THE SPECTRAL GAP
OF THE 2-D STOCHASTIC ISING MODEL
WITH NEARLY SINGLE-SPIN BOUNDARY CONDITIONS

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ABSTRACT. We establish upper bounds for the spectral gap of the stochastic Ising model at low temperature in an $N \times N$ box, with boundary conditions which are "plus" except for small regions at the corners which are either free or "minus." The spectral gap decreases exponentially in the size of the corner regions, when these regions are of size at least of order $\log N$. This means that removing as few as $O(\log N)$ plus spins from the corners produces a spectral gap far smaller than the order N^{-2} gap believed to hold under the all-plus boundary condition. Our results are valid at all subcritical temperatures.

1. INTRODUCTION AND MAIN THEOREM

Let $\Lambda \subset \mathbb{Z}^2$ and let $\eta \in \{-1, 0, 1\}^{\partial\Lambda}$. Here $\partial\Lambda = \{x \in \mathbb{Z}^2 \backslash \Lambda : x$ is adjacent to some site in $\Lambda\}$. The Hamiltonian for the Ising model on Λ with boundary condition η is

$$H_{\Lambda,\eta}(\sigma) = -\sum_{\langle xy\rangle : x,y\in\Lambda} \sigma_x\sigma_y - \sum_{\langle xy\rangle : x\in\Lambda, y\in\partial\Lambda} \sigma_x\eta_y, \quad \sigma \in \{-1, 1\}^\Lambda,$$

where the first sum is over unordered pairs of adjacent sites. Let $\mu = \mu^\beta_{\Lambda,\eta}$ denote the equilibrium measure when the inverse temperature is β:

$$\mu^\beta_{\Lambda,\eta}(\sigma) = (Z^\beta_{\Lambda,\eta})^{-1} e^{-\beta H_{\Lambda,\eta}(\sigma)},$$

where $Z^\beta_{\Lambda,\eta}$ is the partition function. Let $\Sigma_\Lambda = \{-1, 1\}^\Lambda$ be the configuration space, and let σ_Λ denote a generic configuration. (When convenient we write "+" and "-" in place of 1 and -1 for the two spins.)

We consider the time evolution of the dynamic version of the model under Glauber dynamics. Let σ^x denote the configuration given by $\sigma^x_y = -\sigma_y$ for $y = x$, $\sigma^x_y = \sigma_y$ for $y \neq x$. The flip rate at a site x when the configuration is σ is denoted $c(x, \sigma)$ (notationally supressing its dependence on η.) We assume that the flip rates are uniformly bounded:

$$0 < c'_0 \leq c(x, \sigma) \leq c_0 \quad \text{for all } x \text{ and } \sigma.$$

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We also make the usual assumptions that the flip rates are attractive and translation invariant, satisfy detailed balance and have finite range; see e.g. [15] for full descriptions of these properties. The generator $A = A^\beta_{\Lambda,\eta}$ of the corresponding Markov process is given by

$$(Af)(\sigma) = \sum_{x\in\Lambda} c(x,\sigma)\big(f(\sigma^x) - f(\sigma)\big),$$

and the Dirichlet form $\mathcal{D} = \mathcal{D}^\beta_{\Lambda,\eta}$ by

$$\mathcal{D}(f,g) = \langle f, Ag\rangle_\mu,$$

so that

$$\mathcal{D}(f,f) = -\frac{1}{2}\sum_{x\in\Lambda}\sum_{\sigma\in\Sigma_\Lambda} \mu(\sigma)c(x,\sigma)\big(f(\sigma^x) - f(\sigma)\big)^2,$$

where $\mu = \mu^\beta_{\Lambda,\eta}$. The spectral gap $\Delta = \Delta(\Lambda,\eta,\beta)$ for such dynamics, which is the smallest positive eigenvalue of $-A^\beta_{\Lambda,\eta}$, has the following representation:

$$\Delta(\Lambda,\eta,\beta) = \inf_{f\in L^2(\mu)} -\frac{\mathcal{D}(f,f)}{\mathrm{var}_\mu(f)}, \tag{1.1}$$

where $\mathrm{var}_\mu(f)$ denotes the variance of f. The gap Δ describes the rate of exponential convergence in $L^2(\mu^\beta_{\Lambda,\eta})$ to equilibrium, in the sense that for $S(\cdot)$ the semigroup generated by A and $\|\cdot\|_\mu$ the $L^2(\mu)$ norm, Δ is the largest constant such that

$$\|S(t)f - \int f\,d\mu\|_\mu \le \|f - \int f\,d\mu\|_\mu e^{-\Delta t} \quad \text{for all } f\in L^2(\mu) \text{ and } t\ge 0.$$

We say that two configurations $\sigma,\sigma'\in\Sigma_\Lambda$ are *adjacent* if for some $x\in\Lambda$ we have $\sigma_x\neq\sigma'_x$ but $\sigma_y = \sigma'_y$ for all $y\neq x$. For $S\subset\Sigma_\Lambda$ define

$$\partial_{in}S = \{\sigma\in S : \sigma \text{ is adjacent to some site of } S^c\}.$$

Considering only indicator functions we obtain

$$\Delta(\Lambda,\eta,\beta) \le c_0|\Lambda| \inf_{D\subset\Sigma_\Lambda} \frac{\mu^\beta_{\Lambda,\eta}(\partial_{in}D)}{\mu^\beta_{\Lambda,\eta}(D)(1-\mu^\beta_{\Lambda,\eta}(D))} \tag{1.2}$$

Let $\tilde\Lambda_N = [-N,N]^2$ and $\Lambda_N = \tilde\Lambda_N\cap\mathbb{Z}^2$. Let l_h and l_v denote the horizontal and vertical axes, respectively, and consider the boundary condition $\eta^{k,\epsilon}$ given for $k\ge 0$ and $\epsilon\in\{0,-1\}$ by

$$\eta^{k,\epsilon}_x = \begin{cases} 1, & \text{if } d(x,l_h)\le k \quad \text{or} \quad d(x,l_v)\le k; \\ \epsilon, & \text{otherwise.}\end{cases} \tag{1.3}$$

Here $d(\cdot,\cdot)$ denotes Euclidean distance. As a special case of results in [13] we have that for β very large, for some C, λ depending only on δ, β and ϵ,

$$\Delta(\Lambda_N, \eta^{\delta N,\epsilon}, \beta) \leq Ce^{-\lambda N} \quad \text{for all } \delta \leq \frac{1}{2} \text{ and } N \geq 1. \tag{1.4}$$

Here we will generalize this as follows. Let β_c denote the critical inverse temperature of the Ising model on $\mathbb{Z}^2$.

Theorem 1.1. *Let $\beta > \beta_c$.*

(i) For some C, K, λ depending only on β, for all $N \geq 1$ and $k \geq 1$ satisfying $N - k \geq K \log N$,

$$\Delta(\Lambda_N, \eta^{k,0}, \beta) \leq Ce^{-\lambda(N-k)}. \tag{1.5}$$

(ii) For some $C, K\lambda$ depending only on β, for all $N \geq 1$ and $k \geq 1$ satisfying $\min(k, N - k) \geq K \log N$,

$$\Delta(\Lambda_N, \eta^{k,-1}, \beta) \leq Ce^{-\lambda \min(k,N-k)}. \tag{1.6}$$

It is easily verified that the spectral gap changes by at most a constant when a single boundary spin is changed; in particular this can be applied in comparing $k = 1$ to a completely free boundary. Thus Theorem 1.1(i) essentially includes exponential decay of the gap under the free boundary condition at low temperatures, a result obtained by Thomas [20].

Theorem 1.1 generalizes (1.4) to all $\delta < 1$ and $\beta > \beta_c$, and shows that the L^2 rate of convergence to equilibrium in the stochastic Ising model can be quite slow even when the boundary condition is overwhelmingly of a single spin, and otherwise free. For the full "plus" boundary condition ($k = N$) at subcritical temperatures, it is believed [7] that the spectral gap is of order N^{-2}; Martinelli [16] proved that for very low temperatures,

$$\Delta(\Lambda_N, \eta^{N,0}, \beta) \geq \exp(-\lambda(\delta)N^{\frac{1}{2}+\delta}) \quad \text{for all } \delta > 0.$$

Presuming N^{-2} is the correct rate for the "plus" boundary condition, Theorem 1.1(i) shows that removing as few as $O(\log N)$ plus spins from the corners of the box dramatically shrinks the spectral gap.

For $k \approx cN$ with $0 < c < 1$, Theorem 1.1 shows that the spectral gap decreases at least exponentially fast in N. Schonmann [19] showed that for all η, all N and all $\beta > 0$, for some $C = C(\beta)$,

$$\Delta(\Lambda_N, \eta, \beta) \geq \frac{C}{N} e^{-4\beta N},$$

so the gap can never decrease faster than exponentially in N.

Our proof of Theorem 1.1 will use the method suggested by (1.2): we find an event D with $\partial_{in} D$ much smaller than D. This event D is a variant of the event that none of the four strips of "+" spins in $\partial\Lambda_N$ is connected to any of the other strips by a path of "+" spins.

2. Preliminaries

Throughout the paper, $c_0, c_1, \dots$ and $\epsilon_1, \epsilon_2, \dots$ will be used to represent constants which depend only on the temperature (or other parameters) of the model. We use ϵ_i for constants which should be viewed as "small."

Our proof will make use of the Fortuin-Kasteleyn random cluster model, or briefly, the *FK model* ([8], [9], [10]; see also [1], [11]) which is a graphical representation of the Potts model. To discuss this, we need some notation for bond configurations. By a bond we mean an unordered pair $\langle xy \rangle$ of adjacent sites in $\mathbb{Z}^2$. When convenient we view bonds as being open line segments in the plane; this should be clear from the context. Define the sets of bonds

$$\mathcal{B}(\Lambda) = \{\langle xy \rangle : x, y \in \Lambda\}, \quad \overline{\mathcal{B}}(\Lambda) = \{\langle xy \rangle : x \in \Lambda \text{ or } y \in \Lambda\}.$$

For general $A \subset \mathbb{R}^2$, we write $\mathcal{B}(A)$ for $\mathcal{B}(A \cap \mathbb{Z}^2)$. Let $\Omega_\Lambda = \{0,1\}^{\overline{\mathcal{B}}(\Lambda)}$. A *bond configuration* is an element $\omega \in \Omega_\Lambda$; when convenient we alternatively view ω as a subset of $\overline{\mathcal{B}}(\Lambda)$ or as a subgraph of $(\overline{\Lambda}, \overline{\mathcal{B}}(\Lambda))$. Bonds e with $\omega_e = 1$ are *open* in ω; those with $\omega_e = 0$ are *closed.* Let $C(\omega)$ denote the number of open clusters in ω which do not intersect $\partial\Lambda$. For $p \in [0,1]$ and $q > 0$, the FK model $P^{p,q}_{\Lambda,w}$ on $(\overline{\Lambda}, \overline{\mathcal{B}}(\Lambda))$ with parameters (p,q) and wired boundary condition is defined by the weights

$$W(\omega) = p^{|\omega|}(1-p)^{|\overline{\mathcal{B}}(\Lambda)| - |\omega|} q^{C(\omega)} \tag{2.1}$$

Here $|\omega|$ means the number of open bonds in ω. More generally, given $\rho \in \{0,1\}^{\overline{\mathcal{B}}(\Lambda)^c}$ we define $(\omega\rho)$ to be the bond configuration on the full lattice which coincides with ω on $\overline{\mathcal{B}}(\Lambda)$ and with ρ on $\overline{\mathcal{B}}(\Lambda)^c$. Let $C(\omega \mid \rho)$ be the number of open clusters of $(\omega\rho)$ which intersect Λ. The FK model $P^{p,q}_{\Lambda,\rho}$ with bond boundary condition ρ is given by the weights in (2.1) with $C(\omega)$ replaced by $C(\omega \mid \rho)$. Alternately, given $\eta \in \{-1,0,1\}^{\partial\Lambda}$ define

$$\begin{aligned} V(\Lambda,\eta) = \big\{\omega \in \{0,1\}^{\overline{\mathcal{B}}(\Lambda)} :\ & \eta_x = \eta_y \text{ for every } x,y \in \partial\Lambda \text{ for which } x \leftrightarrow y \text{ in } \omega, \\ & \omega_e = 0 \text{ for all } e \in \{\langle xy \rangle : x \in \Lambda, y \in \partial\Lambda, \eta_y = 0\}\big\}. \end{aligned} \tag{2.2}$$

Here $x \leftrightarrow y$ means there is a path of open bonds connecting x to y. The FK model $P^{p,q}_{\Lambda,\eta}$ with site boundary condition η is given by the weights in (2.1), multiplied by $\delta_{V(\Lambda,\eta)}(\omega)$. Taking $\eta_x = 0$ for all x gives the FK model with free boundary condition; we denote it $P^{p,q}_{\Lambda,f}$. For a summary of basic properties of the FK model, see [11]. In particular, since we are in two dimensions, for $p \neq \sqrt{q}/(1+\sqrt{q})$ there is a unique translation-invariant infinite-volume FK measure on $\mathcal{B}(\mathbb{Z}^2)$, which can be obtained as the limit of $P^{p,q}_{\Lambda,w}$ as $\Lambda \nearrow \mathbb{Z}^2$; we denote this measure $P^{p,q}$. For $q \geq 1$, the FK model has the FKG property. For

$$\alpha(p,q) = \frac{p}{p + q(1-p)}$$

and $e \in \mathcal{B}(\mathbb{Z}^2)$, we have

$$P^{p,q}(\omega_e = 1 \mid \omega_b, b \neq e) \geq \alpha(p,q) \quad \text{for every } (\omega_b, b \neq e).$$

Changing a single bond in the boundary condition changes the value of $C(\omega \mid \rho)$ by at most 1. It follows easily that for boundary conditions ρ, ρ' differing at only one bond, we have

$$P^{p,q}_{\Lambda,\rho} \leq q P^{p,q}_{\Lambda,\rho'}. \tag{2.3}$$

As shown in [6], for β given by $p = 1 - e^{-\beta}$, a configuration of the Ising model on Λ with boundary condition η at inverse temperature β can be obtained from a configuration ω of the FK model at $(p, 2)$ with site boundary condition η, by choosing a label for each cluster of ω independently and uniformly from $\{-1, 1\}$; this *cluster-labeling construction* yields a joint site-bond configuration for which the sites are an Ising model and the bonds are an FK model. When the parameters are related in this way, we call the Ising and FK models *corresponding*. Alternately, if one selects an Ising configuration σ and does independent percolation at density p on the set of bonds

$$\{\langle xy \rangle \in \overline{\mathcal{B}}(\Lambda) : \sigma_x = \sigma_y\},$$

the resulting bond configuration is a realization of the corresponding FK model. We call this the *percolation construction* of the FK model. From Onsager's exact solution of the Ising model (see [18]) and basic properties of the FK model (see [11]) the critical point β_c of the Ising model and the percolation critical point p_c for the FK model with $q = 2$ are given by

$$1 - e^{-\beta_c} = p_c = \frac{\sqrt{2}}{1 + \sqrt{2}}. \tag{2.4}$$

The *dual lattice* $(\mathbb{Z}^2)^*$ is $\mathbb{Z}^2$ shifted by $(\frac{1}{2}, \frac{1}{2})$; sites and bonds of this lattice are called *dual sites* and *dual bonds*. x^* denotes $x + (\frac{1}{2}, \frac{1}{2})$. When necessary for clarity, bonds of $\mathbb{Z}^2$ are called *regular bonds*. To each regular bond e there is associated a unique dual bond e^* which is its perpendicular bisector. For $\mathcal{D} \subset \mathcal{B}(\mathbb{Z}^2)$ we write $\mathcal{D}^*$ for $\{e^* : e \in \mathcal{D}\}$. For $\Lambda \subset (\mathbb{Z}^2)^*$, $\partial\Lambda$ is defined as for $\Lambda \subset \mathbb{Z}^2$, but using adjacency in the dual lattice. The dual bond e^* is defined to be open precisely when e is closed, so that for each bond configuration ω on $\mathbb{Z}^2$, there is unique dual configuration ω^* on $(\mathbb{Z}^2)^*$. For $p \in [0, 1]$ the value p^* *dual to* p at level q is given by

$$\frac{p}{q(1-p)} = \frac{1 - p^*}{p^*}.$$

If the regular bonds are distributed as the infinite-volume FK model at (p, q) on $\mathbb{Z}^2$ with wired boundary condition, then the dual bonds form the infinite-volume FK model at (p^*, q) on $(\mathbb{Z}^2)^*$ with free boundary condition (see [11].)

An Ising configuration $\sigma \in \Sigma_\Lambda$ determines a set of contours, each consisting of dual bonds $e^* \in \overline{\mathcal{B}}(\Lambda)^*$ for which the corresponding regular bond $e = \langle xy \rangle$ has $\sigma_x \neq \sigma_y$. In the joint Ising/FK configuration, therefore, contours consist entirely of open dual bonds.

Given sets $\Phi \subset \Lambda$ and a site configuration $\sigma \in \Sigma_\Lambda$, we write σ_Φ for $\{\sigma_x : x \in \Phi\}$ and let $\mathcal{F}_\Phi$ denote the σ-algebra generated by σ_Φ. Similarly for $\mathcal{D} \subset \overline{\mathcal{B}}(\Lambda)$ and a bond configuration $\omega \in \Omega_\Lambda$, we write $\omega_{\mathcal{D}}$ for $\{\omega_e : e \in \mathcal{D}\}$ and let $\mathcal{G}_{\mathcal{D}}$ denote the σ-algebra generated by $\omega_{\mathcal{D}}$.

An infinite-volume FK model $P^{p,q}$ (or other bond percolation model) is said to have the *weak mixing property* if there exist C, λ such that, given finite sets $\Phi \subset \Lambda$ and any two bond boundary conditions ρ_1 and ρ_2 on $\overline{\mathcal{B}}(\Lambda)^c$, we have

$$\mathrm{Var}\big(P^{p,q}_{\Lambda,\rho_1}(\omega_{\mathcal{B}(\Phi)} \in \cdot), P^{p,q}_{\Lambda,\rho_2}(\omega_{\mathcal{B}(\Phi)} \in \cdot)\big) \leq C \sum_{x\in\Phi, y\notin\Lambda} e^{-\lambda|y-x|},$$

Loosely this says that the maximum influence, on a fixed region, of the boundary condition decays exponentially to 0 as the boundary recedes to infinity. Equivalently, for all events $A \in \mathcal{G}_{\Lambda^c}$ and $B \in \mathcal{G}_\Phi$,

$$|P^{p,q}(B \mid A) - P^{p,q}(B)| \leq C \sum_{x\in\Phi, y\notin\Lambda} e^{-\lambda|y-x|}. \tag{2.5}$$

In contrast, $P^{p,q}$ is said to have the *ratio weak mixing property* if there exist C, λ such that, given finite sets $\Phi \subset \Lambda$ and any two bond boundary conditions ρ_1 and ρ_2 on $\overline{\mathcal{B}}(\Lambda)^c$, we have for all events $A \in \mathcal{G}_{\Lambda^c}$ and $B \in \mathcal{G}_\Phi$,

$$\left|\frac{P^{p,q}(A\cap B)}{P^{p,q}(A)P^{p,q}(B)} - 1\right| \leq C \sum_{x\in\Phi, y\notin\Lambda} e^{-\lambda|y-x|}, \tag{2.6}$$

whenever the right side of this inequality is at most 1. Note that (2.6) is much stronger than (2.5) for A, B for which the probabilities on the left side of (2.5) are much smaller than the right side of (2.5). Weak mixing for the Ising model has a variety of useful consequences, particularly in two dimensions; see [17]. It was shown in [4] that for the FK model in two dimensions, exponential decay of either the connectivity (in infinite volume, with wired boundary) or the dual connectivity (in infinite volume, with free boundary) implies ratio weak mixing. In particular, for $q = 2$ and $p > p_c(2)$, exponential decay of dual connectivity follows from the known properties (see [18]) of Gibbs uniqueness and exponential decay of correlations for the Ising model at inverse temperature $\beta^* < \beta_c(2)$ corresponding to $p^* < p_c(2)$. Thus we have the following.

Lemma 2.1. *Suppose $p > p_c(2)$. Then the FK model $P^{p,2}$ on $\mathcal{B}(\mathbb{Z}^2)$ has the ratio weak mixing property.*

The following is an immediate consequence of the definition of ratio weak mixing.

Lemma 2.2. *([3]) Suppose that the FK model $P^{p,q}$ has the ratio weak mixing property. There exists a constant c_1 as follows. Suppose $r > 3$ and $\mathcal{D}, \mathcal{E} \subset \mathcal{B}(\mathbb{Z}^2)$ with $\mathrm{diam}(\mathcal{E}) \leq r$ and $d(\mathcal{D}, \mathcal{E}) \geq c_1 \log r$. Then for all $A \in \mathcal{G}_\mathcal{D}$ and $B \in \mathcal{G}_\mathcal{E}$, we have*

$$\frac{1}{2}P^{p,2}(A)P^{p,2}(B) \leq P^{p,2}(A\cap B) \leq 2P^{p,2}(A)P^{p,2}(B).$$

We write $y \overset{*}{\leftrightarrow} z$ for the event that y is connected to z by a path of open dual bonds. For $q \geq 1$, $P^{p,q}$ has the FKG property (see [11]), so $-\log P^{p,q}(0^* \overset{*}{\leftrightarrow} x^*)$ is a subadditive function of x, and therefore the limit

$$\tau(x) = \lim_{n\to\infty} -\frac{1}{n} \log P^{p,q}(0^* \overset{*}{\leftrightarrow} (nx)^*), \tag{2.7}$$

exists for $x \in \mathbb{Q}^2$, provided we take the limit through values of n for which $nx \in \mathbb{Z}^2$. This definition extends to $\mathbb{R}^2$ by continuity (see [2]); the resulting τ is a norm on $\mathbb{R}^2$, when the dual connectivity decays exponentially (i.e. $\tau(x)$ is positive for all $x \neq 0$, or equivalently by lattice symmetry, $\tau(x)$ is positive for some $x \neq 0$; we abbreviate this by saying τ *is positive.*) By standard subadditivity results,

$$P^{p,q}(0^* \overset{*}{\leftrightarrow} x^*) \leq e^{-\tau(x)} \quad \text{for all } x. \tag{2.8}$$

In the opposite direction, it is known [3] that if τ is positive (so ratio weak mixing holds), then for some ϵ_1 and c_2,

$$P^{p,q}(0^* \overset{*}{\leftrightarrow} x^*) \geq \epsilon_1 |x|^{-c_2} e^{-\tau(x)} \quad \text{for all } x \neq 0. \tag{2.9}$$

It follows from the fact that the surface tension τ is a norm on $\mathbb{R}^2$ with axis symmetry that, letting e_i denote the ith unit coordinate vector, we have

$$\frac{1}{\sqrt{2}}\tau(e_1) \leq \frac{\tau(x)}{|x|} \leq \sqrt{2}\tau(e_1) \quad \text{for all } \ x \neq 0. \tag{2.10}$$

A weakness of Lemma 2.2 is that the locations $\mathcal{D}, \mathcal{E}$ of the two events must be deterministic. The next lemma from [5] applies only to a limited class of events but allows the locations to be partially random. For $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}(\mathbb{Z}^2)$ we say an event $A \subset \{0,1\}^{\mathcal{D}}$ *occurs on* $\mathcal{C}$ (or on $\mathcal{C}^*$) in $\omega \in \{0,1\}^{\mathcal{D}}$ if $\omega' \in A$ for every $\omega' \in \{0,1\}^{\mathcal{D}}$ satisfying $\omega'_e = \omega_e$ for all $e \in \mathcal{C}$. For a possibly random set $\mathcal{F}(\omega)$ we say A *occurs only on* $\mathcal{F}$ (or equivalently, on $\mathcal{F}^*$) if $\omega \in A$ implies A occurs on $\mathcal{F}(\omega)$ in ω. We say events A and B *occur at separation* r in ω if there exist $\mathcal{C}, \mathcal{E} \subset \mathcal{D}$ with $d(\mathcal{C}, \mathcal{E}) \geq r$ such that A occurs on $\mathcal{C}$ and B occurs on $\mathcal{E}$ in ω. Let $A \circ_r B$ denote the event that A and B occur at separation r. Let $\mathcal{D}^r = \{e \in \mathcal{B}(\mathbb{Z}^2) : d(e, \mathcal{D}) \leq r\}$.

For x a (regular or dual) site, we write $C_x = C_x(\omega)$ for the (regular or dual) cluster of x in the bond configuration ω.

Lemma 2.3. *([5]) Let $P^{p,q}$ be an FK model on $\mathcal{B}(\mathbb{Z}^2)$, with τ positive and $q \geq 1$, satisfying the ratio weak mixing property. There exist constants c_i, ϵ_i as follows. Let $\mathcal{D} \subset \mathcal{B}(\mathbb{Z}^2), x \in (\mathbb{Z}^2)^*$ and $r > c_3 \log |\mathcal{D}|$, and let A, B be events such that A occurs only on C_x and $B \in \mathcal{G}_{\mathcal{D}}$. Then*

$$P^{p,q}(A \circ_r B) \leq (1 + c_4 e^{-\epsilon_2 r}) P^{p,q}(A) P^{p,q}(B). \tag{2.11}$$

Let $\overline{xy}$ denote the line through x and y. Let

$$H_a^+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq a\}$$

and let H_{xy} denote the closed halfspace bounded by $\overline{xy}$ which is to the right as one moves from x to y.

Lemma 2.4. *([3]) Let $P^{p,q}$ be an FK model on $\mathcal{B}(\mathbb{Z}^2)$, with τ positive and $q \geq 1$, satisfying the ratio weak mixing property. There exist ϵ_3, c_5 such that for all $x \neq y \in \mathbb{R}^2$ and all dual sites $u, v \in H_{xy}$,*

$$P(u \overset{*}{\leftrightarrow} v \text{ via a path in } H_{xy}) \geq \frac{\epsilon_3}{|v-u|^{c_5}} e^{-\tau(v-u)}.$$

For the τ of (2.7), the next lemma is an easy consequence of the sharp triangle inequality satisfied by τ (see [14]), which is obtained from the exact solution of the Ising model on $\mathbb{Z}^2$. We will not use this method here, however, to help make it apparent that our results are not specific to the Ising model.

For $u \in \mathbb{R}^2$ let D_u^+ and D_u^- denote the diagonal lines through u and $u + (1,1)$ and through u and $u + (1,-1)$, respectively. Let $T_{N,k}^1$ denote the triangle with vertices $(-k,-N), (k,-N)$ and $(0, -(N-k))$. Note the base of $T_{N,k}^1$ is in the bottom side of Λ_N and the other two sides are parallel to the diagonals. Let $T_{N,k}^i, i = 2,3,4$, be the corresponding triangles (obtained by rotation) with bases in the left, top and right sides of Λ_N, respectively.

Lemma 2.5. *Suppose τ is a norm on the plane which has axis and diagonal symmetry. There exists a constant ϵ_4 as follows. Let $0 < k < k + 2m < N$ and $x = (x_1, -N - \frac{1}{2}), y = (y_1, -N - \frac{1}{2})$ with $x_1 \leq -k, y_1 \geq k$, and $z = (z_1, z_2) \in H^+_{-N-\frac{1}{2}} \backslash T^1_{N+\frac{1}{2}, k+2m+\frac{1}{2}}$. Then*

$$\tau(z-x) + \tau(y-z) \geq 2k\tau(e_1) + \epsilon_4 m. \tag{2.12}$$

Proof. If $x_1 \leq -k-m$ or $y_1 \geq k+m$ then $\tau(y-x) \geq \tau((2k+m)e_1)$ and (2.12) follows easily. Hence assume $|x_1|, y_1 \in [k, k+m)$. We may also assume $z_1 \geq 0$.

If $z_1 > y_1$ then from symmetry and convexity we have $\tau(z-x) \geq \tau(y-x)$ and $\tau(y-z) \geq \tau(me_1/2)$, and (2.12) follows easily. Hence we assume $0 \leq z_1 \leq y_1$.

If z is above D_x^+, let z' be the reflection of z across D_x^+. Then $z' \notin T^1_{N,k+2m}$ and $\tau(z'-x) = \tau(z-x)$, and it follows easily from symmetry and convexity that $\tau(y-z) \geq \tau(y-z')$. Thus it is sufficient to prove (2.12) for z'. Hence we may assume z is on or below D_x^+.

Now let $u = (u_1, u_2)$ be the reflection of z across D_y^-, and let $v = (v_1, v_2)$ be the point where $\overline{xz}$ intersects D_y^-. By the above assumptions on z and simple geometry, we have $x_1 \leq u_1 \leq v_1$ and $x_2 \leq u_2 \leq v_2$. Using symmetry and convexity we therefore obtain

$$\tau(z-x) = \tau(z-v) + \tau(v-x) \geq \tau(z-v) + \tau(u-x).$$

Since $\tau(y-z) = \tau(y-u)$, it follows that

$$\begin{aligned}\tau(z-x) + \tau(y-z) &\geq \tau(z-v) + \tau(u-x) + \tau(y-u) \\ &\geq \epsilon_4 m + \tau(y-x),\end{aligned}$$

as desired. □

For $x, y \in (\mathbb{Z}^2)^*, r > 0$ and $G \subset \mathbb{R}^2$, we say there is an *r-near dual connection* from x to y in G if for some $u, v \in (\mathbb{Z}^2)^*$ with $d(u,v) \leq r$, there are open dual paths from x to u and from y to v in G. Let $N(x, y, r, G)$ denote the event that such an r-near dual connection exists. The following result is from [3].

Lemma 2.6. *Let $P^{p,q}$ be an FK model on $\mathcal{B}(\mathbb{Z}^2)$, with $q \geq 1$, for which τ is positive. There exist c_i such that if $|x| > 1$ and $r \geq c_6 \log |x|$ then*

$$P^{p,q}(N(0, x, r, \mathbb{R}^2)) \leq e^{-\tau(x)+c_7 r}.$$

The next lemma shows that a dual connection via a site for which the triangle inequality is strict by an amount $t > 0$ has an excess cost proportional to t.

Lemma 2.7. *Let $P^{p,q}$ be an FK model on $\mathcal{B}(\mathbb{Z}^2)$, with $q \geq 1$, for which τ is positive. There exists c_8 as follows. Suppose $x, y, z \in (\mathbb{Z}^2)^*$ and $t \geq c_8 \log |y - x|$ satisfy $|y - x| > 1$ and*

$$\tau(z - x) + \tau(y - z) \geq \tau(y - x) + t.$$

Then

$$P^{p,q}(x \overset{*}{\leftrightarrow} z \overset{*}{\leftrightarrow} y) \leq e^{-\tau(y-x)-\frac{1}{20}t}.$$

Proof. By Lemma 2.1, $P^{p,q}$ has the ratio weak mixing property. Let $B = B_\tau(z, 3(\tau(y-x)+t))$. Then provided c_8 is large,

$$P^{p,q}(z \overset{*}{\leftrightarrow} B^c) \leq c_9(\tau(y - x) + t)e^{-\tau(y-x)-t} \leq \frac{1}{3}e^{-\tau(y-x)-\frac{1}{2}t}. \tag{2.13}$$

Thus we need only consider paths inside B. Let $\epsilon_5 > 0$ to be specified and consider $\omega \in [x \overset{*}{\leftrightarrow} z \overset{*}{\leftrightarrow} y] \cap N(x, y, \epsilon_5 t, B \backslash B_\tau(z, t/5))^c$. For such ω, there exist $z', z'' \in \partial(B_\tau(z, t/5) \cap (\mathbb{Z}^2)^*)$ and paths $x \overset{*}{\leftrightarrow} z', y \overset{*}{\leftrightarrow} z''$ in B occuring at separation $\epsilon_5 t$. Now

$$\tau(z' - x) + \tau(y - z'') \geq \tau(x - x) + \tau(y - z) - 2\left(\frac{t}{5} + \tau(e_1)\right) \geq \tau(y - x) + \frac{1}{2}t,$$

so using Lemma 2.3, provided c_8 is large,

$$\begin{aligned} &P^{p,q}\big([x \overset{*}{\leftrightarrow} z \overset{*}{\leftrightarrow} y \text{ in } B] \cap N(x, y, \epsilon_5 t, B \backslash B_\tau(z, t/5))^c\big) \\ &\leq \sum_{z', z''} 2P^{p,q}(x \overset{*}{\leftrightarrow} z')\ P^{p,q}(z'' \leftrightarrow y) \\ &\leq 2|\partial(B_\tau(z, t/5) \cap (\mathbb{Z}^2)^*)|^2 e^{-\tau(y-x)-t/2} \\ &\leq \frac{1}{3}e^{-\tau(y-x)-t/4}. \end{aligned} \tag{2.14}$$

Next, provided ϵ_5 is small, an application of Lemma 2.2 gives

$$\begin{aligned} &P^{p,q}\big([x \overset{*}{\leftrightarrow} z \overset{*}{\leftrightarrow} y \text{ in } B] \cap N(x, y, \epsilon_5 t, B \backslash B_\tau(z, t/5))\big) \\ &\leq P^{p,q}\big([z \overset{*}{\leftrightarrow} B_\tau(z, t/10)^c] \cap N(x, y, \epsilon_5 t, B \backslash B_\tau(z, t/5))\big) \\ &\leq 2P^{p,q}\big(z \overset{*}{\leftrightarrow} B_\tau(z, t/10)^c\big) P^{p,q}\big(N(x, y, \epsilon_5 t, B \backslash B_\tau(z, t/5))\big) \\ &\leq 2c_{10} t e^{-t/10} e^{-\tau(y-x)+c_7 \epsilon_5 t} \\ &\leq \frac{1}{3}e^{-\tau(y-x)-t/20}. \end{aligned} \tag{2.15}$$

Together, (2.13), (2.14) and (2.15) complete the proof. □

3. Proof of Theorem 1.1(i)

Let

$$\overline{\Lambda} = \Lambda \cup \partial\Lambda, \quad \Lambda \subset \mathbb{Z}^2.$$

By a *plus path* in a site configuration σ we mean a lattice path on which all sites z have $\sigma_z = 1$; *minus paths* are defined analogously. We write $x \overset{+}{\leftrightarrow} y$ ($x \overset{-}{\leftrightarrow} y$) for the event that x is connected to y by a plus (minus) path. For $\Phi \subset \overline{\Lambda}_N$, the *cluster* of Φ in a bond configuration $\omega \in \{0,1\}^{\overline{\mathcal{B}}(\Lambda_N)}$ is the set

$$C(\Phi, \omega) = \{x \in \Lambda : \text{ in } \omega, x \leftrightarrow \Phi \text{ in } \overline{\mathcal{B}}(\Lambda_N)\}.$$

The *plus cluster* of Φ in a site configuration $\sigma \in \Sigma_{\Lambda_N}$ is the set

$$C_+(\Phi, \sigma) = \{x \in \Lambda : \text{ in } \sigma, x \overset{+}{\leftrightarrow} \Phi \text{ in } \overline{\mathcal{B}}(\Lambda_N)\}.$$

If $\sigma_x = -1$, then of course $C_+(x,\sigma)$ is empty. The *minus cluster* $C_-(\Phi,\sigma)$ is defined analogously.

For $\Phi \subset \mathbb{Z}^2$ we define

$$Q(\Phi) = \cup_{x \in \Phi} \left(x + \left[-\frac{1}{2}, \frac{1}{2} \right]^2 \right).$$

Here $x + [-\frac{1}{2}, \frac{1}{2}]^2$ denotes the translation of $[-\frac{1}{2}, \frac{1}{2}]^2$ by x. For $x \in \mathbb{R}^2$ and $r > 0$ we let $B(x,r)$ and $B_\tau(x,r)$ be the closed Euclidean ball and τ-ball, respectively, of radius r about x.

Fix $\beta > \beta_c, N \geq 1$ and $K \log N \leq k \leq N$, with K to be specified later. Let $p = 1 - e^{-\beta}$ and $j = (N-k)/4$. We assume k and j are integers; the modifications otherwise are trivial. Let $\mathbb{P}^{\epsilon}_{N,k}$ denote the joint site/bond distribution obtained using the percolation construction of the FK model, for which the site marginal distribution is $\mu^{\beta}_{\Lambda_N, \eta^{k,\epsilon}}$ and the bond marginal distribution is $P^{p,2}_{\Lambda_N, \eta^{k,\epsilon}}$. (To avoid ambiguous notation we write $\mathbb{P}^{-}_{N,k}$ for $\mathbb{P}^{-1}_{N,k}$.) We write (σ, ω) for a generic joint configuration in $\Sigma_{\Lambda_N} \times \{0,1\}^{\overline{\mathcal{B}}(\Lambda_N)}$. We call (σ,ω) *allowable* (under $\eta^{k,\epsilon}$) if $\mathbb{P}^{\epsilon}_{N,k}((\sigma,\omega)) > 0$. Define the strip of sites Γ_1 by $\Gamma_1 = ([-k,k] \times \{-N-1\}) \cap \mathbb{Z}^2$, and let $\Gamma_i, i = 2,3,4$, be the corresponding strips of sites, obtained by rotation, in the left, top and right sides of $\partial\tilde{\Lambda}_{N+1}$, respectively. (We will refer to the side corresponding to subscript i as the ith side of $\tilde{\Lambda}_n$, for general n.) For $i = 1,2,3$ let $\Gamma_{i,i+1}$ be the set of sites in $\partial\Lambda_N$ which are between Γ_i and Γ_{i+1}, in the obvious sense, and let $\Gamma_{4,5} = \Gamma_{4,1} = \Gamma_{0,1}$ be the set of sites in $\partial\Lambda_N$ which are between Γ_4 and Γ_1. We also include the appropriate "corner site" as an element of $\Gamma_{i,i+1}$, e.g. $(-N-1,-N-1) \in \Gamma_{1,2}$. Let

$$\Gamma^*_{i,j} = \{x \in (\mathbb{Z}^2)^* \cap \partial\tilde{\Lambda}_{N+\frac{1}{2}} : x \text{ is a corner of } Q(y) \text{ for some } y \in \Gamma_{i,j}\}.$$

Let D_i be the event that there is no plus-path in σ in $\overline{\mathcal{B}}(\Lambda_N)$ from Γ_i to $(T^i_{N,k+3j})^c$, and $D = \cap_{i=1}^4 D_i$. In the FK model, an event closely related to D_i is

$$E_i = \{\omega \in \{0,1\}^{\overline{\mathcal{B}}(\Lambda_N)} : \Gamma^*_{i-1,i} \overset{*}{\leftrightarrow} \Gamma^*_{i,i+1} \text{ in } T^i_{N+\frac{1}{2},k+j+\frac{1}{2}}\}.$$

We begin with a lower bound on the probability of E_i. The main point is that restricting the path to lie in $T^i_{N+\frac{1}{2},k+j+\frac{1}{2}}$ does not excessively alter the probability of an open dual path from $\Gamma^*_{i-1,i}$ to $\Gamma^*_{i,i+1}$.

Lemma 3.1. *Let $p > p_c(2)$. There exist c_i, ϵ_i such that for $N, k \geq 1$ with $c_{11} \log N \leq N - k \leq N$, and E_i as above,*

$$P^{p,2}_{\Lambda_N,\eta^{k,0}}(E_i) \geq \frac{\epsilon_6}{k^{c_{13}}} e^{-2k\tau(e_1)}.$$

Proof. We may assume $i = 1$ and $k \geq c_{12}$, with c_{12} to be specified. Let $m = c_{14} \log k$ and $n = \lfloor c_{15} \log k \rfloor$, where $c_{14} > c_{15}$ are to be specified and $\lfloor \cdot \rfloor$ denotes the integer part. Provided c_{11} is large (depending on c_{14}), we have $m \leq j$. Let $x' = (-k - \frac{1}{2}, -N - \frac{1}{2}), y' = (k + \frac{1}{2}, -N - \frac{1}{2}), x = x' + (n,n), y = y' + (-n,n)$ and let

$$E'_1 = \{\omega \in \{0,1\}^{\overline{\mathcal{B}}(\Lambda)} : x \overset{*}{\leftrightarrow} y \text{ in } T^1_{N,k+2m} \cap H^+_{-N+n}\}.$$

By Lemma 2.4, for some ϵ_7,

$$P^{p,2}(x \overset{*}{\leftrightarrow} y \text{ in } H^+_{-N+n}) \geq \frac{\epsilon_7}{k^{c_5}} e^{-2k\tau(e_1)}. \tag{3.1}$$

Note this probability is for the infinite-volume limit. For each dual site $z \in H^+_{-N+n} \backslash T^1_{N,k+2m}$, let F_z denote the event that there exist open dual paths $x' \overset{*}{\leftrightarrow} z \overset{*}{\leftrightarrow} y'$. By (3.1),

$$P^{p,2}(E'_1) \geq \frac{\epsilon_7}{k^{c_5}} e^{-2k\tau(e_1)} - P^{p,2}\left(\cup_{z \in (\mathbb{Z}^2)^* \cap H^+_{-N+n} \backslash T^1_{N,k+2m}} F_z\right). \tag{3.2}$$

We wish to show that the second term on the right side of (3.2) is at most half of the first term on the right side. Let $\Theta = (\mathbb{Z}^2)^* \cap B_\tau(x, 3k\tau(e_1)) \cap H^+_{-N+n} \backslash T^1_{N,k+2m}$. We have

$$P^{p,2}\left(\cup_{z \in (\mathbb{Z}^2)^* \cap H^+_{-N+n} \backslash T^1_{N,k+2m}} F_z\right) \leq P^{p,2}\big(x \overset{*}{\leftrightarrow} B_\tau(x, 3k\tau(e_1))^c\big) + \sum_{z \in \Theta} P^{p,2}(F_z). \tag{3.3}$$

By (2.8), provided c_{12} is large,

$$P^{p,2}\big(x \overset{*}{\leftrightarrow} B_\tau(x, 3k\tau(e_1))^c\big) \leq c_{16} k e^{-3k\tau(e_1)} \leq \frac{1}{4} \frac{\epsilon_7}{k^{c_5}} e^{-2k\tau(e_1)}. \tag{3.4}$$

Fix $z \in \Theta$. We decompose the event F_z according to whether there is a $(c_6 \log 5k)$-near dual connection from x to y in $B(z, 3c_{17} \log k)^c$, where c_6 is from Lemma 2.6 and c_{17} is to be specified. Let $\Psi_z = B(z, c_{17} \log k) \cap (\mathbb{Z}^2)^*$. Using (2.8) and Lemma 2.2, provided c_{17} and c_{12}

are large we obtain

$$
\begin{aligned}
(3.5) \qquad & P^{p,2}\big(F_z \cap N(x,y,c_6 \log 5k, B(z,2c_{17}\log k)^c)\big) \\
&\leq P^{p,2}\big([z \overset{*}{\leftrightarrow} \partial\Psi_z] \cap N(x,y,c_6 \log 5k, B(z,2c_{17}\log k)^c)\big) \\
&\leq 2P^{p,2}\big(z \overset{*}{\leftrightarrow} \partial\Psi_z\big) P^{p,2}\big(N(x,y,c_6 \log 5k, B(z,2c_{17}\log k)^c)\big) \\
&\leq 2c_{18} k e^{-\frac{1}{2}c_{17}\tau(e_1)\log k} e^{-\tau(y-x)+c_7 c_6 \log 5k} \\
&\leq e^{-\frac{1}{4}c_{17}\tau(e_1)\log k - 2(k-n)\tau(e_1)},
\end{aligned}
$$

where c_7 is from Lemma 2.6. Since $|\Theta| \leq c_{19}k^2$, provided we choose c_{17} large enough (depending on c_{15}) this gives

$$
(3.6) \qquad \sum_{z\in\Theta} P^{p,2}\big(F_z \cap N(x,y,c_6 \log 5k, B(z,2c_{17}\log k)^c)\big) \leq \frac{1}{8}\frac{\epsilon_7}{k^{c_5}} e^{-2k\tau(e_1)}.
$$

Next, let $r = c_6 \log 5k$ and for $z \in \Theta$ let $\Psi'_z = B(z, 2c_{17}\log k) \cap (\mathbb{Z}^2)^*$. We have

$$
(3.7) \qquad F_z \cap N(x,y,c_6 \log 5k, B(z,2c_{17}\log k)^c)^c \subset \cup_{u,v\in\partial\Psi'_z}\big([x \leftrightarrow u] \circ_r [y \leftrightarrow v]\big).
$$

Now for $u, v \in \partial\Psi'_z$,

$$
\tau(u-x) \geq \tau(z-x) - \tau(u-z) \geq \tau(z-x) - 3c_{17}\tau(e_1)\log k
$$

and similarly

$$
\tau(y-v) \geq \tau(y-v) - 3c_{17}\tau(e_1)\log k.
$$

Hence provided c_{14} is large enough (depending on c_{17}), we obtain using Lemma 2.5 that

$$
\tau(u-x) + \tau(y-v) \geq 2(k-n)\tau(e_1) + \epsilon_4 m - 6c_{17}\tau(e_1)\log k \geq 2(k-n)\tau(e_1) + \frac{\epsilon_4}{2}m.
$$

Combining this with (3.7), Lemma 2.3 and (2.8), provided c_{14} and c_{14}/c_{15} are large we get

$$
\begin{aligned}
(3.8) \qquad P^{p,2}\big(F_z \cap N(x,y,c_6 \log 5k, B(z,2c_{17}\log k)^c)^c\big) &\leq \sum_{u,v\in\partial\Psi'_z} 2P^{p,2}(x \overset{*}{\leftrightarrow} u)P^{p,2}(y \overset{*}{\leftrightarrow} v) \\
&\leq |\partial\Psi'_z|^2 e^{-2(k-n)\tau(e_1)-\epsilon_4 m/2} \\
&\leq e^{-2k\tau(e_1)-\epsilon_4 m/4},
\end{aligned}
$$

and then

(3.9)

$$
\sum_{z\in\Theta} P^{p,2}\big(F_z \cap N(x,y,c_6 \log 5k, B(z,2c_{17}\log k)^c)^c\big) \leq c_{19}k^2 e^{-2k\tau(e_1)-\epsilon_4 m/4} \leq \frac{1}{8}\frac{\epsilon_7}{k^{c_5}} e^{-2k\tau(e_1)}.
$$

Combining (3.2), (3.3), (3.6) and (3.9) we obtain

$$
P^{p,2}(E'_1) \geq \frac{1}{2}\frac{\epsilon_7}{k^{c_5}} e^{-2k\tau(e_1)}.
$$

Then from Lemma 2.2, provided c_{15} is large,

$$P^{p,2}_{\Lambda_N,\eta^{k,0}}(E_1') \geq \frac{1}{4}\frac{\epsilon_7}{k^{c_5}}e^{-2k\tau(e_1)}.$$

Let γ_x and γ_y be dual paths of (minimal) length $2n$ from x to x' and from y to y', respectively, in $T^1_{N,k+2m}$. Let E_1'' denote the event that all dual bonds in γ_x and γ_y are open. From the FKG inequality,

$$P^{p,2}_{\Lambda_N,\eta^{k,0}}(E_1) \geq P^{p,2}_{\Lambda_N,\eta^{k,0}}(E_1' \cap E_1'') \geq P^{p,2}_{\Lambda_N,\eta^{k,0}}(E_1')P^{p,2}_{\Lambda_N,\eta^{k,0}}(E_1'') \geq \frac{1}{4}\frac{\epsilon_7}{k^{c_5}}e^{-2k\tau(e_1)}\alpha(p,2)^{4n}$$

and the lemma follows. $\square$

For $\omega \in E_i$, $\partial Q(C(\Gamma_i,\omega))$ includes a unique open dual path $\gamma_i(\omega)$ in $T^i_{N+\frac{1}{2},k+j+\frac{1}{2}}$ from $\Gamma^*_{i-1,i}$ to $\Gamma^*_{i,i+1}$. This path is "closer to Γ_i" than any other open dual path in $\tilde{\Lambda}_{N+\frac{1}{2}}$ from $\Gamma^*_{i-1,i}$ to $\Gamma^*_{i,i+1}$. Further, for fixed ν the event $[\gamma_i = \nu]$ depends only on the bond/dual bond configuration in the closed region, which we denote $I(\nu)$, between ν and the side of $\partial\tilde{\Lambda}_{N+\frac{1}{2}}$ to which ν is attached.

Let $E = \cap_{i=1}^4 E_i$, and suppose $\omega \in E$. Let

$$R(\omega) = \tilde{\Lambda}_{N+\frac{1}{2}} \backslash \cup_{i=1}^4 I(\gamma_i(\omega)).$$

Note $0 \in R(\omega)$, and (under boundary condition $\eta^{k,0}$) all the dual bonds forming $\partial R(\omega)$ are open in ω. The latter means that for fixed U, conditionally on $R(\omega) = U$ the configuration on $\mathcal{B}(U)$ is the FK model with free boundary condition.

Let

$$Y_N = \tilde{\Lambda}_{N+\frac{1}{2}} \backslash \cup_{i=1}^4 T^i_{N+\frac{1}{2},k+j+\frac{1}{2}}, \quad Y_N' = \{x \in Y_N : d(x,\partial Y_N) \geq j\}.$$

Let $0 < h < j$ to be specified, let $w_{12} = (k+2j, -N+h)$, let λ_{12} be the vertical line from w_{12} down to $\partial\tilde{\Lambda}_N$ at $(k+2j,-N)$ and let χ_{12} be the vertical line from w_{12} up to the diagonal D_0^- at $(k+2j,-k-2j)$. Using axis symmetry we obtain 7 more corresponding points w_{ij} and paths χ_{ij}, λ_{ij}, for $i = 1,2,3,4$ and $j = 1,2$, with w_{ij} at distance h from side i of $\tilde{\Lambda}_N$.

We want to show that with high probability, there are no open dual paths starting from Y_N', or from near χ_{ij}, which reach ∂Y_N. Let y_{ijl1}, y_{ijl2} be the endpoints of the dual bond which is dual to the lth bond of χ_{ij}. Then

$$d(y_{ijlm}, \partial Y_N) \geq h + \frac{l}{\sqrt{2}} \quad \text{for all } i,j,l,m.$$

For $x \in Y_N$ let

$$G_x = B\left(x, \frac{1}{2}d(x,\partial Y_N)\right) \cap (\mathbb{Z}^2)^*,$$

and define

$$\Psi_N = \left(Y_N' \cap (\mathbb{Z}^2)^*\right) \cup \{y_{ijlm} : 1 \leq i \leq 4; j = 1,2; 1 \leq l \leq N-h-k-2j; m = 1,2\}.$$

Suppose $U \supset Y_N$. Then $d(G_x, \partial U) \geq j/2$ for all $x \in Y_N'$. Hence using Lemma 2.2, provided K and h are large enough we get

$$
\begin{aligned}
(3.10) \qquad & P^{p,2}_{\Lambda_N,\eta^k,0}(x \overset{*}{\leftrightarrow} \partial G_x \text{ for some } x \in \Psi_N \mid \omega \in E, R(\omega) = U) \\
&= P^{p,2}_{U\cap\mathbb{Z}^2,f}(x \overset{*}{\leftrightarrow} \partial G_x \text{ for some } x \in \Psi_N) \\
&\leq \sum_{x\in Y_N'\cap(\mathbb{Z}^2)^*} P^{p,2}_{U\cap\mathbb{Z}^2,f}(x \overset{*}{\leftrightarrow} \partial G_x) + \sum_{i,j,l,m} P^{p,2}_{U\cap\mathbb{Z}^2,f}(y_{ijlm} \overset{*}{\leftrightarrow} \partial G_{y_{ijlm}}) \\
&\leq \sum_{x\in Y_N'\cap(\mathbb{Z}^2)^*} 2P^{p,2}(x \overset{*}{\leftrightarrow} \partial G_x) + \sum_{i,j,l,m} 2P^{p,2}(y_{ijlm} \overset{*}{\leftrightarrow} \partial G_{y_{ijlm}}) \\
&\leq |Y_N' \cap (\mathbb{Z}^2)^*| e^{-j\tau(e_1)/2} + c_{20}\sum_{l\geq 1}(h+l)e^{-\frac{1}{4}\tau(e_1)(h+l)} \\
&\leq \frac{1}{2}.
\end{aligned}
$$

Let F denote the event that $x \overset{*}{\leftrightarrow} \partial G_x$ for no $x \in \Psi_N$, and all bonds in λ_{ij} are open for all i, j. If $\omega \in E \cap F$, then there is an open circuit in Y_N surrounding Ψ_N and for each i, a portion of this open circuit, together with λ_{i1} and λ_{i2}, forms an open path in $T^i_{N,k+3j}\backslash T^i_{N,k+j}$ from a site adjacent to $\Gamma_{i-1,i}$ to a site adjacent to $\Gamma_{i,i+1}$. When this occurs (with $\omega \in E\cap F$), we call this circuit together with all λ_{ij} a *blocking pattern.* Note the blocking pattern is contained in $R(\omega)$. We have using (3.10) and the FKG inequality that for all $U \supset Y_N$,

$$
\begin{aligned}
(3.11) \qquad & P^{p,2}_{\Lambda_N,\eta^k,0}(\text{there is a blocking pattern in } R(\omega) \mid \omega \in E, R(\omega) = U) \\
&= P^{p,2}_{U\cap\mathbb{Z}^2,f}(\text{there is a blocking pattern in } U) \\
&\geq P^{p,2}_{U\cap\mathbb{Z}^2,f}(F) \\
&\geq \frac{1}{2}P^{p,2}_{U\cap\mathbb{Z}^2,f}(\text{all bonds in } \lambda_{ij} \text{ are open for all } i, j) \\
&\geq \frac{1}{2}\alpha(p,2)^{8h}.
\end{aligned}
$$

But considering the cluster-labeling construction of the joint Ising/FK configuration, we see that if the configuration $\omega \in E$ has a blocking pattern in $R(\omega)$ and all sites x in the blocking pattern have $\sigma_x = -1$ (which occurs with probability 1/2, given such ω), then $\sigma \in D$. Thus from (3.11), the FKG property and Lemma 3.1, for some ϵ_8,

$$
(3.12) \qquad \mu^\beta_{\Lambda_N,\eta^k,0}(D) \geq \frac{1}{4}\alpha(p,2)^{8h} P^{p,2}_{\Lambda_N,\eta^k,0}(E) \geq \frac{\epsilon_8}{k^{4c_{13}}} e^{-8k\tau(e_1)}.
$$

We turn now to upper bounds on $\mu^\beta_{\Lambda_N,\eta^k,0}(\partial_{in} D)$. Analogously to $\gamma_i(\omega)$, for $\sigma \in D$, $\partial Q(C_+(\Gamma_i,\sigma))$ includes a unique open dual path $\gamma_i^+(\sigma)$ in $T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}}$ from $\Gamma^*_{i,i+1}$ to $\Gamma^*_{i-1,i}$, for each i. For fixed ν the event $[\gamma_i^+ = \nu]$ depends only on the site configuration in $\overline{I(\nu) \cap \mathbb{Z}^2}$.

Suppose $\sigma \in \partial_{in} D$ and $\sigma^x \notin D$. Then for some i we have $x \in \partial C_+(\Gamma_i, \sigma)$, and either $x \in \partial T^i_{N,k+3j+1}$ or there is an open dual circuit γ in ω outside $I(\gamma_i^+(\sigma))$ which includes an edge of $Q(x)$ and surrounds some site outside $T^i_{N,k+3j}$. We can choose γ to be the outer boundary of $Q(\Phi)$ for some plus-cluster Φ in σ. In this case we call γ an *appendable circuit attachable at* x.

According to Lemma 2.2, we can choose a constant c_{21} as follows. Let $Z_N^{i,i+1} = \{x \in \mathbb{R}^2 : d(x, \Gamma_{i,i+1}) \leq c_{21} \log N\}$ and $Z_N = \cup_i Z_N^{i,i+1}$. Let V_{closed} be the event that all bonds in $\{\langle xy \rangle : y \in \Lambda_N, x \in \Gamma_{i,i+1}$ for some $i\}$ are closed. Let V_{open} be the event that all bonds in $\mathcal{B}(\mathbb{Z}^2) \backslash \overline{\mathcal{B}}(\Lambda_N)$ are open. (Note the boundary condition $\eta^{k,0}$ conditions ω on $V_{open} \cap V_{closed}$.) Then for all events $A \in \mathcal{G}_{\overline{\mathcal{B}}(\Lambda_N) \backslash \mathcal{B}(Z_N)}$, we have

$$\frac{1}{2} P^{p,2}(A) \leq P^{p,2}(A \mid V_{closed}) \leq 2P^{p,2}(A). \tag{3.13}$$

We therefore call Z_N the *free-boundary influence region.* In particular, provided K is large (depending on c_{21}), using the FKG inequality and (3.13) we have

$$\begin{aligned} P^{p,2}_{\Lambda_N, \eta^{k,0}}(Z_N^{i-1,i} \overset{*}{\leftrightarrow} Z_N^{i,i+1}) &= P^{p,2}(Z_N^{i-1,i} \overset{*}{\leftrightarrow} Z_N^{i,i+1} \mid V_{open} \cap V_{closed}) \\ &\leq P^{p,2}(Z_N^{i-1,i} \overset{*}{\leftrightarrow} Z_N^{i,i+1} \mid V_{closed}) \\ &\leq 2P^{p,2}(Z_N^{i-1,i} \overset{*}{\leftrightarrow} Z_N^{i,i+1}) \\ &\leq c_{22} (\log N) e^{-2(k - c_{21} \log N)\tau(e_1)} \\ &\leq N^{c_{23}} e^{-2k\tau(e_1)}. \end{aligned} \tag{3.14}$$

Our main task is roughly to show, using Lemma 2.5, that the probability for a connection $\Gamma^*_{i-1,i} \overset{*}{\leftrightarrow} \Gamma^*_{i,i+1}$ (specifically, part of $\gamma_i^+(\sigma)$) which does not stay inside $T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}}$ is smaller than the right side of (3.12) by at least a factor of $e^{-\epsilon_9 j}$, for some ϵ_9. We must decompose the event $\partial_{in} D$ into several pieces according to the geometry of the sets $C_+(\Gamma_i, \sigma)$ and $C(\Gamma_i, \omega)$. The most difficult case is that of leakage along the (free) boundary, in which $\gamma_i^+(\sigma)$ goes outside $T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}}$ by way of Z_N.

We define one more special dual path as follows. For $\Phi \subset \Lambda_N$ let

$$\hat{C}(\Phi, \omega) = \{x \in \Lambda_N \backslash Z_N : \text{in } \omega, x \leftrightarrow \Phi \text{ in } \overline{\mathcal{B}}(\Lambda_N) \backslash \mathcal{B}(Z_N)\}.$$

If (σ, ω) is allowable and $\sigma \in D$, then $\partial Q(\hat{C}(\Gamma_i, \omega))$ includes a unique open dual path from $Z_N^{i,i+1}$ to $Z_N^{i-1,i}$ in $\overline{\mathcal{B}}(\Lambda_N) \backslash \mathcal{B}(Z_N)$; we denote this path $\hat{\gamma}_i(\omega)$. We have

$$\hat{\gamma}_i(\omega) \subset I(\gamma_i(\omega)) \subset I(\gamma_i^+(\omega)) \subset T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}}.$$

Let $u_i(\omega)$ and $v_i(\omega)$ be the starting and ending sites, respectively, of $\hat{\gamma}_i(\omega)$ in $Z_N^{i,i+1}$ and $Z_N^{i-1,i}$, respecitvely. Also define

$$W_N^{i,i+1} = \{x \in \mathbb{R}^2 : d(x, \Gamma_{i,i+1}) \leq 2\epsilon_{10} j\}, \qquad W_N = \cup_i W_N^{i,i+1},$$

where ϵ_{10} is to be specified. Provided K is large enough (depending on ϵ_{10} and c_{21}), we have $Z_N^{i,i+1} \subset W_N^{i,i+1}$.

Case 1. Consider $\sigma \in \partial_{in} D$, and ω with (σ,ω) allowable, for which for some i there exist (in order) dual sites $x, z, y \in \hat{\gamma}_i(\omega)$ with

$$\tau(y-x) \geq (2k - 2c_{21}\log N)\tau(e_1), \qquad \tau(z-x) + \tau(y-z) \geq (2k + 4\epsilon_{10} j)\tau(e_1). \tag{3.15}$$

We let A_1 denote the set of (σ,ω) for which this occurs, let J_i denote the set of all $(x,y,z) \in (T^i_{N+\frac12,k+3j+\frac32} \cap (\mathbb{Z}^2)^*)^3$ for which (3.15) holds, and let J_i' denote the set of all $(x,y,z) \in J_i$ which also satisfy

$$\tau(y-x) \geq (2k + 2\epsilon_{10} j)\tau(e_1).$$

As in (3.14), provided K is large enough (depending on ϵ_{10} and c_{21}), using (3.13), (3.14) and Lemmas 2.2 and 2.7 we get

(3.16)

$$\begin{aligned}
\mathbb{P}^0(A_1) &\leq \sum_{i=1}^4 \sum_{(x,y,z)\in J_i} P^{p,2}_{\Lambda_N,\eta^{k,0}}\big(x \overset{*}{\leftrightarrow} z \overset{*}{\leftrightarrow} y \text{ in } T^i_{N+\frac12,k+3j+\frac32}\backslash \mathcal{B}(Z_N); \\
&\qquad\qquad Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1} \text{ in } T^l_{N+\frac12,k+3j+\frac32}\backslash\mathcal{B}(Z_N) \text{ for all } l \neq i\big) \\
&\leq \sum_{i=1}^4 \sum_{(x,y,z)\in J_i} 2P^{p,2}\big(x \overset{*}{\leftrightarrow} z \overset{*}{\leftrightarrow} y \text{ in } T^i_{N+\frac12,k+3j+\frac32}\backslash \mathcal{B}(Z_N); \\
&\qquad\qquad Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1} \text{ in } T^l_{N+\frac12,k+3j+\frac32}\backslash\mathcal{B}(Z_N) \text{ for all } l \neq i\big) \\
&\leq \sum_{i=1}^4 \sum_{(x,y,z)\in J_i} 16P^{p,2}\big(x \overset{*}{\leftrightarrow} z \overset{*}{\leftrightarrow} y \text{ in } T^i_{N+\frac12,k+3j+\frac32}\backslash \mathcal{B}(Z_N)\big) \\
&\qquad\qquad \cdot \prod_{l\neq i} P^{p,2}\big(Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1} \text{ in } T^l_{N+\frac12,k+3j+\frac32}\backslash\mathcal{B}(Z_N)\big) \\
&\leq 16\left(\sum_{i=1}^4 \sum_{(x,y,z)\in J_i'} e^{-\tau(y-x)} + \sum_{i=1}^4 \sum_{(x,y,z)\in J_i\backslash J_i'} e^{-\tau(y-x)-\frac{1}{10}\epsilon_{10} j\tau(e_1)}\right)\big(N^{c_{23}}e^{-2k\tau(e_1)}\big)^3 \\
&\leq 16\left(4|J_1'|e^{-(2k+2\epsilon_{10}j)\tau(e_1)} + 4|J_1\backslash J_1'|e^{-(2k+\frac{1}{10}\epsilon_{10}j - 2c_{21}\log N)\tau(e_1)}\right)\big(N^{c_{23}}e^{-2k\tau(e_1)}\big)^3 \\
&\leq e^{-(8k+\frac{1}{20}\epsilon_{10}j)\tau(e_1)}.
\end{aligned}$$

Case 2. Let

$$R^i_N = T^i_{N+\frac12,k+3j+\frac32} \cap (T^i_{N+\frac12,k+j+\frac12} \cup W_N).$$

We may think of R^i_N as a "triangle with feet." Let A_2 denote the set of all $(\sigma,\omega) \in \partial_{in} D\backslash A_1$ for which, for some i, there exists a dual site $z \in T^i_{N+\frac12,k+3j+\frac32}\backslash R^i_N$ which is either in $\gamma_i^+(\sigma)$ or in some appendable circuit attachable at some $x \in \partial C_+(\Gamma_i,\sigma)$. (In particular, this means that $z \overset{*}{\leftrightarrow} \partial(B(z,\epsilon_{10}j) \cap (\mathbb{Z}^2)^*)$ and $B(z, 2\epsilon_{10}j) \subset \tilde{\Lambda}_{N+1}$.) Suppose $(\sigma, \omega) \in A_2$. We claim that $B(z, 2\epsilon_{10}j) \cap \hat{\gamma}_i(\omega) = \phi$. For all $u \in \Gamma^*_{i,i+1}$ and $v \in \Gamma^*_{i-1,i}$, by Lemma 2.5 we have

$$\tau(z-u) + \tau(v-z) \geq 2k\tau(e_1) + \epsilon_4 j.$$

Therefore for all $u' \in Z_N^{i,i+1}, v' \in Z_N^{i-1,i}$ and $z' \in B(z, 2\epsilon_{10}j)$, provided K is large (depending on c_{21}) and ϵ_{10} is small (depending on ϵ_4),

$$\tau(z'-u') + \tau(v'-z') \geq 2k\tau(e_1) + \frac{1}{2}\epsilon_4 j - 2c_{21}(\log N)\tau(e_1) - 4\epsilon_{10}j\tau(e_1) \geq (2k + 4\epsilon_{10}j)\tau(e_1).$$

Taking $u' = u(\omega), v' = v(\omega)$ and comparing to (3.15) we see that since $(\sigma,\omega) \notin A_1$, we cannot have $z' \in \hat{\gamma}_i(\omega)$, proving our claim. Therefore as in (3.16),

(3.17)

$$\begin{aligned}
\mathbb{P}^0_{N,k}(A_2) \leq \sum_{i=1}^4 \sum_{z \in (T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}} \backslash R^i_N) \cap (\mathbb{Z}^2)^*} & P^{p,2}_{\Lambda_N, \eta^{k,0}}\big(z \overset{*}{\leftrightarrow} B(z,\epsilon_{10}j)^c \text{ in } B(z,\epsilon_{10}j+1), \\
& Z_N^{i-1,i} \overset{*}{\leftrightarrow} Z_N^{i,i+1} \text{ in } T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}} \cap B(z,2\epsilon_{10}j)^c \backslash \mathcal{B}(Z_N), \\
& Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1} \text{ in } T^l_{N+\frac{1}{2},k+3j+\frac{3}{2}} \backslash \mathcal{B}(Z_N) \text{ for all } l \neq i\big) \\
\leq \sum_{i=1}^4 \sum_{z \in (T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}} \backslash R^i_N) \cap (\mathbb{Z}^2)^*} & 2P^{p,2}\big(z \overset{*}{\leftrightarrow} B(z,\epsilon_{10}j)^c \text{ in } B(z,\epsilon_{10}j+1), \\
& Z_N^{i-1,i} \overset{*}{\leftrightarrow} Z_N^{i,i+1} \text{ in } T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}} \cap B(z,2\epsilon_{10}j)^c \backslash \mathcal{B}(Z_N), \\
& Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1} \text{ in } T^l_{N+\frac{1}{2},k+3j+\frac{3}{2}} \backslash \mathcal{B}(Z_N) \text{ for all } l \neq i\big) \\
\leq 32 \sum_{i=1}^4 \sum_{z \in (T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}} \backslash R^i_N) \cap (\mathbb{Z}^2)^*} & P^{p,2}(z \overset{*}{\leftrightarrow} B(z,\epsilon_{10}j)^c) \prod_{l=1}^4 P^{p,2}\big(Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1}\big) \\
\leq c_{24} |T^1_{N+\frac{1}{2},k+3j+\frac{3}{2}} \cap (\mathbb{Z}^2)^*| j e^{-\frac{1}{2}\epsilon_{10}j\tau(e_1)} & \big(N^{c_{23}} e^{-2k\tau(e_1)}\big)^4 \\
\leq e^{-(8k+\frac{1}{4}\epsilon_{10}j)\tau(e_1)}. &
\end{aligned}$$

Case 3. Let $A_3 = \partial_{in} D \backslash (A_1 \cup A_2)$, and suppose $(\sigma,\omega) \in A_3$. In this case, plus spins are "leaking along the boundary," in the following sense: for some i, we have $\gamma_i^+(\sigma) \subset R^i_N$, and either $\gamma_i^+(\sigma)$ or some appendable circuit γ contains a dual site $z \in W_N$ at one of the "toes" of R^i_N, that is, in the right or left side of $T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}}$. Note that such a γ necessarily has $\gamma \cap T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}} \subset R^i_N$, since $(\sigma,\omega) \notin A_2$. We will assume z is in an appendable circuit, attachable at some site x; the case of $z \in \gamma_i^+(\sigma)$ is similar but slightly simpler. From symmetry, we may also assume $i = 1$ and z is in the right side of $T^1_{N+\frac{1}{2},k+3j+\frac{3}{2}}$. We let $A_3'(x,z)$ denote the event that A_3 occurs with a specified choice of x, z, with $i = 1$ and with

z in an appendable circuit, and let $A_3' = \cup_{x,z} A_3'(x,z)$. We define $A_3''(z)$ and A_3'' analogously for the case of $z \in \gamma_i^+(\sigma)$.

The Ising model has the following "bounded energy" property:

$$\mu^{\beta}_{\Lambda_N,\eta^{k,0}}(\sigma_y = 1 \mid \sigma_w, w \neq y) \geq \frac{1}{1+e^{8\beta}} \quad \text{for all } (\sigma_w, w \neq y). \tag{3.18}$$

Given a site x, let ω^x denote the configuration given by

$$\omega_e^x = \begin{cases} 0, & \text{if } x \text{ is an endpoint of } e; \\ \omega_e, & \text{otherwise.} \end{cases}$$

Note that if (σ, ω) is allowable, then so is (σ^x, ω^x). Let

$$B_m^{trunc} = B \cap \{(y_1, y_2) \in \mathbb{R}^2 : y_1 < m\} \quad \text{for } m > 0, B \subset \mathbb{R}^2,$$

$$\hat{\Gamma}_{1,2} = \big([k, k+2j) \times \{-N\}\big) \cap \mathbb{Z}^2,$$

$$\psi_{1,2} = \{k+2j\} \times [-N, -N + 2\epsilon_{10} j]$$

and

$$\hat{C}_-(\hat{\Gamma}_{1,2}, \sigma^x) = \quad \{y \in (\Lambda_N)^{trunc}_{k+2j} : \text{ in } \sigma^x, y \overset{-}{\leftrightarrow} \hat{\Gamma}_{1,2} \text{ in } (\tilde{\Lambda}_N)^{trunc}_{k+2j}\}.$$

For $J \subset \mathbb{Z}^2$ define the boundary condition $\eta^{k,0,J}$ by

$$\eta_u^{k,0,J} = \begin{cases} 1, & \text{if } u \in J; \\ \eta_u^{k,0}, & \text{otherwise.} \end{cases} \tag{3.19}$$

Fix x, z and suppose $(\sigma, \omega) \in A_3'(x,z)$. There is then a plus path in σ^x from Γ_1 to $\psi_{1,2}$ in R_N^1, and $\hat{C}_-(\hat{\Gamma}_{1,2}, \sigma^x)$ is contained in the region between this plus path and $\partial(\Lambda_N)^{trunc}_{k+2j}$. Since

$$|\partial \hat{C}_-(\hat{\Gamma}_{1,2}, \sigma^x) \cap \psi_{1,2}| \leq 2\epsilon_{10} j,$$

using the "bounded energy" property (3.18) of the Ising model we have for fixed J

(3.20)

$$\begin{aligned}
&\mathbb{P}^0_{N,k}\big((\sigma,\omega) \in A_3'(x,z), \hat{C}_-(\hat{\Gamma}_{1,2},\sigma^x) = J\big) \\
&\quad \le \mathbb{P}^0_{N,k}\big(\hat{C}_-(\hat{\Gamma}_{1,2},\sigma^x) = J, \text{ and in } \omega^x, \psi_{1,2} - \tfrac{1}{2}e_1 \overset{*}{\leftrightarrow} Z_N^{4,1} \text{ in } \overline{\mathcal{B}}(\Lambda_N \backslash \overline{J})^* \cap (R_N^1)^{trunc}_{k+2j-\frac{1}{2}} \\
&\qquad\qquad \text{and } Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1} \text{ in } T^l_{N+\frac{1}{2},k+j+\frac{1}{2}} \backslash \mathcal{B}(Z_N) \text{ for } l = 2,3,4\big) \\
&\quad \le c_{25}^{2\epsilon 10 j} \mathbb{P}^0_{N,k}\big(\hat{C}_-(\hat{\Gamma}_{1,2},\sigma^x) = J, \text{ and in } \omega^x, \psi_{1,2} - \tfrac{1}{2}e_1 \overset{*}{\leftrightarrow} Z_N^{4,1} \text{ in } \overline{\mathcal{B}}(\Lambda_N \backslash \overline{J})^* \cap (R_N^1)^{trunc}_{k+2j-\frac{1}{2}} \\
&\qquad\qquad \text{and } Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1} \text{ in } T^l_{N+\frac{1}{2},k+j+\frac{1}{2}} \backslash \mathcal{B}(Z_N) \text{ for } l = 2,3,4 \\
&\qquad\qquad \text{and } \sigma_y = 1 \text{ for all } y \in \psi_{1,2} \cap \partial J\big) \\
&\quad = c_{25}^{2\epsilon 10 j} \mathbb{P}^0_{N,k}\big(\text{in } \omega^x, \psi_{1,2} - \tfrac{1}{2}e_1 \overset{*}{\leftrightarrow} Z_N^{4,1} \text{ in } \overline{\mathcal{B}}(\Lambda_N \backslash \overline{J})^* \cap (R_N^1)^{trunc}_{k+2j-\frac{1}{2}} \\
&\qquad\qquad \text{and } Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1} \text{ in } T^l_{N+\frac{1}{2},k+j+\frac{1}{2}} \backslash \mathcal{B}(Z_N) \text{ for } l = 2,3,4 \mid \hat{C}_-(\hat{\Gamma}_{1,2},\sigma^x) = J \\
&\qquad\qquad \text{and } \sigma_y = 1 \text{ for all } y \in \psi_{12} \cap \partial J\big) \\
&\qquad\quad \cdot \mathbb{P}^0_{N,k}\big(\hat{C}_-(\hat{\Gamma}_{1,2},\sigma^x) = J \text{ and } \sigma_y = 1 \text{ for all } y \in \psi_{1,2} \cap \partial J\big).
\end{aligned}$$

Here $\psi_{1,2} - \frac{1}{2}e_1$ means the translate of $\psi_{1,2}$ by $-\frac{1}{2}e_1$. When $\hat{C}_-(\hat{\Gamma}_{1,2},\sigma^x) = J$, we have by definition $\sigma^x_y = 1$ for all $y \in \partial J \cap (\Lambda_N)^{trunc}_{k+2j}$ and for all $y \in \hat{\Gamma}_{1,2} \backslash J$, so from the Markov property of the Ising model, the conditioning on the right side of (3.20) is equivalent to conditioning on $\sigma^x_y = 1$ for all $y \in \overline{J} \cup \hat{\Gamma}_{1,2}$. Therefore the first probability on the right side of (3.20) is

$$\begin{aligned}
P^{p,2}_{\Lambda_N \backslash (\overline{J} \cup \hat{\Gamma}_{1,2}), \eta^{k,0,J}}\big(&\text{in } \omega^x, \psi_{1,2} - \tfrac{1}{2}e_1 \overset{*}{\leftrightarrow} Z_N^{4,1} \text{ in } \overline{\mathcal{B}}(\Lambda_N \backslash (\overline{J} \cup \hat{\Gamma}_{1,2}))^* \cap (R_N^1)^{trunc}_{k+2j-\frac{1}{2}} \\
&\text{and } Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1} \text{ in } T^l_{N+\frac{1}{2},k+j+\frac{1}{2}} \backslash \mathcal{B}(Z_N) \text{ for } l = 2,3,4\big).
\end{aligned} \tag{3.21}$$

Let

$$\hat{Z}_N^{1,2} = \{x \in \mathbb{R}^2 : d(x, \Gamma_{1,2} \backslash \hat{\Gamma}_{1,2}) \le c_{21} \log N\}, \quad \hat{Z}_N = \hat{Z}_N^{1,2} \cup Z_N^{2,3} \cup Z_N^{3,4} \cup Z_N^{4,1},$$

where c_{21} is from the definition of Z_N. One effect of changing the measure from $P^{p,2}_{\Lambda_N, \eta^{k,0}}$ to $P^{p,2}_{\Lambda_N \backslash (\overline{J} \cup \hat{\Gamma}_{1,2}), \eta^{k,0,J}}$ is to shrink the free-boundary influence region from Z_N to $\hat{Z}_N$. More precisely, as in (3.16), we have using Lemma 2.2 that (3.21) is at most

$$
\begin{aligned}
(3.22) \qquad & P^{p,2}_{\Lambda_N\backslash(\overline{J}\cup\hat{\Gamma}_{1,2}),\eta^{k,0,J}}\Big(\text{in } \omega^x, Z_N^{4,1} \overset{*}{\leftrightarrow} \Big(\psi_{1,2}-\frac{1}{2}e_1\Big) \cup \hat{Z}_N^{1,2} \\
&\qquad \text{in } \overline{\mathcal{B}}(\Lambda_N\backslash(\overline{J}\cup\hat{\Gamma}_{1,2}))^* \cap (R_N^1)^{trunc}_{k+2j-\frac{1}{2}}\backslash\mathcal{B}(\hat{Z}_N) \text{ and} \\
&\qquad Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1} \text{ in } T^l_{N+\frac{1}{2},k+j+\frac{1}{2}}\backslash\mathcal{B}(\hat{Z}_N) \text{ for } l=2,3,4\Big) \\
&\leq 16P^{p,2}\Big(\text{in } \omega^x, Z_N^{4,1} \overset{*}{\leftrightarrow} \Big(\psi_{1,2}-\frac{1}{2}e_1\Big) \cup \hat{Z}_N^{1,2} \\
&\qquad \text{in } \overline{\mathcal{B}}(\Lambda_N\backslash(\overline{J}\cup\hat{\Gamma}_{1,2}))^* \cap (R_N^1)^{trunc}_{k+2j-\frac{1}{2}}\backslash\mathcal{B}(\hat{Z}_N)\Big) \\
&\quad \cdot \prod_{l=2}^{4} P^{p,2}\big(Z_N^{l-1,l} \overset{*}{\leftrightarrow} Z_N^{l,l+1} \text{ in } T^l_{N+\frac{1}{2},k+j+\frac{1}{2}}\backslash\mathcal{B}(\hat{Z}_N)\big).
\end{aligned}
$$

To bound the first probability on the right side of (3.22), we observe that, provided K is large, if $u \in \hat{Z}_N^{4,1}$ and $v \in (\psi_{1,2} - \frac{1}{2}e_1) \cup \hat{Z}_N^{1,2}$, then

$$\tau(v-u) \geq (2k+2j-\tfrac{1}{2}-2c_{21}\log N)\tau(e_1) \geq (2k+j)\tau(e_1).$$

Further, we can replace ω^x with ω at the expense of at most a constant factor. Therefore as in (3.16), the right side of (3.22) is at most

$$c_{26}je^{-(2k+j)\tau(e_1)}\big(N^{c_{23}}e^{-2k\tau(e_1)}\big)^3 \leq e^{-(8k+\frac{1}{2}j)\tau(e_1)}.$$

Plugging this into (3.20), provided $\epsilon_{10} < 1/8$ and K is large we obtain

$$
\begin{aligned}
(3.23) \qquad \mathbb{P}^0_{N,k}(A_3') &\leq \sum_{x,z,J} c_{25}^{2\epsilon_{10}j}e^{-(8k+\frac{1}{2}j)\tau(e_1)}\mathbb{P}^0_{N,k}\big(\hat{C}_-(\hat{\Gamma}_{1,2},\sigma^x)=J\big) \\
&\leq e^{-(8k+\frac{1}{4}j)\tau(e_1)}.
\end{aligned}
$$

A similar proof gives the same bound for $\mathbb{P}^0_{N,k}(A_3'')$. Combining this with (3.16) and (3.17) gives

$$(3.24) \qquad \mu^\beta_{\Lambda_N,\eta^{k,0}}(\partial_{in}D) = \mathbb{P}^0_{N,k}(A_1\cup A_2\cup A_3) \leq e^{-(8k+\epsilon_{11}j)\tau(e_1)}.$$

It follows easily from (3.14) that $\mu^\beta_{\Lambda_N,\eta^{k,0}}(D) \leq 1/2$. Combining this with (3.12), (3.24) and (1.2) yields

$$\Delta(\Lambda_N,\eta^{k,0},\beta) \leq c_{27}|\Lambda_N|k^{4c_{13}}e^{-\epsilon_{11}j} \leq e^{-\epsilon_{11}j/2} = e^{-\epsilon_{11}(N-k)/8},$$

which proves Theorem 1.1 for $\epsilon = 0$.

4. Proof of Theorem 1.1(ii)

The FK measure corresponding to the boundary condition $\eta^{k,-1}$ is given (cf. (2.2)) by

$$P^{p,2}_{\Lambda_N,\eta^{k,-1}} = P^{p,2}_{\Lambda_N,w}(\cdot \mid V(\Lambda_N,\eta^{k,-1})),$$

where $V(\Lambda_N,\eta^{k,-1})$ is the event that there is no open dual path from Γ_i to $\Gamma_{j,j+1}$ for any $i \neq j$. Our calculations, however, are facilitated by using a different conditioning, as follows. Consider bond configurations on $\overline{\mathcal{B}}(\Lambda_{N+1})$. Let U^{FK} denote the event that for all i, all bonds $\langle xy\rangle$ with $x,y \in \Gamma_i$ are open, all bonds $\langle xy\rangle$ with $x,y\in\Gamma_{i,i+1}$ are open, and all other bonds in $\overline{\mathcal{B}}(\Lambda_{N+1})\backslash\overline{\mathcal{B}}(\Lambda_N)$ are closed. The FK model $P^{p,2}_{\Lambda_{N+1},f}(\cdot \mid U^{FK})$ corresponds to an Ising model $\mu^{\beta}_{\Lambda_{N+1},f}(\cdot \mid U^{Ising})$, where U^{Ising} is the event that for all i, all sites in Γ_i have the same spin, and all sites in $\Gamma_{i,i+1}$ have the same spin. Let L denote the event that for all i, all sites in Γ_i have spin 1 and all sites in $\Gamma_{i,i+1}$ have spin -1. Then

$$\mu^{\beta}_{\Lambda_{N+1},f}(\sigma_{\Lambda_N}\in\cdot \mid L) = \mu_{\Lambda_N,\eta^{k,-1}}.$$

The measure $\mathbb{P}^0_{N+1,k}(\cdot \mid \omega \in U^{FK}) = \mathbb{P}^0_{N+1,k}(\cdot \mid \sigma\in U^{Ising})$ gives the joint construction, coupling $P^{p,2}_{\Lambda_{N+1,k},f}(\cdot \mid U^{FK})$ and $\mu^{\beta}_{\Lambda_{N+1},f}(\cdot\mid U^{Ising})$.

For $\sigma\in L$, let

$$\mathcal{J} = \{(i,j) : 1\le i,j\le 4, i<j\},$$

$$\mathcal{A}_N(\sigma) = \{(i,j)\in\mathcal{J} : \Gamma_i \overset{+}{\leftrightarrow} \Gamma_j \text{ in } \overline{\mathcal{B}}(\Lambda_N)\}.$$

As motivation, note we expect that, roughly,

$$\begin{aligned}
(4.1)\qquad & \mu^{\beta}_{\Lambda_{N+1},f}(\mathcal{A}_N = \mathcal{J}\mid L)\approx 1 \quad \text{if } 2k\tau(e_1) > (N-k)\tau(e_1+e_2),\\
& \mu^{\beta}_{\Lambda_{N+1},f}(\mathcal{A}_N = \phi\mid L)\approx 1 \quad \text{if } 2k\tau(e_1) < (N-k)\tau(e_1+e_2).
\end{aligned}$$

In the case $2k\tau(e_1)\ge (N-k)\tau(e_1+e_2)$, we will bound the spectral gap using in (1.2) the same event D as in Section 3, but in the opposite case we replace it with a different event $\hat{D} = \cap_{i=1}^4 \hat{D}_{i,i+1}$. Here $\hat{D}_{1,2}$ is the event that there is no minus-path in σ in $\overline{\mathcal{B}}(\Lambda_N)$ from Γ_1 to $(S^{1,2}_{N,k/4})^c$, where $S^{1,2}_{N,m}$ is the square $[m,N+1]\times[-N-1,-m]$, and $\hat{D}_{i,i+1}, S^{i,i+1}_{N,m}$ are the corresponding event and square obtained by rotation, for $i=2,3,4$.

Suppose first that $2k\tau(e_1)\ge(N-k)\tau(e_1+e_2)$. Let $x_{1,1} = (-k-\frac{1}{2}, -N-\frac{1}{2})$ and $x_{1,2} = (k+\frac{1}{2},-N-\frac{1}{2})$. These dual sites are approximately at the ends of Γ_1. We define corresponding sites x_{ij} for $i=2,3,4$ and $j=1,2$. In place of the event E_i of Lemma 3.1, we will use

$$\hat{E}_i = \{\omega : x_{1,1}\overset{*}{\leftrightarrow} x_{1,2} \text{ in } T^i_{N+\frac{1}{2},k+j+\frac{1}{2}}\}.$$

Lemma 3.1 and its proof remain valid for $\hat{E}_i$ in place of E_i, and the proof of the lower bound (3.12) for $\mu^{\beta}_{\Lambda_N,\eta^{k,0}}(D)$ goes through with minimal changes to give

$$\mu^{\beta}_{\Lambda_{N+1},f}(D\cap L\mid U^{Ising}) \ge \frac{\epsilon_{12}}{k^{c_{28}}}e^{-8k\tau(e_1)}. \tag{4.2}$$

The proof of (3.24) also goes through with minimal changes; in fact Case 3 can be made simpler using the fact that the boundary regions $\Gamma_{i,i+1}$ are each wired. (We will not do so here, since it is unnecessary.) The result is that

$$\mu^{\beta}_{\Lambda_{N+1},f}(\partial_{in} D \cap L \mid U^{Ising}) \leq e^{-(8k+\epsilon_{13}j)\tau(e_1)}. \tag{4.3}$$

Combining (4.2) and (4.3) gives

$$\begin{aligned} \frac{\mu^{\beta}_{\Lambda_N,\eta^{k,-1}}(\partial_{in} D)}{\mu^{\beta}_{\Lambda_N,\eta^{k,-1}}(D)} &= \frac{\mu^{\beta}_{\Lambda_{N+1},f}(\partial_{in} D \mid L)}{\mu^{\beta}_{\Lambda_{N+1},f}(D \mid L)} \\ &= \frac{\mu^{\beta}_{\Lambda_{N+1},f}(\partial_{in} D \cap L \mid U^{Ising})}{\mu^{\beta}_{\Lambda_{N+1},f}(D \cap L \mid U^{Ising})} \\ &\leq \frac{\epsilon_{12}}{k^{c_{28}}} e^{-\epsilon_{13} j \tau(e_1)}. \end{aligned} \tag{4.4}$$

In Section 3 we easily obtained the lower bound $\mu^{\beta}_{\Lambda_N,\eta^{k,0}}(D^c) \geq 1/2$ to complete the proof. Here the situation is a little more complex. A lower bound of the form

$$\frac{\mu^{\beta}_{\Lambda_{N+1},f}(D^c \cap L \mid U^{Ising})}{\mu^{\beta}_{\Lambda_{N+1},f}(D \cap L \mid U^{Ising})} \geq \theta \tag{4.5}$$

for some θ is equivalent to the statement

$$\mu^{\beta}_{\Lambda_N,\eta^{k,-1}}(D^c) = \mu^{\beta}_{\Lambda_{N+1},f}(D^c \mid L) \geq \frac{\theta}{1+\theta}. \tag{4.6}$$

Hence we consider bounds for the numerator and denominator of (4.5). Let $F_{i,i+1}$ denote the event that $x_{i,2} \overset{*}{\leftrightarrow} x_{i+1,1}$ via a path in $S^{i,i+1}_{N,k/4}$, and $F = \cap_{i=1}^{4} F_{i,i+1}$. We have

$$\begin{aligned} \mu^{\beta}_{\Lambda_{N+1},f}(D^c \cap L \mid U^{Ising}) &\geq \mu^{\beta}_{\Lambda_{N+1},f}(\hat{D} \cap L \mid U^{Ising}) \\ &\geq P^{p,2}_{\Lambda_{N+1},f}(F \mid U^{FK}) \mathbb{P}^{0}_{N+1,k}(\hat{D} \cap L \mid F \cap U^{FK}). \end{aligned} \tag{4.7}$$

It is straightforward to prove an analog of Lemma 2.5 for $S^{1,2}_{N+\frac{1}{2},2m+\frac{1}{2}}$ in place of $T^{1}_{N+\frac{1}{2},2m+\frac{1}{2}}$. Therefore mimicking the proof of Lemma 3.1, we obtain

$$P^{p,2}_{\Lambda_{N+1},f}(F_{i,i+1} \mid U^{FK}) \geq \frac{\epsilon_{14}}{(N-k)^{c_{29}}} e^{-(N-k)\tau(e_1+e_2)}. \tag{4.8}$$

Then, analogously to (3.12), from (4.7),

$$\mu^{\beta}_{\Lambda_{N+1},f}(D^c \cap L \mid U^{Ising}) \geq \frac{\epsilon_{15}}{(N-k)^{c_{29}}} e^{-4(N-k)\tau(e_1+e_2)}. \tag{4.9}$$

Next we have, using (2.3) and Lemma 2.2,

$$\begin{aligned}(4.10)\qquad \mu^{\beta}_{\Lambda_{N+1},f}(D\cap L \mid U^{Ising}) &\leq P^{p,2}_{\Lambda_{N+1},f}(x_{i,2} \overset{*}{\leftrightarrow} x_{i+1,1} \text{ in } T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}} \text{ for all } i \mid U^{FK}) \\ &\leq 2^8 P^{p,2}_{\Lambda_N,w}(x_{i,2} \overset{*}{\leftrightarrow} x_{i+1,1} \text{ in } T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}} \text{ for all } i) \\ &\leq 2^8 P^{p,2}(x_{i,2} \overset{*}{\leftrightarrow} x_{i+1,1} \text{ in } T^i_{N+\frac{1}{2},k+3j+\frac{3}{2}} \text{ for all } i) \\ &\leq 2048 e^{-8k\tau(e_1)}.\end{aligned}$$

Since $2k\tau(e_1) \geq (N-k)\tau(e_1+e_2)$, (4.9) and (4.10) give

$$\frac{\mu^{\beta}_{\Lambda_{N+1},f}(D^c\cap L \mid U^{Ising})}{\mu^{\beta}_{\Lambda_{N+1},f}(D\cap L \mid U^{Ising})} \geq \frac{\epsilon_{15}}{8(N-k)^{c_{29}}}.$$

With (4.5) and (4.6), this shows

$$\mu^{\beta}_{\Lambda_N,\eta^{k,-1}}(D^c) \geq \frac{\epsilon_{15}}{16(N-k)^{c_{29}}},$$

which with (4.4) completes the proof of Theorem 1.1 for $\epsilon = -1$, as in Section 3.

The proof when $2k\tau(e_1) < (N-k)\tau(e_1+e_2)$ is similar, with the roles of D and $\hat{D}$ interchanged, using squares $S^{i,i+1}_{\cdot,\cdot}$ in place of the triangles $T^i_{\cdot,\cdot}$.

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References

[1] Aizenman, M., Chayes, J.T., Chayes, L., and Newman, C.M., *Discontinuity of the magnetization in the $1/|x-y|^2$ Ising and Potts models*, J. Stat. Phys. **50** (1988), 1-40.

[2] Alexander, K.S., *Approximation of subadditive functions and rates of convergence in limiting shape results*, Ann. Probab. **25** (1997), 30-55.

[3] Alexander, K.S., *Power-law corrections to exponential decay of connectivities and correlations in lattice models*, preprint (1997).

[4] Alexander, K.S., *On weak mixing in lattice models*, Probab. Theory Rel. Fields **110** (1998), 441-471.

[5] Alexander, K.S., *Cube-root boundary fluctuations for droplets in random cluster models*, preprint (2000).

[6] Edwards, R.G. and Sokal, A.D., *Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo algorithm*, Phys. Rev. D **38** (1988), 2009-2012.

[7] Fisher, D.S. and Huse, D.A., *Dynamics of droplet fluctuations in pure and random Ising systems*, Phys. Rev. B **35** (1987), 6841-6846.

[8] Fortuin, C.M., *On the random cluster model. II. The percolation model*, Physica **58** (1972), 393-418.

[9] Fortuin, C.M., *On the random cluster model. III. the simple random-cluster process*, Physica **59** (1972), 545-570.

[10] Fortuin, C.M. and Kasteleyn, P.W., *On the random cluster model. I. Introduction and relation to other models*, Physica **57** (1972), 536-564.

[11] Grimmett, G.R., *The stochastic random-cluster process and uniqueness of random-cluster measures*, Ann. Probab. **23** (1995), 1461-1510.

[12] Grimmett, G.R. and Piza, M.S.T., *Decay of correlations in random-cluster models*, Commun. Math. Phys. **189** (1997), 465-480.

[13] Higuchi, Y. and Yoshida, N., *Slow relaxation of $2-D$ stochastic Ising models with random and non-random boundary conditions*, in: *New Trends in Stochastic Analysis* (K.D. Elworthy, S. Kusuoka and I. Shigekawa, eds.), 153-167, World Scientific, Singapore (1997).
[14] Ioffe, D., *Large deviations for the 2D Ising model: A lower bound without cluster expansions*, J. Stat. Phys. **74** (1994), 411-432.
[15] Liggett, T.M., *Interacting Particle Systems*, Springer-Verlag, New York (1985).
[16] Martinelli, F.,*On the two dimensional dynamical Ising model in the phase coexistence region*, J. Stat. Phys. **76** (1994), 1179-1246.
[17] Martinelli, F., Olivieri, E., and Schonmann, R.H., *For 2-D lattice spin systems weak mixing implies strong mixing*, Commun. Math. Phys. **165** (1994), 33-47.
[18] McCoy, B.M. and Wu, T.T., *The Two-Dimensional Ising Model*, Harvard University Press, Cambridge, USA (1973).
[19] Schonmann, R.H., *Slow droplet-driven relaxation of the stochastic Ising models in the vicinity of the phase coexistence region*, Commun. Math. Phys. **161** (1994), 1-49.
[20] Thomas, L.E., *Bound on the mass gap for finite volume stochastic Ising models at low temperature*, Commun. Math. Phys. **126** (1989), 1-11.

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