in J$:
5. $\|\rho(z) - \rho(z^*)\| < \eta$.
6. $\|D^k h(\xi, u(\xi)) - D^k h(\xi, v(\xi))\| < \eta, h = f, g$.
7. $\|S_k^U(\xi) - S_k^V(\xi)\| < \eta$.
8. $\|T_k^U(\xi) - T_k^V(\xi)\| < \eta$.
9. $\|C_k^U(\xi) - C_k^V(\xi)\| < \eta$.
10. $\|\mathcal{K}^U\rho(\xi) - \mathcal{K}^V\rho(\xi)\| < \eta$.

Proof. Notice that in each term of $\Lambda^{\phi_j}\hat{\rho}$, the factors that need special attention are ρ and components of $D^k f$ and $D^k g$. The rest are at least uniformly Lipschitz continuous. But since $\mathcal{W}$ is compact, $D^k f, D^k g$ and ρ are uniformly continuous on $\mathcal{W}$.

Assertions (1) and (2) follow from Lemma 3.7. Assertion (3) follows from assertion (5) of Lemma 5.4 since $p_{k-1} = 1$.

Assertion (4) is true because $Dh, h = f, g$, is at least uniformly Lipschitz continuous and $\sigma_{j,1}$ is also uniformly Lipschitz continuous since $s_1 = 1$.

To prove assertion (5) notice that ρ is uniformly continuous on $\mathcal{W}$ since $\mathcal{W}$ is compact. Thus for all $\eta > 0$ there is $c > 0$ such that: if $x, x' \in \mathcal{W}$ and $|x - x'| < c$, then $\|\rho(x) - \rho(x')\| < \eta$. Let $\|\phi - \psi\|_o < c$. Since $z, z^* \in \mathcal{W}$, it follows from (1) that $|z - z^*| \le \|\phi - \psi\|_o < c$. Thus $\|\rho(z) - \rho(z^*)\| < \eta$.

Similar arguments prove assertions (6) and (7) since $D^k h$ is uniformly continuous on $\mathcal{W}$ for $h = f, g$.

Assertion (8) follows from (6) and (7) and the fact that $\sigma_{j,1}$ is uniformly Lipschitz continuous because $s_1 = 1$.

Assertion (9) follows from assertion (6).

As for assertion (10), notice that in view of (5) above, $\rho(z)$ in $\mathcal{K}^U\rho(\xi)$ is continuous in u. The term $\mathcal{H}^u\sigma_{j,1}$ is obviously uniformly Lipschitz continuous in u and $\sigma_{j,1}$. □

Proposition 6.7. *$\Lambda(\phi, \hat\rho)$ is continuous in ϕ.*

Proof. We know that $\Lambda_o\phi$ is continuous in ϕ. Therefore we only need to show that $\Lambda^\phi\hat\rho$ is continuous in ϕ.

Notice that in each term of $\Lambda^{\phi_j}\hat\rho$, the factors that need special attention are D^kf, D^kg and ρ. The rest are at least uniformly Lipschitz continuous. Moreover, since $\mathcal{W}$ is compact, D^kf, D^kg and ρ are uniformly continuous on $\mathcal{W}$.

The last four terms in (6.3) are continuous in ϕ in view of assertions (3) and (6)-(8) of Lemma 6.6.

The first four terms (6.3) are continuous in ϕ in view of assertions (4) and (9)-(10) of the same Lemma 6.6.

Therefore, for all $\eta > 0$ there is a $c > 0$ such that if $\|\phi - \psi\|_o < c$, then

$$\|\Lambda^\phi\hat\rho - \Lambda^\psi\hat\rho\| \le \eta,$$

which proves that $\Lambda(\phi, \hat\rho)$ is continuous in ϕ. □

Proposition 6.8. *For sufficiently small $\epsilon > 0, \Lambda : \mathbb{V}\times\mathbb{Y} \to \mathbb{V}\times\mathbb{Y}$ has a unique attractive fixed point $(\phi_\infty, \mu|_J)$.*

Proof. We know from Proposition 6.4 that $\Lambda : \mathbb{V}\times\mathbb{Y} \to \mathbb{V}\times\mathbb{Y}$ is a fiber contraction, that the fiber $\mathfrak{Y}_\infty = \{\phi_\infty\}\times\mathbb{Y}$ is invariant under Λ and that Λ^{ϕ_∞} has a unique attractive fixed point $\tau = \mu|_J \in \mathfrak{Y}_\infty$. By Proposition 6.7, Λ is continuous in both arguments. Thus by the Fiber Contraction Theorem, $(\phi_\infty, \mu|_J)$ is a unique attractive fixed point for $\Lambda : \mathbb{V}\times\mathbb{Y} \to \mathbb{V}\times\mathbb{Y}$. □

Proof of Proposition 6.1. Each u_n is C^k. Therefore

$$\mathcal{L}_n(x,h) := D^{k-1}u_n(x+h) - D^{k-1}u_n(x) = \int_0^1 D^ku_n(x+th)hdt =: \mathcal{R}_n(x,h). \tag{6.7}$$

We also know that $N_{k,k-1}(D^{k-1}u_n - D^{k-1}w) \to 0$ as $n\to\infty$, which implies uniform convergence on all of $B_1(0)$. Thus $\mathcal{L}_n(x,h)$ converges uniformly to the left hand side of (6.1). It remains to show that $\mathcal{R}_n(x,h)$ converges to the right hand side of (6.1).

Recall that $(\phi_\infty, \mu|_J)$ is the unique attractive fixed point of Λ. Thus, by (6.5),

$$\|\mu|_J - D^ku_n|_J\| \to 0,$$

as $n \to 0$. It follows that,

$$\|\int_0^1 D^ku_n(x+th)h - \mu(x+th)hdt\| \le |h|\|D^ku_n|_J - \mu|_J\| \to 0$$

as $n \to 0$. □

Remark 6.9. In Section 5 we proved Theorem 4 for a $C^{n,\delta}$ map with $n \le k$ and $0 < \delta \le 1$. In this Section 6 we proved Theorem 4 for a $C^{k,0}$ map with $n = k$ and $\delta = 0$. It is not possible to consider the $\delta = 0$ case when $n < k$. The reason is that Lemma 10.9 allows us to replace assertion (5) of Lemma 6.6 by only

$$\begin{aligned}\|\rho(z) - \rho(z^*)\| &\le \eta^{\gamma/p_n}[N_{k,n}(\rho)]^{1-(\gamma/p_n)}(|z| + |z^*|)^{p_n-\gamma},\\ &\le \eta^{\gamma/p_n}[N_{k,n}(\rho)]^{1-(\gamma/p_n)}a^{p_n-\gamma}(1+t)|x|^{p_n-\gamma},\end{aligned}$$

for any $0 < \gamma \leq 1$. We can replace assertions (6)-(10) of the same lemma by similar estimates. These estimates are not good enough because the factor $|x|^{p_n-\gamma}$ is not of the right order. We need a factor of $|x|^{p_n}$ in order to be able to use the norm $N_{k,n}$. We do not encounter this problem when $n = k$ and $\delta = 0$ because $N_{k,k} = \|\cdot\|$, the standard sup-norm.

7. Proof of Theorem 1

Remarks 7.1. We would like to start this section by emphasizing a few points and making the notation more precise.

1. We started with the map of Theorem 1 which we denote by
$$\Phi(z) = D\Phi(0) + F(z).$$
We are looking for an invariant manifold for Φ of the form
$$y = w(x), \qquad |x| \leq \gamma_k, \tag{7.1}$$
alternatively, $x \in B_k := B_{\gamma_k}(0)$.
2. In Lemma 2.8 we eliminated all j-monomials in $x, 2 \leq j \leq k, k \geq L \geq 2$, from $g(x, y)$. We denote the new map by
$$\Phi_{(k)}(z) = D\Phi(0) + F_{(k)}(z).$$
We denote the invariant manifold for $\Phi_{(k)}$ by
$$y = w^{(k)}(x), \qquad |x| \leq \gamma_k. \tag{7.2}$$
Notice that when we eliminate $g_2(x)$ we change the remainder of $g(x, y)$ in particular $g_i(x), 3 \leq i \leq k$. The same thing happens at each step. In order not to complicate the notation, at each step we will continue to denote the new y-component of the map by $g(x, y)$ and the new j-monomials in x by $g_j(x)$.
3. In Lemma 2.9 we created , so to speak, smallness assumptions on $F_{(k)}(z)$, measured by $\epsilon = \epsilon_k$, (H(5)-H(9)) which correspond to small $\gamma_k > 0$ in Theorem 1. We denote the new map by
$$\Phi_{(k)}^{(\gamma_k)}(z) = D\Phi(0) + F_{(k)}^{(\gamma_k)}(z).$$
Theorem 4 was proven for $\Phi_{(k)}^{(\gamma_k)}$. We denote the invariant manifold of $\Phi_{(k)}^{(\gamma_k)}$ obtained in Theorem 4 by
$$y = w_{(\gamma_k)}(x), \qquad |x| \leq 1. \tag{7.3}$$
We also know that, $D^j w_{(\gamma_k)} \in \mathfrak{X}_{r,j}, 0 \leq j \leq k, r = k + \delta$.
4. It follows from Lemma 2.9 that
$$w^{(k)}(x) = \gamma_k w_{(\gamma_k)}(x/\gamma_k), \qquad |x| \leq \gamma_k. \tag{7.4}$$
Equivalently,
$$w_{(\gamma_k)}(x) = \frac{1}{\gamma_k} w^{(k)}(\gamma_k x), \qquad |x| \leq 1. \tag{7.5}$$
Strictly speaking, we should denote $w_{(\gamma_k)}$ by $w_{(\gamma_k)}^{(k)}$. We do not do that to keep the notation simple.

5. By (7.5) we can see that for $0 \le j \le k$

$$N_{r,j}(D^j w_{(\gamma_k)}) \le 1 \Longleftrightarrow N_{r,j}(D^j w^{(k)}) \le \frac{1}{\gamma_k^{k+\delta-1}}, \tag{7.6}$$
$$L_{r,j}(D^j w_{(\gamma_k)}) \le 1 \Longleftrightarrow L_{r,j}(D^j w^{(k)}) \le \frac{1}{\gamma_k^{k+\delta-1}},$$

where the sups on the left-hand side are taken over $x \in B_1(0)$, while the sups on on the righthand side are taken over $x \in B_{\gamma_k}(0)$. This equivalence allows us to translate estimates (2.24)-(2.27) (for $w_{(\gamma_k)}$) of Theorem 4 to estimates (2.3)-(2.6) (for $w^{(k)}$) of Theorem 1. Assertion (5) of theorem 1 will be dealt with separately in Proposition 7.4.
6. It follows that the following is true for the map of Theorem 1 after performing the elimination, that is for the map $\Phi_{(k)}$: For every $k \ge L \ge 2$, there is an invariant $C^{k,\delta}$ manifold given by $y = w^{(k)}(x), x \in B_k := B_{\gamma_k}(0)$, satisfying assertions (3)-(5) of Theorem 1. And possibly $\gamma_k \to 0$ as $k \to \infty$. However, $w^{(k)}|_{B_{k+1}} \ne w^{(k+1)}$ because we obtain $w^{(k+1)}$ after eliminating $g_{k+1}(x)$ from $\Phi_{(k)}$. That is, $w^{(k+1)}$ corresponds to $\Phi_{(k+1)}$, while $w^{(k)}$ corresponds to $\Phi_{(k)}$.
7. In Proposition 7.2 we show that

$$w^{(k+1)} = [w^{(k)}|_{B_{k+1}}] - p_{k+1},$$

where p_{k+1} is as in Lemma 2.8.
8. In Proposition 7.3 we show that if $2 \le L < k \le \infty$, then $w^{(L)} \in C^{k,\delta}(B_L, Y)$ and that $w^{(L)}$ is unique in the class of functions $v : B_L \to Y$ which satisfy (2.3) and (2.4). In view of assertion (4) of Lemma 2.8 we have

$$w(x) = p_2(x) + \cdots + +p_L(x) + w^{(L)}(x), \qquad |x| \le \gamma_L.$$

This proves assertions(1)-(3) of Theorem 1. Assertion (4) follows from the fact that $D^j w_{(\gamma_L)} \in \mathfrak{X}_{r,j}, 0 \le j \le L, r = L + s$, and equivalence (7.6).
9. In Proposition 7.4 we prove assertion (5) of Theorem 1. That is, we show that $w^{(L)}$ is unique in the class of $C^{L,\beta}$ functions $v : B_L \to Y$ which satisfy (2.7) and (2.8).
10. In Proposition 7.6 we prove assertion (6) of Theorem 1 : $y = w(x), |x| \le \gamma_L$, is unique in $C^{L,\beta}(B_L, Y)$. Basically, $w = p_2 + \cdots + p_L + w^{(L)}$, where each term is unique.
11. We would like to point out that it is essential for the uniqueness of w in $C^{L,\beta}(B_\gamma(0), Y)$ to know that $w^{(L)}$ is unique in the class of functions that satisfy (2.7) and (2.8). It is not enough that $w^{(L)}$ is unique in the class of functions that satisfy (2.3) and (2.4) because if we are given a manifold $y = v^{(L)}(x), v \in C^{L,\beta}(B_L, Y)$ which is invariant under $\Phi_{(L)}$, we cannot conclude that $v^{(L)}$ satisfies (2.3) and (2.4). All what we can conclude is that $v^{(L)}$ satisfies (2.7) and (2.8). We elaborate on this issue in Remarks 7.5.

Proposition 7.2. *Let $w^{(k)}|_{B_{k+1}}$ be the restriction of $w^{(k)}$ to B_{k+1}. Let $p_{k+1} \in \mathcal{S}(X^{k+1}, Y)$ be as in Lemma 2.8. I.e. the change of variables $(x, y) \mapsto (x, \bar{y}), \bar{y} = y - p_{k+1}(x)$, eliminates the $(k+1)$-monomials $g_{k+1}(x)$. Then,*

$$w^{(k+1)}(x) = [w^{(k)}|_{B_{k+1}}](x) - p_{k+1}(x), \qquad |x| \le \gamma_{k+1}.$$

Thus

$$[w^{(k)} \mid B_{k+1}] \in C^{k+1,\delta}(B_{k+1}, Y).$$

Proof. To simplify notation we will write $f(x;u) := f(x, u(x)), \cdots$. Notice that $w^{(k)}$ and p_{k+1} satisfy the following:

$$w^{(k)}(Ax + f(x; w^{(k)})) = Bw^{(k)}(x) + g_{k+1}(x) + \bar{h}_{k+2}(x; w^{(k)}), \tag{7.7}$$

$$p_{k+1}(Ax) = Bp_{k+1}(x) + g_{k+1}(x). \tag{7.8}$$

It follows from (2.20) that after eliminating $g_{k+1}(x)$ with the change of variables $(x, y) \mapsto (x, \bar{y})$ the map takes the form

$$x^* = Ax + f(x, \bar{y} + p_{k+1}(x)), \tag{7.9}$$

$$\bar{y}^* = B\bar{y} + \bar{h}_{k+2}(x, \bar{y} + p_{k+1}(x)) - p_{k+1}(x^*) + p_{k+1}(Ax), \tag{7.10}$$

with invariant manifold $\bar{y} = w^{(k+1)}(x), |x| \leq \gamma_{k+1}$, that satisfies

$$\begin{aligned} w^{(k+1)}(Ax + f(x; w^{(k+1)} + p_{k+1})) = Bw^{(k+1)}(x) + \bar{h}_{k+2}(x; w^{(k+1)} + p_{k+1}) \\ - p_{k+1}(Ax + f(x; w^{(k+1)} + p_{k+1})) \quad + p_{k+1}(Ax). \end{aligned} \tag{7.11}$$

Let $\tilde{v}(x) = w^{(k)}(x) - p_{k+1}(x), x \in B_{k+1}$. Straightforward computations, using (7.7) and (7.8), show that $\tilde{v}$ satisfies the same equation (7.11) that $w^{(k+1)}$ satisfies, namely,

$$\begin{aligned} \tilde{v}(Ax + f(x; \tilde{v} + p_{k+1})) =& B\tilde{v}(x) + \bar{h}_{k+2}(x; \tilde{v} + p_{k+1}) \\ &- p_{k+1}(Ax + f(x; \tilde{v} + p_{k+1})) + p_{k+1}(Ax). \end{aligned} \tag{7.12}$$

Straightforward calculations, using (2.23), show that (7.12) can be rewritten as

$$\Gamma_F \tilde{v}_{(\gamma_{k+1})} = \tilde{v}_{(\gamma_{k+1})},$$

where $F = F^{(\gamma_{k+1})}$. Notice that

$$|z| = |Ax + f(x; [\tilde{v} + p_{k+1}]_{(\gamma_{k+1})})| = a + |f(x; w_{(\gamma_{k+1})})| \leq a(1+t)|x|.$$

The same calculations that led to assertions (1) and (2) of Proposition 3.5 lead to the estimates

$$N_{r,o}(\tilde{v}_{(\gamma_{k+1})}) = N_{r,o}(\Gamma_F \tilde{v}_{(\gamma_{k+1})}) \leq \theta N_{r,o}(\tilde{v}_{(\gamma_{k+1})}) + t,$$
$$L_{r,o}(\tilde{v}_{(\gamma_{k+1})}) = L_{r,o}(\Gamma_F \tilde{v}_{(\gamma_{k+1})}) \leq \theta L_{r,o}(\tilde{v}_{(\gamma_{k+1})}) + t.$$

Thus,

$$N_{r,o}(\tilde{v}_{(\gamma_{k+1})}) \leq t/(1-\theta) < 1,$$
$$L_{r,o}(\tilde{v}_{(\gamma_{k+1})}) \leq t/(1-\theta) < 1.$$

It follows that $\tilde{v}_{(\gamma_{k+1})} \in \mathfrak{X}_{r,o}$. But $w_{(\gamma_{k+1})}$ is a unique fixed point for $\Gamma_F : \mathfrak{X}_{r,o} \to \mathfrak{X}_{r,o}$. Hence $\tilde{v}_{(\gamma_{k+1})} = w_{(\gamma_{k+1})}$. By (2.23), $\tilde{v} = w^{(k+1)}$. □

Proposition 7.3 (Proof of assertions (1)-(4) of Theorem 1). *The following are true:*

1. $w^{(L)} \in C^{k,\delta}(B_L, Y)$ *for all* $k = L, L+1, L+2, \cdots$, *as long as the map is* $C^{k,\delta}$.
2. *If* $k = \infty$, *then* $w^{(L)} \in C^\infty(B_L, Y)$.
3. $w^{(L)}$ *is unique in the class of functions* $v : B_L \to Y$ *which satisfy (2.3) and (2.4).*
4. $w^{(L)}$ *satisfies (2.5) and (2.6).*

5. *In view of assertion (4) of Lemma 2.8, we have an invariant manifold*

$$y = w(x) = p_2(x) + \cdots + +p_L(x) + w^{(L)}(x), \qquad |x| \leq \gamma_L.$$

This proves assertions (1)-(4) of Theorem 1.

Proof. (1) We know that $w^{(L)} \in C^{L,s}(B_L, Y), s = s_L$. Assume that $k \geq L+1$. Let B_k^o be the interior of B_k. By Proposition 7.2, for $k > L$

$$u := w^{(L)}|_{B_k^o} = w^{(k)} + p_k + \cdots + p_{L+1}$$

is $C^{k,\delta}$ in B_k^o. Let $v(x) := w^{(L)}(x), x \in B_L \setminus B_k^o$. Let $x_o \in B_L \setminus B_k^o$ be fixed but arbitrary. Since $A + f \circ (id, v)$ is a contraction on the invariant manifold, it follows that after a finite number of iterations x_o will land in B_k^o. Without loss of generality we may assume that this happens after the first iteration. Let $z_o = Ax_o + f(x_o, v(x_o))$. Since $u(z_o) = w^{(L)}(z_o)$ and $w^{(L)}(x_o) = v(x_o)$ we have

$$u(z_o) = Bv(x_o) + g(x_o, v(x_o)). \tag{7.13}$$

Let $\alpha_o = v(x_o)$. Consider the following function in a neighbourhood of (x_o, α_o):

$$\mathcal{F}(x, \alpha) = B\alpha + g(x, \alpha) - u(z), \tag{7.14}$$
$$z = Ax + f(x, \alpha),$$
$$\mathcal{F}(x_o, \alpha_o) = 0,$$
$$D_\alpha \mathcal{F}(x, \alpha) = B + D_y g(x, \alpha) - Du(z) D_y f(x, \alpha). \tag{7.15}$$

Recall that B is invertible and Df and Dg are small. Thus $D_\alpha \mathcal{F}(x_o, \alpha_o)$ is nonsingular. By the Implicit Function Theorem we have, in an open neighbourhood of (x_o, α_o), a unique $C^{k,\delta}$ solution and a unique $C^{L,s}$ solution of

$$\mathcal{F}(x, \alpha(x)) = 0, \tag{7.16}$$
$$\alpha(x_o) = \alpha_o.$$

The two solutions coincide because $k \geq L$. But $v = w^{(L)}$ is a $C^{L,s}$ solution of (7.16) near (x_o, α_o) since

$$\mathcal{F}(x, v(x)) = 0, \tag{7.17}$$
$$v(x_o) = \alpha_o.$$

Thus $w^{(L)}(x) = v(x) = \alpha(x)$ in a small neighbourhood of x_o. Since x_o is fixed but arbitrary in $B_L \setminus B_k^o$, it follows that $w^{(L)}$ is $C^{k,\delta}$ in all of B_L. This finishes the proof of part (1).
(2) In the C^∞ case repeated applications of part (1) show that $w^{(L)}$ is C^k in B_L for all $k \geq L$.
(3) By Proposition 3.9 we know that $w_{(\gamma_L)}$ is unique in $\mathfrak{X}_{L+\beta,0}, 0 \leq \beta \leq s_L$. In view of (7.6) and Lemma 2.9 with $k = L$ and $j = 0$, we can see that $w^{(L)}$ is unique in the class of functions that satisfy estimates (2.3) and (2.4) with $\gamma = \gamma_L$.
(4) Recall that $D^j w_{(\gamma_L)} \in \mathfrak{X}_{L+\beta,j}, 0 \leq j \leq L$. It follows from (7.6) that $w^{(L)}$ satisfies (2.5) and (2.6). ∎

Proposition 7.4 (Proof of assertion (5) of Theorem 1). *Let $\gamma = \gamma_L$. Let $0 \leq \beta \leq s_L$. The invariant manifold $y = w^{(L)}(x)$, $|x| \leq \gamma$, is unique in the class of $C^{L,\beta}$ functions which satisfy (2.7) and (2.8). More precisely: Suppose that the graph of $u \in C^{L,\beta}(B_\gamma(0), Y)$ is invariant under the map of Theorem 1 but after eliminating all j-monomials in x from $g(x,y), 2 \leq j \leq L$. Also suppose that $u(x)$ satisfies (2.7) and (2.8).*

1. *Then the manifold*
$$y = u_{(\gamma)}(x) = \gamma^{-1}u(\gamma x), \qquad |x| \leq 1,$$
is invariant under the map of Theorem 4. That is, $\Gamma_F u_{(\gamma)} = u_{(\gamma)}, F = F^{(\gamma)}$.
2. *$u_{(\gamma)}(x) = w_{(\gamma)}(x), |x| \leq 1$.*
3. *Thus $u(x) = w^{(L)}(x), |x| \leq \gamma$.*

Proof. 1. This assertion follows from Lemma 2.9.

2. Notice that $u_{(\gamma)}$ satisfies (5.1) for $0 \leq m \leq L$. Thus,
$$\|D^L u_{(\gamma)}\| \leq \frac{t}{1-\theta}$$
with $\theta = b^{-1}a^L(1+t) + t < 1$, and hence, $\|D^L u_{(\gamma)}\| < 1$. Since $u(x)$ satisfies (2.7), it follows that $D^j u_{(\gamma)}(0) = 0, 0 \leq j \leq L$. Thus
$$\text{(i)} \qquad \|D^j u_{(\gamma)}(x)\| \leq |x|^{L-j}, \qquad 0 \leq j \leq L.$$
Moreover, for $0 \leq j \leq L-1$, for some $0 \leq c \leq 1$
$$\text{(ii)} \qquad \begin{aligned}\|D^j u_{(\gamma)}(x) - D^j u_{(\gamma)}(x')\| &\leq D^{j+1}u_{(\gamma)}(x + c(x'-x)) \\ &\leq |x - x'|(|x| + |x'|)^{L-j-1}.\end{aligned}$$
Thus $N_{L,o}(u_{(\gamma)}) \leq 1$ and $L_{L,o}(u_{(\gamma)}) \leq 1$. Hence $u_{(\gamma)} \in \mathfrak{X}_{L,o}$. Thus $u_{(\gamma)} = \tilde{w}_{(\gamma)}$, where $\tilde{w}_{(\gamma)} \in \mathfrak{X}_{L,o}$ is the unique fixed point for $\Gamma_F : \mathfrak{X}_{L,o} \to \mathfrak{X}_{L,o}$. By Proposition 3.9,$w_{(\gamma)} = \tilde{w}_{(\gamma)}$. Thus $u_{(\gamma)} = \tilde{w}_{(\gamma)} = w_{(\gamma)}$.

3. This assertion follows from the definition of $u_{(\gamma)}$ and $w_{(\gamma)}$ in terms of u and $w^{(L)}$ respectively.

□

Remarks 7.5. 1. We had to take this round about way to prove that $u_{(\gamma)} = w_{(\gamma)}$ because all what we know is that $N_{L+\beta,o}(u_{(\gamma)}) < \infty$ and $L_{L+\beta,o}(u_{(\gamma)}) < \infty$. That is, the class of u's that satisfy (2.7) and (2.8) is not bounded. Thus, we cannot show that $N_{L+\beta,o}(u_{(\gamma)}) \leq t/(1-\theta) \leq 1$, just by knowing that $u_{(\gamma)}$ satisfies (3.2). The reason is that estimates (3.4) and (3.9) do not allow us to conclude that
$$N_{L+\beta,o}(u_{(\gamma)}) \leq \frac{t}{1-\theta}.$$
with t independent of u. We do not have this problem when we deal with $D^L u_{(\gamma)}$ because we start by taking $r = L$ and using the sup-norm.

2. Notice also that we do not face this problem when we deal with assertion (3) of Proposition 7.3 because the corresponding $w_\gamma \in \mathfrak{X}_{r,o}$, and hence is unique by Proposition 3.8. Recall that $\mathfrak{X}_{r,o}$ is bounded in $N_{r,o}$ and $L_{r,o}$.

Proposition 7.6 (Proof of assertion (6) of Theorem 1). *The invariant manifold*

$$y = w(x), |x| \leq \gamma_L,$$

is unique in $C^{L,\beta}(B_L, Y), 0 \leq \beta \leq s_L$.

Proof. Suppose $v \in C^{L,\beta}(B_L, Y)$ is invariant under the map of Theorem 1. Let $\gamma = \gamma_L$.

1. By assertion (4) of Lemma 2.8, with $k = L$, $y = v(x), |x| \leq \gamma$, is invariant under the original map Φ (of Theorem 1) iff $v_1(x) \equiv 0$ and

$$y = v^{(L)}(x) := v(x) - p_2(x) - \cdots - p_L(x), \qquad |x| \leq \gamma,$$

is invariant under the mapping $\Phi_{(L)}$ (of Theorem 1 but after eliminating all j-monomials in x from $g(x, y), 2 \geq j \geq L$). Moreover, $|v^{(L)}(x)| = O(|x|^{L+\beta})$ when $\beta \neq 0$ and $|v^{(L)}(x)| = o(|x|^L)$ when $\beta = 0$.
2. From Proposition 7.4, it follows that $v^{(L)} = w^{(L)}$ and hence

$$v(x) = w^{(L)}(x) + p_2(x) + \cdots + p_L(x) = w(x), \qquad |x| \leq \gamma.$$

□

8. Dependence on Parameters

In this section we study the dependence of the invariant manifold we obtained on a parameter that lies in a Banach space. We will prove Theorem 2.

Remarks 8.1.
1. Notice that we need to consider only the case $0 < \delta \leq 1$ since assertion (3) follows from assertion (2).
2. Assertion 1 was proved immediately after Theorem 2.
3. Let $\mu \in \mathcal{U} = B^o_c(\mu) \subset \Lambda$ be as in Theorem 2.
4. Recall that in order to reduce Theorem 1 to Theorem 4 we used Lemma 2.8 to eliminate j-monomials in x from $g(x, y)$ for $2 \leq j \leq k$. We know from assertion (5) of Lemma 2.8 that as $\lambda \to \mu$ $\|u_{\lambda,j} - u_{\mu,j}\|_{C^{k,\delta}} \to 0$. It follows from the continuous dependence of Φ_λ on λ that for $c > 0$ sufficiently small, we can choose $\epsilon > 0$ in Lemma 2.9 uniformly for $\lambda \in \mathcal{U}$. Thus, terms of the form $b^{-1}a^r(1 + t) + t$ can be made less then 1 uniformly for $\lambda \in \mathcal{U}$. Recall that the small $\epsilon > 0$ is obtained by taking $\gamma > 0$ small. This proves assertion (2) of Theorem 2.
5. It follows that for finite k it is enough to consider the dependence on parameters for maps satisfying the hypotheses of Theorem 4 with $n = k$. The C^∞ case will be studied at the end of this section.
6. Our first step is to study the dependence of $f_\lambda \circ (id, w_\lambda)$ and $g_\lambda \circ (id, w_\lambda)$ on λ in the norm $N_{k+\beta,o}$ for all $0 \leq \beta < \delta$. To do that we use estimate (10.2) in Lemma 10.2. Then we will show that $N_{k+\beta,o}(w - u) = O(\|\Phi_\lambda - \Phi_\mu\|^{1-\alpha})$ for $\alpha = (k + \beta)/(k + \delta)$. Recall that the norm $N_{k+\beta,o}$ controls a function at the C^0 level. So, we use some classical results, which we state and prove in the appendix for completeness, to show that $w_\lambda \to w_\mu$ in

the $C^{k,\beta}$ topology. In fact, estimate (Hk) of Lemma 10.4 shows that

$$\|D^k w_\lambda - D^k w_\mu\| = O(\|\Phi_\lambda - \Phi_\mu\|^{\beta_k(1-\alpha)}),$$
$$\beta_k = \frac{\beta}{(k+\beta)(k!)}.$$

7. Let $y = w(x)$ and $y = u(x), |x| \le 1$, be the unique invariant manifolds in $\mathfrak{X}_{k+\delta,o}$ which Theorem 4 establishes for Φ_λ and Φ_μ respectively. Then,

$$\begin{aligned} w(x) &= B_\lambda^{-1} w(z) - B_\lambda^{-1} g_\lambda(x, w(x)), \\ u(x) &= B_\mu^{-1} u(\zeta) - B_\mu^{-1} g_\mu(x, u(x)), \\ z &= A_\lambda x + f_\lambda(x, w(x)), \\ \zeta &= A_\mu x + f_\mu(x, u(x)). \end{aligned} \tag{8.1}$$

8. Notice that when $0 < \delta \le 1$ all our estimates and conclusions of the previous sections are valid for any $\beta \in (0, \delta]$.
9. It follows from Proposition 3.9 that $y = w(x)$ and $y = u(x), |x| \le 1$ are also the unique invariant manifolds in $\mathfrak{X}_{k+\beta,o}$ for all $0 \le \beta < \delta$.

Lemma 8.2. *Let $0 < \beta < \delta \le 1$. Let $\alpha = (k+\beta)/r, r = k+\delta$. Then $\alpha r = k+\beta$ and*

$$\frac{2}{3} \le \frac{k}{k+1} \le \frac{k}{k+\delta} \le \frac{k+\beta}{k+\delta} = \alpha < 1, \qquad k \ge 2.$$

Let

$$\|\psi\|_* := \sup_{|x|+|y|>0} \frac{|\psi(x)|}{|x|^r + |y|},$$
$$\|\psi\|_\# := \sup_{|x|+|y|>0} \frac{|\psi(x)|}{|x|^2 + |y|}.$$

It is obvious that $\|g_\lambda - g_\mu\|_* = O(\epsilon)$ and $\|f_\lambda - f_\mu\|_\# = O(\epsilon)$. The following are true:

$$|f_\lambda(x,y) - f_\mu(x,y)| \le \|f_\lambda - f_\mu\|_\#^\alpha \|f_\lambda - f_\mu\|^{1-\alpha}[|x|^2 + |y|]^\alpha, \tag{8.2}$$
$$|f_\lambda(x,w(x)) - f_\mu(x,u(x))| \le [tN_{r,o}(w-u) + 2^\alpha \|f_\lambda - f_\mu\|_\#^\alpha \|f_\lambda - f_\mu\|^{1-\alpha}]|x|^{2\alpha}, \tag{8.3}$$
$$|g_\lambda(x,y) - g_\mu(x,y)\| \le \|g_\lambda - g_\mu\|_*^\alpha \|g_\lambda - g_\mu\|^{1-\alpha}[|x|^{k+\delta} + |y|]^\alpha, \tag{8.4}$$
$$|g_\lambda(x,w(x)) - g_\mu(x,u(x))| \le [tN_{r,o}(w-u) + 2^\alpha \|g_\lambda - g_\mu\|_*^\alpha \|g_\lambda - g_\mu\|^{1-\alpha}]|x|^{k+\beta}, \tag{8.5}$$

$$|z - \zeta| \le [\|A_\lambda - A_\mu\| + tN_{r,o}(w-u) + 2^\alpha \|f_\lambda - f_\mu\|_\#^\alpha \|f_\lambda - f_\mu\|^{1-\alpha}]|x| \tag{8.6}$$
$$\begin{aligned} |w(z) - u(\zeta)| \le |x|^{k+\beta}[&a^{k+\beta}(1+t)N_{k+\beta,o}(w-u) \\ &+ \eta(\|A_\lambda - A_\mu\| + \|f_\lambda - f_\mu\|_\#^\alpha \|f_\lambda - f_\mu\|^{1-\alpha})], \\ \eta = a^{k-1+\beta}&L_{k+\beta,o}(w-u). \end{aligned} \tag{8.7}$$

Proof. Notice that $|x|^{2\alpha} < |x| \le 1$ since $2\alpha > 1$. We use Lemma 10.2, with $s = 1$, to prove the first four estimates. To prove estimate (8.4) and (8.5) we use estimates (10.1) and (10.2) respectively with $\|\cdot\|_{*,1} = \|\cdot\|_*$. Estimate (8.4) follows from estimate (10.1) immediately. To prove estimate (8.5) we use estimate (10.2) with $\psi = g_\lambda - g_\mu$ and $v = w$. Since $N_{k+\delta,o}(w) \le 1$, we have

$$|g_\lambda(x, w(x)) - g_\mu(x, u(x))| = |g_\mu(x, w(x)) - g_\mu(x, u(x))| + |g_\lambda(x, w(x)) - g_\mu(x, w(x))|$$
$$\le [tN_{k+\delta,o}(w-u) + 2^\alpha \|g_\lambda - g_\mu\|_*^\alpha \|g_\lambda - g_\mu\|^{1-\alpha}]|x|^{k+\beta}$$

To prove estimates (8.2) (8.3) we repeat the same argument with with $k = 2$ and $\|\cdot\|_{*,1} = \|\cdot\|_\#$.

Estimate (8.6) follows from estimate (8.3). Estimate (8.7) follows from (8.6) as follows

$$\begin{aligned}
|w(z) - u(\zeta)| &\le |w(z) - u(z)| + |u(z) - u(\zeta)| \\
&\le N_{k+\beta,o}(w-u)|z|^{k+\beta} + L_{k+\beta,o}(w-u)|z-\zeta|(|z| + |\zeta|)^{k-1+\beta} \\
&\le [a^{k+\beta}(1+t)N_{k+\beta,o}(w-u) \\
&\quad + \eta(\|A_\lambda - A_\mu\| + \|f_\lambda - f_\mu\|_\#^\alpha \|f_\lambda - f_\mu\|^{1-\alpha})]|x|^{k+\beta}, \\
\eta &= a^{k-1+\beta} L_{k+\beta,o}(w-u).
\end{aligned}$$

□

Proposition 8.3. *Let $F_\lambda = (f_\lambda, g_\lambda)$. Let $0 < \beta < \delta \le 1$. Let $\alpha = (k+\beta)/(k+\delta) = (k+\beta)/r$. Then for $k \ge 2, 2/3 < \alpha < 1$ and*

$$N_{k+\beta,o}(w-u) \le O(\|D\Phi_\lambda(0) - D\Phi_\mu(0)\| + \|F_\lambda - F_\mu\|_*^\alpha \|F_\lambda - F_\mu\|^{1-\alpha}).$$

Proof. Notice that $\|\psi\|_\# \le \|\psi\|_*$ since we are dealing with $|x| \le 1$. We have the following for sufficiently small $\epsilon > 0$:

$$|w(x) - u(x)| \le \|B_\lambda^{-1} - B_\mu^{-1}\||u(z)| + b^{-1}|w(z) - u(\zeta)|$$
$$+ \|B_\lambda^{-1} - B_\mu^{-1}\||g_\mu(x, u(x))| + b^{-1}|g_\lambda(x, w(x)) - g_\mu(x, u(x))|.$$

By Lemma 8.2 we have

$$|w(x) - u(x)| \le [\theta N_{k+\beta,o}(w-u) + Q]|x|^{k+\beta},$$
$$Q = Q(\Phi_\lambda - \Phi_\mu) = O(\|D\Phi_\lambda(0) - D\Phi_\mu(0)\| + \|F_\lambda - F_\mu\|_*^\alpha \|F_\lambda - F_\mu\|^{1-\alpha}),$$

where $\theta = b^{-1}a^r(1+t)+t < 1$ for sufficiently small $\epsilon > 0$. Thus $N_{k+\beta,o}(w-u) \le Q/(1-\theta) \to 0$ as $\|\Phi_\lambda - \Phi_\mu\|^{1-\alpha} \to 0$. □

Corollary 8.4. *The following are true as $\|\Phi_\lambda - \Phi_\mu\|^{1-\alpha} \to 0$:*

1. *$\|w - u\| \le Q/(1-\theta) \to 0$, where Q, α and θ are as in Proposition 8.3.*
2. *Let $U_d = \{x \in X \mid |x| \le d < 1\}$ and $0 \le \beta < \delta \le 1$. Then $\|w - u\|_{k,\beta} \to 0$ uniformly on U_d. In fact*

$$(*) \qquad \|D^k w_\lambda - D^k w_\lambda\| = O(\|\Phi_\lambda - \Phi_\mu\|^{\beta_k(1-\alpha)}),$$
$$\beta_k = \frac{\beta}{(k+\beta)(k!)}.$$

3. *If $\delta = 0, \|w - u\|_{k-1,\beta}$ for $0 \le \beta < 1$ on U_d.*

Proof. 1. Since $|x| \leq 1$, it follows that

$$\|w-u\| = \sup_{0<|x|\leq 1} |w(x)-u(x)| \leq \sup_{0<|x|\leq 1} N_{k+\beta,o}(w-u)|x|^r \leq N_{k+\beta,o}(w-u).$$

2. The assertion follows from Lemma 10.8 of the appendix. Estimate (*) follows from estimate (Hk) of Lemma 10.4.
3. This assertion follows if we apply the previous assertion to $C^{k-1,1}$ maps.

□

Proof of Theorem 2. For finite k the assertion follows from Corollary 8.4 and assertion (5) of Lemma 2.8.

For the C^∞ case we use the metric

$$d(w,u) := \sum_{k=0}^{\infty} 2^{-k} \frac{\|w-u\|_{C^k}}{1+\|w-u\|_{C^k}}.$$

It is obvious that the sum is convergent and that $d(w,u) \to 0$ as $\lambda \to \mu$. □

9. **Appendix**
On The Homological Equation in Banach Spaces

In order to eliminate a nonresonant symmetric j-multilinear term $g_j(x)$ from a map of Banach spaces, we need to solve the homological equation (2.19). In doing so, we use Lemma 2.7. The technical part of that lemma is the first assertion which is simple for maps of finite dimensional spaces. The proof for maps of Banach spaces can be found in [17]. We present the proof in our notation for completeness.

Definition 9.1. Let X and Y be Banach spaces.

- Let $\mathcal{L}(X^n, Y)$ be the Banach space of n-multilinear functions from X^n to Y. For $u \in \mathcal{L}(X^n, Y)$ we use the norm

$$\|u\| = \sup\left\{ \frac{|u(x_1,\cdots,x_n)|}{|x_1|\cdots|x_n|} \;\middle|\; 0<|x_i|\leq 1, 1\leq i\leq n \right\}$$

- Let $\mathcal{S}(X^n, Y)$ be the Banach space of symmetric n-multilinear functions from X^n to Y.
- Let $A \in \mathcal{L}(X,X)$ and $C \in \mathcal{L}(Y,Y)$. Define a linear transformation on $\mathcal{S}(X^n,Y)$ by

$$Q_{A,C} : \mathcal{S}(X^n,Y) \to \mathcal{S}(X^n,Y), \tag{9.1}$$
$$[Q_{A,C}u](x_1,\cdots,x_n) = Cu(Ax_1,\cdots,Ax_n).$$

  Notice that for Lemma 2.7, $K_{A,B} = Q_{A,B^{-1}}$.
- For A and C as above, define

$$\begin{aligned} L_C &: \mathcal{L}(X^n,Y) \to \mathcal{L}(X^n,Y), \\ L_C u &= Cu, \\ R_A^i &: \mathcal{L}(X^n,Y) \to \mathcal{L}(X^n,Y), \\ [R_A^i u](x_1,\cdots,x_n) &= u(x_1,\cdots,x_{i-1},Ax_i,x_{i+1},\cdots,x_n). \end{aligned}$$

- For a linear map $T \in \mathcal{L}(E,E)$ let $\rho(T;E)$ and $\sigma(T;E)$ be its resolvent and spectrum respectively. When there is no ambiguity we write $\rho(T)$ and $\sigma(T)$.

- Let $\sigma_i \subset \mathbb{C}, i = 1, \cdots n$. Let
$$\sigma_1\sigma_1 \cdots \sigma_n = \{\alpha_1\alpha_2 \cdots \alpha_n \mid \alpha_i \in \sigma_i, 1 \le i \le n\}$$
- Let I_E denote the identity map on a space E.

Lemma 9.2. *The following are true:*

1. *The collection*
$$\mathfrak{B} = \left\{L_C, R^i_A : A \in \mathcal{L}(X, X), C \in \mathcal{L}(Y, Y), 1 \le i \le n\right\}$$
is a Banach algebra over the complex field $\mathbb{C}$.
2. *Let* $A \in \mathcal{L}(X, X)$ *and* $C \in \mathcal{L}(Y, Y)$. *Then*

$$\sigma(L_C; \mathcal{S}(X^n, Y)) \subset \sigma(L_C; \mathcal{L}(X^n, Y)) \subset \sigma(C), \tag{9.2}$$
$$\sigma(R^i_A; \mathcal{S}(X^n, Y)) \subset \sigma(R^i_A; \mathcal{L}(X^n, Y)) \subset \sigma(A).$$

Proof. (1) Let $A_1, A_2 \in \mathcal{L}(X, X)$ and $C_1, C_2 \in \mathcal{L}(Y, Y)$. Let $1 \le i < j \le n$. Then

$$\begin{aligned} R^i_{A_2} R^i_{A_1} &= R^i_{A_1 A_2}, \\ R^i_{aA_1+bA_2} &= aR^i_{A_1} + bR^i_{A_2}, \\ L_{C_2}L_{C_1} &= L_{C_2C_1}, \\ L_{aC_1+bC_2} &= aL_{C_1} + bL_{C_2}, \\ R^i_{I_X} &= I_{\mathcal{L}(X^n,Y)}, \\ L_{I_Y} &= I_{\mathcal{L}(X^n,Y)}, \\ R^i_{A_2} R^j_{A_1} &= R^j_{A_1} R^i_{A_2}, \\ L_C R^i_A &= R^i_A L_C. \end{aligned} \tag{9.3}$$

Notice also that $\|R^i_A\| \le \|A\|$ and $\|L_C\| \le \|C\|$. Now it is obvious that $\mathfrak{B}$ is a Banach algebra.

(2) It is obvious that

$$\sigma(L_C; \mathcal{S}(X^n, Y)) \subset \sigma(L_C; \mathcal{L}(X^n, Y))$$
$$\sigma(R^i_A; \mathcal{S}(X^n, Y)) \subset \sigma(R^i_A; \mathcal{L}(X^n, Y))$$

If $[C - \gamma I_Y]$ is invertible, then by (9.3)

$$[L_C - \gamma I_{\mathcal{L}(X^n,Y)}]^{-1} = [L_C - \gamma L_{I_Y}]^{-1} = L_{[C-\gamma I_Y]^{-1}}.$$

If $[A - \alpha I_X]$ is invertible, then by (9.3)

$$[R^i_A - \alpha I_{\mathcal{L}(X^n,Y)}]^{-1} = [R^i_A - \alpha R^i_{I_X}]^{-1} = R^i_{[A-\alpha I_X]^{-1}}.$$

It follows that $\rho(C) \subset \rho(L_C; \mathcal{L}(X^n, Y))$ and $\rho(A) \subset \rho(R^i_A; \mathcal{L}(X^n, Y))$, which proves (9.2). ☐

Lemma 9.3. *Let* $Q_{A,C} : \mathcal{S}(X^n, Y) \to \mathcal{S}(X^n, Y)$ *be as in (9.1). Then*

$$\sigma(Q_{A,C}) \subset \sigma(C)\sigma(A)^n \tag{9.4}$$

Proof. 1. Theorem 11.23 of [22] states that if two elements T and S of a Banach algebra commutes, then $\sigma(TS) = \sigma(T)\sigma(S)$. It follows that

$$\sigma(R^1_A \cdots R^n_A; \mathcal{L}(X^n, Y)) \subset \sigma(A)^n. \tag{9.5}$$

2. Now we would like to show that

$$\rho(R_A^1 \cdots R_A^n; \mathcal{L}(X^n, Y)) \subset \rho(R_A^1 \cdots R_A^n; \mathcal{S}(X^n, Y)), \tag{9.6}$$

which implies that

$$\sigma(R_A^1 \cdots R_A^n; \mathcal{S}(X^n, Y)) \subset \sigma(R_A^1 \cdots R_A^n; \mathcal{L}(X^n, Y)) \subset \sigma(A)^n. \tag{9.7}$$

In order to simplify notation we let $R_A^{[n]} = R_A^1 \cdots R_A^n$, $\mathcal{L}^n = \mathcal{L}(X^n, Y)$ and $\mathcal{S}^n = \mathcal{S}(X^n, Y)$. Let $\alpha \in \rho(R_A^{[n]}; \mathcal{L}^n)$. Thus $[R_A^{[n]} - \alpha I_{\mathcal{L}^n}]$ is a bijection with a continuous inverse. In order to show that $\alpha \in \rho(R_A^{[n]}; \mathcal{S}^n)$ we need to show that $[R_A^{[n]} - \alpha I_{\mathcal{S}^n}]$ is a bijection with a continuous inverse. Since $\mathcal{S}^n \subset \mathcal{L}^n$, $[R_A^{[n]} - \alpha I_{\mathcal{S}^n}]$ is injective. Moreover, by the open mapping theorem, *a continuous bijective linear transformation of one Banach space onto another has a continuous inverse.* See [23, pages 82,83,188]. Thus, it suffices to show that $[R_A^{[n]} - \alpha I_{\mathcal{S}^n}]$ is surjective. That is to show the following: Given any $w \in \mathcal{S}^n$, there is $\phi \in \mathcal{S}^n$ such that

$$R_A^{[n]}\phi - \alpha\phi = w \tag{9.8}$$

Since $\alpha \in \rho(R_A^{[n]}; \mathcal{L}^n)$, there is $\psi \in \mathcal{L}^n$ such that

$$\begin{aligned} R_A^{[n]}\psi - \alpha\psi &= w \\ \|\psi\| &\le K\,\|w\| \\ K &= \|[R_A^{[n]} - \alpha I_{\mathcal{L}^n}]^{-1}\| \end{aligned} \tag{9.9}$$

For any permutation τ of $\{1, 2, \cdots, n\}$, since $w \in \mathcal{S}^n$, we have

$$R_A^{[n]}\psi \circ \tau - \alpha\psi \circ \tau = w \circ \tau = w. \tag{9.10}$$

On averaging over all such permutations, we obtain the coveted $\phi \in \mathcal{S}^n$:

$$\phi := \frac{1}{n!}\sum_{\tau} \psi \circ \tau.$$

In view of (9.10), ϕ satisfies (9.8). This finishes the proof of (9.6) which implies (9.7).

3. Since $R_A^{[n]}$ and L_C commute, (9.2) and (9.7) shows that

$$\sigma(Q_{A,C}; \mathcal{S}^n) = \sigma(L_C R_A^{[n]}; \mathcal{S}^n) \subset \sigma(L_C; \mathcal{S}^n)\,\sigma(R_A^{[n]}; \mathcal{S}^n) \subset \sigma(C)\,\sigma(A)^n.$$

which finishes the proof of the lemma. □

10. **Appendix**
Embedding Lemmas

The following lemmas are from [4] and [5]. We include them here for completeness.

Definition 10.1. Let X, Y and F be Banach spaces. Let $U \oplus V \subset X \oplus Y$ be such that W is open or $cl(int(W)) = W$ for $W = U, V$. For $k = 0, 1, 2, \cdots$ and $0 \leq \delta \leq 1$ let $r = k + \delta$. Let $0 < s \leq 1$ and define the following for $\psi : U \oplus V \to F$:

$$\|\psi\|_{*,s} = \sup_{|x|+|y|>0} \frac{|\psi(x)|}{(|x|^r + |y|)^s}.$$

Lemma 10.2. 1. *Let $0 < s \leq 1$. Let $\psi : U \oplus V \to F$ be such that $\|\psi\| < \infty$ and $\|\psi\|_{*,s} < \infty$. Then, for any $0 < \alpha < s$*

$$\|\psi\|_{*,\alpha} \leq \|\psi\|_{*,s}^{\alpha/s} \, \|\psi\|^{1-(\alpha/s)}. \tag{10.1}$$

2. *Let $v : U \to V$ be such that $N_{r,o}(v) < \infty, r = k + \delta$. Let ψ be as in (10.1). Then, for any $0 < \beta < \delta \leq 1$, there is $0 < \alpha < 1$ sufficiently close to 1, such that $\alpha(k + \delta) = k + \beta$ and*

$$|\psi(x, v(x))| \leq \|\psi\|_{*,s}^{\alpha/s} \, \|\psi\|^{1-(\alpha/s)} [1 + N_{r,o}(v)]^\alpha |x|^{k+\beta}. \tag{10.2}$$

In other words,

$$N_{k+\beta,o}(\psi \circ (id, v)) \leq \|\psi\|_{*,s}^{\alpha/s} \, \|\psi\|^{1-(\alpha/s)} [1 + N_{r,o}(v)]^\alpha. \tag{10.3}$$

3. *Let $\{\phi_n : U \oplus V \to F \mid n = 1, 2, 3, \cdots\}$ be a sequence such that $\|\phi_n\| \leq c$ and $\|\phi_n\|_{*,s} \leq c$ for some $c > 0$. Let $\phi : U \to F$. Assume that $\|\phi_n - \phi\| \to 0$ as $n \to \infty$. Then $\|\phi\| \leq c$ and $\|\phi\|_{*,s} \leq c$. Moreover, for all $0 < \alpha < 1$, we have $\|\phi_n - \phi\|_{*,\alpha} \to 0$ as $n \to \infty$.*
4. *Let $\{\phi_n \mid n \geq 1\}$ and ϕ be as in assertion 3. Let v be as in assertion 2. Let $0 < \alpha < 1$, be such that $\alpha(k + \delta) = k + \beta$. Then $N_{k+\beta}((\phi_n - \phi) \circ (id, v)) \to 0$ as $n \to \infty$.*

Proof. 1. Let $q = |x|^r + |y|$. Notice that

$$\Delta := \frac{|\psi|}{(|x|^r + |y|)^\alpha} \leq \min \left\{ \|\psi\|_{*,s} \, q^{s-\alpha} \, , \, \frac{\|\psi\|}{q^\alpha} \right\}. \tag{*}$$

The two quantities on the right hand side of (*) are equal iff

$$q = A := \left(\frac{\|\psi\|}{\|\psi\|_{*,s}} \right)^{1/s}.$$

There are two cases. Case 1: If $q \leq A$, then

$$\Delta \leq \|\psi\|_{*,s} \, q^{s-\alpha} = \|\psi\|_{*,s}^{\alpha/s} \|\psi\|^{1-(\alpha/s)}.$$

Case 2: If $q > A$, then

$$\Delta \leq \frac{\|\psi\|}{q^\alpha} = \|\psi\|_{*}^{\alpha/s} \|\psi\|^{1-(\alpha/s)}.$$

Thus, $\Delta \leq \|\psi\|_{*,s}^{\alpha/s} \|\psi\|^{1-(\alpha/s)}$. On taking the supremum, we obtain estimate (10.1)

2. Apply estimate (10.1) with $r = k + \delta$:

$$\begin{aligned} |\psi(x, v(x))| &\leq \|\psi\|_{*,\alpha} [|x|^r + |v(x)|]^\alpha \\ &\leq \|\psi\|_{*,s}^{\alpha/s} \, \|\psi\|^{1-(\alpha/s)} [1 + N_{r,o}(v)]^\alpha |x|^{k+\beta} \end{aligned}$$

3. The assertions that $\|\phi\| \le c$ and $\|\phi\|_{*,s} \le c$ follow from
$$|\phi(x)| \le c + \|\phi - \phi_n\| \to c,$$
$$|\phi(x)| \le c(|x|^r + |y|)^s + \|\phi - \phi_n\| \to c(|x|^r + |y|)^s,$$
respectively. Now, $\|\phi - \phi_n\|_{*,s} \le 2c$. Applying estimate (10.1) to $\psi_n = \phi - \phi_n$, we see that $\|\phi - \phi_n\|_{*,\alpha} \to 0$.
4. This assertion follows from (10.3).

□

Definition 10.3 below is needed to state a lemma due to D. Henry [11, page 151]. See also [2, pages 305, 319]. More detailed and important estimates are stated in Lemma 10.4 whose proofs are given in [6]. See also Remark 10.6 below.

Definition 10.3. Let X and Y be Banach spaces. Let $U \subset X$ be an open subset. Let $k = 0, 1, 2, \cdots$ and $0 < \delta \le 1$. Let $b > 0$ and
$$C_b^{k,\delta}(U, Y) := \{u \in C^{k,\delta}(U, Y) \mid \|u\|_{k,\delta} \le b\},$$
where
$$\|u\|_{k,\delta} = \max\{\|u\|, \|Du\|, \cdots, \|D^k u\|, H_\delta(D^k u)\},$$
$$H_\delta(D^k u) = \sup\left\{\frac{\|D^k u(x) - D^k u(x')\|}{|x - x'|^\delta} \mid x \ne x', x \in U, x' \in U\right\}.$$

Lemma 10.4 (Estimate H). *Let X, Y, U, k, δ and b be as in Definition 10.3. Let ∂U be the boundary of U. For $x \in U$, define $d(x) := \min\{1, dist(x, \partial U)\}$. For all $f \in C^{k,\delta}(U, Y)$, for all $x \in U$ the following estimates are true:*

$$d(x)\|Df(x)\| \le N_1 \|f\|^{\frac{\delta}{1+\delta}} \|f\|_{1,\delta}^{\frac{1}{1+\delta}}, \tag{H1}$$
$$d(x)^{a_k}\|D^k f(x)\| \le N_k \|f\|^{\beta_k} \|f\|_{k,\delta}^{1-\beta_k}, \tag{Hk}$$

where $N_1 = 4$ and $a_1 = 1$. If $k = 1, \beta_1 = \delta/(1+\delta)$. If $k \ge 2, \beta_1 = 1/2$ and for $j = 2, \cdots, k$
$$\begin{aligned} N_j &= 2(j!)[2 + \frac{N_1}{1!} + \frac{N_2}{2!} \cdots + \frac{N_{j-1}}{(j-1)!}] \\ &= 4(j!)(3^{j-1}), \qquad j = 2, \cdots, k, \\ a_j &= a_{j-1} + j = \frac{j(j+1)}{2}, \qquad j = 2, \cdots, k, \\ \beta_j &= \frac{1}{j+1}\beta_{j-1} = \frac{1}{(j+1)!}, \qquad j = 2, \cdots, k-1, \\ \beta_k &= \frac{\delta}{k+\delta}\beta_{k-1} = \frac{\delta}{(k+\delta)(k!)}. \end{aligned}$$

Proof. See [5] □

Lemma 10.5 (Henry's lemma)**.** *Let $0 < \delta \le 1$. The following are true:*

1. *Any closed bounded subset* $C_b^{k,\delta}(U,Y)$ *is a closed and bounded subset of* $(C^0(U,Y), \|\cdot\|)$. *That is, if* $\{u_n \mid 0 \le n < \infty\}$ *is a sequence in* $C_b^{k,\delta}(U,Y), u : U \to Y$ *and* $\|u_n - u\| \to 0$ *as* $n \to \infty$, *then* $u \in C_b^{k,\delta}(U,Y)$. *Moreover,* $D^k u_n(x) \to D^k u(x)$ *point-wise as* $n \to \infty$.
2. *Let* $U_o \subset U$ *be an open subset which is uniformly bounded away from the boundary of* U. *Then* $D^k u_n(x) \to D^k u(x)$ *uniformly on* U_o.

Proof. (1) If $k = 0$, then for any $x \neq x'$,

$$\begin{aligned}|u(x) - u(x')| &\le |u_n(x) - u_n(x')| + 2\|u_n - u\| \\ &\le c|x - x'|^\delta + 2\|u_n - u\| \to c|x - x'|^\delta,\end{aligned}$$

as $n \to \infty$.

When $k \ge 1$, let $f := u_n - u_m$. It follows from estimate (Hk) of Lemma 10.4 that

$$(*) \qquad d(x)^{a_k}\|D^k u_n(x) - D^k u_m(x)\| \le N_k \|u_n - u_m\|^{\beta_k} \|u_n - u_m\|_{k,\delta}^{1-\beta_k}.$$

Thus u is $C^{k,\delta}$and $D^k u_n(x) \to D^k u(x)$ for each $x \in U$. It also follows that $\|u\|_{k,\delta} \le b$.

Part (2) follows immediately from estimate (*) and the fact that $U_o \subset U$ is uniformly bounded away from the boundary of U. □

Remarks 10.6. 1. The proof of Henry's Lemma 10.5 is based on estimates (H1) and (Hk) of Lemma 10.4.
2. Estimate (H1) and the proof of Henry's Lemma for $k = 1$ are given in [11, page 151] and [2, page 319] without the explicit dependence on $\|f\|_{1,\delta}$.
3. The $k \ge 2$ case of Henry's Lemma does not follow by induction from the $C^{1,\delta}$ case without estimate (Hk) because the convergence in U is point-wise and uniform convergence holds only in a subset $U_o \subset U$ which is uniformly bounded away from the boundary of U.
4. Moreover, estimates (H1) and (Hk) of Lemma 10.4 are of interest by themselves. In particular, notice that in estimates (H1) and (Hk), as well as all the estimates in this appendix, the sum of the exponents of the different norms of the function that appear on the righthand side is always one. In other words, these inequalities are homogenous so to speak. This is not the case with the estimates in [11, page 151] and [2, page 319].

The following is a classical result which we will need latter. We present a proof here for completeness.

Lemma 10.7. 1. *Let* $g \in C^{0,\delta}(U_o, Y)$ *and* $0 < \beta < \delta \le 1$. *Then*

$$(10.4) \qquad \|g\|_{0,\beta} \le 2^{1-(\beta/\delta)} \|g\|^{1-(\beta/\delta)} \|g\|_{0,\delta}^{\beta/\delta}.$$

2. *Let* f_n *and* f *be* $C_b^{0,\delta}(U_o, Y)$ *and assume that* $\|f_n - f\| \to 0$. *Then* $\|f_n - f\|_{0,\beta} \to 0$ *for all* $0 \le \beta < \delta$.

Proof. Notice that part (2) follows from part (1) by setting $g = f_n - f$. So, we prove part (1).

Notice that

$$(\#) \qquad \Delta := \frac{|g(x+h) - g(x)|}{|h|^\beta} \le \min\left\{2\frac{\|g\|}{|h|^\beta}\ ,\ |h|^{\delta-\beta}\|g\|_{0,\delta}\right\}.$$

The two terms on the righthand side of (#) are equal iff

$$|h| = \left(\frac{2\|g\|}{\|g\|_{0,\delta}} \right)^{1/\delta} := A.$$

When $|h| > A$, we use the first term on the righthand side of (#) to estimate Δ. When $|h| \le A$ we use the second term. In either case we obtain

$$\Delta \le 2^{1-(\beta/\delta)} \|g\|^{1-(\beta/\delta)} \|g\|_{0,\delta}^{\beta/\delta},$$

which proves (1). □

Lemma 10.8. *Let u_n be a sequence in $C_b^{k,\delta}(U,Y)$ and $u : U \to Y$. Assume that $\|u_n - u\| \to 0$. Let $U_o \subset U$ be an open subset which is uniformly bounded away from the boundary of U. Let $0 < \beta < \delta$. Then $u \in C_b^{k,\delta}(U,Y)$. Moreover, $\|u_n - u\|_{k,\beta} \to 0$ as $n \to \infty$, uniformly on U_o, .*

Proof. Applying Henry's Lemma 10.5,for $j = 0, 1, \cdots, k$, we can see that $\|D^j u_n - D^j u\| \to 0$ uniformly on U_o as $n \to \infty$.

Notice that for $f \in C^{k,\delta}(U_o, Y)$ we have $D^k f \in C^{0,\delta}(U_o, \mathcal{S}(X^k, Y))$ where $\mathcal{S}(X^k, Y)$ is the Banach space of symmetric k-multilinear maps from X^k to Y. Thus we can apply assertion (2) of Lemma 10.7 to $D^k u_n$ and $D^k u$ and obtain

$$\|D^k u_n - D^k u\|_{0,\beta} \to 0,$$

as $n \to \infty$ on U_o. □

The following lemma shows the relationship between the N-norms we are using and uniform continuity on compact sets. It is stated in more general terms so that we can use it to prove the first part of Lemma 10.10.

Lemma 10.9. *Let $f \in \mathfrak{Z}_{r,o}(K, Y)$ and $s < r = k + \delta \le k + 1$.*

1. *Assume that K satisfies the following condition :*
   - *There are two real valued continuous functions $\eta(x, x') > 0$ and $\chi(x, x') > 0$ such that if $x, x' \in K$ and $|x - x'| \le \chi(x, x')$, then $|f(x) - f(x')| \le \eta(x, x')$.*

   Then the following is true: If $x, x' \in K$ and $|x - x'| < \chi(x, x')$, then

$$|f(x) - f(x')| \le [\eta(x, x')]^{1-(s/r)} [N_{r,o}(f)]^{s/r} (|x| + |x'|)^s \tag{10.5}$$

2. *Assume K is compact. Then for all $\epsilon > 0$, there is $c > 0$, such that if $|x - x'| < c$, (10.5) holds with $\eta(x, x') = \epsilon$.*

Proof. 1. Let $q = |x| + |x'|$. If $|x - x'| < \chi(x, x')$, then

$$\Delta := \frac{|f(x) - f(x')|}{q^s} \le \min \left\{ \frac{\eta(x, x')}{q^s} \, , \, q^{r-s} N_{r,o}(f) \right\}. \tag{10.6}$$

The two terms on the right hand side of (10.6) are equal iff

$$|x| + |x'| = A := \left(\frac{\eta(x, x')}{N_{r,o}(f)} \right)^{1/r}$$

There are two cases: either $|x|+|x'|>A$ or $|x|+|x'|\le A$. In the first case we use the first term on the righthand side of (10.6) to estimate Δ; in the second case we use the second term. In either case estimate (10.5) follows.

2. If K is compact, then $f(x)$ is uniformly continuous on K. Thus for all $\epsilon>0$, there is $c>0$, such that if $x,x'\in K$ and $|x-x'|<c$, then $|f(x)-f(x')|<\epsilon$. Let $\eta(x,x')=\epsilon$ and $\chi(x,x')=c$. Now the assertion follows from part (1). ☐

The following lemma shows the relationship between the N-norms we are using and uniform Hölder continuity.

Lemma 10.10. *Let $f\in C^0(U,Y), N_{r,o}(f)<\infty$ and $H_\delta(f)<\infty$. Let $s=r-\beta<r,\beta<\delta$. For all $x,x'\in U, x\ne x'$ let $h=|x-x'|$ and $q=|x|+|x'|$. Then*

$$|f(x)-f(x')|\le[N_{r,o}(f)]^{s/r}[H_\delta(f)]^{1-(s/r)}h^{\delta(1-(s/r))}q^s \tag{10.7}$$

$$|f(x)-f(x')|\le[N_{r,o}(f)]^{1-(\beta/\delta)}[H_\delta(f)]^{\beta/\delta}h^\beta q^{r(1-(\beta/\delta))} \tag{10.8}$$

Proof. Let $K=U$ in assertion (1) of Lemma 10.9. Since $H_\delta(f)<\infty$, for all $x,x'\in U$,

$$|f(x)-f(x')|\le\eta(x,x'):=H_\delta(f)|x-x'|^\delta,\qquad x,x'\in U.$$

We can take $\chi(x,x')=|x|+|x'|$. Thus the condition $|x-x'|\le\chi(x,x')$ is trivially satisfied. Thus estimate (10.7) follows from estimate (10.5) of Lemma 10.9.

We can prove estimate (10.7) directly as follows: Notice that

$$\begin{aligned}\Delta&=\frac{|f(x)-f(x')|}{|x-x'|^\beta(|x|+|x'|)^{r-\beta}}\\&\le\min\left\{\frac{|x-x'|^{\delta-\beta}H_\delta(f)}{(|x|+|x'|)^{r-\beta}}\,,\ \frac{(|x|+|x'|)^\beta N_{r,o}(f)}{|x-x'|^\beta}\right\}\end{aligned}\tag{10.9}$$

The two terms on the right hand side of (10.9) are equal iff

$$|x|+|x'|=\left(\frac{H_\delta(f)|x-x'|^\delta}{N_{r,o}(f)}\right)^{1/r}:=A$$

There are two cases: either $|x|+|x'|>A$ or $|x|+|x'|\le A$. In the first case we use the first term on the righthand side of (10.9) to estimate Δ; in the second case we use the second term. In either case,

$$\Delta\le[N_{r,o}(f)]^{1-(\beta/r)}[H_\delta(f)]^{\beta/r}\frac{|x-x'|^\alpha}{|x-x'|^\beta},\qquad\alpha=\frac{\beta\delta}{(k+\delta)},$$

which implies estimate (10.7).

To prove estimate (10.8) notice that the two terms on the right hand side of (10.9) are equal iff

$$|x-x'|=B:=\left(\frac{N_{r,o}(f)(|x|+|x'|)^r}{H_\delta(f)}\right)^{1/\delta}$$

Whether $|x-x'|\le B$ or $|x-x'|>B$ we have

$$\Delta\le[N_{r,o}(f)]^{1-(\beta/\delta)}[H_\delta(f)]^{\beta/\delta}\frac{(|x|+|x'|)^\gamma}{(|x|+|x'|)^{r-\beta}},\qquad\gamma=r(1-(\beta/\delta)),$$

which implies estimate (10.8). □

Remark 10.11. We can see from Lemma 10.10 that H_δ is not the appropriate norm to measure Hölder continuity for functions that vanish up to order r at the origin. This is because the exponents of $|x-x'|$ and $(|x|+|x'|)$ do not add up to r. Notice that in proving assertion 2 of Proposition 3.5 using $L_{r,o}, r = k+\delta$, the fact that these exponents add up to r is essential in obtaining $\theta = b^{-1}a^r < 1$. Also notice that the exponent of $|x-x'|$ is less that δ in both estimates. The following lemma gives the relation between $L_{r,o}$ and $L_{k+\beta,o}, 0<\beta<\delta$. In fact, we give a more general version of the norm $L_{r,o}$. The proof and other results for $C^{k,\delta}$ maps that vanishes up to order k at the origin are in [4].

Lemma 10.12. *Let $f \in C^0(U,Y)$ where $r = k+\delta$ and $0<\gamma\le 1$. Define*

$$L_{r,\gamma}(f) = \sup\left\{ \frac{|f(x)-f(x')|}{|x-x'|^\gamma[|x|+|x'|]^{r-\gamma}} \;\middle|\; x \neq x', x\in U, x'\in U \right\}$$
$$\mathfrak{Z}^0_r(U,Y) = \{f \in C^0(U,Y) \mid N_{r,o}(f) < \infty\}$$
$$\mathfrak{Z}^{0,\gamma}_r(U,Y) = \{f \in \mathfrak{Z}^0_r(U,Y) \mid L_{r,\gamma}(f) < \infty\}$$
$$\mathfrak{Z}^{0,\gamma}_{r,c}(U,Y) = \{f \in \mathfrak{Z}^{0,\gamma}_r(U,Y) \mid L_{r,\gamma}(f) \le c\}$$

Then, the following are true:

1. *Since $f(0)=0$, it follows that for $f\in \mathfrak{Z}^{0,\gamma}_r(U,Y)$: $N_{r,o}(f) \le L_{r,\gamma}(f), 0<\gamma\le 1$.*
2. *$\mathfrak{Z}^{0,\gamma}_{r,c}(U,Y)$ is closed under point-wise convergence.*
3. *$\mathfrak{Z}^{0,\gamma}_{r,c}(U,Y)$ is a bounded closed subset of $(\mathfrak{Z}^0_r(U,Y), N_r)$.*
4. *$(\mathfrak{Z}^{0,\gamma}_{r,c}(U,Y), \|\cdot\|)$ is a complete metric space.*
5. *$(\mathfrak{Z}^{0,\gamma}_r(U,Y), L_{r,\gamma})$ is a Banach space.*
6. *Let $f\in \mathfrak{Z}^{0,\gamma}_r(U,Y)$. Let $0<\alpha<\gamma\le 1$. Then*

$$L_{r,\alpha}(f) \le [L_{r,\gamma}(f)]^{\alpha/\gamma}[N_r(f)]^{1-(\alpha/\gamma)} \tag{10.10}$$

7. *Let $\{u_n\}$ be a sequence in $\mathfrak{Z}^{0,\gamma}_{r,c}(U,Y)$ and $u: U\to Y$. Assume that $N_r(u-u_n)\to 0$ as $n\to\infty$. Then $L_{r,\alpha}(u-u_n)\to 0$ as $n\to\infty$ for $0<\alpha<\gamma$.*
8. *Let $\{u_n \mid n\ge 1\}$ be a sequence in $\mathfrak{Z}^{0,\gamma}_{r,c}(U,Y)$ and $u: U\to Y$. Assume that $\|u-u_n\|\to 0$ as $n\to\infty$. Then $u\in \mathfrak{Z}^{0,\gamma}_{r,c}(U,Y)$. Moreover, for all $s<r$, $L_{r,\alpha}(u_n-u)\to 0$ as $n\to\infty$.*

References

[1] P. W. Bates, K. Lu, C. Zeng, *Existence and persistence of invariant manifolds for semi-flows in Banach spaces*, preprint

[2] S-N Chow, K. Lu, *C^k centre unstable manifolds*, Proc. Roy. Soc. Edinburgh, 108A, 1988, 303-320.

[3] S-N Chow, K. Lu, *Invariant manifolds for flows in Banach spaces*, J. Differential Equations, 74, 1988, 285-317.

[4] M. S. ElBialy, *Spaces of functions which vanish up to higher order terms*, preprint.

[5] M. S. ElBialy, *On sequences of $C^{k,\delta}_b$ maps which converges in the C^0 norm*, Proc. AMS, to appear.

[6] M. S. ElBialy, *On pseudo-stable and pseudo-unstable manifolds for maps*, J. Math. Anal. App. 232, 1999, 229-258

[7] N. Fenichel, *Persistence and smoothness of invariant manifolds for flows*, Ind. Math. Journal, 21, 1973, 193-226.

[8] N. Fenichel, *Asymptotic stability with rate conditions*, Ind. Math. Journal, 23, 1973, 1109-1137.

[9] S. A. van Gils, A. Vanderbauwhede, *Center manifolds and contractions on scales of Banach spaces*, Journal of Functional Analysis, 72, 1987, 209-224.
[10] J. Hadamard, *Sur l'itrération et les solutions asymptotiques des équations differentielles*, Bull. Soc. Math, France, 29 1901, 224-228.
[11] D. Henry, *Geometric theory of parabolic equations*, Lecture Notes in Mathematics 840, Springer, New York, 1981.
[12] M. W. Hirsh, C. C. Pugh, *Stable manifolds and hyperbolic sets*, Proc. Symp. In Pure Math., vol. 14, AMS, Providence, RI, 1970, 133-164.
[13] M. W. Hirsh, C. C. Pugh, M. Shub, *Invariant manifolds* Lecture notes in mathematics, vol. 583, Springer Verlag, New York 1977.
[14] M. C. Irwin, *A new proof of the pseudo-stable manifold theorem*, J. London Math. Soc. 21 1980, 557-556
[15] N. Kurzweil, *On approximation in real Banach spaces*, Studia Math. 14, 213-231, 1954.
[16] R. de la Llave, *Invariant manifolds associated to non-resonant spectral subspaces*, J. Stat. Phys. 87, No. 1-2, 211-249, 1997.
[17] R. de la Llave, C. E. Wayne, *Invariant manifolds associated to non-resonant invariant subspaces for some non-invertible maps and semi-flows*, preprint.
[18] R. de la Llave, C. E. Wayne, *On Irwin's proof of the pseudo-stable manifold theorem*, Mathematische Zeitschrift, 219, 1995 301-321.
[19] E.B. Leach, J. H. M. Whitfiels, *Differentiable functions and rough norms on Banach spaces* Proc. A. M. S., 33, 120-126, 1979.
[20] O. Perron, *Über Stbilität und asymptotisches Verhalten der Integrale von Differentialgleichungssystemen*, Mathematische Zeitschrift, 29, 1928, 129-190.
[21] Ya. B. Pesin, *On the existence of invariant fiberings for diffeomorphism of a smooth manifold*, USSR Sbornik vol. 20, No. 2, 1973, 213-222.
[22] W. Rudin, *Functional Analysis*, second edition, McGraw-Hill, 1991.
[23] *Methods of Modern Mathematical Physics, I :Functional Analysis*, M. Reed, B. Simon, Academic Press, 1980.

Mathematics Department, University of Toledo, Toledo, Ohio 43606, U.S.A.
E-mail address: `melbialy@math.utoledo.edu`