Multifractal spectrum via return times

J. Luévano[*], V. Penné[†], S. Vaienti[†]

Abstract

A multifractal spectrum related to return times is exhibited: the method consists in computing the sum of the first returns of the center of balls in a covering of the invariant set. This spectrum partly recovers that of the generalized dimensions for the invariant measure. Some numerical examples are presented and the comparison with the usual multifractal analysis for measures is discussed.

In this letter we define a new multifractal spectrum to characterize the invariant measures arising from dynamical systems and we compare it with the spectrum of generalized dimensions; let us first give a short review of the latter. Suppose that T is a transformation applying the space X into itself and leaving the probability measure μ invariant. The generalized dimensions $D_\mu(q)$ of such a measure are defined by the following scaling:

$$\int_X \mu(B(x,r))^{q-1} d\mu \sim r^{D_\mu(q)(q-1)} \tag{1}$$

where $B(x,r)$ is the ball of center x and radius r, with r going to zero (in the following all the dimensions will be referred to the measure considered in the context, so that we will write $D(q)$ instead of $D_\mu(q)$). The scaling (1) can be

[*]Centre de Physique Théorique, Université d'Aix Marseille II, Universidad Autónoma Metropolitana, Azcapotzalco, México

[†]Centre de Physique Théorique, Luminy, Marseille and PHYMAT, Université de Toulon et du Var, France

rigorously justified for large classes of dynamical systems including notably conformal mixing repellers [5], maps of the intervals [1], [2], [3], [4], [8], [9], Axiom-A attractors [6], [7] and parabolic Julia sets [10]. These systems exhibit some sort of hyperbolicity which considerably helps and supports the proofs; however the scaling (1) is revealed also in the numerical investigations of non-hyperbolic systems, like the Henon-map and for maps with singularities. We recall the meaning of the generalized dimensions $D(q)$: they provide a fine description of the measure μ, in the sense that the Legendre transform of the function $\tau(q) = D(q)(q-1)$ is the Hausdorff dimension $f(\alpha)$ of the level sets of points x where $\mu(B(x,r))$ scales like r^{α} in the limit of small r. In particular on a set of full measure $\mu(B(x,r)) \sim r^{D(1)}$, where $D(1)$ is the Hausdorff dimension of the measure μ (information dimension). This latter dimension has even a dynamical meaning being, in the one-dimensional case, the ratio between the metric entropy and the (positive) Lyapunov exponent, and for more general hyperbolic measures, a suitable combination of metric entropies and Lyapunov exponents along invariant subspaces, a formula better known in the physical literature as the Kaplan-Yorke's one [20], [21], [14]. The other dimensions reveal the existence of subsets of X of zero measure and with different local scaling of the measure of balls; among these dimensions, $D(2)$ plays a particular role, since it often coincides with the correlation dimension introduced by Grassberger and Procaccia [11]. The importance of the spectrum of the generalized dimensions and of its Legendre transform, the "$f(\alpha)$" function, are nowadays recognized as basic tools in the investigation of chaotic dynamics epecially for the identification and reconstruction of strange attractors through the analysis of time series and embedding techniques and for the caracterization of spectral measures in quantum mechanics and transport phenomenas [16], [15], [17], [18], [19].

The new method that we propose in this letter is based on the general idea of *systematically replacing the measure of a ball with the inverse of the first return time of the center of the ball into itself*.

Our first motivation in this direction was the classical result by M. Kac [22]; at this regard we introduce the first return time of the point $x \in A \subset X$ as: $\tau_A(x) = \inf(k > 0, T^k(x) \in A)$. Kac's theorem says that whenever the measure μ is ergodic: $\int_A \tau_A(x) d\mu = 1$, which is equivalently to say that the average of $\tau_A(x)$ over A is the inverse of the measure of A. Our second motivation was the Theorem of Ornstein and Weiss [12]. In the context of measurable dynamical systems introduced above, this theorem can be stated in the following way. We introduce a generating partition $\mathcal{A}$ of X, and then we refine it around all the points $x \in X$ by defining the cylinder of order n, $A_n(x)$, as the intersection of all the elements of $\mathcal{A}$, $T^{-1}\mathcal{A}$,...,$T^{-n+1}\mathcal{A}$ containing x. Ornstein-Weiss's theorem states that, whenever the measure μ is ergodic, the following limit exists μ-almost everywhere and is equal to the metric entropy of μ:

$$\lim_{n\to\infty} \frac{\log \tau_{A_n(x)}(x)}{n} = h(\mu) \tag{2}$$

This remarkable result is the first that allows to compute a thermodynamic quantity, like metric entropy, by using return times. This theorem works with cylinders, in the same way as the Shannon-Mc Millan-Breiman theorem gives the metric entropy by looking at the exponential decay of the measure of cylinders around almost all points. It is well known, however, that one can consider variant of the Shannon-Mc Millan-Breiman theorem by replacing the measure of cylinders with the measure of balls: this is in particular done in the Brin-Katok's theorem [13].

One could follow the same prescrition for Ornstein-Weiss by considering the quantity $\tau(x,r) \equiv \tau_{B(x,r)}(x)$ and by looking for a power law decay of the type $\tau(x,r) \sim r^{-d}$, where the exponent d depends on the point x. An easy argument for cookie-cutter Cantor sets gives the meaning of such an exponent: to make the computation easier, we consider cookies-cutter for which the lenghts of the "holes" are larger than the "full" subsets for any generation in the construction of the invariant Cantor set. Suppose that r is replaced by the countable sequence $r_n = |DT^n(x)|^{-1}$, where DT is the derivative of T and x is any point in a "full" subset of the n-th generation, that is a cylinder of the refined partition of order n derived from a generating partition $\mathcal{A}$ of the interval with elements of diameter of order 1. Then the ball $B(x, r_n)$ will be bounded from above and below respectively by a cylinder of order $n-1$ and n, say $A_{n-1}(x)$ and $A_n(x)$ as a consequence of our preceding assumption. Moreover the lenghts of $A_{n-1}(x)$ and $A_n(x)$ will be of order $|DT^{n-1}(x)|^{-1}$ and $|DT^n(x)|^{-1}$ respectively; we thus have: $\tau_{A_{n-1}(x)}(x) \le \tau_{B(x,r_n)}(x) \le \tau_{A_n(x)}(x)$. Taking the logarithm and dividing by $r_n = |DT^n(x)|^{-1}$ and by using the Ornstein-Weiss theorem applied to cylinders we finally get:

$$\lim_{r\to 0} -\frac{\log \tau(x,r)}{\log r} = \lim_{n\to\infty} \frac{\log \tau_{B(x,r_n)}(x)}{\log |DT^n(x)|} = \lim_{n\to\infty} \frac{\log \tau_{A_n(x)}(x)}{n \frac{\log |DT^n(x)|}{n}} = \frac{h(\mu)}{\lambda(\mu)} = D(1) \tag{3}$$

The quantity $\lambda(\mu)$ denotes the Lyapunov exponent of the measure μ and the limits hold μ-a.e.. The relation (3) has been rigourosly derived for Gibbs measures of Axiom-A diffeomorphisms in [24] and for a wide class of maps of the interval in [23]. These results mean that $\tau(x,r)$ scales likes $r^{-D(1)}$ for μ-almost all points x.
Another power law, whose derivation and justification will be presented elsewhere, gives the scaling of the q-moments of $\tau(x,r)$ when r goest to zero, namely:

$$\int_X \tau(x,r)^{1-q} d\mu \sim r^{d(q)(q-1)} \tag{4}$$

This scaling law is the basic announcement of this letter. The exponents $d(q)$ will be compared in the following with the generalized dimensions $D(q)$ introduced before. We want to point out that formula (4) is very easy to compute numerically as soon as one works with physical (SBR) measures on attractors. In this case the integral can be replaced with a Birkhoff sum along the orbit of a generic Lebesgue point x in the basin of attraction:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} [\tau(T^l(x),r)]^{1-q} \sim r^{d(q)(q-1)} \tag{5}$$

In the case of one-dimensional mixing repellers equipped with the equilibrium measure for the potential $-\beta \log |DT(x)|$ with pressure $P(\beta)$, the preceding formula can be replaced by averaging along the preimages of Lebesgue any point x in the interval, as stated by the following limit derived by the usual Perron-Frobenius theory:

$$\frac{\log \int_X \tau(x,r)^{1-q} d\mu}{\log r^{-1}} \sim \frac{\log[\lim_{n\to\infty} e^{-nP(\beta)} \sum_{y\in T^{-n}(x)} \frac{\tau(y,r)^{1-q}}{|DT^n(y)|^{\beta}}]}{\log r^{-1}}, \text{for } r \to 0 \tag{6}$$

An analogous formula holds for Iterated Function Systems (IFS); if $\Phi_1, ...\Phi_k$ are k affine maps defined on R^n with the measure μ associated to the weights $p_1, ..., p_k$, it is well known that for any $x \in R^n$ [25]:

$$\int_{R^n} \tau(x,r)^{1-q} d\mu = \lim_{n\to\infty} \sum_{j_1=1,k} ... \sum_{j_n=1,k} p_{j_1}...p_{j_n} \tau(\Phi_{j_1}(x),r)^{1-q} ... \tau(\Phi_{i_n}(x),r)^{1-q} \tag{7}$$

Finally our integral could be easily adapted to *signals*; let us take the time series $Y(1), Y(2), ..., Y(n), ...$ which we suppose to be a generic realization of some ergodic process. By defining $\tau(Y(j), r) = \inf(q \geq 1, |Y(j+q) - Y(j)| \leq r)$, we could investigate the scaling for r going to zero of the following ergodic sum:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \tau(Y(j), r)^{1-q} \tag{8}$$

We want to emphasize that both formulas (5)-(8) require the computation of only *one* sum along the orbits, while in the usual Grassberger-Procaccia (G.B.) technique for the q-correlation integrals, one must work with *two* sums where all the couples of points along the same (or different) orbit are compared [11].
Moreover the return times of the center of the balls are *integer* numbers and their computation is therefore unaffected by approximation errors; we have olny to take care of the lenght of this times which grows exponentially fast.

Another advantage of the method is that, in the numerical computation of the sum, it turns out that it is possible to calculate in one pass the same sum for many different values of r. To give an idea, we get so many different values of r in one pass that the critical computation time become to fit the log/log curve that we obtain.

We now present a few numerical computations on some classical dynamical systems for which the generalized dimension have been evaluated using standard techniques, like the q-correlation integral quoted above. Applications to IFS and signals will be presented in a forthcoming paper.

The first example that we consider it the Baker map, defined as

$$\begin{aligned} x_{n+1} &= \begin{cases} \gamma_a x_n, & \text{if } y_n < \alpha, \\ \frac{1}{2} + \gamma_b x_n, & \text{if } y_n \geq \alpha, \end{cases} \\ y_{n+1} &= \begin{cases} \frac{1}{\alpha} y_n, & \text{if } y_n < \alpha, \\ \frac{1}{1-\alpha}(y_n - \alpha), & \text{if } y_n \geq \alpha, \end{cases} \end{aligned}$$

with $0 \leq x_n, y_n \leq 1$ and $0 < \gamma_a < \gamma_b < 1/2, \alpha \leq 1/2$. This map has a strange attractor Λ which is the product of a Cantor set and the vertical interval $[0, 1]$. An analytical expression of $D(q)$ has is given in [26], we will use it to compare with the function $d(q)$ computed with return times.

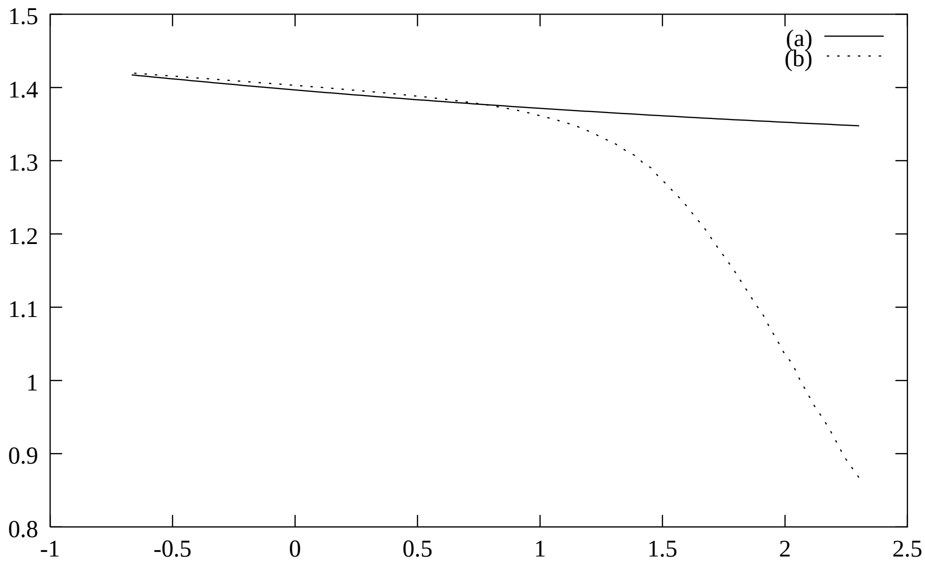

Figure 1: Baker map, with parameters $\gamma_a = 0.1$, $\gamma_b = 0.3$, $\alpha = 0.3$, (a) : dimension spectrum for return times, $d(q)$, with an orbit of 100000 points. (b) : theoretical dimension spectrum of the invariant measure, $D(q)$.

We observe that $D(q)$ and $d(q)$ behave very similarly for $q < 1$, instead, standard technique would give a better approximation to $D(q)$ in the range $q > 1$.

The second example that we will use is the Bernouilli map

$$x_{n+1} = \begin{cases} \frac{1}{\alpha} x_n, & \text{if } x_n < \alpha, \\ \frac{1}{1-\alpha}(x_n - \alpha), & \text{if } x_n \geq \alpha, \end{cases}$$

with $0 \leq x_n < 1$ and $0 < \alpha < 1$. Again we will compare the function $d(q)$ with an analytical expression of $D(q)$, and additionally, we also compare with a function $D_{\text{exp.}}(q)$ computed numerically with the traditional G.P. algorithm (counting the pair (i, j) such that $d(x_i, x_j) < \epsilon$). We see that $d(q)$ and $D((q)$ fit well in the range $q < 1$.

The conclusion of this note is that the new method we introduce allows one to compute the generalized dimension for $q < 1$, which is the domain of parameter where other methods usually fail. We came to these conclusions, by comparing our results to some analytical expression of $D(q)$ in some special cases, and also to numerically computed $D_{\text{exp.}}(q)$ with standard method.

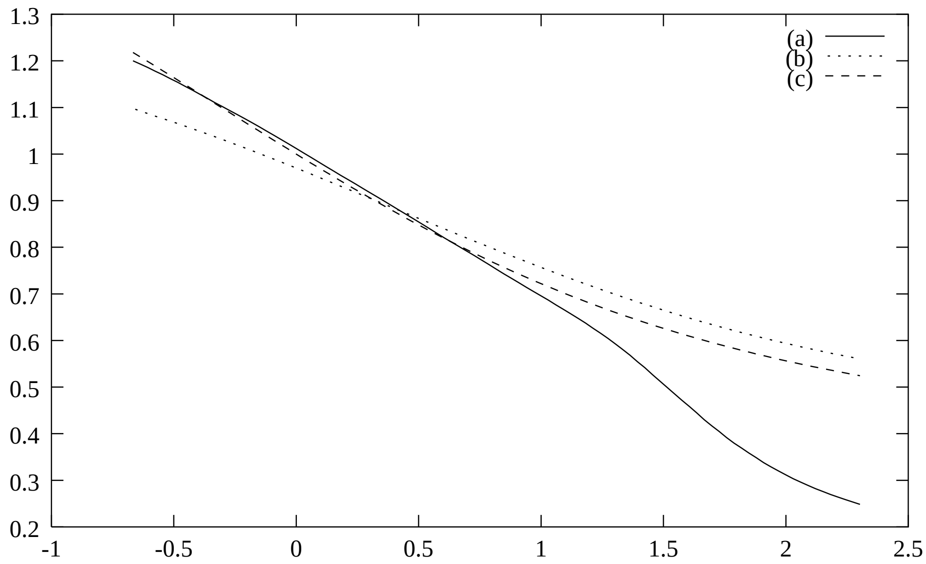

Figure 2: Tent map, with parameters $\alpha = 0.2$, (a) : dimension spectrum for return times, $d(q)$, with an orbit of 100000 points. (b) : numerically computed dimension spectrum of the invariant measure, $D_{\text{exp.}}(q)$, (c) : theoritical dimension spectrum of the invariant measure, $D(q)$.

These results adress two main questions:

1. the meaning of the spectrum $d(q)$ as the Legendre transform of some level set functions (precisely the hausdorff dimension of the set of points with a given local scaling of $\tau(x, r)$).

2. the relationship with the spectrum of $D(q)$: in particular the coincidence for some interval of values of q.

We finally think that the $d(q)$ spectrum has an intrinsic theoretical interest, whose implication deserve a better understanding ; its easy numerical implementation makes it a flexible and performing tool in the investigation of chaotic dynamics.

Acknowledgments. We thanks B. Saussol and S. Troubetzkoy for the common and shared passion in the "thermodynamics of return times".

References

[1] P. Collet, J.L. Lebowitz, A. Porzio, J. Stat. Phys., **47** (1987) 609–644

[2] T. Bohr, D. Rand, Phys. D, **25D** (1987), no. 1-3, 387–398

[3] D. Rand, Erg. Th. Dyn. Sys. **9** (1989) 527–541

[4] L.Olsen, Advances in Mathematics, **116** (1995) 82–196

[5] Ya. Pesin, "Dimension theory in dynamical systems", The University of Chicago Press, 1997

[6] A. Porzio, J. Statist. Phys. **58** (1990), no. 5-6, 923–937.

[7] D. Simpalaere, J. Stat. Phys., **76** (1994) 1359–1375

[8] F. Hofbauer, Erg. Th. Dyn. Sys. **15** (1995) 1119–1142

[9] F Hofbauer, P. Raith, Canad. Math. Bull., **35**(1992) 84–98

[10] F. Przytychi, M. Urbanski, A. Zdunik, Ann. of Math., **130** (1989) 1–40

[11] P. Grassberger, I. Procaccia, Phys. rev. Lett., **84** (1983) 346–349

[12] D. Ornstein, B. Weiss, IEEE Trans. inf. Theory, **39** (1993) 78–83

[13] M. Brin, A. Katok, "On local entropy", Geometric dynamics (Rio de Janeiro, 1981), 30–38, Lecture Notes in Math., 1007, Springer, Berlin-New York, 1983

[14] L. Barreira, Ya. Pesin, J. Schmeling, Ann. of Math.

[15] H.G.E. Hentschel, I. Procaccia, Physica D, **8** (1983) 435–444

[16] T.C.Halsey, M. Jensen, L.Kadanoff, I. Procaccia, B.Shraiman, Phys. Rev. A, **33:2** (1986) 1141–1151

[17] G. Paladin, A. Vulpiani, Phys. Rep. **156** (1987), no. 4, 147–225

[18] P Badii, A. Politi, Phys. Lett. A **104** (1984), no. 6-7, 303–305

[19] P Grassberger, A. Politi, R. Badii, J. Statist. Phys. **51** (1988), no. 1-2, 135–178.

[20] L.S. Young, Ergodic Theory Dynamical Systems **2** (1982), no. 1, 109–124

[21] F. Ledrappier, L-S. Young, Ann. of Math., **122** (1985) 540–574

[22] M. Kac, "Probability and related topics in physical sciences" (1959) AMS

[23] B. Saussol, S. Troubetzkoy, S. Vaienti, *No small returns*, in preparation

[24] L. Barreira, B. Saussol, *Hausdorff dimension of measures via Poincaré recurrence*, preprint 2000.

[25] M. Barnsely, S. Demko, "Chaotic dynamics and fractals" (1986) Acad. Press

[26] D. Bessis, G. Paladin, G. Turchetti, S. Vaienti, Journal of Statistical Physics, **51**, Nos. 1/2, 1988.