Scaling law for the critical function of an approximate renormalization

C Chandre† and P Moussa‡

Service de Physique Théorique, CEA Saclay, F-91191 Gif-sur-Yvette Cedex, France

Abstract. We construct an approximate renormalization for Hamiltonian systems with two degrees of freedom in order to study the break-up of invariant tori with arbitrary frequency. We derive the equation of the critical surface of the renormalization map, and we compute the scaling behavior of the critical function of one-parameter families of Hamiltonians, near rational frequencies. For the forced pendulum model, we find the same scaling law found for the standard map in [Carletti and Laskar, preprint (2000)]. We discuss a conjecture on the link between the critical function of various types of forced pendulum models, with the Bruno function.

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† E-mail: `chandre@spht.saclay.cea.fr`
‡ E-mail: `moussa@spht.saclay.cea.fr`

1. Introduction

In this article, we analyze the critical function of one parameter families of Hamiltonians near a resonance, by an approximate renormalization group method. We consider the following Hamiltonians with two degrees of freedom written in actions $\boldsymbol{A} = (A_1, A_2) \in \mathbb{R}^2$ and angles $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \mathbb{T}^2$:

$$H(\boldsymbol{A}, \boldsymbol{\varphi}) = H_0(\boldsymbol{A}) + \varepsilon V(\boldsymbol{\varphi}). \tag{1}$$

where the integrable part H_0 of the Hamiltonian is

$$H_0(\boldsymbol{A}) = \boldsymbol{\omega} \cdot \boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2. \tag{2}$$

The vector $\boldsymbol{\omega} = (\omega, -1)$ is the frequency vector of the considered torus which is located at $\boldsymbol{\Omega} \cdot \boldsymbol{A} = 0$ for H_0. For ε sufficiently small and for V sufficiently smooth, the KAM theorem ensures the persistence of this torus for a diophantine frequency ω [1, 2, 3] (see Ref. [4] for a proof in the case of Hamiltonians (1)-(2)). When we increase the parameter up to a critical value, the torus breaks up. The critical function $\omega \mapsto \varepsilon_c(\omega)$ is the value of the parameter ε for which the invariant torus with frequency ω breaks up. We notice that this function vanishes at all rational values of ω. For ω close to P/Q satisfying a diophantine condition, Carletti and Laskar [5] found numerically, using the frequency map analysis, the following scaling law for the standard map critical function :

$$\log \varepsilon_c(\omega) \approx \alpha_{P,Q} \log |\omega - P/Q| + c_{P,Q} \qquad \text{for } \omega \approx P/Q, \tag{3}$$

with $\alpha_{P,Q} = 1/Q$. For the main resonances $0/1$ and $1/1$, they proved analytically that this exponent is equal to one.

We derive an approximate renormalization transformation $\mathcal{R} : H \mapsto H'$ acting in some space of Hamiltonians, in order to analyze the scaling law for different one parameter families of Hamiltonians (1). This transformation gives an approximate value for the critical function $\varepsilon_c(\omega)$. The approximate renormalization scheme we use is similar to the one set up by Escande and Doveil [6, 7] (see also Refs. [8, 9]). Complete versions of these transformations have been defined and studied numerically in Refs. [10, 11, 12, 13, 14, 15]. All these transformations are composed by a sequence of canonical changes of coordinates : a partial elimination of the Fourier modes of the Hamiltonian and a rescaling procedure (shift of the Fourier modes, rescaling of time and of the actions). The aim of this renormalization is to change the scale of phase space such that the renormalized system describes the original one on a smaller scale in phase space and longer time scale.

In the space of Hamiltonians there are two main domains for the renormalization map : a domain where the iterations of $\mathcal{R}$ converge to the integrable Hamiltonian H_0, and another domain where the iterations diverge to infinity. The two domains are separated by a surface $\mathcal{S}$. We assume that $\mathcal{S}$ is a good approximation of the critical surface which is the set of Hamiltonians that have a torus at the threshold of the break-up (see Ref. [9] for numerical evidence). In what follows, we will identify $\mathcal{S}$ with the critical surface.

The approximations we perform consist in mainly two approximations : we project the transformation on the two main Fourier modes of the perturbation and we neglect higher orders in ε (the amplitude of the perturbation). These approximations reduce the renormalization transformation $\mathcal{R}$ to a 4 dimensional map from which we can derive analytical results.

Following Ref [8], we derive analytically the equation of the critical surface $\mathcal{S}$ for the approximate renormalization transformation, and we analyze the precise shape of this surface near a resonance P/Q, i.e., when the frequency ω is close to P/Q. The result we found is that the scaling law (3) is satisfied. The characteristic exponent $\alpha_{P,Q}$ of the scaling law (3) depends on the one parameter family we consider. For the following forced pendulum model (Escande-Doveil Hamiltonian [6, 7])

$$H(p,x,t) = p^2/2 - \varepsilon\left(\cos x + \cos(x-t)\right), \tag{4}$$

which shares the same symmetries as the standard map, the characteristic exponent is the same as the one for the standard map critical function : $\alpha_{P,Q} = 1/Q$ for $0 < P < Q$. However, this exponent $\alpha_{P,Q}$ is not the same for all one-parameter family critical function since we will exhibit examples of Hamiltonians with different values (see Eq. (25) below).

In Sec. 2, we derive the equivalence between the critical function of a simple Hamiltonian model and the critical function of the following forced pendulum models :

$$H(p,x,t) = \frac{1}{2}p^2 - \varepsilon(\cos(ax-bt) + \cos(cx-dt)),$$

where $(a,b,c,d) \in \mathbb{Z}^4$, $a,c > 0$ and $ad - bc = 1$. In Sec. 3, we construct the renormalization transformation by defining the two generators $\mathcal{R}_C$ and $\mathcal{R}_S$ of this transformation. We give the explicit formulas of this transformation as a function of the continued fraction expansion of the frequency of the torus. In Sec. 4, we derive the equation of the critical surface and we analyze its shape near a resonance by computing the characteristic exponents of the scaling laws of the critical function. In Sec 5, we analyze the scaling law for the forced pendulum models in order to see the generality of the scaling law and its characteristic exponent. We also discuss a conjecture on the link between the critical function and the Brjuno function.

2. Critical function of various types of forced pendulum models

We consider the following families of Hamiltonian systems :

$$H = \frac{1}{2}p^2 - \varepsilon_1\cos(ax-bt) - \varepsilon_2\cos(cx-dt), \tag{5}$$

with $(a,b,c,d) \in \mathbb{Z}^4$, $a,c > 0$ and $ad - bc = 1$. In particular, we notice that for $a = c = d = 1$ and $b = 0$, the Hamiltonian (5) with $\varepsilon_1 = \varepsilon_2 = \varepsilon$ is the forced pendulum (4). For these models, the invariant torus with frequency ω is located at $p = \omega$ for the unperturbed motion $\varepsilon_1 = \varepsilon_2 = 0$. It is equivalent to study the torus located at $p = 0$ of the following Hamiltonian :

$$H = \frac{1}{2}p^2 + \omega p - \varepsilon_1\cos(ax-bt) - \varepsilon_2\cos(cx-dt).$$

This Hamiltonian is also mapped into a time-independent two degrees of freedom Hamiltonian :

$$H(\boldsymbol{A},\boldsymbol{\varphi}) = \boldsymbol{\omega}\cdot\boldsymbol{A} + \frac{1}{2}A_1^2 - \varepsilon_1\cos(a\varphi_1 + b\varphi_2) - \varepsilon_2\cos(c\varphi_1 + d\varphi_2), \tag{6}$$

where $\boldsymbol{\omega} = (\omega, -1)$. This Hamiltonian belongs to the family (1), and it is canonically conjugated with the following Hamiltonian :

$$\hat{H}(\boldsymbol{A},\boldsymbol{\varphi}) = \boldsymbol{\omega}\cdot\boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2 - f_1\cos\varphi_1 - f_2\cos\varphi_2, \tag{7}$$

where $\boldsymbol{\omega} = (\omega, -1)$ and $\boldsymbol{\Omega} = (1, k)$ for some $k > 0$. The exact generalized canonical transformation which links the two models is given by three successive steps. The first step is a shift of the Fourier modes $(\boldsymbol{A}', \boldsymbol{\varphi}') = (\tilde{T}^{-1}\boldsymbol{A}, T\boldsymbol{\varphi})$, where $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its transposed matrix $\tilde{T}$ have integer coefficients and determinant 1 in order to preserve the periodic structure of the angles. After the shift of the Fourier modes, the Hamiltonian (6) becomes :

$$H'(\boldsymbol{A}, \boldsymbol{\varphi}) = (d - c\omega)\boldsymbol{\omega}' \cdot \boldsymbol{A} + \frac{a^2}{2}(\boldsymbol{\Omega}' \cdot \boldsymbol{A})^2 - \varepsilon_1 \cos\varphi_1 - \varepsilon_2 \cos\varphi_2,$$

where $\boldsymbol{\omega}' = ((a\omega - b)/(d - c\omega), -1)$ and $\boldsymbol{\Omega}' = (1, c/a)$. The second step is a rescaling of time by a factor $1/(d - c\omega)$: we multiply globally the Hamiltonian by a factor $1/(d - c\omega)$ such that the linear term becomes $\boldsymbol{\omega}' \cdot \boldsymbol{A}$. The third step is a rescaling of the actions by a factor $\lambda = a^2/(d - c\omega)$: we replace the Hamiltonian H' by $\lambda H'(\boldsymbol{A}/\lambda, \boldsymbol{\varphi})$ (this transformation does not change the equations of motion), with $\lambda = a^2/(d - c\omega)$ such that the quadratic term is equal to $(\boldsymbol{\Omega}' \cdot \boldsymbol{A})^2/2$. Thus the image of the Hamiltonian H given by Eq. (6) by this transformation is the following one :

$$H''(\boldsymbol{A}, \boldsymbol{\varphi}) = \boldsymbol{\omega}' \cdot \boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega}' \cdot \boldsymbol{A})^2 - \frac{a^2}{(d - c\omega)^2}(\varepsilon_1 \cos\varphi_1 + \varepsilon_2 \cos\varphi_2), \quad (8)$$

which is of the same form as $\hat{H}$ with $\omega' = (a\omega - b)/(d - c\omega)$, $k = c/a$, $f_1 = a^2\varepsilon_1/(d - c\omega)^2$ and $f_2 = a^2\varepsilon_2/(d - c\omega)^2$.

We consider the one-parameter family of Hamiltonians (5) with $\varepsilon_1 = \varepsilon_2 = \varepsilon$ which is denoted F_1, and the one-parameter family of Hamiltonians (7) with $f_1 = f_2 = \varepsilon$, denoted F_2. We denote $\varepsilon_{fp}(\omega)$ the critical function of the one-parameter family F_1, and $\varepsilon_H(\omega)$ the one of the family F_2. According to the equivalence between Hamiltonians (5) and Hamiltonians (7), the critical functions are linked by the following law :

$$\varepsilon_{fp}(\omega) = a^{-2}(d - c\omega)^2 \varepsilon_H(\omega'), \quad (9)$$

for $\omega \in [b/a, d/c]$, where $\omega' \geq 0$ is given by

$$\omega' = -\frac{a\omega - b}{c\omega - d}. \quad (10)$$

Consequently, we emphasize the role of modular transformations in the study of critical functions. One already knows that the classical diophantine conditions are invariant under modular transformations. Furthermore, the modular properties have been previously considered in the modular smoothing approach [16] which analyzes the compensations between the singular behaviors at ω and $1/\omega$.

3. Renormalization operators

In this section, we derive the approximate renormalization map for a given irrational frequency $\omega > 0$ by defining two elementary operators $\mathcal{R}_C$ and $\mathcal{R}_S$. This construction follows the approximate renormalization developed by MacKay in Ref. [8]. However the resulting renormalization transformation is different since in Ref. [8], the scheme is based on the analysis of stability of nearby periodic orbits by looking at the residues. We perform $\mathcal{R}_C$ if $\omega > 1$ and $\mathcal{R}_S$ if $\omega < 1$. The renormalization operator $\mathcal{R}_C$ is exact (it is generalized canonical transformation without any truncature), and $\mathcal{R}_S$ contains a projection on some relevant Fourier modes of the Hamiltonian, and we neglect the

high orders in the perturbation parameter.
Each of these two elementary renormalization operators acts on the following family of Hamiltonians :

$$H(\boldsymbol{A},\boldsymbol{\varphi}) = \boldsymbol{\omega}\cdot\boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2 - f_1\cos\varphi_1 - f_2\cos\varphi_2, \qquad (11)$$

which belong to the family of Hamiltonians (1). The vector $\boldsymbol{\omega} = (\omega,-1)$ is the frequency vector of the considered torus, and the vector $\boldsymbol{\Omega} = (1,k)$ is the direction of twist in the angles. We notice that if $kf_1f_2 = 0$, the Hamiltonian (11) is integrable.

3.1. Definition of $\mathcal{R}_C$

The renormalization operator $\mathcal{R}_C$ contains a shift of the Fourier modes and a rescaling of time and of the actions. The shift of the Fourier modes is constructed such that it exchanges the mode $(1,0)$ and $(0,1)$. More precisely, we require that the new angles $\boldsymbol{\varphi}'$ satisfy :

$$\cos\varphi_1' = \cos\varphi_2,$$
$$\cos\varphi_2' = \cos\varphi_1.$$

This is performed by the following linear canonical transformation :

$$(\boldsymbol{A},\boldsymbol{\varphi}) \mapsto (\boldsymbol{A}',\boldsymbol{\varphi}') = (C\boldsymbol{A}, C\boldsymbol{\varphi}),$$

where C is the following symmetric orthogonal matrix :

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The vectors $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are changed into $C\boldsymbol{\omega} = (-1,\omega)$ and $C\boldsymbol{\Omega} = (k,1)$. We impose that the images $\boldsymbol{\omega}'$ and $\boldsymbol{\Omega}'$ of the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ by the renormalization $\mathcal{R}_C$ satisfy the following normalization conditions : The second component of $\boldsymbol{\omega}'$ must be equal to -1, and the first component of $\boldsymbol{\Omega}'$ must be equal to 1. We rescale the time by a factor ω, i.e. we multiply the Hamiltonian by $1/\omega$, and we change the sign of phase space coordinates $(\boldsymbol{A}',\boldsymbol{\varphi}') = (-\boldsymbol{A},-\boldsymbol{\varphi})$. Then $\boldsymbol{\omega}'$ is given by $\boldsymbol{\omega}' = (1/\omega,-1)$, and $\boldsymbol{\Omega}' = (1,1/k)$.
The quadratic part of the Hamiltonian becomes $k^2(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2/(2\omega)$. In order that this quadratic term is equal to $(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2/2$, we rescale the actions by a factor $\lambda_C = k^2/\omega$, i.e. we change the Hamiltonian H into $\lambda_C H(\boldsymbol{A}/\lambda_C,\boldsymbol{\varphi})$.
In summary, a Hamiltonian H given by Eq. (11) is mapped into

$$H'(\boldsymbol{A},\boldsymbol{\varphi}) = \boldsymbol{\omega}'\cdot\boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2 - f_1'\cos\varphi_1 - f_2'\cos\varphi_2,$$

where $\omega' = 1/\omega$, $k' = 1/k$, and

$$f_1' = f_2 k^2/\omega^2,$$
$$f_2' = f_1 k^2/\omega^2.$$

The renormalization operator $\mathcal{R}_C$ is equivalent to the 4-dimensional map given by the above equations :

$$(f_1, f_2, \omega, k) \mapsto (f_1', f_2', 1/\omega, 1/k).$$

In order to simplify the computations, we denote $\tilde{f}_2 = k^2 f_2/\omega^2$. The renormalization operators is given by :

$$
\begin{aligned}
f_1' &= \tilde{f}_2, \\
\tilde{f}_2' &= f_1.
\end{aligned}
$$

We notice that the new frequency $\omega' = 1/\omega$ is smaller than one.

3.2. Definition of $\mathcal{R}_S$

The renormalization operator $\mathcal{R}_S$ acts on the family of Hamiltonians (11) for $\omega < 1$. It contains an elimination of the mode $(0,1)$ of the perturbation, and a rescaling procedure (shift of the resonances, rescaling of time and of the actions) such that the image of a Hamiltonian H given by Eq. (11) is of the same general form as H and describes the system on a smaller scale in phase space and on a longer time scale.
We eliminate the mode $(0,1)$ of the perturbation, by a near-identity canonical transformation. We perform a Lie transformation generated by a function $F(\boldsymbol{A},\boldsymbol{\varphi})$. The image of a Hamiltonian H is given by :

$$
H' = \exp(\hat{F})H = H + \{F,H\} + \frac{1}{2}\{F,\{F,H\}\} + \cdots, \tag{12}
$$

where $\{\,,\,\}$ denotes the Poisson bracket :

$$
\{f,g\} = \frac{\partial f}{\partial \boldsymbol{\varphi}} \cdot \frac{\partial g}{\partial \boldsymbol{A}} - \frac{\partial g}{\partial \boldsymbol{\varphi}} \cdot \frac{\partial f}{\partial \boldsymbol{A}},
$$

and the operator $\hat{F}$ acts on H like $\hat{F}H = \{F,H\}$. The generating function is chosen linear in the actions and of the form :

$$
F(\boldsymbol{A},\boldsymbol{\varphi}) = -f_2(1 + k\boldsymbol{\Omega}\cdot\boldsymbol{A})\sin\varphi_2. \tag{13}
$$

The generating function F satisfies :

$$
\{F,H_0\} = f_2\cos\varphi_2 + O(\varepsilon\boldsymbol{A}^2),
$$

where ε denotes the order of f_1 and f_2, and H_0 is the integrable part of H given by Eq. (2). The image of a Hamiltonian H given by Eq. (11) is :

$$
\begin{aligned}
H' = H_0 &- f_1\cos\varphi_1 - f_2\cos\varphi_2 + \{F,H_0\} \\
&- f_1\{F,\cos\varphi_1\} - f_2\{F,\cos\varphi_2\} + \frac{1}{2}\{F,\{F,H_0\}\} + O(\varepsilon^3).
\end{aligned}
$$

We neglect the order $O(\varepsilon\boldsymbol{A}^2)$ produced by the transformation together with the orders $O(\varepsilon^3)$. These approximations are consistent since for small ε, $\boldsymbol{A}(t)$ is of order ε near the torus with frequency ω. The term $-f_2\{F,\cos\varphi_2\}$ is independent of the actions and contains the modes $\mathbf{0}$ and $(0,2)$ that we neglect. The term $-f_1\{F,\cos\varphi_1\}$ generates the modes $(1,1)$ and (1,-1). We neglect the last one and we keep the next relevant Fourier mode $-\frac{1}{2}kf_1f_2\cos(\varphi_1+\varphi_2)$. Then after the elimination of the mode $(0,1)$, the new Hamiltonian is equal to :

$$
H' = \boldsymbol{\omega}\cdot\boldsymbol{A} + (\boldsymbol{\Omega}\cdot\boldsymbol{A})^2/2 - f_1\cos\varphi_1 - \frac{1}{2}kf_1f_2\cos(\varphi_1+\varphi_2). \tag{14}
$$

In this elimination procedure, we have neglected the Fourier modes $(1,-1)$ and $(0,2)$ which are of the same order ε^2 as the mode $(1,1)$ that we kept. The justification for this choice is that the mode $(1,1)$ is associated with the smallest small denominator $\boldsymbol{\omega}\cdot\boldsymbol{k}$

among these three modes : $|\boldsymbol{\omega}\cdot(1,1)| = 1-\omega < |\boldsymbol{\omega}\cdot(1,-1)| = 1+\omega < |\boldsymbol{\omega}\cdot(0,2)| = 2$. Thus we expect that the dynamics is mainly influenced by the mode $(1,1)$ at the order ε^2 of the perturbation. In phase space, this means that the periodic orbit with frequency vector $(1,1)$ is closer to the torus (with frequency ω) than the periodic orbits with frequency vectors $(1,-1)$ and $(0,2)$.
We shift the Fourier modes according to the following linear canonical transformation :

$$(\boldsymbol{A},\boldsymbol{\varphi}) \mapsto (\boldsymbol{A}',\boldsymbol{\varphi}') = (S^{-1}\boldsymbol{A}, S\boldsymbol{\varphi}),$$

where

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

such that $\cos\varphi_1 = \cos\varphi_2'$ and $\cos(\varphi_1+\varphi_2) = \cos\varphi_1'$. The vectors $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are changed into $S\boldsymbol{\omega} = (\omega-1,\omega)$ and $S\boldsymbol{\Omega} = (k+1,1)$. Thus the normalization conditions on these vectors give the image of ω and k :

$$\omega' = (1/\omega) - 1,$$
$$k' = 1/(k+1).$$

We rescale time in such a way that the linear term in the actions of H_0 is equal to $\boldsymbol{\omega}'\cdot\boldsymbol{A}$ with $\boldsymbol{\omega}' = (\omega',-1)$: we multiply the Hamiltonian H' by a factor $1/\omega$. The quadratic term of the new Hamiltonian is equal to $(k+1)^2(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2/(2\omega)$. In order to map this quadratic part into $(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2/2$, we rescale the actions by a factor

$$\lambda_S = (k+1)^2/\omega,$$

by changing the Hamiltonian H' into $\lambda_S H'(\boldsymbol{A}/\lambda_S,\boldsymbol{\varphi})$.
After this elimination and rescaling procedures, a Hamiltonian H is mapped into

$$H'' = \boldsymbol{\omega}'\cdot\boldsymbol{A} + \frac{1}{2}(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2 - f_1'\cos\varphi_1 - f_2'\cos\varphi_2,$$

where

$$f_1' = \frac{k(k+1)^2}{2\omega^2} f_1 f_2,$$
$$f_2' = \frac{(k+1)^2}{\omega^2} f_1.$$

The renormalization operator $\mathcal{R}_S$ is equivalent to the 4-dimensional map

$$(f_1, f_2, \omega, k) \mapsto (f_1', f_2', 1/\omega - 1, 1/(k+1)),$$

given by the above equations. In the variables $(f_1, \tilde{f}_2 = k^2 f_2/\omega^2)$ the renormalization operator is given by :

$$f_1' = \frac{(k+1)^2}{2k} f_1 \tilde{f}_2,$$
$$\tilde{f}_2' = \frac{1}{(1-\omega)^2} f_1.$$

3.3. Renormalization operator $\mathcal{R}$

We define the renormalization operator $\mathcal{R}$ according to the value of the frequency ω in the following way :

$$\mathcal{R} = \mathcal{R}_C \quad \text{if } \omega > 1,$$
$$\mathcal{R} = \mathcal{R}_S \quad \text{if } \omega < 1.$$

In order to know the sequence of operators $\mathcal{R}_C$ and $\mathcal{R}_S$, we look at the continued fraction expansion of the frequency ω. We denote a_i the entries in the continued fraction of ω :

$$\omega = [a_0, a_1, \ldots] = \frac{1}{a_0 + \frac{1}{a_1 + \cdot}}.$$

The sequence of renormalization operators is $(\mathcal{R}_S\mathcal{R}_C)^{a_0-1}\mathcal{R}_S(\mathcal{R}_S\mathcal{R}_C)^{a_1-1}\mathcal{R}_S\cdots$. We denote $\mathcal{R}_a = (\mathcal{R}_S\mathcal{R}_C)^{a-1}\mathcal{R}_S$. This renormalization operator changes the frequency $\omega = [a_0, a_1, \ldots]$ into $\omega' = [a_1, a_2, \ldots]$, and $k = [b_0, b_1, \ldots]$ into $k' = [a_0, b_0, b_1, \ldots]$. The explicit formulas for $\mathcal{R}_a$ are the following ones :

$$f_1' = \rho_1(\omega, k) f_1^a \tilde{f}_2,$$
$$\tilde{f}_2' = \rho_2(\omega) f_1,$$
$$\omega' = 1/\omega - a,$$
$$k' = 1/(k+a),$$

where the coefficients ρ_1 and ρ_2 are given by

$$\rho_1(\omega, k) = 2^{-a}(1 + a/k) \prod_{l=0}^{a-1} (k + l + 1)(1 - l\omega)^{-2}, \tag{15}$$
$$\rho_2(\omega) = (1 - a\omega)^{-2}. \tag{16}$$

For $a = 1$, we check that $\rho_1(\omega, k)$ and $\rho_2(\omega)$ are equal to the coefficients of $\mathcal{R}_S$. The relations (15)-(16) can be proven recursively, using the fact that $\mathcal{R}_{a+1} = \mathcal{R}_S\mathcal{R}_C\mathcal{R}_a$. Moreover, the rescaling in the actions after the renormalization $\mathcal{R}_a$ is equal to $\lambda_a = (k+a)^2/\omega$, and the rescaling in time is equal to $1/\omega$. We notice that $\mathcal{R}_a$ has two fixed points with $k = \omega = (\sqrt{a^2+4} - a)/2$: a trivial one $f_1 = \tilde{f}_2 = 0$, and a non-trivial one $f_1 = (\rho_1\rho_2)^{-1/a}$, $\tilde{f}_2 = \rho_2(\rho_1\rho_2)^{-1/a}$.

Remark : In the construction of $\mathcal{R}_a$, we have neglected several terms, some of them are of order $O(\varepsilon^3)$. Therefore this approximation will be valid for ε small, and in particular in the case where the frequency is close to a rational.

4. Critical surface and scaling behavior

4.1. Renormalization trajectories

First, we derive the general solution for the renormalization trajectories. We compute the renormalization trajectories in the variables $\boldsymbol{x} = (x = \log f_1, y = \log \tilde{f}_2)$. We denote $\gamma_1(\omega, k) = \log \rho_1(\omega, k)$, $\gamma_2(\omega) = \log \rho_2(\omega)$, and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$. The action of the renormalization operator $\mathcal{R}_a$ becomes :

$$\boldsymbol{x}' = N_a \boldsymbol{x} + \boldsymbol{\gamma}(\omega, k), \tag{17}$$

where

$$N_a = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}.$$

By iteration, we obtain the general equation for a renormalization trajectory :

$$\boldsymbol{x}_n = N_{a_{n-1}} \cdots N_{a_0} \boldsymbol{x}_0 + \boldsymbol{\gamma}(\omega_{n-1}, k_{n-1}) + \sum_{l=0}^{n-2} N_{a_{n-1}} \cdots N_{a_{l+1}} \boldsymbol{\gamma}(\omega_l, k_l), \quad (18)$$

where

$$\omega_n = 1/\omega_{n-1} - a_{n-1},$$
$$k_n = 1/(k_{n-1} + a_{n-1}).$$

We denote $p_n/q_n = [a_0, \ldots, a_{n-1}]$ the nth convergent of the frequency ω (with $p_0 = 0$ and $q_0 = 1$). Then

$$M_n \equiv N_{a_{n-1}} \cdots N_{a_0} = \begin{pmatrix} q_n & p_n \\ q_{n-1} & p_{n-1} \end{pmatrix}.$$

The matrix $N_{a_{n-1}} \cdots N_{a_{l+1}}$ is equal to $M_n M_{l+1}^{-1}$:

$$N_{a_{n-1}} \cdots N_{a_{l+1}} = (-1)^{l+1} \begin{pmatrix} q_n p_l - p_n q_l & p_n q_{l+1} - q_n p_{l+1} \\ q_{n-1} p_l - p_{n-1} q_l & p_{n-1} q_{l+1} - q_{n-1} p_{l+1} \end{pmatrix}.$$

4.2. Critical surface

Following Ref. [8], we determine the critical surface that separates points for which $\boldsymbol{x}_n \to -\infty$ like $-q_n$ and from those for which $\boldsymbol{x}_n \to +\infty$ like q_n. Thus the critical surface is obtained by the condition $\boldsymbol{x}_n/q_n \to 0$ when $n \to \infty$. The first component of Eq. (18) becomes :

$$\frac{x_n}{q_n} = x + \frac{p_n}{q_n} y + \sum_{l=0}^{n-2} (-1)^{l+1} \left[\left(p_l - \frac{p_n}{q_n} q_l \right) \gamma_1(\omega_l, k_l) + \left(\frac{p_n}{q_n} q_{l+1} - p_{l+1} \right) \gamma_2(\omega_l) \right] + \frac{\gamma_1(\omega_{n-1}, k_{n-1})}{q_n}.$$

We use the fact that $p_n/q_n \to \omega$. Then we have :

$$x + \omega y + \sum_{l=0}^{\infty} \left[\beta_l \gamma_1(\omega_l, k_l) + \beta_{l+1} \gamma_2(\omega_l) \right] = 0, \quad (19)$$

where $\beta_l = (-1)^{l+1}(p_l - \omega q_l)$. We notice that the second component of Eq. (18) gives the same equation (19) for the critical surface. Following Ref. [8], one can prove that the sum in Eq. (19) converges if and only if ω satisfies the Bruno condition :

$$\sum_{l=0}^{\infty} \frac{\log q_{l+1}}{q_l} < +\infty.$$

4.3. Scaling law near ω rational

In order to investigate the behavior of the critical surface near a rational frequency $P/Q = [a_0, \ldots, a_{J-1}]$, we assume that ω has a large entry in its continued fraction expansion such that :

$$\omega^{(n)} = [a_0, \ldots, a_{J-1}, n, a_{J+1}, \ldots].$$

We consider the behavior of Eq. (19) when n goes to infinity. In the sum, we have two terms : one referring to γ_1 and the other one to γ_2. The dominant term for the first sum is obtained when $l = J$ and is equal to

$$\beta_J \gamma_1(\omega_J, k_J) = \beta_J \log \left[\frac{(k_J+1)\cdots(k_J+n)}{(1-\omega_J)^2 \cdots (1-(n-1)\omega_J)^2} \frac{1+n/k_J}{2^n} \right].$$

The remainder of the series is of order $O(\log n/n)$. The continued fraction expansion of ω_J is equal to $[n, a_{J+1}, \ldots]$. Then we have $\omega_J = 1/n + O(1/n^2)$. Furthermore, k_J is independent of n since its continued fraction expansion is $[a_{J-1}, \ldots, a_0, b_0, b_1, \ldots]$. We use the following asymptotic expansions :

$$\begin{aligned} &\sum_{l=1}^{n} \log(k_J + l) = n \log n - n + O(\log n), \\ &-2 \sum_{l=1}^{n-1} \log(1 - l\omega_J) = 2n + O(\log n). \end{aligned}$$

These expansions are obtained by using Stirling's formula. Concerning the expansion of β_J, we use the formula :

$$\beta_J = \frac{\omega_J}{q_J + q_{J-1}\omega_J},$$

which follows from the fact that $\omega_J = -(q_J\omega - p_J)/(q_{J-1}\omega - p_{J-1})$. Since $\omega_J = 1/n + O(n^{-2})$ and $q_J = Q$, we have :

$$\beta_J = \frac{1}{nQ} + O(n^{-2}). \tag{20}$$

Thus the expansion of the first sum is the following one :

$$\sum_{l=0}^{\infty} \beta_l \gamma_1(\omega_l, k_l) = \frac{1}{Q}(\log n - \log 2 + 1) + O\left(\frac{\log n}{n}\right).$$

We rewrite the second sum as :

$$-2 \sum_{l=0}^{\infty} \beta_{l+1} \log(1 - a_l \omega_l).$$

We use the fact that $1 - a_l\omega_l = \omega_l \omega_{l+1}$. The dominant terms of the second sum are the ones for $l = J-1$ and $l = J$ (terms containing $\log n$). Since $\omega_J \approx 1/n$, these terms are of order $O(\log n/n)$.

In summary, near a resonance P/Q, the asymptotic form of the critical surface is :

$$x + \frac{P}{Q} y = -\frac{1}{Q}(\log n - \log 2 + 1) + O\left(\frac{\log n}{n}\right).$$

In order to express this behavior in terms of the distance from ω to P/Q, we have from Eq. (20) :

$$|\omega - P/Q| = \frac{1}{nQ^2} + O(n^{-2}).$$

Then we have :

$$x + \frac{P}{Q}y \approx \frac{1}{Q}\log|\omega - P/Q| + c_Q.$$

where $c_Q = (\log 2 - 1 + 2\log Q)/Q$. We consider the following one-parameter family of Hamiltonians (11) with $f_1 = f_2 = \varepsilon$ (denoted F_2 in Sec. 2). We have the following scaling law for the critical function of this one parameter family :

$$\log \varepsilon_c(\omega) \approx \frac{1}{P+Q}\log|\omega - P/Q| + c_{P,Q}(k), \tag{21}$$

where $c_{P,Q}(k) = (\log 2 - 1 + 2\log Q - 2P\log(kQ/P))/(P+Q)$ for $P \neq 0$, and $c_{0,1} = \log 2 - 1$. We notice that near the main resonance $0/1$ the characteristic exponent of the scaling law is equal to one as it has been analytically found for the standard map in Ref. [5], but for a resonance P/Q the scaling law is different from the one of the standard map, and is characterized by a characteristic exponent $1/(P+Q)$. Moreover we notice that the exponent $\alpha_{P,Q}$ of the scaling law (3) is independant of k in that case, whereas $c_{P,Q}$ has a term in $\log k$. In the integrable limit $k = 0$, the constant $c_{P,Q}(k)$ and consequently $\varepsilon_c(\omega)$ tend to infinity.
In summary, we have obtained a characteristic exponent $\alpha_{P,Q} = 1/(P+Q)$ of the scaling law (3) for the one-parameter family F_2 by an approximate renormalization transformation. In what follows, we analyze the scaling law (3) for other types of one-parameter families of Hamiltonians which are linked to the family of Hamiltonians F_2 by simple canonical transformations.

5. Forced pendulum critical functions

The critical function of the forced pendulum (4) is linked with the one of the one-parameter family of Hamiltonians (5) according to Eq. (9) :

$$\varepsilon_{fp}(1-\omega) = \varepsilon_{fp}(\omega) = \omega^2 \varepsilon_H(1/\omega - 1). \tag{22}$$

Near a resonance P/Q where $0 < P < Q$ for Hamiltonians (4), the corresponding frequency for ε_H which is $1/\omega - 1$, is near a resonance $(Q-P)/P$ for Hamiltonians (7). Thus the scaling law is the following one :

$$\log \varepsilon_{fp}(\omega) \approx \frac{1}{Q}\log|\omega - P/Q| + C_{P/Q}, \tag{23}$$

where

$$C_{P/Q} = \frac{1}{Q}(\log 2 - 1 + 2\log Q) + 2\frac{P}{Q}\log\frac{P}{Q} + 2\left(1 - \frac{P}{Q}\right)\log\left(1 - \frac{P}{Q}\right).$$

We notice that $C_{P/Q}$ is symmetric : $C_{P/Q} = C_{1-P/Q}$.
The scaling law (23) has the same characteristic exponent $1/Q$ as the scaling law found for the standard map by Carletti and Laskar [5]. This exponent is consistent with the fact that the standard map and the forced pendulum Hamiltonian (4) have the following symmetry for the critical function : $\varepsilon_c(1-\omega) = \varepsilon_c(\omega)$.

The singular behavior of the critical function for the standard map and for the forced pendulum model (4) is linked to the singular behavior of the Bruno function [17] $B : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}^+ \cup \{\infty\}$ which is defined as follows : Let x denotes a real number, $\langle x \rangle$ is its nearest integer (i.e. $\langle x \rangle = [x + 1/2]$, where $[\cdot]$ is the integer part), and $\|x\|$ is the distance between x and $\langle x \rangle$. For an irrational frequency ω, we consider the sequences $\{\theta_k\}$ and $\{\beta_k\}$ defined by

$$\theta_k = \|\theta_{k-1}^{-1}\|,$$
$$\beta_{k-1} = \prod_{i=0}^{k-1} \theta_i,$$

for $k \geq 1$ and $\theta_0 = \|\omega\|$, and we define $B(\omega)$ by

$$B(\omega) = -\sum_{k=0}^{+\infty} \beta_{k-1} \log \theta_k,$$

with $\beta_{-1} = 1$.
For the standard map, the following conjecture is supported by numerical calculations [18, 5] :

$\mathcal{C}$ *: The function* $\varepsilon_c(\omega)/\exp(-B(\omega))$ *is a bounded function on* $[0, 1]$.

A strongest statement has been conjectured in Ref. [18], by requiring that this function is continuous. These conjectures come from the study of the radius of convergence of the Lindstedt series of the standard map or complex analytic maps (see for instance, Refs. [19, 20, 21]), and supported by renormalization arguments [22, 17]. For the critical function of the standard map, this problem is not solved even if recent numerical results obtained in Ref. [5] suggest that the conjecture $\mathcal{C}$ holds. For the approximate critical function we derive for the forced pendulum (4), we expect that this conjecture is also true since it has the same singular behavior near rationals as the one of the function $\exp(-B(\omega))$.
However, the conjecture $\mathcal{C}$ is not satisfied in the above statement, and this comes from the fact that one has to take into account the right argument for the Bruno function, which is not always ω. In what follows, we analyze the critical function of the forced pendulum (5) in order to clarify the right choice of the argument, and to reformulate the conjecture $\mathcal{C}$ in a more appropriate form.

The critical function of the one-parameter family of Hamiltonians (5), denoted $\varepsilon'_{fp}(\omega)$, is linked to the critical function of the Hamiltonian (7) according to Eq. (9) :

$$\varepsilon'_{fp}(\omega) = a^{-2}(d - c\omega)^2 \varepsilon_H(\omega'),$$

for $\omega \in [b/a, d/c]$, where $\omega' = -(a\omega - b)/(c\omega - d)$. Moreover the critical function $\varepsilon'_{fp}(\omega)$ is linked with the critical function $\varepsilon_{fp}(\omega)$ by the following relation (see Eq. (22)) :

$$\varepsilon'_{fp}(\omega) = a^{-2}((a - c)\omega + d - b)^2 \varepsilon_{fp}(\omega''),$$

where

$$\omega'' = \frac{a\omega - b}{(a - c)\omega + d - b}. \tag{24}$$

Near a rational frequency P/Q, the scaling law (3) is satisfied with

$$\alpha_{P,Q} = 1/((a - c)P + (d - b)Q), \tag{25}$$

since for the one-parameter family of Hamiltonians (7), this exponent is equal to the inverse of the sum of the numerator and the denominator of the rational frequency. Thus the singularities of these functions ε'_{fp} are of a different nature than the ones of the Bruno function B. However, $\varepsilon'_{fp}(\omega)$ has the same singularities as $\exp(-B(\omega''))$ where ω'' is given by Eq. (24). The conjecture $\mathcal{C}$ has to be replaced by the following one :

$\mathcal{C}'$: *The function* $\varepsilon_c(\omega)/\exp(-B(\omega''))$ *is a bounded function on* $[b/a, d/c]$.

Practically the nature of the singularities is not changed, and the scaling law (3) is general with the exponent $1/Q$ for a typical one-parameter family of Hamiltonians : The two main resonances for the Hamiltonian (5) are b/a and d/c. They correspond to $\omega''=0$ and $\omega''=1$ which are the two main resonances of the forced pendulum (4). The characteristic exponent $\alpha_{P,Q}$ close to these main resonances is equal to 1, since $(a-c)P+(d-b)Q=1$ for $P/Q=a/b$ or d/c. Thus the characteristic exponent of the scaling law (3) close to the main resonances is equal to one for all the Hamiltonians (5) even if the main resonances are not at the same place as the ones of the forced pendulum model (4). Moreover, the characteristic exponent of the secondary resonances is equal to $1/Q$ where Q if the order of the resonance in the parameter ε in the perturbation expansion. For instance, the resonance $(d+b)/(a+c)$ is of order ε^2. Its characteristic exponent is equal to $1/2$ since $(a-c)P+(d-b)Q=2$ for $P/Q=(d+b)/(a+c)$.

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