C^k INVARIANT MANIFOLDS FOR MAPS OF BANACH SPACES

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ABSTRACT. In this work we give a unified proof for the existence, under the appropriate gap conditions, of the standard invariant manifolds for a C^k map of a Banach space near a fixed point where $k \geq 1$ is an integer.

1. Introduction

In [4] we proved a fixed point theorem for a class of $C^{k,\delta}, 0 < \delta \leq 1$, self-maps of a Banach space of functions. We used the theorem we proved to give a unified proof for the existence, under the appropriate gap conditions, of pseudo-stable, pseudo-unstable, centre-stable, centre-unstable weak-stable, weak-unstable, strong-stable, strong-unstable, stable and unstable manifolds for $C^{k,\delta}, 0 < \delta \leq 1$, maps of a Banach space. In this work we consider the case $\delta = 0$.

The case $\delta = 0$ needs to be considered separately because the standard fiber contraction theorem [12] does not apply when $\delta = 0$. We still have a fiber contraction and we have a candidate for the k^{th}-derivative but we cannot conclude that it is a unique attractive fixed point because, when $\delta = 0$, the fiber contraction is not necessarily continuous in the base points, a condition needed for applying the fiber contraction theorem. As a result, if we use the fiber contraction theorem we can only show that these invariant manifolds are $C^{k-1,1}$. The only places in the literature, known to the author, where the possible loss of differentiability of an invariant manifold when the map is $C^k, k \in \mathbb{N}$ is avoided are [2], [5], [9] and [17].

Since in this article we use the results of [4] , we keep the style notation and definitions of [4] . For the sake of completeness we include all definitions and notation. Moreover, ([4] 5.2) will mean equation (5.2) in [4] . Also, Lemma [4] 3.5 will mean Lemma 3.5 of [4] .

In Section 2 we give some definitions and state our main theorem. In Section 3 we give all the definitions, notation and setup used in [4] and indicate which conclusions in [4] are still valid when $\delta = 0$. We also reduce the main theorem to Proposition 3.3 which we prove in Section 4.

The idea of the proof is as follow: By applying the results in [4] we know that we have a $C^{k-1,1}$ map $y = w(x)$ which is an attractive fixed point for a transformation Γ defined on the appropriate function space. We also

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know that $\Delta^{k-1}w := (w, Dw, \cdots, D^{k-1}w)$ is an attractive fixed point for the $(k-1)^{\underline{st}}$ lift of Γ which we denote by Γ_{k-1}. Moreover, we know that $k^{\underline{st}}$ lift Γ_k is a fiber contraction which is not necessarily continuous in the base points. Thus, we can apply the standard contraction theorem to the invariant fiber above $\Delta^{k-1}w$ and obtain a fixed point, say μ. This μ is our candidate for $D^k w$. We cannot use the fiber contraction theorem to conclude that $D^k w = \mu$ because Γ_k is not necessarily continuous in the base points. To show that $D^k w = \mu$, it suffices to show that given x and h

$$(*) \qquad D^{k-1}w(x+h) - D^{k-1}w(x) = \int_0^1 \mu(x+th)hdt$$

Now let u_o be any C^k function (such that $\Delta^k u_o$ is in the function space defined below) and let $u_n = \Gamma u_n, n \geq 1$. Now, $\|\Delta^{k-1}u_n - \Delta^{k-1}w\| \to 0$ and

$$(\#) \qquad D^{k-1}u_n(x+h) - D^{k-1}u_n(x) = \int_0^1 D^k u_n(x+th)hdt$$

The left hand side of (#) converges uniformly (everywhere) to the left hand side of (*). Let $I = \{x + th \mid 0 \leq h \leq 1\}$. To show that the right hand side of (#) converges uniformly on I to the right hand side of (*) we restrict the fiber contraction Γ to a set $\mathcal{W}$ which consists of the line segment I and its images under $D^i u_n, 1 \leq i \leq k-1, n \geq 0$ and a few other sets given explicitly in Lemma 4.4 where we show that $\mathcal{W}$ is in fact compact. We all know that a continuous function on a compact set is uniformly continuous. We use this fact to show that the restriction of Γ_k to $\mathcal{W}$ is continuous in the *restrictions of the base points*. Then we can apply the fiber contraction theorem to the restricted Γ_k and conclude that $D^k u_n \mid_I \to \mu \mid_I$ as $n \to \infty$.

1.1. Remarks. We refer the reader to [4] for the details on how to obtain the various invariant manifolds at a fixed point of a map from our main Theorem 1 below. We would like to recall a few points from [4] here:

1. In [4, Section 1.7, page 235] we show how to obtain what we call the PU invariant manifolds from our main Theorem 1. The PU manifolds are the pseudo-unstable, centre-unstable, unstable, strong-stable and weak-stable manifolds. The last one is for contracting maps. Each of these invariant manifolds corresponds to part of the spectrum (of the linear part of the map at the fixed point) that lies outside a circle of radius ρ in the complex plane $\mathbb{C}$. To define the graph transform for PU manifolds we need to use the Lipschitz inverse function theorem [22, page 49], [4, Lemma 1.8 page 236].
2. In [4, Section 1.9, page 238] we show how to obtain what we call the PS invariant manifolds from our main Theorem 1. The PS manifolds are the pseudo-stable, centre-stable, stable, strong-stable and weak-unstable manifolds. The last one is considered for expanding maps. Each of these invariant manifolds corresponds to part of the spectrum (of the linear part of the map at the fixed point) that lies inside a circle

of radius ρ in $\mathbb{C}$. We do not need to use the Lipschitz inverse function theorem to define the graph transform for PS manifolds.

3. In [4, Section 1.10, page 239] we discuss local invariant manifolds. There, we divide these invariant manifolds into two groups: S-manifolds and W-manifolds. There is a fundamental difference between these two groups when it comes to the issue of local manifolds. First we explain what they are: S-manifolds are the strong-stable, strong-unstable, stable and unstable manifolds. Notice the the part of the spectrum that corresponds to each these manifolds does not include the unit circle. They satisfy condition (S) in Definition 2.1.2 below and hypothesis (H1) of Theorem 1 below. W-manifolds are the pseudo-unstable, centre-stable, centre-unstable, weak-stable and weak-unstable manifolds. The part of the spectrum that correspond to a W-manifolds either include the unit circle (the first 3) or lies in an annulus with radii 1 and ρ (the weak ones). They satisfy condition (W).
4. The fundamental difference between S-manifolds and W-manifolds is that the graph transform for S-manifolds can be formulated on the unit sphere $B_1(0) \subset X$. Thus, the smallness assumption needed can be obtained by replacing the original map say Φ by
$$\Phi_\lambda(x) = \frac{1}{\lambda}\Phi(\lambda x)$$
Notice that the linear part does not change. Moreover, $y = w_\lambda(x)|x| \leq 1$, is invariant under Φ_λ iff $y = w(x) := \lambda w_\lambda(x/\lambda), |x| \leq \lambda$, is invariant under Φ.

   On the other hand, the graph transform for a W-manifold has to be studied on all of the Banach space X. In this case one would like to obtain the smallness assumptions on the nonlinear part of the map by means of a smooth cut-off functions. And this is where trouble starts: Infinite dimensional Banach spaces does not always admit smooth cut-off functions. In [15], it is shown that the space of continuous functions on an interval does not admit a C^1 cut-off function. In [19], it is shown that a separable Banach space admits a Fréchet differentiable cut-off function iff its dual space is separable.

   Now we can see that for W-manifolds, the smallness assumptions are very restrictive because they *truly* have to satisfied *globally.*

2. Main Result

We start by recalling some definitions from [4] and restating Theorem [4] C with "$0 < \delta \leq q$" replaced by "$0 \leq \delta \leq q$".

2.1. Definition. 1. Let X and Y be Banach spaces. Let $Z = X \oplus Y$ be endowed with the box norm. For a linear operator $L, \|L\|$ will denote the operator norm of L.

2. Let $X_o \subset X$ and $q = 0, 1$. We will consider two cases:
   (W): $X_o = X$ and $q = 0$.

(S): $X_o = B_1(0)$ and $q = 1$, where $B_1(0)$ is the closed unit ball in X centred at the origin

3. Let $C^{0,1}(X_o, Y)$ the space of uniformly Lipschitz functions. Let
$$\|u\|_* = \sup_{x \neq 0} \frac{|u(x)|}{|x|^q}, \qquad u \in C^{0,1}(X_o, Y)$$
4. Let $id_{X_o} : X_o \to X_o$ be given by $id_{X_o}(x) = x$. Consider a map
$$Q : C^{0,1}(X_o, Y) \to C^{0,1}(X_o, X_o),$$
$$u \mapsto Q_u.$$
If $Q_u = id_{X_o}$ for all $u \in C^{0,1}(X_o, Y)$ we say that Q is of type (a) and write $Q \equiv id_{X_o}$.
5. Let $T : X \to X$ be linear and continuous and $h \in C^{0,1}(Z, X)$. Suppose that for all $u \in C^{0,1}(X_o, Y)$ we can solve the implicit relation
$$z = Tx + h(z, u(z))$$
uniquely in z in terms of $x \in X_o$. Thus, for all $u \in C^{0,1}(X_o, Y)$, we have a function
$$Q_u : X_o \to X_o,$$
$$x \mapsto z = Q_u(x)$$
$$Q_u = T + h \circ (id, u) \circ Q_u$$
Assume that for all $u \in C^{0,1}(X_o, Y), Q_u \in C^{0,1}(X_o, X_o)$. Thus we have a map
$$Q : C^{0,1}(X_o, Y) \to C^{0,1}(X_o, X_o),$$
$$u \mapsto Q_u.$$
In this case we say that Q is of type $b(T, h)$.

The following lemma is straightforward.

2.2. Lemma. *Let $Q : C^{0,1}(X_o, Y) \to C^{0,1}(X_o, X_o)$ be of type $b(T, h)$. Assume that $Lip(h) < 1$ and $h(0, 0) = 0$. Let $u \in C^{0,1}(X_o, Y)$ be such that $u(0) = 0$ and $Lip(u) \leq 1$. Then $Lip((id, u)) = 1, Q_u(0) = 0$ and*
$$Lip(Q_u) \leq \frac{\|T\|}{1 - Lip(h)}.$$

2.3. Definition. Let E_1 and E_2 be Banach spaces and $X_o \subset E_1$. For a function $h \in C^k(X_o, E_2)$ define the following for $0 \leq j \leq k$ and $0 < \gamma \leq 1$:
$$\|D^j h\| := \sup\left\{\|D^j h(x)\| \;\mid\; x \in X_o\right\}$$
$$\beta_{j,\gamma}(h) := \beta_\gamma(D^j h) := \sup\left\{\frac{\|D^j h(x) - D^j h(x')\|}{|x - x'|^\gamma} \;\mid\; x \neq x', x, x' \in X_o\right\},$$
$$\Delta^j h(x) := (h(x), Dh(x), \cdots, D^j h(x)).$$
If $\beta_{k,\delta}(h) < \infty$, we say that $h \in C^{k,\delta}(X_o, E_2)$.

Notice that for $0 \le j \le k-1, \beta_{j,1}(h) \le \|D^{j+1}h\|$.
In this work we prove the following Theorem 1:

Theorem 1. *Let X, Y, Z and X_o be as in definition 2.1. Let $k = 1, 2, \cdots$ and $0 \le \delta \le 1$, Assume the following:*

- **(H1):** *Let $M : Y \to Y; B_1, B_2, C : X \to X$ be continuous linear maps such that $\|M\| \, \|C\| < 1$ and assume that either (W) or (S) holds:*
  - **(W):** *$\|M\| \le 1$. (Thus $\|M\| \, \|C\|^q < 1$ since $q = 0$).*
  - **(S):** *$\|C\| < 1$. (Thus $\|M\| \, \|C\|^q < 1$, since $q = 1$).*
- **(C1):** *Let $F = (f, g, h_1, h_2)$ where $f, h_1, h_2 \in C^{k,\delta}(Z, X), g \in C^{k,\delta}(Z, Y)$, $F(0,0) = 0$ and $DF(0,0) = 0$.*
- **(C2):** *$\|D^j F\| \le \epsilon, j = 0, 1, \cdots, k$.*
- **(C3):** *$\beta_{j,1}(F) := Lip(D^j F) < \epsilon, j = 0, 1, \cdots, k-1$. If $0 < \delta \le 1$, assume that $\beta_{k,\delta}(D^j F) < \epsilon$.*
- **(H2):** *Let $J : C^{k,\delta}(X_o, Y) \to C^{k,\delta}(X_o, X_o)$ be of type (a) or $b(B_1, h_1)$. Let*

$$\begin{aligned} H : C^{k,\delta}(X_o, Y) &\to C^{k,\delta}(X_o, X_o) \\ u &\mapsto H_u := C + S_F u \\ S_F u &:= f \circ (id, u) \circ J_u. \end{aligned}$$

- **(H3):** *Let $K : C^{k,\delta}(X_o, Y) \to C^{k,\delta}(X_o, X_o)$ be of type (a) or $b(B_2, h_2)$. Let*

$$\begin{aligned} R_F : C^{k,\delta}(X_o, Y) &\to C^{k,\delta}(X_o, X_o) \\ u &\mapsto R_F u := g \circ (id, u) \circ K_u. \end{aligned}$$

For $u \in C^{k,\delta}(X_o, Y)$ let

$$\Gamma_F u := Mu \circ H_u + R_F u. \tag{2.1}$$

Then, for sufficiently small $\epsilon > 0, \Gamma_F$ has a fixed point $w \in C^{k,\delta}(X_o, Y)$ which is unique in the class of functions that satisfies

$$\begin{aligned} w(0) &= 0 \\ \|w\|_* &\le 1 \\ \|D^j w\| &\le 1, \qquad 0 \le j \le k \\ \beta_1(D^j w) &\le 1, \qquad 0 \le j \le k-1 \end{aligned}$$

If $0 < \delta \le 1, w$ also satisfies

$$\beta_\delta(D^k w) \le 1$$

In [4] we proved Theorem 1 for $0 < \delta \le 1$. Therefore we know that Theorem 1 is true for $C^{k-1,1}$ maps. In this work we consider the case $\delta = 0$.

In what follows we need the *fiber contraction theorem* which is proved in [12, page 136].

2.4. The Fiber Contraction Theorem. *Let $\mathfrak{B}$ be a metric space and $\phi : \mathfrak{B} \to \mathfrak{B}$ a map having an attractive fixed point p (i.e. $\forall u \in \mathfrak{B} : \phi^n(u) \to p$ as $n \to \infty$.) Let $\mathfrak{Y}$ be a metric space and $\{\psi^u \mid u \in \mathfrak{B}\}$ a family of maps $\psi^u : \mathfrak{Y} \to \mathfrak{Y}$ such that the formula $\Gamma(u,\sigma) = (\phi(u), \psi^u(\sigma))$ defines a continuous map $\Gamma : \mathfrak{B} \times \mathfrak{Y} \to \mathfrak{B} \times \mathfrak{Y}$. Let $q \in \mathfrak{Y}$ be a fixed point for ψ^p. Then, (p,q) is an attractive fixed point for Γ provided that*

$$\limsup_{n\to\infty} Lip(\psi^{\phi^n(u)}) < 1, \qquad \forall u \in \mathfrak{B}. \tag{2.2}$$

3. The Setup

In this section we give all the definitions, notation and setup used in [4] which we need for the case $\delta = 0$. We also indicate which conclusions in [4] are still valid when $\delta = 0$.

3.1. Definition. 1. Let $\|u\|_*$ be as in Definition 2.1 and define the following spaces:

$$\mathfrak{X}^o = \{u \in C^0(X_o, Y) \mid \|u\|_* < \infty\},$$
$$\mathfrak{Y}^o = \{u \in \mathfrak{X}^o(X_o, Y) \mid u(0) = 0, \|u\|_* \le 1, Lip(u) \le 1\}.$$

Let $\Gamma_o := \Gamma_F : \mathfrak{Y}^o \to \mathfrak{Y}^o$ be given by (2.1).

2. For $1 \le j \le k$, let $\mathcal{L}(X^j, Y)$ be the Banach space of symmetric continuous j-multilinear maps from X^j to Y with the standard norm

$$\|A\| := \sup\left\{\frac{|A\xi|}{|\xi|} \mid 0 \ne \xi \in X\right\}.$$

For $\sigma \in C^0(X_o, \mathcal{L}(X^j, Y))$ and $0 < \gamma \le 1$, let

$$\|\sigma\| := \sup_{x \in X_o} \|\sigma(x)\|,$$
$$\beta_\gamma(\sigma) := \sup\left\{\frac{\|\sigma(x) - \sigma(x')\|}{|x - x'|^\gamma} \mid x \ne x', x, x' \in X_o\right\}.$$

Notice that $\beta_1(\sigma) = Lip(\sigma)$.

3. For $1 \le j \le k$, let

$$\mathfrak{X}^j := \left\{\sigma_j \in C^0(X_o, \mathcal{L}(X^j, Y) \mid \|\sigma_j\| < \infty\right\},$$
$$\mathfrak{Y}^j := \left\{\sigma_j \in \mathfrak{X}^j \mid \|\sigma_j\| \le 1, \beta_1(\sigma_j) \le 1\right\}, \quad 1 \le j \le k-1$$
$$\mathfrak{Y}^k := \left\{\sigma_k \in \mathfrak{X}^k \mid \|\sigma_k\| \le 1\right\},$$
$$\Sigma_o := (u) = \sigma_o,$$
$$\Sigma_j := (u, \sigma_1, \cdots, \sigma_j), \qquad 1 \le j \le k-1,$$
$$\Sigma := \Sigma_{k-1},$$
$$\|\Sigma_j\|_* := \max\{\|u\|_*, \|\sigma_1\|, \cdots, \|\sigma_j\|\}, \qquad 1 \le j \le k-1.$$

It is obvious that each $(\mathfrak{X}^j, \|\cdot\|)$ is a Banach space and that $\mathfrak{Y}^j$ is a closed subset.

4. Let $z := H_u(x), \zeta := J_u(x)$ and $\xi := K_u(x)$.

5. Let $k=1$ and $\sigma_1 \in \mathfrak{Y}^1$. Let $\mathcal{J}^1_\Sigma \sigma_1(x) := \mathcal{J}^1_u \sigma_1(x)$ where

$$\mathcal{J}^1_\Sigma \sigma_1(x) := \begin{cases} I, & J \text{ is of type (a)}, \\ \left[I - Dh_1(\zeta, u(\zeta))(I, \sigma_1(\zeta))\right]^{-1} B_1, & J \text{ is of type } b(B_1, h_1) \end{cases}$$

Let $\mathcal{K}^1_\Sigma \sigma_1(x) := \mathcal{K}^1_u \sigma_1(x)$ where

$$\mathcal{K}^1_\Sigma \sigma_1(x) := \begin{cases} I, & K \text{ is of type (a)}, \\ \left[I - Dh_2(\xi, u(\xi))(I, \sigma_1(\xi))\right]^{-1} B_2, & K \text{ is of type } b(B_2, h_2) \end{cases}$$

Let

$$\begin{aligned} \mathcal{H}^1_\Sigma \sigma_1(x) &:= \mathcal{H}^1_u \sigma_1(x) := C + \mathcal{S}^1_u \sigma_1(x) \\ &:= C + Df(\zeta, u(\zeta))(I, \sigma_1(\zeta)) \mathcal{J}^1_u \sigma_1(x). \end{aligned}$$

Let

$$\begin{aligned} &\Gamma_1 : \mathfrak{Y}^o \times \mathfrak{Y}^1 \to \mathfrak{Y}^o \times \mathfrak{Y}^1, \\ &\Gamma_1(u, \sigma_1) := \Gamma_{\Delta F}(u, \sigma_1) := (\Gamma_o u, \Gamma^u_1 \sigma_1), \\ &\Gamma^u_1 \sigma_1(x) := M \sigma_1(z) \mathcal{H}^1_u \sigma_1(x) + Dg(\xi, u(\xi))(I, \sigma_1(\xi)) \mathcal{K}^1_u \sigma_1(x). \end{aligned} \tag{3.1}$$

6. Let $k \geq 2$ and $\sigma_k \in \mathfrak{Y}^k$. Let

$$\mathcal{J}^k_\Sigma \sigma_k(x) := \begin{cases} I, & J \text{ is of type (a)}, \\ \left[I - Dh_1(\zeta, u(\zeta))(I, \sigma_1(\zeta))\right]^{-1} \mathcal{A}(x) & J, \text{is of type } b(B_1, h_1) \end{cases}$$

$$\mathcal{A}(x) := Dh_1(\zeta, u(\zeta))(0, \sigma_k(\zeta))[\mathcal{J}^1_\Sigma \sigma_1(x)]^k + Q^{h_1;J}_k(\Sigma, x)$$

$$\mathcal{K}^k_\Sigma \sigma_k(x) := \begin{cases} I & K \text{ is of type (b)}, \\ \left[I - Dh_2(\xi, u(\xi))(I, \sigma_1(\xi))\right]^{-1} \mathcal{B}(x) & K, \text{is of type } b(B_2, h_2) \end{cases}$$

$$\mathcal{B}(x) := Dh_2(\xi, u(\xi))(0, \sigma_k(\xi))[\mathcal{K}^1_\Sigma \sigma_1(x)]^k + Q^{h_2;K}_k(\Sigma, x)$$

$$\begin{aligned} \mathcal{H}^k_\Sigma \sigma_k(x) := {} & Df(\zeta, u(\zeta))(0, \sigma_k(\zeta))[\mathcal{J}^1_\Sigma(x) \sigma_1(x)]^k \\ & + Df(\zeta, u(\zeta))(I, \sigma_1(\zeta)) \mathcal{J}^k_\Sigma \sigma_k(x) + Q^{f;J}_k(\Sigma, x), \end{aligned}$$

where

$$\begin{aligned} &P_k(\Sigma, x) := P_k(\Delta^k f(\zeta, u(\zeta)), \Sigma(z), \Sigma(\zeta), \mathcal{H}^{(k-1)}_\Sigma(x)), \\ &Q^{h;L}_k(\Sigma, x) := Q^{h;L}_k(\Delta^k h(\eta, u(\eta)), \Sigma_{k-1}(\eta), \mathcal{N}^{(k-1)}_\Sigma(x)), \\ &\qquad \eta := L_u(x), \qquad h = f, g, h_1, h_2, \qquad L = J, K, \\ &\mathcal{N}^{(k-1)}_\Sigma(x) := (\mathcal{N}^1_\Sigma \sigma_1(x), \cdots, \mathcal{N}^{k-1}_\Sigma \sigma_{k-1}(x)), \qquad \mathcal{N} = \mathcal{H}, \mathcal{J}, \mathcal{K}, \\ &P_k(0, \cdot, \cdot, \cdot) \equiv 0, \qquad P_k(\cdot, 0, \cdot, \cdot) \equiv 0, \qquad P_k(\cdot, \cdot, \cdot, 0) \equiv 0, \\ &Q^{h;L}_k(0, \cdot, \cdot) \equiv 0, \qquad Q^{h;L}_k(\cdot, \cdot, 0) \equiv 0. \end{aligned}$$

Moreover, $P_k(t_1, t_2, t_3, t_4)$ and $Q^{h;L}_k(t_1, t_2, t_3)$ are polynomials in the components of their arguments and linear in the components of t_1.

7. For $k \geq 1$, let $\rho = \sigma_k$ and

$$\begin{aligned}
\mathfrak{B}_{k-1} &:= \prod_{i=0}^{k-1} \mathfrak{Y}^i \\
\mathit{\Gamma}_k &: \mathfrak{B}_{k-1} \times \mathfrak{Y}^k \to \mathfrak{B}_{k-1} \times \mathfrak{Y}^k, \\
\mathit{\Gamma}_k(\Sigma, \rho) &:= \mathit{\Gamma}_{\Delta^k F}(\Sigma, \rho) \\
&:= (\mathit{\Gamma}_{k-1}(\Sigma), \mathit{\Gamma}_k^{\Sigma} \rho), \\
\mathit{\Gamma}_k^{\Sigma} \rho(x) &:= M\rho(z)[\mathcal{H}_{\Sigma}^1 \sigma_1(x)]^k + M\sigma_1(z)\mathcal{H}_{\Sigma}^k \rho(x) \\
&\quad + P_k(\Sigma, x) + Q_k^{g;K}(\Sigma, x) \\
&\quad + Dg(\xi, u(\xi))(0, \rho(\xi))[\mathcal{K}_{\Sigma}^1 \sigma_1(x)]^k \\
&\quad + Dg(\xi, u(\xi))(I, \sigma_1(\xi))\mathcal{K}_{\Sigma}^k \rho(x).
\end{aligned}$$

8. Notice that if w is a C^k fixed point for $\mathit{\Gamma}_o$, then

$$\mathit{\Gamma}_k(\Delta^k w) = \Delta^k w$$

3.2. Information about $\mathit{\Gamma}_k$ which can be obtained from [4] **.** Most of the estimates in [4] are valid for the case $\delta = 0$. The only estimates that are not valid are the ones that lead to the conclusion that $\mathit{\Gamma}_k^{\Sigma} \rho$ is continuous in Σ. These are the estimates in [4, Lemma 4.7]. More precisely we know the following for sufficiently small $\epsilon > 0$:

1. Theorem 1 is proven in [4] for the case $0 < \delta \leq 1$. If we apply it, with smoothness $C^{k-1,1}$, to $\mathit{\Gamma}_{k-1} : \mathfrak{B}_{k-1} \to \mathfrak{B}_{k-1}$, we conclude that $\mathit{\Gamma}_{k-1}$ is well defined, (i.e. $\mathit{\Gamma}_{k-1}$ maps $\mathfrak{B}_{k-1}$ to itself) and has a unique attractive fixed point $\Sigma_* = (w, \tau_1, \cdots, \tau_{k-1}) \in \mathfrak{B}_{k-1}$ and $\tau_j = D^j w$ for $j = 1, \cdots, k-1$.
2. For all $\Sigma \in \mathfrak{B}_{k-1}$, the map $\mathit{\Gamma}_k^{\Sigma}$ maps $\mathfrak{Y}^k$ to itself and hence $\mathit{\Gamma}_k$ maps $\mathfrak{B}_{k-1} \times \mathfrak{Y}^k$ to itself. This is shown [4, Lemma 4.4] where the proof is valid for $\delta = 0$.
3. For sufficiently small $\epsilon > 0$, there is $0 < \theta = [\|M\| \|C\|^k + O(\epsilon)] < 1$ such that for all $\Sigma \in \mathfrak{B}_{k-1}$, for all $\rho, \lambda \in \mathfrak{Y}^k$

$$\|\mathit{\Gamma}_k^{\Sigma} \rho - \mathit{\Gamma}_k^{\Sigma} \lambda\| \leq \theta \|\rho - \lambda\|. \tag{3.2}$$

   This is shown in [4, Proposition 4.6] where the proof is valid for $\delta = 0$.
4. It follows that $\mathit{\Gamma}_k : \mathfrak{B}_{k-1} \times \mathfrak{Y}^k \to \mathfrak{B}_{k-1} \times \mathfrak{Y}^k$ is well defined and is a fiber contraction which is not necessarily continuous in Σ. Since $\Sigma_* \in \mathfrak{B}_{k-1}$ is fixed under $\mathit{\Gamma}_{k-1}$, the fiber $\mathfrak{F}_* := \{\Sigma_*\} \times \mathfrak{Y}^k$ is invariant under $\mathit{\Gamma}_* := \mathit{\Gamma}_{k-1} \times \mathit{\Gamma}_k^{\Sigma_*}$. By (3.2) and the standard contraction mapping theorem, $\mathit{\Gamma}_*$ has a unique attractive fixed point $(\Sigma_*, \mu) \in \mathfrak{F}_*$.
5. However, we do not know that $\mathit{\Gamma}_k^{\Sigma} \rho$ is continuous in Σ, which is one of the hypotheses of the Fiber Contraction Theorem 2.4. Therefore we cannot conclude that (Σ_*, μ) is a unique attractive fixed point for $\mathit{\Gamma}_k : \mathfrak{B}_{k-1} \times \mathfrak{Y}^k \to \mathfrak{B}_{k-1} \times \mathfrak{Y}^k$, which is essential in showing that $\mu = D^k w$.

6. Despite this temporary setback, we have a candidate for $D^k w$, namely, μ the fixed point of Γ_*. We show that $\mu = D^k w$ directly using Proposition 3.3 below.
7. Before we proceed to show that in fact $\mu = D^k w$, let's see why we can prove that $\Gamma_k^{\Sigma}\rho$ is continuous in Σ only when $0 < \delta \le 1$ but not when $\delta = 0$: When $0 < \delta \le 1$, we include the requirement $\beta_\delta(\sigma_k) \le 1$ in the definition of $\mathfrak{Y}^k$. Then, the map F and $\rho \in \mathfrak{Y}^k$ are uniformly δ-Hölder continuous. This precisely what we used in the proof of Lemma 4.7 of [4] to show that $\Gamma_k^{\Sigma}\rho$ is continuous in Σ.

3.3. Proposition. *If x and h are in X such that both x and $x+h$ are in X_o, then*

$$D^{k-1}w(x+h) - D^{k-1}w(x) = \int_0^1 \mu(x+th)hdt. \tag{3.3}$$

It follows that $D^{k-1}w$ is continuously differentiable and $D^k w = \mu$.

4. **Proof of Proposition 3.3.**

4.1. Definition. Let x and h be as in Proposition 3.3 and define the following:

1. Let $u_o \in \mathfrak{Y}^o$ be such that $\Delta^k u_o \in \mathfrak{B}_{k-1} \times \mathfrak{Y}^k$. Let $u_n := \Gamma_o u_{n-1}, n \ge 1$. Thus, for $n \ge 0, \Delta^k u_n \in \mathfrak{B}_{k-1} \times \mathfrak{Y}^k$. Let $u_\infty := w$.
2. Let $U_n := \Delta^{k-1}u_n$ and $U_\infty := W := \Delta^{k-1}w$. Thus, for $n \ge 0, U_n \in \mathfrak{B}_{k-1}, U_n = (\Gamma_{k-1})^n(U_o)$, and $\|U_n - W\|_* \to 0$ as $n \to \infty$.
3. Let $\chi : [0,1] \to X_o$ be given by $\chi(t) := x + th$. Let $I := \chi([0,1])$.
4. Let $\phi_n := U_n|_I$, for $n \ge 0$, and $\phi_\infty := W|_I = U_\infty|_I$.
5. Let

$$\begin{aligned} \mathbb{V} &:= \{\phi_n \mid 0 \le n \le \infty\}, \\ \mathbb{Y} &:= \{\hat\rho = \rho|_I \ \mid \ \rho \in \mathfrak{Y}^k\}, \\ \|\phi_n\|_o &:= \|U_n|_I\|_*, \\ \|\hat\rho\|_1 &:= \|\rho|_I\| = \sup_{\chi \in I}\|\rho(\chi)\|, \qquad \hat\rho = \rho|_I \in \mathbb{Y}. \end{aligned}$$

6. Let $\Lambda : \mathbb{V}\times\mathbb{Y} \to \mathbb{V}\times\mathbb{Y}$ be the obvious restriction of Γ_k to $\mathbb{V}\times\mathbb{Y}$ given by:

$$\Lambda(\phi_j, \hat\rho) := (\Lambda_o\phi_j \ , \ \Lambda^{\phi_j}\hat\rho) := [\Gamma_k(U_j, \rho)]|_I. \tag{4.1}$$

Let $z_j = H_{u_j}(\chi), \zeta_j = J_{u_j}(\chi)$ and $\xi_j = K_{u_j}(\chi)$. Thus for $\chi \in I$,

$$\Lambda_o\phi_j(\chi) := \Lambda_o(U_j|_I)(\chi) := [\Gamma_{k-1}U_j]|_I(\chi),$$

$$\begin{aligned} \Lambda^{\phi_j}\hat\rho(\chi) &:= [\Gamma_k^{U_j}\rho]|_I(\chi) \\ &= M\rho(z_j)[\mathcal{H}^1_{U_j}Du_j(\chi)]^k + MDu_j(z_j)\mathcal{H}^k_{U_j}\rho(\chi) \\ &\quad + P_k(U_j,\chi) + Q_k^{g;K}(U_j,\chi) \\ &\quad + Dg(\xi_j, u_j(\xi_j))(0, \rho(\xi_j))[\mathcal{K}^1_{U_j}Du_j(\chi)]^k \end{aligned} \tag{4.2}$$

$$+ Dg(\xi_j, u(\xi_j))(I, Du_j(\xi_j))\mathcal{K}^k_{U_j}\rho(\chi).$$

4.2. Remarks. 1. It is obvious that $(\mathbb{V}, \|\cdot\|_o)$ is a complete metric space since it consists of one sequence and its limit point. Also $(\mathbb{Y}, \|\cdot\|_1)$ is a metric space. Recall that the fiber contraction theorem is stated for metric spaces.

2. Recall that

$$(\Gamma_{k-1}U_j\ ,\ \Gamma_k^{U_j}D^k u_j) = \Gamma_k(U_j, D^k u_j) = \Gamma_k(\Delta^k u_j) \\ = \Delta^k(\Gamma_o u_j) = \Delta^k u_{j+1} = (U_{j+1}, D^k u_{j+1}). \tag{4.3}$$

Restricting (4.3) to I, we obtain

$$\Lambda_o\phi_j = [\Gamma_{k-1}U_j]|_I = U_{j+1}|_I = \phi_{j+1}, \\ \Lambda^{\phi_j}(D^k u_j|_I) = [\Gamma^{U_j}D^k u_j)]|_I = [D^k u_{j+1}]|_I. \tag{4.4}$$

4.3. Proposition. *The following are true for sufficiently small $\epsilon > 0$:*

1. *Λ maps $\mathbb{V}\times\mathbb{Y}$ to itself.*
2. *The fiber $\mathbb{Y}_\infty := \{\phi_\infty\}\times\mathbb{Y}$ is invariant under Λ.*
3. *$\Lambda : \mathbb{V}\times\mathbb{Y} \to \mathbb{V}\times\mathbb{Y}$ is a fiber contraction (not necessarily continuous in U_j yet).*
4. *$\Lambda^{\phi_\infty} : \mathbb{Y}_\infty \to \mathbb{Y}_\infty$ has a unique attractive fixed point τ.*
5. *In fact, $\tau = \mu|_I$ where μ is the unique attractive fixed point of $\Gamma_k^W : \{W\}\times\mathfrak{Y}^k \to \{W\}\times\mathfrak{Y}^k$.*

Proof. Assertions (1)-(4) follow from Section 3.2. As for (5), it is obvious that $\mu|_I \in \mathbb{Y}$ is a fixed point for $\Lambda^{\phi_\infty} = \Gamma^w\ |_I$. But τ is the unique fixed point of Λ^{ϕ_∞}. Thus $\tau = \mu|_I$. □

In order to be able to apply the Fiber Contraction Theorem we need to show that $\Lambda(\phi_j, \hat\rho)$ is continuous in ϕ_j. We need the following elementary but important lemma.

4.4. Lemma. *Let $0 \le i \le k-1, 0 \le j \le \infty$ and $L = H, J, K$. The following sets are compact subsets of Y:*

$$\mathcal{U}_j = u_j(I), \qquad \mathcal{Z}_j^L = L_{u_j}(I), \qquad \mathcal{V}_j^{L,i} = D^i u_j(L_{u_j}(I)),$$

$$\mathcal{U} = \bigcup_{j=0}^{j=\infty}\mathcal{U}_j, \qquad \mathcal{Z}^L = \bigcup_{j=0}^{j=\infty}\mathcal{Z}_j^L, \qquad \mathcal{V}^{L,i} = \bigcup_{j=0}^{j=\infty}\mathcal{V}_j^{L,i},$$

$$\mathcal{V}^L = \bigcup_{i=0}^{k-1}\mathcal{V}^{L,i},$$

$$\mathcal{W} = \mathcal{U}\cup\mathcal{Z}^H\cup\mathcal{Z}^J\cup\mathcal{Z}^K\cup\mathcal{V}^H\cup\mathcal{V}^J\cup\mathcal{V}^K.$$

Proof. Since a finite union of compact sets is compact, and since a continuous image of the interval $[0,1]$ is compact, we need to consider only $\mathcal{U}, \mathcal{Z}^L$ and $\mathcal{V}^{L,i}$. We will prove that $\mathcal{V}^{L,i}$ is compact. The other two cases are easier.

Recall that X and Y are Banach. Since $\mathcal{V}_\infty^{L,i}$ is compact, for all $\alpha > 0$ there is a finite set of points $\{x_1, \cdots, x_{n_\alpha}\} \subset \mathcal{V}_\infty^{L,i}$, such that

$$\mathcal{V}_\infty^{L,i} \subset \mathcal{O}(\alpha) := \bigcup_{i=1}^{n_\alpha} \mathcal{O}(x_i, \alpha), \qquad \mathcal{O}(x,\alpha) := \{x' \in X_o \mid |x' - x| < \alpha\}.$$

Fix $\chi \in I$ and let $\eta_j := L_{u_j}(\chi), 0 \le j \le \infty$. By assertion (1) and (2) of [4, Lemma 2.3],

$$|\eta_j - \eta_\infty| = |L_{u_j}(\chi) - L_w(\chi)| \le O(\epsilon)\|u_j - w\|_* \to 0, \tag{4.5}$$

as $j \to \infty$, uniformly in $\chi \in X_o$. Moreover,

$$\begin{aligned} \|D^i u_j(\eta_j) - D^i w(\eta_\infty)\| &\le \|D^i u_j(\eta_j) - D^i w(\eta_j)\| \\ &\quad + \|D^i w(\eta_j) - D^i w(\eta_\infty)\| \\ &\le \|D^i u_j - D^i w\| + \beta_1(w)|\eta_j - \eta_\infty| \\ &\le 2\|U_j - W\|_* \to 0, \end{aligned} \tag{4.6}$$

as $j \to \infty$, uniformly in $\chi \in X_o$.

It follows that for all $\alpha > 0$, there is j_o such that: If $j \ge j_0$, then $\mathcal{V}_j^{L,i} \subset \mathcal{O}(\alpha)$. The finite union of compact sets $\mathcal{V}_0^{L,i} \cup \mathcal{V}_1^{L,i} \cup \cdots \cup \mathcal{V}_{j_o-1}^{L,i}$, is compact. Thus, for all $\alpha > 0$, $\mathcal{V}^{L,i}$ has a finite cover of open balls of radius α. Since Y is also a metric space, it follows that $\mathcal{V}^{L,i}$ is compact. $\square$

4.5. Lemma. *As above, let $L = H, J, K; \mathcal{N} = \mathcal{H}, \mathcal{J}, \mathcal{K}; h = f, g, h_1, h_2;$ $A = C, B_1, B_2;$ and $\eta_j = L_{u_j}(\chi), \chi \in I, 0 \le j \le \infty$, where the correspondence between $L, \mathcal{N}, h$ and A should be obvious from hypotheses (H1) and (H2). The following are true for sufficiently small $\epsilon > 0$:*

$$\begin{aligned} |\eta_j - \eta_n| &\le O(\epsilon)\|(u_j - u_n)\mid_I\|_* \le \|(u_j - u_n)\mid_I\|_* \\ &\le \|\phi_j - \phi_n\|_o, \end{aligned} \tag{4.7}$$

$$\begin{aligned} |u_j(\eta_j) - u_n(\eta_n)| &\le [\|A\|^q + O(\epsilon)]\|(u_j - u_n)\mid_I\|_* \\ &\le [\|A\|^q + O(\epsilon)]\|\phi_j - \phi_n\|_o. \end{aligned} \tag{4.8}$$

For all $\alpha > 0$, there is $c > 0$ such that: if $\|\phi_j - \phi_n\|_o < c$, then for all $\chi \in I$, the following are true for $1 \le j, n \le \infty$:

$$\|D^k h(\eta_j, u_j(\eta_j)) - D^k h(\eta_n, u_n(\eta_n))\| < \alpha, \qquad k \ge 1 \tag{4.9}$$

$$\|\rho(\eta_j) - \rho(\eta_n)\| < \alpha \tag{4.10}$$

$$\|\mathcal{L}^1_{U_j}\rho(\chi) - \mathcal{L}^1_{U_n}\rho(\chi)\| < \alpha \tag{4.11}$$

$$\|P_k(\phi_j, \chi) - P_k(\phi_n, \chi)\| < \alpha, \qquad k \ge 2 \tag{4.12}$$

$$\|Q_k^{h;L}(\phi_j, \chi) - Q_k^{h;L}(\phi_n, \chi)\| < \alpha, \qquad k \ge 2 \tag{4.13}$$

$$\|\mathcal{L}^k_{U_j}\rho(\chi) - \mathcal{L}^k_{U_n}\rho(\chi)\| < \alpha, \qquad k \ge 2 \tag{4.14}$$

Proof. Notice that Lemma 2.3 of [4] is applicable for $\delta = 0, k \ge 1$ since it is proved at the $C^{0,1}$ level. Now, estimate (4.7) follows from [4, Lemma 2.3(1,2)] and estimate (4.8) follows from [4, Lemma 2.3(3)].

To prove estimate (4.9) Let $G = D^k h$. Since G is continuous and $\mathcal{W}$ is compact, it follows that $G|_{\mathcal{W}}$ is uniformly continuous. Thus, for all $\alpha > 0$, there is $d > 0$, such that : if $\xi, \xi', \nu, \nu' \in \mathcal{W}, |\xi - \xi'| < d$ and $|\nu - \nu'| < d$, then $|G(\xi, \nu) - G(\xi', \nu')| < \alpha$. Let $c = min\{d, d[\|A\|^q + O(\epsilon)]^{-1}\}$. Assume that $\|\phi_j - \phi_n\|_o < c$. Then, by (4.7),$|\eta_j - \eta_n| < c < d$, and by (4.8),

$$|u_j(\eta_j) - u_n(\eta_n)| < [\|A\|^q + O(\epsilon)]c < d$$

Thus, $G(\eta_j, u_j(\eta_j)) - G(\eta_n, u_n(\eta_n))| < \alpha$.

The proof of estimate (4.10) is similar since ρ is continuous and $\mathcal{W}$ is compact. In fact it is simpler since we need only estimate (4.7).

To prove estimate (4.11) notice that all the functions involved in $\mathcal{L}^1_{U_i}\rho(\chi)$ are continuous and hence uniformly continuous on $\mathcal{W}$. Now the estimate follows from estimates (4.7)-(4.10) using an argument similar to the one used to prove (4.9). Notice that in the case $k = 1, \Gamma^1_u\rho$ does not include P and Q terms.

As for estimates (4.12) and (4.13), notice that for $k \geq 2$, all the terms in $P_k(\phi_i, \chi)$ and $Q^{h;L}_k(\phi_i, \chi)$, except $(\Delta^k h) \circ (id, u_i)$, are at least C^1 and have uniformly bounded first derivatives which implies uniform continuity on all of X_o. The term $(\Delta^k h) \circ (id, u_i)$ is treated in (4.9). Thus, $P_k(\phi_i, \chi)$ and $Q^{h;L}_k(\phi_i, \chi)$ are uniformly continuous on $\mathcal{W}$. In view of estimates (4.7)-(4.10), an argument similar to that used in the proof of estimate (4.9) shows that estimates (4.12) and (4.13) are true.

Now estimate (4.14) follows from estimates (4.7)-(4.13) using an argument similar to that used to prove estimate (4.9). □

4.6. Proposition. *For sufficiently small $\epsilon > 0, \Lambda(\phi, \hat{\rho}) = (\Lambda_o\phi, \Lambda^\phi\hat{\rho})$ is continuous in ϕ.*

Proof. We know that $\Lambda_o\phi$ is continuous in ϕ. We need to show that $\Lambda^\phi\hat{\rho}$ is continuous in ϕ. Considering (4.2), we can see that all terms are at least uniformly Lipschitz continuous except the terms $\rho(z), , \rho(\xi), \mathcal{H}^k_U\rho(\chi), P_k(U, \chi)$, $Q^{g,K}_k(U, \chi)$ and $\mathcal{K}^k_U\rho(\chi)$. However, all these terms are continuous and hence uniformly continuous on the compact set $\mathcal{W}$. In view of Lemma 4.5, for all $\alpha > 0$, there is $c > 0$, such that: if $\|\phi - \psi\|_o < c$, then $\|\Lambda^\phi\hat{\rho} - \Lambda^\phi\hat{\rho}\|_1 < \alpha$. This shows that $\Lambda^\phi\hat{\rho}$ is continuous in ϕ. □

4.7. Proposition. *For sufficiently small $\epsilon > 0$, the fiber contraction $\Lambda : \mathbb{V} \times \mathbb{Y} \to \mathbb{V} \times \mathbb{Y}$ has a unique attractive fixed point given by $(\phi_\infty, \mu \mid_I)$.*

Proof. In Proposition 4.3 we showed that $\Lambda : \mathbb{V} \times \mathbb{Y} \to \mathbb{V} \times \mathbb{Y}$ is a fiber contraction and that on the invariant fiber $\mathbb{Y}_\infty$, the map $\Lambda^{\phi_\infty} : \mathbb{Y}_\infty \to \mathbb{Y}_\infty$ has a unique attractive fixed point $\tau = \mu \mid_I$. By Proposition 4.6, $\Lambda^\phi\hat{\rho}$ is continuous in ϕ. By the fiber contraction theorem, $(\phi_\infty, \mu \mid_I)$ is a unique attractive fixed point for $\Lambda : \mathbb{V} \times \mathbb{Y} \to \mathbb{V} \times \mathbb{Y}$. □

Proof of Proposition 3.3. Let $\{u_n \mid n = 0, 1, 2, \cdots\}$ be as in Definition 4.1. Each u_n is C^k. Therefore

$$\mathcal{L}_n(x, h) := D^{k-1}u_n(x + h) - D^{k-1}u_n(x)$$

$$= \int_0^1 D^k u_n(x+th)hdt =: \mathcal{R}_n(x,h)$$

We also know that $\|D^{k-1}u_n - D^{k-1}w\| \to 0$ as $n \to \infty$ on X_o. Thus $\mathcal{L}_n(x,h)$ converges uniformly to the left hand side of (3.3) on X_o. It remains to show that $\mathcal{R}_n(x,h)$ converges to the right hand side of (3.3).

Recall that $(\phi_\infty, \mu|_I)$ is the unique attractive fixed point of Λ. Thus, by (4.4),

$$\|\mu|_I - D^k u_n|_I\| \to 0,$$

as $n \to 0$. It follows that,

$$\|\int_0^1 D^k u_n(x+th)h - \mu(x+th)hdt\| \leq |h|\|D^k u_n|_J - \mu|_J\| \to 0$$

as $n \to 0$. $\square$

This finishes the proof of Proposition 3.3 and Theorem 1.

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