MEAN FIELD BOUNDS ON LYAPUNOV EXPONENTS IN $\mathbf{Z}^d$ AT THE CRITICAL ENERGY

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Abstract. We further develop the supersymmetric formalism initiated in [W1] (see also [SjW]). We obtain the optimal mean field bounds at the critical energy for Lyapunov exponents of random walks in random potentials in $\mathbf{Z}^d$ at weak disorder. This extends some of the results in [W1].

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I. Introduction

In this paper we continue our quantitative study of Lyapunov exponents for random walks in random potentials started in [W1]. We use the analytical framework introduced there.

Let

$$H = \Delta + \gamma V, \text{ on } \ell^2(\mathbf{Z}^d), \tag{1.1}$$

where

$$\begin{aligned} \Delta_{ij} &= 1 \quad |i-j|_{\ell^1} = 1 \\ &= 0 \quad \text{otherwise;} \end{aligned} \tag{1.2}$$

γ is a positive parameter, the potential function V is a diagonal matrix: $V = \operatorname{diag}(v_j)$, $j \in \mathbf{Z}^d$, where $\{v_j\}$ is a family of independently identically distributed (iid) real random variables with distribution dg. (dg is only assumed to be a measure.) From now on, we write $|\,|$ for $|\,|_{\ell^1}$. ℓ^2 norms will be denoted by $\|\,\|$. The probability measure is taken to be $P = \prod_{j\in\mathbf{Z}^d} dg(v_j)$. We use the notation $\langle\,\rangle$ to denote the expectation value with respect to P.

For hypercubes Λ: $\Lambda = [-L, L]^d$ ($L \in \mathbf{N}$), we define

$$H_\Lambda = \Delta_\Lambda + \gamma V, \text{ on } \ell^2(\Lambda), \tag{1.3}$$

where the matrix elements of Δ_Λ are defined as in (1.2) for $i, j \in \Lambda$ and 0 otherwise.

As is well known, the spectrum $\sigma(\Delta)$ is $[-2d, 2d]$. (Note that the Laplacian defined here differs from the one usually used by the probabilists by the constant $2d$ times the identity. So the spectrum differs by $2d$.) Let supp dg be the support of dg, then we have the well established fact (See e.g., [CFKS,PF].) that

$$\sigma(H) = [-2d, 2d] + \gamma \operatorname{supp} dg \tag{1.4}$$

almost surely. For all Λ, we also have

$$\sigma(H_\Lambda) \subset [-2d, 2d] + \gamma \operatorname{supp} dg \tag{1.5}$$

For $E \notin \sigma(H)$, we define $G(E) = (E-H)^{-1}$ and its kernel

$$G(E, i, j) = (\delta_i, (E-H)^{-1}\delta_j), \qquad i, j \in \mathbf{Z}^d, \tag{1.6}$$

to be the Green's function of H at energy E. Similarly we define

$$G_\Lambda(E, i, j) = (E - H_\Lambda)^{-1}(i, j), \qquad E \notin \sigma(H_\Lambda), \tag{1.7}$$

to be the finite volume Green's function.

As elaborated in [W1], H defines a random walk in the random potential γV. For a given realization of V, $G(E, \mu, \nu)$ is the probability of finding the random walk at site ν conditioned that it starts at μ. If $\operatorname{dist}(E, \sigma(H)) > 0$, then the corresponding random walk is the so called "killed" random walk in the random potential γV.

We assume dg has semi-bounded support with finite first and second moments, i.e.,

(H1) $\text{supp } dg \subseteq (-\infty, 0]$ or $\text{supp } dg \subseteq [0, +\infty)$;

(H2) $|\int v dg(v)| < \infty, \quad \int v^2 dg(v) < \infty$.

(Due to the presence of the parameter E, we could assume (H1) without loss of generality.)

Under the assumption (H1), $\sigma(H) \subseteq (-\infty, 2d]$ or $\sigma(H) \subseteq [-2d, \infty)$. We call $E = 2d$ or $-2d$, the critical energy. They are respectively the supremum or infimum of $\sigma(H)$. In [W1], assuming $E \notin \sigma(H)$, (H1) and in addition some regular variation conditions on dg, we *computed* the asymptotics of $\langle G(E, \mu, \nu)\rangle$ as $|\mu - \nu| \to \infty$ for $\gamma << 1$. In this paper we study $\langle G(E, \mu, \nu)\rangle$ and $G(E, \mu, \nu)$ for almost all realizations of V, as $|\mu - \nu| \to \infty$ for $\gamma << 1$, at the critical energy $E = 2d$ or $-2d$. The estimates obtained here can be further used to deduce spectral properties of H near $E = 2d$ or $-2d$. (See [W2].)

The appropriate mathematical notion for studying long distance behavior of Green's functions is the notion of Lyapunov exponents as introduced by Sznitman in [Sz]. By using the FKG inequality [FKG, Li] as stated in Remark 3.2 on p. 241-242 of [Sz] and subadditivity ergodic theorem, it can be shown, similar to [Sz, Z] that for all $E \geq 2d$ or $E \leq -2d$, the limit

$$\beta_E(j) = \lim_{n\to\infty} -\frac{\log\langle G(E, 0, nj)\rangle}{n}, \tag{1.8}$$

exists. Moreover by patching of limits:

$$\lim_{j\to\infty} \frac{|\beta_E(j) + \log\langle G(E, 0, j)\rangle|}{\|j\|} = 0. \tag{1.9}$$

The $\beta_E(j)$ are called the annealed Lyapunov exponents. For a fixed E, β_E defines a norm on $\mathbf{R}^d$ [Sz, Z].

Similarly it is shown in [Z] that for all $E \geq 2d$ or $E \leq -2d$, the limit

$$\alpha_E(j) = \lim_{n\to\infty} -\frac{\log G(E, 0, nj)}{n}, \tag{1.10}$$

exists a.s. and that

$$\lim_{j\to\infty} \frac{|\alpha_E(j) + \log G(E, 0, j)|}{\|j\|} = 0, \tag{1.11}$$

a.s. The $\alpha_E(j)$ are called the quenched Lyapunov exponents. Like β_E, for a fixed E, α_E defines a norm on $\mathbf{R}^d$ [Sz, Z].

As mentioned earlier, in [W1] for $E \notin \sigma(H)$ we computed the asymptotics of β_E for $\gamma << 1$. Our main tools were supersymmetry and Witten Laplacians. (For previous uses of supersymmetry in the context of localization, see e.g., [BCKP,KS]. See also [K].) However we were unable to let $E \to 2d^+$ or $E \to -2d^-$ due to the loss of convexity and the occurance of degeneracy at infinity. Their absence is crucial for the Witten Laplacian formulation. (For more details, see sect. III of [W1]. See also [BJS,Jo,Sj1,2].)

In this paper, by further exploiting the supersymmetric structure and replacing the (so far) single scale Witten Laplacian approach by a simple multi-scale induction process, we prove

Theorem 4.2'. *Let $\hat{j} = \frac{j}{\|j\|} \in \mathcal{S}^{d-1}$. Assume dg satisfies (H1,2). For sufficiently small γ, $\alpha_E(\hat{j})$, $\beta_E(\hat{j})$ satisfy the mean field bound i.e.,*

$$\alpha_E(\hat{j}),\, \beta_E(\hat{j}) > a\sqrt{\gamma} \quad (a > 0) \tag{1.12}$$

for all $E \geq 2d$ if $\operatorname{supp} dg \subseteq (-\infty, 0]$; for all $E \leq -2d$ if $\operatorname{supp} dg \subseteq [0, \infty)$.

Remark. Previous general bounds established in [Sz,Z], which are valid for all $\gamma > 0$ gives only $\alpha_E(\hat{j})$, $\beta_E(\hat{j}) > a\gamma$, $(a > 0)$ in the case $\gamma << 1$. Theorem 4.2' is the same as Theorem 4.2 in sect. IV, except that the a appearing in Theorem 4.2' is not made as explicit as in Theorem 4.2.

If we further assume that dg has compact support, then

Corollary. *Assume dg has compact support. For sufficiently small γ,*

$$(1/\mathbf{O}(1))\sqrt{\gamma} \leq \alpha_E(\hat{j}),\, \beta_E(\hat{j}) \leq \mathbf{O}(1)\sqrt{\gamma}, \tag{1.13}$$

where $E = 2d$ if $\operatorname{supp} dg \subseteq (-\infty, 0]$, and $E = -2d$ if $\operatorname{supp} dg \subseteq [0, \infty)$.

Remark. The condition that dg has compact support can in fact be weakened, see [W3].

Without loss of generality, we may assume $\operatorname{supp} dg \subseteq (-\infty, 0]$. This is because the replacement of V by $-V$ followed by the unitary map $u(n) \to (-1)^{|n|}u(n)$ takes H into $-H$. The Corollary follows from Theorem 4.2' and the observation that for all hypercubes Λ, deterministically:

(1.14)

$$|(E - \Delta_\Lambda - \gamma V)^{-1}_{\mu\nu}| > (E + \gamma|v|_\infty - \Delta_\Lambda)^{-1}_{\mu\nu} \sim e^{-\mathbf{O}(1)\sqrt{\gamma}|\mu-\nu|}, \qquad E \geq 2d,\ \gamma << 1$$
$$|(E - \Delta_\Lambda)^{-1}_{\mu\nu}| \sim e^{-\mathbf{O}(1)\sqrt{\gamma}|\mu-\nu|}, \qquad E \leq -2d - \gamma|v|_\infty,\ \gamma << 1.$$

Theorem 4.2' shows that at $E = 2d$ or $-2d$ and as $|\mu - \nu| \to \infty$, $\langle G(E, \mu, \nu)\rangle$ and almost surely $G(E, \mu, \nu)$ behave in a distinctly different way from the free Green's function ($\gamma = 0$). Recall that at $E = \pm 2d$,

$$0 < |(E - \Delta)^{-1}_{ij}| \begin{cases} \text{not defined} & d \leq 2 \\ \leq \dfrac{\mathbf{O}_d(1)}{(|i-j|+1)^{d-2}} & d \geq 3, \end{cases} \tag{1.15}$$

for all $i,\ j \in \mathbf{Z}^d$, and in hypercubes $\Lambda = [-L, L]^d \subset \mathbf{Z}^d$ $(L \in \mathbf{N})$,

$$0 < |(E - \Delta_\Lambda)^{-1}_{ij}| \leq \begin{cases} \mathbf{O}(L), & d = 1 \\ \mathbf{O}(\log(\dfrac{L}{|i-j|+1})) & d = 2 \\ \dfrac{\mathbf{O}_d(1)}{(|i-j|+1)^{d-2}} & d \geq 3, \end{cases} \tag{1.16}$$

for all $i,\ j \in \Lambda$ and $\mathbf{O}_d(1)$ is uniform in L. So Theorem 4.2' is *not* a result of perturbation theory about the free Laplacian. In fact the resolvent expansion of $(E - \Delta_\Lambda - \gamma V)^{-1}$ about $(E - \Delta_\Lambda)^{-1}$ is almost surely *divergent*.

Theorem 4.2' should be interpreted as stating that the probability of finding an eigenvalue in an interval of $\mathbf{O}(\gamma)$ about $E = 2d$ or $-2d$ is "very" small, *uniformly* in Λ. Since the eigenvalues of $H_\Lambda = \Delta_\Lambda + \gamma V$ strongly depends on the configurations of V, this is a problem which has an important combinatorial aspect.

In terms of resolvent series and in view of (1.13), the above observation means that one should perturbe about $(E - \Delta_\Lambda - \gamma\langle v\rangle)^{-1}$ instead. But pointwise (in ω) the perturbation series of $(E - \Delta_\Lambda - \gamma V)^{-1}$ about $(E - \Delta_\Lambda - \gamma\langle v\rangle)^{-1}$ is not convergent either, all the terms being of the same order. By developing further the supersymmetric formalism introduced in [W1], we show that $\langle (E - \Delta_\Lambda - \gamma V)^{-1}_{\mu\nu}\rangle$ can be expressed as an asymptotic series in γ about $(E - \Delta_\Lambda - \gamma\langle v\rangle)^{-1}$ with an explicit control over the remainder. This is achieved in Proposition 3.1 by using certain Jacobian-free "integration by parts" special to supersymmetry, determinant estimates of the free Laplacian: $E - \Delta_\Lambda$ ($E \geq 2d$ or $\leq -2d$), and an induction on the order of γ.

We note that Proposition 3.1 has *direct* control over the Green's function and is the main reason why we obtain the optimal mean-field bound in (1.12). Estimates on Green's functions as detailed as in Proposition 3.1 were not known before. The strategy used in the proof is novel. It completely avoids the difficult counting problem which would arise from a possible direct combinatorial estimate a la cluster expansion. This is because by using supersymmetry, only certain *algebraic combinations* of terms appear in the expansion. They are the determinants of various co-matrices in the problem. We *never* need to estimate term by term and *count* the number of terms in each order. (See the proof of Poposition 3.2.)

Using Proposition 3.1 for a sufficiently large (depending on γ) but *fixed* Λ, we devise a simple induction process over the scale L and prove Theorem 4.2' (Theorem 4.2). (Proposition 3.1 is not sufficient as $|\mu - \nu| \to \infty$, as it is only an asymptotic series in γ.) This last induction process is inspired by a talk of Aizenman on [AFHS].

II. The supersymmetric representation

For definitiveness, we assume $\operatorname{supp} dg \subseteq (-\infty, 0]$. The other case clearly works in the same way by the remark made earlier in the introduction. Let $E = 2d$. Let $\Lambda = [-L, L]^d$, $L \in \mathbf{N}$. We define H_Λ, G_Λ as in (1.3), (1.7). It is well known that for all cubes Λ in $\mathbf{Z}^d$,

$$E - \Delta_\Lambda = 2d - \Delta_\Lambda > 0 \tag{2.1}$$

as an operator. Hence

$$E - \Delta_\Lambda - \gamma V > 0 \tag{2.2}$$

when $\operatorname{supp} dg \subseteq (-\infty, 0]$. The finite volume Green's function

$$G_\Lambda = (E - \Delta_\Lambda - \gamma V)^{-1} \tag{2.3}$$

is well defined at $E = 2d$.

Our aim in this section is to use supersymmetry to express $\langle G_\Lambda(E,\mu,\nu) = \langle (E - \Delta_\Lambda - \gamma V)^{-1}_{\mu\nu}\rangle$, μ, $\nu \in \Lambda$, in a convenient form, paving the way for the estimate in sect. III. The supersymmetric representation of $\langle G_\Lambda\rangle$ was previously derived in [W1]. Below we recall its derivation.

We start by presenting a brief introduction to the supersymmetric formalism, culmulating in the representation for $\langle G_\Lambda\rangle$ in (2.29). This is essentially a reorganization of sect. II of [W1]. For a general proper formulation, see e.g., [Be,V].

For any symmetric $n\times n$ matrix A with $\Re A > 0$, we recall the following well-known formula

$$A^{-1}_{\mu\nu} = \int x_\mu \cdot x_\nu \det A e^{-\sum_{i,j=1}^n A_{ij} x_i \cdot x_j} \prod_{i=1}^n \frac{d^2 x_i}{\pi}, \tag{2.4}$$

where $x_i \in \mathbf{R}^2$ and $x_i \cdot x_j$ is the usual scalar product in $\mathbf{R}^2$. Substituting $E - H_\Lambda$ for A in the above formula, and drop the subscript Λ for simplicity, we then have

$$G(E,\mu,\nu) = \int x_\mu \cdot x_\nu \det(E-H) e^{-\sum_{i,j\in\Lambda}(E-H)_{ij} x_i\cdot x_j} \prod_{i\in\Lambda} \frac{d^2 x_i}{\pi}. \tag{2.5}$$

(In this section, since we only work in Λ, if it is convenient, we will continue to drop the subscript Λ without further notice.)

Let $|\Lambda|$ be the number of points in Λ. We use the Grassmann algebra of $2|\Lambda|$ generators to express $\det[E-H]$. This algebra is generated by $2|\Lambda|$ anticommuting variables ξ_i, η_i, $i \in \Lambda$ satisfying the relations:

$$\begin{aligned} [\xi_i, \eta_j] &= \xi_i \eta_j + \eta_j \xi_i = 0, \\ [\xi_i, \xi_j] &= \xi_i \xi_j + \xi_j \xi_i = 0, \\ [\eta_i, \eta_j] &= \eta_i \eta_j + \eta_j \eta_i = 0, \end{aligned} \tag{2.6}$$

where we write $[a,b] = ab+ba$ for the anti-commutator. It is denoted by $\varLambda[\xi_1, \eta_1, .., \xi_{|\Lambda|}, \eta_{|\Lambda|}]$ (if we identify Λ with $\{1, .., |\Lambda|\}$). From (2.4), we see in particular that $\xi_i^2 = \eta_i^2 = 0$. We write these anticommutative variables collectively as ξ, η.

To make connections with better known objects, ξ_j, η_j can be thought of as the two 1-forms on the j^{th} copy of $\mathbf{R}^2$. "C^∞ functions" $F(\xi_i, \eta_j)$ of these anticommuting variables are defined by Taylor's formula at $(0,0)$ which contains a finite number of terms because of nilpotency. In this way $F(\xi,\eta)$ becomes an element of the Grassmann algebra. For example if

$$F(\xi,\eta) = e^{A_{ij}\xi_i\eta_j}, \tag{2.7}$$

then

$$F(\xi,\eta) = 1 + A_{ij}\xi_i\eta_j. \tag{2.8}$$

This is the function that we need in writing the determinant. We also need to define the notions of differentiation and integration. Define:

$$\begin{aligned} \frac{\partial}{\partial \xi_i}(\xi_i) &= 1, \\ \frac{\partial}{\partial \eta_i}(\eta_i) &= 1. \end{aligned} \tag{2.9}$$

We require further that these differentiations be linear operators and that Leibnitz' rule hold. We can then define integrals (with respect to ∂) as follows:

$$\int 1 d\xi_i = 0, \quad \int \xi_i d\xi_i = 1, \quad \int 1 d\eta_i = 0, \quad \int \eta_i d\eta_i = 1. \tag{2.10}$$

A multiple integral is defined to be a repeated integral. For example,

$$\int \xi_i \eta_j d\xi_i d\eta_j = -\int \eta_j \xi_i d\xi_i d\eta_j = -\int \eta_j d\eta_j = -1. \tag{2.11}$$

Using (2.7, 2.8), (2.10, 2.11), we obtain

$$\det(E-H) = \int e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}\eta_j\xi_k} \prod_{j\in\Lambda}(d\eta_j d\xi_j), \tag{2.12}$$

$$(E-H)^{-1}_{\mu\nu}\det(E-H) = \int \xi_\mu \eta_\nu e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}\eta_j\xi_k} \prod_{j\in\Lambda}(d\eta_j d\xi_j), \tag{2.13}$$

We illustrate (2.12) in the case where $(E-H) = M$ is a 2×2 matrix. The integrand in the RHS of (2.12) is

$$e^{-\sum_{j,k=1,2} M_{jk}\eta_j\xi_k} = \prod_{j,k=1,2}(1 - M_{jk}\eta_j\xi_k), \tag{2.14}$$

where we used the commutativity of $e^{-M_{jk}\eta_j\xi_k}$ and (2.7, 2.8). Doing the integral in the RHS of (2.14) upon using (2.10) and (2.11), we obtain

$$\text{RHS} = M_{11}M_{22} - M_{12}M_{21} = \det M, \tag{2.15}$$

as expected. Similarly, we obtain for the RHS of (2.13):

$$\text{RHS} = (-1)^{|\mu|+|\nu|}\det(M^{\mu\nu}) = M^{-1}_{\mu\nu}\det M, \qquad \mu,\, \nu = 1,\, 2, \tag{2.16}$$

where $M^{\mu\nu}$ is the co-matrix with the μth row and νth column removed. (In the 2×2 case it is just the one matrix element that is left.)

Combining (2.5) with (2.12), we obtain the following expression:

$$G(E,\mu,\nu) = \int x_\mu \cdot x_\nu e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}X_j\cdot X_k} \prod_{j\in\Lambda}\frac{d^2X_j}{\pi}, \tag{2.17}$$

where

$$\begin{aligned} x_j &\in \mathbf{R}^2, \\ X_j &\stackrel{\text{def}}{=} (x_j, \xi_j, \eta_j), \\ X_j \cdot X_k &\stackrel{\text{def}}{=} x_j \cdot x_k + \frac{1}{2}(\eta_j\xi_k + \eta_k\xi_j), \\ d^2X_j &\stackrel{\text{def}}{=} d^2x_j d\eta_j d\xi_j, \end{aligned} \tag{2.18}$$

and we used that $(E-H)$ is a symmetric matrix.

Let $\hat{g}$ be the Laplace transform of dg:

$$\hat{g}(t)=\int e^{tv}dg(v), \qquad t\geq 0 \tag{2.19}$$

Write

$$\hat{g}(t)=e^{k(t)}, \tag{2.20}$$

as $\hat{g}>0$. For example, for the Bernoulli measure:

$$dg(v)=\frac{\delta(v+1)+\delta(v)}{2}, \tag{2.21}$$

$\hat{g}(t)=e^{-t/2}\cosh(t/2)$. Using (2.17) and taking the expectation value with respect to dg, we obtain

(2.22)

$$\begin{aligned}\langle G(E,\mu,\nu)\rangle &= \int x_\mu\cdot x_\nu e^{-(\sum_{j\in\Lambda}EX_j\cdot X_j-\sum_{j,k\in\Lambda,|j-k|_1=1}X_j\cdot X_k-\gamma\sum_{j\in\Lambda}v_jX_j\cdot X_j)} \\ &\qquad \prod_{j\in\Lambda}dg(v_j)\prod_{j\in\Lambda}\frac{d^2X_j}{\pi} \\ &= \int x_\mu\cdot x_\nu e^{-(\sum_{j\in\Lambda}EX_j\cdot X_j-\sum_{j,k\in\Lambda,|j-k|_1=1}X_j\cdot X_k-\sum_{j\in\Lambda}k(\gamma X_j\cdot X_j))}\prod_{j\in\Lambda}\frac{d^2X_j}{\pi} \\ &\overset{\text{def}}{=} \int x_\mu\cdot x_\nu e^{-2\Phi(X)}\prod_{j\in\Lambda}\frac{d^2X_j}{\pi},\end{aligned}$$

where

$$\begin{aligned}k(\gamma X_j\cdot X_j) &= k(\gamma x_j\cdot x_j+\gamma\eta_j\xi_j) \\ &= k(\gamma x_j\cdot x_j)+\gamma k'(x_j\cdot x_j)\eta_j\xi_j,\end{aligned} \tag{2.23}$$

by using (2.8) to expand $e^{\gamma v_j\xi_j\eta_j}$ and direct computations, and

$$2\Phi(X)\overset{\text{def}}{=}\sum_{j\in\Lambda}EX_j\cdot X_j-\sum_{i,j\in\Lambda,|i-j|_1=1}X_i\cdot X_j-\sum_{j\in\Lambda}k(\gamma X_j\cdot X_j). \tag{2.24}$$

(Strictly speaking the LHS of (2.23) is defined only for k's which are C^∞ functions. Under (H1,2), k is only C^2. This is, however, sufficient to define "functions" of the form $k(X_i\cdot X_j)$, as the formal Taylor series terminates at order 2 due to nilpotency.)

We notice in (2.22), (2.24) that apart from the factor $x_\mu\cdot x_\nu$, the integrand is only a "function" of $X_j\cdot X_k$. More precisely, we consider the algebra $C^\infty(\mathbf{R}^{2|\Lambda|})\otimes\Lambda[\xi_1,\eta_1,..,\xi_{|\Lambda|},\eta_{|\Lambda|}]$. We call elements of this algebra, (super)functions. Clearly,

$$\begin{aligned}D(X)=X_i\cdot X_j &= x_i\cdot x_j+\frac{1}{2}(\eta_i\xi_j+\eta_j\xi_i) \\ &\in C^\infty(\mathbf{R}^{2|\Lambda|})\otimes\Lambda[\xi_1,\eta_1,..,\xi_{|\Lambda|},\eta_{|\Lambda|}],\end{aligned}$$

and is hence a (super)function. We call the scalars (which are functions in $C^\infty(\mathbf{R}^{2|\Lambda|})$) in front of the Grassmann generators ξ, η, coefficients. In the case of $X_i \cdot X_j$, the coefficients are $x_i \cdot x_j$ and $1/2$. Those (super)functions which are only functions of $X_i \cdot X_j$ $(i, j \in \Lambda)$ are called supersymmetric functions. More precisely, supersymmetries are defined to be the set of transformations (in the alegebraic sense) that leave the dot products $X_i \cdot X_j$ $(i, j \in \Lambda)$ invariant. Two obvious transformations that leave the dot product invariant are the usual rotations O in $\mathbf{R}^2$,

$$x_i = x_i' O \qquad (i = 1, \cdots, n)$$

and the transformations $A \in Sp(2)$ acting on ξ_i, η_i $(i = 1, \cdots, n)$ such that

$$(\xi_i, \eta_i) = (\xi_i', \eta_i')A,$$

where $\{x_i', \xi_i', \eta_i'\}_{i=1}^n$ is another set of coordinates, x_i' being the even ones and ξ_i', η_i' the odd ones. We put $X_i' = (x_i', \xi_i', \eta_i') \qquad (i = 1, \cdots, n)$. Aside from these two linear transformations, supersymmetries also include transformations generated by (super)vector fields of the type:

$$\begin{aligned} V &= \sum_i V_i \\ &= \sum_i (\xi_i a + \eta_i b)\frac{\partial}{\partial x_i} + 2(b \cdot x_i)\frac{\partial}{\partial \xi_i} - 2(a \cdot x_i)\frac{\partial}{\partial \eta_i}, \end{aligned}$$

where a, $b \in \mathbf{R}^2$, and

$$\begin{aligned} a\frac{\partial}{\partial x_i} &\overset{\text{def}}{=} a^{(1)}\frac{\partial}{\partial x_i^{(1)}} + a^{(2)}\frac{\partial}{\partial x_i^{(2)}}, \\ b\frac{\partial}{\partial x_i} &\overset{\text{def}}{=} b^{(1)}\frac{\partial}{\partial x_i^{(1)}} + b^{(2)}\frac{\partial}{\partial x_i^{(2)}}. \end{aligned}$$

(Note that it is the same transformation in all the X_i.) The above transformation is to be understood in the algebraic sense. We check that $VD(X_i, X_j) = 0$. Analogous to the usual change of variables, the (super) change of variables also generates a Jacobian, which is called a Berezinian. We will not go into the details, except mentioning that the Berezinian here is 1.

Let τ be a supersymmetric transformation. Let $X_i = \tau X_i'$ $(i = 1, \cdots, n)$.

Definition. *A superfunction F is supersymmetric if it is invariant under all supersymmetries:*

$$F(X_1, \cdots, X_n) = F(X_1', \cdots, X_n'),$$

for all τ supersymmetric transformations.

Clearly, supersymmetric functions belong to a rather restricted class of functions. For example, in $\mathbf{R}^2$, with two generators ξ, η, F is supersymmetric if and only if there exists f: $[0, \infty) \mapsto \mathbf{R}$ of class C^∞, such that

$$F(X) = f(X \cdot X) = f(x \cdot x) + f'(x \cdot x)\eta\xi. \tag{2.25}$$

For the general case, see *e.g.*, [KS].

Define $dX_i = (d^2x_i/\pi)d\eta_i d\xi_i$ $(i = 1, \cdots, n)$. One of the most useful properties of the supersymmetric functions, which we rely on for the proof of Proposition 2.4, is the following:

Proposition 2.1. *If F is supersymmetric with all of its coefficients in $\mathcal{S}(\mathbf{R}^{2n})$, then*

$$\int F(X_1, \cdots, X_n) dX_1 \cdots dX_n = F(0, \cdots 0). \tag{2.26}$$

For a proof, see e.g., [K,KS]. Proposition 2.1 is related to the so called "localization formula" in geometry, see e.g., [BGV].

Proposition 2.2.

$$\begin{aligned}\langle G(E, \mu, \nu)\rangle &= \langle (E - \Delta - \gamma V)^{-1}_{\mu\nu}\rangle \\ &= \int \xi_\mu \eta_\nu e^{-2\Phi} \prod_{j\in\Lambda} d^2 X_j \\ &= \int (E - \Delta - \gamma \operatorname{diag} k')^{-1}_{\mu\nu} e^{-2\phi} \prod_{j\in\Lambda} \frac{d^2 x_j}{\pi},\end{aligned} \tag{2.27}$$

where

(2.28)

$$\phi(x) = \frac{1}{2}[\sum_{i,j}(E - \Delta)_{ij} x_i \cdot x_j - \sum_{j\in\Lambda} k(\gamma x_j \cdot x_j) - \log\det(E - \Delta - \gamma \operatorname{diag} k'(\gamma x_j \cdot x_j))],$$

$x_i \in \mathbf{R}^2$,

k is as defined in (2.20), and $\operatorname{diag} k'$ *denotes the diagonal matrix with entries* $(\operatorname{diag} k')_{jj} = k'(\gamma x_j \cdot x_j)$.

Remark. It is instructive to compare the first and the third equalities in (2.27).

Proof. Let $F(X) = X_\mu \cdot X_\nu e^{-2\Phi}$. Using Proposition 2.1, (2.23), $F(0) = 0$, (2.18), (2.24) and integrating over the anti-commuting variables as in (2.13), we have:

$$\begin{aligned}\langle G(E, \mu, \nu)\rangle &= \int x_\mu \cdot x_\nu e^{-2\Phi} \prod_{j\in\Lambda} d^2 X_j \\ &= \int \xi_\mu \eta_\nu e^{-2\Phi} \prod_{j\in\Lambda} d^2 X_j \\ &= \int (E - \Delta - \gamma \operatorname{diag} k')^{-1}_{\mu\nu} e^{-2\phi} \prod_{j\in\Lambda} \frac{d^2 x_j}{\pi}.\end{aligned} \tag{2.29}$$

□

We end this section by remarking that the measure $e^{-2\phi}$ with ϕ defined in (2.28) is normalized:

$$\int e^{-2\phi} = 1 \qquad \text{for all } \Lambda, \tag{2.30}$$

as $\prod_{j\in\Lambda} dg(v_j)$ is normalized. This can also be verified by using (2.24,2.26) and Proposition 2.1

III. An asymptotic series for $\langle G_\Lambda(E,\mu,\nu)\rangle$

Using the representation in Proposition 2.2, we arrive at our main estimate of this section:

Proposition 3.1. *Assume dg satisfies (H1,2). There exists $\gamma_0 > 0$, such that for all $0 < \gamma < \gamma_0$, all $n \in \mathbf{N}$, all cubes $\Lambda = [-L, L]^d$ in $\mathbf{Z}^d$, all μ, $\nu \in \Lambda$,*

$$
\begin{aligned}
\langle (E-\Delta_\Lambda-\gamma V)^{-1}_{\mu\nu}\rangle =&(E-\Delta_\Lambda-\gamma\langle v\rangle)^{-1}_{\mu\nu}+\mathbf{O}(\gamma\tilde{\gamma})(E-\Delta_\Lambda-\gamma\langle v\rangle)^{-2}_{\mu\nu}\\
&+\mathbf{O}(\gamma\tilde{\gamma})^2(E-\Delta_\Lambda-\gamma\langle v\rangle)^{-3}_{\mu\nu}+\cdots\\
&+\mathbf{O}(\gamma\tilde{\gamma})^{n-1}(E-\Delta_\Lambda-\gamma\langle v\rangle)^{-n}_{\mu\nu}\\
&+\mathbf{O}_n(\tilde{\gamma}^n)L^2\\
\le&(E-\Delta_\Lambda-\gamma\langle v\rangle+\mathbf{O}_n(\gamma\tilde{\gamma}))^{-1}_{\mu\nu}+\mathbf{O}_n(\tilde{\gamma}^n)L^2,\\
&\qquad\qquad (E\ge 2d),
\end{aligned}
\tag{3.1}
$$

where

$$
\tilde{\gamma}=\begin{cases}\gamma L & d=1\\ \gamma\log L & d=2\\ \gamma & d\ge 3.\end{cases}
\tag{3.2}
$$

Remark. We note that the error term is independent of $|\mu-\nu|$. (Compare with Theorem 7.2 of [W1].) (3.1) is therefore useful only for fixed Λ and μ, ν. The dependence of $\tilde{\gamma}$ on the spatial dimension d reflects the well known dependence of the free Green's function $(E-\Delta_\Lambda)^{-1}_{ij}$ $(E\ge 2d)$, on d. See Lemma 3.3 below.

To prove Proposition 3.1, we use the third equality in (2.27). Using the resolvent equation, we expand the integrand about the constant potential $k'(0)=\langle v\rangle$:

$$
\begin{aligned}
&(E-\Delta_\Lambda-\gamma\operatorname{diag}k')^{-1}_{\mu\nu}\\
&\quad=(E-\Delta_\Lambda-\gamma k'(0))^{-1}_{\mu\nu}\\
&\qquad+\gamma\sum_{j_1\in\Lambda}(E-\Delta_\Lambda-\gamma k'(0))^{-1}_{\mu j_1}(k'(\gamma x_{j_1}\cdot x_{j_1})-k'(0))(E-\Delta_\Lambda-\gamma k'(0))^{-1}_{j_1\nu}\\
&\qquad+\gamma^2\sum_{j_1,j_2\in\Lambda}(E-\Delta_\Lambda-\gamma k'(0))^{-1}_{\mu j_1}(k'(\gamma x_{j_1}\cdot x_{j_1})-k'(0))(E-\Delta_\Lambda-\gamma k'(0))^{-1}_{j_1 j_2}\\
&\qquad\qquad(k'(\gamma x_{j_2}\cdot x_{j_2})-k'(0))(E-\Delta_\Lambda-\gamma k'(0))^{-1}_{j_2\nu}\\
&\qquad+\cdots\\
&\qquad+\gamma^n\sum_{j_1\cdots j_n\in\Lambda}(E-\Delta_\Lambda-\gamma k'(0))^{-1}_{\mu j_1}(k'(\gamma x_{j_1}\cdot x_{j_1})-k'(0))\cdots\\
&\qquad\qquad(E-\Delta_\Lambda-\gamma k'(0))^{-1}_{j_{n-1}j_n}(k'(\gamma x_{j_n}\cdot x_{j_n})-k'(0))(E-\Delta_\Lambda-\gamma\operatorname{diag}k')^{-1}_{j_n\nu}.
\end{aligned}
\tag{3.3}
$$

Note that the last factor in (3.3) is $(E-\Delta_\Lambda-\gamma\operatorname{diag}k')^{-1}$ itself. In view of (3.3), the proof of the Proposition 3.1 depends on the following:

Proposition 3.2. *For all $m \in \mathbf{N}$, if dg satisfies (H1,2), then*

$$
\begin{aligned}
&\int \prod_{i=1}^{m} (k'(\gamma x_{\ell_i} \cdot x_{\ell_i}) - k'(0)) e^{-2\phi} \prod_{j \in \Lambda} \frac{d^2 x_j}{\pi} \\
&= \int \prod_{i=1}^{m} |(k'(\gamma x_{\ell_i} \cdot x_{\ell_i}) - k'(0))| e^{-2\phi} \prod_{j \in \Lambda} \frac{d^2 x_j}{\pi} \\
&= \mathbf{O}_m(1) \begin{cases} (\gamma L)^m & d = 1 \\ (\gamma \log L)^m & d = 2 \\ \gamma^m & d \geq 3. \end{cases} \\
&\stackrel{\text{def}}{=} \mathbf{O}_m(1) \tilde{\gamma}^m.
\end{aligned} \tag{3.4}
$$

The proof of Proposition 3.2 uses Proposition 2.1, induction and the following standard estimate on the free Green's function, which we state without proof. See e.g., [La]. Let $\Lambda = [-L, L]^d$ as before.

Lemma 3.3. *For $E \geq 2d$,*

$$
0 < (E - \Delta_\Lambda)^{-1}_{ij} \leq \begin{cases} \mathbf{O}(L), & d = 1 \\ \mathbf{O}(\log(\dfrac{L}{|i-j|+1})) & d = 2 \\ \dfrac{\mathbf{O}_d(1)}{(|i-j|+1)^{d-2}} & d \geq 3, \end{cases} \tag{3.5}
$$

for all $i, j \in \Lambda$ and $\mathbf{O}_d(1)$ is uniform in L.

Corollary 3.4. *(maximum principle) Let $S \subseteq \Lambda$, be any subset of $\Lambda = [-L, L]^d$. Define*

$$
\tilde{\Delta}_\Lambda(i, j) = \begin{cases} \Delta_\Lambda(i, j), & i, j \in \Lambda \backslash S, \\ 0 & \textit{otherwise}; \end{cases} \tag{3.6}
$$

$$
\tilde{M} = E - \tilde{\Delta}_\Lambda - \gamma \operatorname{diag} k' \qquad (E \geq 2d).
$$

Then

$$
0 < \tilde{M}^{-1}_{ij} \leq \begin{cases} \mathbf{O}(L), & d = 1 \\ \mathbf{O}(\log(\dfrac{L}{|i-j|+1})) & d = 2 \\ \dfrac{\mathbf{O}_d(1)}{(|i-j|+1)^{d-2}} & d \geq 3, \end{cases} \tag{3.7}
$$

for all $i, j \in \Lambda \backslash S$ and $\mathbf{O}_d(1)$ is uniform in L.

Proof. By the definition (2.20),

$$
\hat{g}(t) = \int e^{tv} dg(v) = e^{k(t)} \qquad (t \geq 0, \operatorname{supp} dg \subseteq (-\infty, 0]). \tag{3.8}
$$

So

$$k'(t) = \frac{\int v e^{tv} dg(v)}{\int e^{tv} dg(v)} \stackrel{\text{def}}{=} \langle v \rangle_t. \tag{3.9}$$

From (H2),

$$-\infty < \langle v \rangle_0 = \langle v \rangle < 0, \tag{3.10}$$

and since

$$\frac{d\langle v \rangle_t}{dt} = \langle v^2 \rangle_t - \langle v \rangle_t^2 \geq 0, \tag{3.11}$$

we have that

$$-\infty < k'(t) < 0 \tag{3.12}$$

for all $t \geq 0$ by using (3.9-3.11).

We first prove the lower bound. Since

$$\frac{\partial}{\partial E} \tilde{M}^{-1}(E)_{ij} = \frac{\partial}{\partial E} (E - \tilde{\Delta}_\Lambda - \gamma \operatorname{diag} k')_{ij}^{-1} = -(E - \tilde{\Delta}_\Lambda - \gamma \operatorname{diag} k')_{ij}^{-2}, \tag{3.13}$$

in order to prove $\tilde{M}(E)_{ij}^{-1} > 0$, for all $E \geq 2d$, it suffices to prove $\tilde{M}(E)_{ij}^{-1} > 0$ for some sufficiently large E. For such large E, we obtain by using the resolvent equation:

$$\begin{aligned}
&(E - \tilde{\Delta}_\Lambda - \gamma \operatorname{diag} k')^{-1} \\
=&(E - \gamma \operatorname{diag} k')^{-1} + (E - \gamma \operatorname{diag} k')^{-1} \tilde{\Delta}_\Lambda (E - \gamma \operatorname{diag} k')^{-1} \\
&+ \cdots + [(E - \gamma \operatorname{diag} k')^{-1} \tilde{\Delta}_\Lambda]^n (E - \gamma \operatorname{diag} k')^{-1} \\
&+ \cdots
\end{aligned} \tag{3.14}$$

The above series is convergent for E sufficiently large. Since $\tilde{\Delta}_\Lambda$ has non-negative entries, we obtain that

$$(E - \tilde{\Delta}_\Lambda - \gamma \operatorname{diag} k')_{ij}^{-1} > 0 \tag{3.15}$$

for large E. Using (3.13), we then obtain the lower bound in (3.7) for $E \geq 2d$.

Set $\delta_S = \Delta_\Lambda - \tilde{\Delta}_\Lambda$. δ_S has non-negative entries. Using the resolvent equation again, we have

(3.16)

$$\begin{aligned}
(E - \tilde{\Delta}_\Lambda - \gamma \operatorname{diag} k')^{-1} =&(E - \Delta_\Lambda + \delta_S - \gamma \operatorname{diag} k')^{-1} \\
=&(E - \Delta_\Lambda)^{-1} \\
&- (E - \Delta_\Lambda)^{-1} (\delta_S - \gamma \operatorname{diag} k')(E - \tilde{\Delta}_\Lambda - \gamma \operatorname{diag} k')^{-1}.
\end{aligned}$$

Since δ_S has non-negative entries and $-\infty < k'(t) < 0$ from (3.12), we obtain

$$(E - \tilde{\Delta}_\Lambda - \gamma \operatorname{diag} k')_{ij}^{-1} < (E - \Delta_\Lambda)_{ij}^{-1}. \tag{3.17}$$

Using Lemma 3.3, we obtain the upper bound in (3.7). □

Proof of Proposition 3.2. We first show that under (H1,2)

$$0 \le k''(t) < \infty \qquad (t \ge 0), \tag{3.18}$$

which implies that $k'(t) - k'(0) \ge 0$ $(t \ge 0)$ and hence prove the first equality in (3.4). Since

$$\begin{aligned} \hat{g}(t) &= \int e^{tv} dg(v) = e^{k(t)}, \\ \hat{g}'(t) &= \int v e^{tv} dg(v) = k'(t) e^{k(t)}, \\ \hat{g}''(t) &= \int v^2 e^{tv} dg(v) = (k'^2(t) + k''(t)) e^{k(t)}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} k''(t) &= \frac{\int v^2 e^{tv} dg(v)}{\int e^{tv} dg(v)} - \left(\frac{\int v e^{tv} dg(v)}{\int e^{tv} dg(v)}\right)^2 \\ &= \langle v^2 \rangle_t - \langle v \rangle_t^2 \\ &\ge 0. \end{aligned} \tag{3.20}$$

We therefore obtain the lower bound in (3.18).

From (H2), $\langle v^2 \rangle < \infty$. Since

$$\frac{d}{dt} \langle v^2 \rangle_t = \langle v^3 \rangle_t - \langle v^2 \rangle_t \langle v \rangle_t \le 0 \tag{3.21}$$

by a trivial application of the FKG inequality [FKG,Li], as v^2, v are monotone in v, we obtain that k'' is bounded.

We now use induction to prove the estimate in (3.4). For $m = 1$, write ℓ for ℓ_1. Recall from (2.25) that

$$k'(\gamma X_\ell \cdot X_\ell) = k'(\gamma x_\ell \cdot x_\ell) + \gamma k''(\gamma x_\ell \cdot x_\ell) \eta_\ell \xi_\ell \tag{3.22}$$

and that $k'(\gamma X_\ell \cdot X_\ell) e^{-2\Phi}$ is a supersymmetric function. Using Proposition 2.1, we therefore have

$$\begin{aligned} &\int (k'(\gamma x_\ell \cdot x_\ell) - k'(0)) e^{-2\phi} \prod_{j \in \Lambda} \frac{d^2 x_j}{\pi} \\ &= \int (k'(\gamma x_\ell \cdot x_\ell) - k'(\gamma X_\ell \cdot X_\ell)) e^{-2\Phi} \prod_{j \in \Lambda} \frac{d^2 X_j}{\pi} \\ &= \int \gamma k''(\gamma x_\ell \cdot x_\ell) \xi_\ell \eta_\ell e^{-2\Phi} \prod_{j \in \Lambda} \frac{d^2 X_j}{\pi}, \end{aligned} \tag{3.23}$$

where Φ is as in (2.24), $d^2 X_j$ is as defined in (2.18) and we used (3.22). Note the change of ϕ, $d^2 x_j$ to Φ, $d^2 X_j$ in the second and third line of (3.23).

Integrating over the anticommuting variables ξ, η in (3.23), we obtain

$$
\begin{aligned}
&\int (k'(\gamma x_\ell \cdot x_\ell) - k'(0)) e^{-2\phi} \prod_{j\in\Lambda} \frac{d^2 x_j}{\pi} \\
=&\gamma \int k''(\gamma x_\ell \cdot x_\ell)(E - \Delta_\Lambda - \gamma \operatorname{diag} k')^{-1}_{\ell\ell} e^{-2\phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi} \\
=&\mathbf{O}(1) \begin{cases} \gamma L & d = 1 \\ \gamma \log L & d = 2 \\ \gamma & d \geq 3. \end{cases} \\
=&\mathbf{O}(\tilde{\gamma}),
\end{aligned} \tag{3.24}
$$

where we used (3.18), (2.30) and Corollary 3.4.

To illustrate the mechanism, we also work out the case $m = 2$ explicitly. For simplicity of notation, we write ℓ for ℓ_1, n for ℓ_2. From Proposition 2.1, we have

$$
\begin{aligned}
&\int (k'(\gamma X_\ell \cdot X_\ell) - k'(0))(k'(\gamma X_n \cdot X_n) - k'(0)) e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi} \\
=&\int (k'(\gamma x_\ell \cdot x_\ell) - k'(0) + \gamma k''(\gamma x_\ell \cdot x_\ell)\eta_\ell \xi_\ell) \\
&\qquad (k'(\gamma x_n \cdot x_n) - k'(0) + \gamma k''(\gamma x_n \cdot x_n)\eta_n \xi_n) e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi} \\
=&0.
\end{aligned} \tag{3.25}
$$

Hence

$$
\begin{aligned}
&\int (k'(\gamma x_\ell \cdot x_\ell) - k'(0))(k'(\gamma x_n \cdot x_n) - k'(0)) e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi} \\
=&\int (k'(\gamma x_\ell \cdot x_\ell) - k'(0))(k'(\gamma x_n \cdot x_n) - k'(0)) e^{-2\phi} \prod_{j\in\Lambda} \frac{d^2 x_j}{\pi} \\
=&\gamma \int (k'(\gamma x_\ell \cdot x_\ell) - k'(0)) k''(\gamma x_n \cdot x_n) \xi_n \eta_n e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi} \\
&+ \gamma \int (k'(\gamma x_n \cdot x_n) - k'(0)) k''(\gamma x_\ell \cdot x_\ell) \xi_\ell \eta_\ell e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi} \\
&- \gamma^2 \int k''(\gamma x_\ell \cdot x_\ell) k''(\gamma x_n \cdot x_n) \xi_\ell \eta_\ell \xi_n \eta_n e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi}
\end{aligned} \tag{3.26}
$$

Writing Δ for Δ_Λ, we perform the Grassmann integrations:

$$
\begin{aligned}
&\int \xi_\ell \eta_\ell e^{-2\Phi} \prod_{j\in\Lambda} d\eta_j d\xi_j \\
=&\det(E - \Delta - \gamma \operatorname{diag} k')^{\ell\ell} e^{-2\phi_0} \\
=&\frac{\det(E - \Delta - \gamma \operatorname{diag} k')^{\ell\ell}}{\det(E - \Delta - \gamma \operatorname{diag} k')} e^{-2\phi_0} \det(E - \Delta - \gamma \operatorname{diag} k') \\
=&(E - \Delta - \gamma \operatorname{diag} k')^{-1}_{\ell\ell} e^{-2\phi},
\end{aligned} \tag{3.27}
$$

where $(E-\Delta-\gamma\,\mathrm{diag}\,k')^{\ell\ell}$ is the minor obtained from $(E-\Delta-\gamma\,\mathrm{diag}\,k')$ by deleting the $\ell^{\rm th}$ row and the $\ell^{\rm th}$ column,

(3.28)
$$\phi_0=\frac{1}{2}[\sum_{i,j}(E-\Delta)_{ij}x_i\cdot x_j-\sum_{j\in\Lambda}k(\gamma x_j\cdot x_j)],$$

and we used (2.28) to reach the last equality in (3.27); and

(3.29)
$$\begin{aligned}&\int\xi_\ell\eta_\ell\xi_n\eta_n e^{-2\Phi}\prod_{j\in\Lambda}d\eta_j d\xi_j\\
=&(1-\delta_{n\ell})\det(E-\Delta-\gamma\,\mathrm{diag}\,k')^{nn,\ell\ell}e^{-2\phi_0}\\
=&(1-\delta_{n\ell})\frac{\det(E-\Delta-\gamma\,\mathrm{diag}\,k')^{nn,\ell\ell}}{\det(E-\Delta-\gamma\,\mathrm{diag}\,k')^{nn}}\frac{\det(E-\Delta-\gamma\,\mathrm{diag}\,k')^{nn}}{\det(E-\Delta-\gamma\,\mathrm{diag}\,k')}e^{-2\phi_0}\det(E-\Delta-\gamma\,\mathrm{diag}\,k')\\
=&(1-\delta_{n\ell})(E-\Delta^{(n)}-\gamma\,\mathrm{diag}\,k')^{-1}_{\ell\ell}(E-\Delta-\gamma\,\mathrm{diag}\,k')^{-1}_{nn}e^{-2\phi},\end{aligned}$$

where $(E-\Delta-\gamma\,\mathrm{diag}\,k')^{nn,\ell\ell}$ is the minor obtained from $(E-\Delta-\gamma\,\mathrm{diag}\,k')$ by deleting the $n^{\rm th}$ row, $n^{\rm th}$ column and the $\ell^{\rm th}$ row, $\ell^{\rm th}$ column, (The order is irrelevant.) and

(3.30)
$$\Delta^{(n)}_{ij}=\begin{cases}\Delta_{ij} & i,\,j\in\Lambda\backslash\{n\},\\ 0 & \text{otherwise},\end{cases}$$

The Kronecker δ function originates from anti-commutativity.

Substituting (3.27) (3.29) in (3.26), we obtain

(3.31)
$$\begin{aligned}&\int(k'(\gamma x_\ell\cdot x_\ell)-k'(0))(k'(\gamma x_n\cdot x_n)-k'(0))e^{-2\Phi}\prod_{j\in\Lambda}\frac{d^2X_j}{\pi}\\
=&\gamma\int(k'(\gamma x_\ell\cdot x_\ell)-k'(0))k''(\gamma x_n\cdot x_n)(E-\Delta-\gamma\,\mathrm{diag}\,k')^{-1}_{nn}e^{-2\phi}\prod_{j\in\Lambda}\frac{d^2x_j}{\pi}\\
&+\gamma\int(k'(\gamma x_n\cdot x_n)-k'(0))k''(\gamma x_\ell\cdot x_\ell)(E-\Delta-\gamma\,\mathrm{diag}\,k')^{-1}_{\ell\ell}e^{-2\phi}\prod_{j\in\Lambda}\frac{d^2x_j}{\pi}\\
&-\gamma^2(1-\delta_{n\ell})\int k''(\gamma x_\ell\cdot x_\ell)k''(\gamma x_n\cdot x_n)(E-\Delta^{(n)}-\gamma\,\mathrm{diag}\,k')^{-1}_{\ell\ell}\\
&\qquad(E-\Delta-\gamma\,\mathrm{diag}\,k')^{-1}_{nn}e^{-2\phi}\prod_{j\in\Lambda}\frac{d^2x_j}{\pi}\\
=&\mathbf{O}(\tilde\gamma)\int(k'(\gamma x_\ell\cdot x_\ell)-k'(0))e^{-2\phi}\prod_{j\in\Lambda}\frac{d^2x_j}{\pi}\\
&+\mathbf{O}(\tilde\gamma)\int(k'(\gamma x_n\cdot x_n)-k'(0))e^{-2\phi}\prod_{j\in\Lambda}\frac{d^2x_j}{\pi}+\mathbf{O}(\tilde\gamma^2)\\
=&\mathbf{O}_2(\tilde\gamma^2)\end{aligned}$$

where we used (3.18,3.24,3.7) to reach the last estimate.

Assume (3.4) holds up to m, for $m+1$ we have from Proposition 2.1

$$\begin{aligned}
&\int \prod_{i=1}^{m+1} (k'(\gamma X_{\ell_i} \cdot X_{\ell_i}) - k'(0)) e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi} \\
&= \int \prod_{i=1}^{m+1} (k'(\gamma x_{\ell_i} \cdot x_{\ell_i}) - k'(0) + \gamma k''(\gamma x_{\ell_i} \cdot x_{\ell_i}) \eta_{\ell_i} \xi_{\ell_i}) e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi} \\
&= 0.
\end{aligned} \tag{3.32}$$

So

$$\begin{aligned}
&-\int \prod_{i=1}^{m+1} (k'(\gamma x_{\ell_i} \cdot x_{\ell_i}) - k'(0)) e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi} \\
&= -\int \prod_{i=1}^{m+1} (k'(\gamma x_{\ell_i} \cdot x_{\ell_i}) - k'(0)) e^{-2\phi} \prod_{j\in\Lambda} \frac{d^2 x_j}{\pi} \\
&= \gamma \sum_{\alpha=1}^{m} \int \prod_{i\neq\alpha} (k'(\gamma x_{\ell_i} \cdot x_{\ell_i}) - k'(0)) k''(\gamma x_{\ell_\alpha} \cdot x_{\ell_\alpha}) \eta_{\ell_\alpha} \xi_{\ell_\alpha} e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi} \\
&\quad + \gamma^2 \sum_{\alpha=1}^{m} \sum_{\beta=1,\, \beta\neq\alpha}^{m} \int \prod_{i\neq\alpha,\, i\neq\beta} (k'(\gamma x_{\ell_i} \cdot x_{\ell_i}) - k'(0)) k''(\gamma x_{\ell_\alpha} \cdot x_{\ell_\alpha}) \eta_{\ell_\alpha} \xi_{\ell_\alpha} \\
&\qquad\qquad k''(\gamma x_{\ell_\beta} \cdot x_{\ell_\beta}) \eta_{\ell_\beta} \xi_{\ell_\beta} e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi} \\
&\quad + \cdots + \gamma^{m+1} \int \prod_{i=1}^{m+1} k''(\gamma x_{\ell_i} \cdot x_{\ell_i}) \eta_{\ell_i} \xi_{\ell_i} e^{-2\Phi} \prod_{j\in\Lambda} \frac{d^2 X_j}{\pi}.
\end{aligned} \tag{3.33}$$

We make the important observation that in the last equality of (3.33), there are *at most* m factors of $k'(\gamma x_{\ell_i} \cdot x_{\ell_i}) - k'(0)$ under the product sign $\prod$. Combining with the estimate

(3.34)

$$\begin{aligned}
&\int \xi_{\ell_1} \eta_{\ell_1} \cdots \xi_{\ell_p} \eta_{\ell_p} e^{-2\Phi} \prod_{j\in\Lambda} d\eta_j d\xi_j \\
&= \det(E - \Delta - \gamma \operatorname{diag} k')^{\ell_1\ell_1, \cdots \ell_p\ell_p} e^{-2\phi_0} \\
&= \frac{\det(E - \Delta - \gamma \operatorname{diag} k')^{\ell_1\ell_1, \cdots \ell_p\ell_p}}{\det(E - \Delta - \gamma \operatorname{diag} k')^{\ell_1\ell_1, \cdots \ell_{p-1}\ell_{p-1}}} \frac{\det(E - \Delta - \gamma \operatorname{diag} k')^{\ell_1\ell_1, \cdots \ell_{p-1}\ell_{p-1}}}{\det(E - \Delta - \gamma \operatorname{diag} k')^{\ell_1\ell_1, \cdots \ell_{p-2}\ell_{p-2}}} \\
&\quad \cdots \frac{\det(E - \Delta - \gamma \operatorname{diag} k')^{\ell_1\ell_1}}{\det(E - \Delta - \gamma \operatorname{diag} k')} \det(E - \Delta - \gamma \operatorname{diag} k') e^{-2\phi_0} \\
&= (E - \Delta^{(\ell_1 \cdots \ell_{p-1})} - \gamma \operatorname{diag} k')^{-1}_{\ell_p\ell_p} \cdots (E - \Delta - \gamma \operatorname{diag} k')^{-1}_{\ell_1\ell_1} e^{-2\phi} \\
&\leq (E - \Delta - \gamma \operatorname{diag} k')^{-1}_{\ell_p\ell_p} \cdots (E - \Delta - \gamma \operatorname{diag} k')^{-1}_{\ell_1\ell_1} e^{-2\phi} \\
&= \mathbf{O}(\tilde{\gamma}^p),
\end{aligned}$$

where $p \leq |\Lambda|$, $\ell_1 \neq \ell_2 \cdots \neq \ell_p$ (otherwise the integral is identically 0), the various notations invoving the superscrips $\ell_1, \cdots, \ell_p$ are just the natural extensions of the corresponding notations used in (3.27), (3.29). (3.34) is the p^{th} repeated version of (3.27), (3.29). We obtain (3.4), by (3.33), (3.34) and induction. □

Proof of Proposition 3.1. Since

$$\langle v\rangle = \int v dg(v) = k'(0), \tag{3.35}$$

the first n terms follow by using (3.4). To estimate the last term, we use the bound

$$(E-\Delta_\Lambda - \gamma \operatorname{diag} k')^{-1}_{j_n\nu} < (E-\Delta_\Lambda)^{-1}_{j_n\nu}, \tag{3.36}$$

we have

(3.37)

$$\begin{aligned}
&\gamma^n \sum_{j_1\cdots j_n\in\Lambda} \int (E-\Delta_\Lambda-\gamma k'(0))^{-1}_{\mu j_1}(k'(\gamma x_{j_1}\cdot x_{j_1})-k'(0))\cdots \\
&\qquad (E-\Delta_\Lambda-\gamma k'(0))^{-1}_{j_{n-1}j_n}(k'(\gamma x_{j_n}\cdot x_{j_n})-k'(0))(E-\Delta_\Lambda-\gamma \operatorname{diag} k')^{-1}_{j_n\nu} \\
&\qquad\qquad e^{-2\phi}\prod_{\ell\in\Lambda}\frac{d^2X_\ell}{\pi} \\
\le&\gamma^n[(E-\Delta_\Lambda-\gamma k'(0))^{-n}(E-\Delta_\Lambda)^{-1}](\mu,\nu)\cdot \mathbf{O}_n(\tilde\gamma^n) \\
\le&\gamma^n \mathbf{O}_n(\tilde\gamma^n)\|(E-\Delta_\Lambda-\gamma k'(0))^{-n}\|_{\ell^2}\|(E-\Delta_\Lambda)^{-1}\|_{\ell^2} \\
=&\mathbf{O}_n(\tilde\gamma^n)L^2,
\end{aligned}$$

where we used

$$\|(E-\Delta_\Lambda-\gamma k'(0))^{-n}\|_{\ell^2} = \mathbf{O}(\gamma^{-n}) \tag{3.38}$$

and

$$\|(E-\Delta_\Lambda)^{-1}\|_{\ell^2} = \mathbf{O}(L^2). \tag{3.39}$$

The last inequality in (3.1) is obtained by comparing the resolvent expansion of $(E-\Delta_\Lambda-\gamma\langle v\rangle+\mathbf{O}_n(\gamma\tilde\gamma))^{-1}_{\mu\nu}$ with the preceding series. Note that $\mathbf{O}_n(\gamma\tilde\gamma)<0$ here. □

IV. Mean field lower bounds on the Lyapunov exponents

Let L_0, $L\in\mathbf{N}$, $\Lambda_0=[-L_0,L_0]^d$, $\tilde\Lambda_0=[-(L_0+2),L_0+2]^d$, $\Lambda=[-L,L]^d$. Define $\partial\Lambda_0=\{j|\,|j|_\infty=L_0\}$ and similarly $\partial\tilde\Lambda_0$. We also extend the notation Δ_S to denote the operator with

$$\begin{cases} \Delta_{i,j} = 1, & i,\ j\in S, \text{ or } i,\ j\in \mathbf{Z}^d\backslash S \\ \quad 0 & \text{otherwise.}\end{cases} \tag{4.1}$$

(Assume $L >> L_0$.) We have the following decoupling lemma:

Lemma 4.1. *Assume* $\mu \in \Lambda_0$, $\nu \notin \tilde{\Lambda}_0$.

$$\begin{aligned} &\langle (E-\Delta_\Lambda-\gamma V)^{-1}_{\mu\nu}\rangle \\ \le&\bar{L}L_0^{2(d-1)} \sup_{j\in\partial\Lambda_0,\, j'\in\partial\tilde{\Lambda}_0} \langle (E-\Delta_{\Lambda_0}-\gamma V)^{-1}_{\mu j}\rangle\langle (E-\Delta_\Lambda-\gamma V)^{-1}_{j'\nu}\rangle, \end{aligned} \tag{4.2}$$

where $E \ge 2d$ *and*

$$\bar{L} = \begin{cases} \mathbf{O}(1)L, & d=1 \\ \mathbf{O}(1)\log L & d=2 \\ \mathbf{O}_d(1) & d\ge 3, \end{cases} \tag{4.3}$$

Proof. $\partial\Lambda_0$ separates μ from ν, so by the resolvent equation we have

$$(E-\Delta_\Lambda-\gamma V)^{-1}_{\mu\nu} = \sum_{<jj''>} (E-\Delta_{\Lambda_0}-\gamma V)^{-1}_{\mu j}(E-\Delta_\Lambda-\gamma V)^{-1}_{j''\nu}, \tag{4.4}$$

where $< jj'' >$ denotes the set $\{j \in \partial\Lambda_0,\, j'' \notin \partial\Lambda_0 |\, |j''-j|_1 = 1\}$. Since $\partial\tilde{\Lambda}_0$ separates ν from j'', using the resolvent equation on the second factor in the sum in (4.4), we have

$$\begin{aligned} &(E-\Delta_\Lambda-\gamma V)^{-1}_{\mu\nu} \\ =&\sum_{<jj''><j'j'''>} (E-\Delta_{\Lambda_0}-\gamma V)^{-1}_{\mu j}(E-\Delta_\Lambda-\gamma V)^{-1}_{j''j'''}(E-\Delta_{\tilde{\Lambda}_0}-\gamma V)^{-1}_{j'\nu} \\ \le&\bar{L}\sum_{j\in\partial\Lambda_0,\, j'\in\partial\tilde{\Lambda}_0} (E-\Delta_{\Lambda_0}-\gamma V)^{-1}_{\mu j}(E-\Delta_{\tilde{\Lambda}_0}-\gamma V)^{-1}_{j'\nu}, \end{aligned} \tag{4.5}$$

where $\bar{L}$ is as defined in (4.3), $< j'j''' >$ denotes the set $\{j' \in \partial\tilde{\Lambda}_0,\, j''' \notin \partial\tilde{\Lambda}_0 |\, |j'''-j'|_1 = 1\}$ and we used Lemma 3.3 to bound $(E-\Delta_\Lambda-\gamma V)^{-1}_{j''j'''}$.

Since $(E-\Delta_{\tilde{\Lambda}_0}-\gamma V)^{-1}_{j'\nu}$ is independent of $\{v_i\}_{i\in\Lambda}$, we have

$$\begin{aligned} &\langle (E-\Delta_\Lambda-\gamma V)^{-1}_{\mu\nu}\rangle \\ \le&\bar{L}\sum_{j\in\partial\Lambda_0} \langle (E-\Delta_{\Lambda_0}-\gamma V)^{-1}_{\mu j}\rangle \sum_{j'\in\partial\tilde{\Lambda}_0} \langle (E-\Delta_{\tilde{\Lambda}_0}-\gamma V)^{-1}_{j'\nu}\rangle \\ \le&\bar{L}L_0^{2(d-1)} \sup_{j\in\partial\Lambda_0,\, j'\in\partial\tilde{\Lambda}_0} \langle (E-\Delta_{\Lambda_0}-\gamma V)^{-1}_{\mu j}\rangle\langle (E-\Delta_\Lambda-\gamma V)^{-1}_{j'\nu}\rangle, \end{aligned} \tag{4.6}$$

where we used Corollary 3.4 to reach the last inequality and absorbed some d dependent constant into $\bar{L}$. □

Let

$$L_0 = [\frac{|\log\gamma^p|}{\sqrt{\gamma}}] + 1 \qquad (\gamma << 1), \tag{4.7}$$

where $[\,]$ denotes the integer part, and $p >$ const. d to be determined later. Define $\Lambda_0(\mu) = \mu + [-L_0, L_0]^d$. (To be consistent with the previous notation, we set $\Lambda_0(0) = \Lambda_0$.) Taking large enough n (depending on p) in (3.1), we have for some $a_0 > 0$, which only depends on $\langle v\rangle$ and d,

(4.8)

$$\langle (E-\Delta_{\Lambda_0(\mu)}-\gamma V)^{-1}_{\mu\nu}\rangle \le e^{-a_0\sqrt{\gamma}|\mu-\nu|_{\ell^\infty}} \le \left(\gamma^{pa_0}\right)^{\frac{|\mu-\nu|_{\ell^\infty}}{L_0}} \qquad (\gamma << 1,\, E\ge 2d),$$

where $\nu \in \Lambda_0$ is such that $|\mu-\nu|_{\ell^\infty} \ge L_0/3$. Using Lemma 4.1, we then arrive at our main theorem:

Theorem 4.2. *Let $\hat{j} = \frac{j}{\|j\|} \in \mathcal{S}^{d-1}$. Assume dg satisfies (H1,2). For sufficiently small γ, $\alpha_E(\hat{j})$, $\beta_E(\hat{j})$ satisfy the mean field bound i.e.,*

$$\alpha_E(\hat{j}),\, \beta_E(\hat{j}) > a\sqrt{\gamma}, \qquad \textit{for any}\, a \in (0, c_d a_0) \tag{4.9}$$

where $c_d > 0$ only depends on the dimension d, and a_0 on $\langle v \rangle$ and d, for all $E \geq 2d$ if $\operatorname{supp} dg \subseteq (-\infty, 0]$; for all $E \leq -2d$ if $\operatorname{supp} dg \subseteq [0, \infty)$.

Proof. The lower bound on α_E follows from that of β_E. To prove the lower bound on β_E, we use Lemma 4.1 and induction. (For $d \geq 3$, we do not even need to use induction. The lower bound follows directly from Lemma 4.1.)

Assume $\gamma << 1$. Take L_0 as in (4.7) and $L_1 = L_0^2$, $\Lambda_1(\mu) = \mu + [-L_1, L_1]^d$. Let $\nu \in \Lambda_1(\mu)$ be such that $|\mu - \nu|_{\ell^\infty} \geq L_1/3$. Using Lemma 4.1 with $\Lambda_0(\mu)$ replacing Λ_0, $\tilde{\Lambda}_0(\mu)$ replacing $\tilde{\Lambda}_0$, $\Lambda_1(\mu)$ replacing Λ, we have for $\gamma << 1$,

(4.10)
$$\begin{aligned}
&\langle (E - \Delta_{\Lambda_1(\mu)} - \gamma V)^{-1}_{\mu\nu} \rangle \\
\leq & \bar{L}_1 L_0^{2(d-1)} \sup_{j_1 \in \partial\Lambda_0(\mu),\, j_1' \in \partial\tilde{\Lambda}_0(\mu)} \langle (E - \Delta_{\Lambda_0(\mu)} - \gamma V)^{-1}_{\mu j_1} \rangle \langle (E - \Delta_{\Lambda_1(\mu)} - \gamma V)^{-1}_{j_1'\nu} \rangle \\
\leq & (C_d L_0)^{2d} e^{-a_0\sqrt{\gamma}L_0} \sup_{j_1' \in \partial\tilde{\Lambda}_0(\mu)} \langle (E - \Delta_{\Lambda_1(\mu)} - \gamma V)^{-1}_{j_1'\nu} \rangle,
\end{aligned}$$

where C_d is some constant which only depends on the dimension d.

We now use Lemma 4.1 on $\langle (E - \Delta_{\Lambda_1(\mu)} - \gamma V)^{-1}_{j_1'\nu} \rangle$ with $\Lambda_0(j_1')$ in place of $\Lambda_0(\mu)$... etc., so that we "pave" a path from μ toward ν using cubes congruent to $\Lambda_0(\mu)$, centered at j_1' ... etc. This iteration process can be repeated at least $n = \left[\frac{|\mu-\nu|_{\ell_\infty}}{L_0+2} - 1\right]$ times before $\operatorname{dist}(j_n', \partial\Lambda_1(\mu)) < L_0$. We therefore obtain

$$\begin{aligned}
&\langle (E - \Delta_{\Lambda_1(\mu)} - \gamma V)^{-1}_{\mu\nu} \rangle \\
\leq & \bar{L}_1 \left((C_d L_0)^{2d} e^{-a_0\sqrt{\gamma}L_0} \right)^{\left[\frac{|\mu-\nu|_{\ell_\infty}}{L_0+2}\right]-1} \\
\leq & e^{-a_1\sqrt{\gamma}|\mu-\nu|_{\ell_\infty}},
\end{aligned} \tag{4.11}$$

where $\bar{L}_1$ comes from the apriori estimate on $(E - \Delta_{\Lambda_1(\mu)} - \gamma V)^{-1}_{j_n'\nu}$ and

$$a_1\sqrt{\gamma} = (a_0\sqrt{\gamma} - \frac{2d\log(C_d L_0)}{L_0})(1 - \frac{5}{L_0}) - \frac{6}{L_0^2}\log(C_d L_0). \tag{4.12}$$

So for $\gamma << 1$, $a_1 \geq a_0 - (d+1)/p > 0$ with the choice of L_0 in (4.7), (assuming $p > (d+1)/a_0$) Let $L_n = L_{n-1}^2$ $(n \geq 1)$, $\Lambda_n = [-L_n, L_n]^d$. Repeating the above iteration procedure with the scale L_{n-1} replacing L_0 and the scale L_n replacing L_1, we obtain

$$\langle (E - \Delta_{\Lambda_n(\mu)} - \gamma V)^{-1}_{\mu\nu} \rangle \leq e^{-a_n\sqrt{\gamma}|\mu-\nu|_{\ell_\infty}}, \qquad (|\mu - \nu|_{\ell_\infty} \geq L_n/3) \tag{4.13}$$

with

(4.14)
$$a_n\sqrt{\gamma} = (a_{n-1}\sqrt{\gamma} - \frac{2d\log(C_d L_{n-1})}{L_{n-1}})(1 - \frac{5}{L_{n-1}}) - \frac{6}{L_{n-1}^2}\log(C_d L_{n-1}) \qquad (n \geq 1).$$

Since $L_n = L_{n-1}^2$, and L_0 is as in (4.7), it is easy to see that for all $n \geq 1$, $a_n \geq a_0 - 2(d+1)/p > 0$, (assuming $p > 2(d+1)/a_0$).

To prove the theorem, assume $j \in \Lambda_n$ for some n, $|j|_{\ell^\infty} \geq L_n/3$ and $\text{dist}(j, \partial\Lambda_n) \geq |j|_{\ell^\infty}$. Let $G_n(E, i, j) = (E - \Delta_{\Lambda_n} - \gamma V)^{-1}_{ij}$. We write

$$G_{n+1}(E, 0, j) = G_n(E, 0, j) + \sum_{<j'j''>} G_n(E, 0, j')G_{n+1}(E, j'', j), \tag{4.15}$$

where $< j'j'' >$ is the set $\{j' \in \partial\Lambda_n,\, j'' \notin \partial\Lambda_n | \, |j'' - j'|_1 = 1\}$. From (4.13),

$$\langle G_n(E, 0, j)\rangle \leq e^{-a_n\sqrt{\gamma}|\mu-\nu|_{\ell_\infty}}. \tag{4.16}$$

Using (4.16) in (4.15),

$$\begin{aligned} &\langle G_{n+1}(E, 0, j)\rangle - \langle G_n(E, 0, j)\rangle \\ \leq& e^{-a_n\sqrt{\gamma}L_n} L_{n+1}(C_d L_n)^{d-1} \\ \leq& e^{-a_n\sqrt{\gamma}L_n/2} \end{aligned} \tag{4.17}$$

for large enough L_n (and hence $|j|_{\ell_\infty}$).

Similarly

$$\begin{aligned} &\langle G_{n+2}(E, 0, j)\rangle - \langle G_{n+1}(E, 0, j)\rangle \\ \leq& e^{-a_{n+1}\sqrt{\gamma}L_{n+1}/2} \end{aligned} \tag{4.18}$$

... etc. So $\langle G_n(E, 0, j)\rangle$ form a Cauchy sequence and

$$\langle G(E, 0, j)\rangle = \lim_{n\to\infty} \langle G_n(E, 0, j)\rangle \leq e^{-a_\infty\sqrt{\gamma}|j|_{\ell^\infty}} \tag{4.19}$$

for some $a_\infty > 0$. Hence for $\gamma << 1$,

$$\beta_E(\hat{j}) \geq a\sqrt{\gamma} \tag{4.20}$$

for any $a \in (0, c_d a_0)$, where c_d comes from changing the ℓ^∞ norm to ℓ^2 norm and a_∞ to a_0.

The lower bound on α_E follows from (4.20), since (4.20) implies that for all $\tilde{a} \in (0, c_d a)$

$$\langle \sum_j (E - \Delta - \gamma V)^{-1}_{0j} e^{\tilde{a}\sqrt{\gamma}\|j\|_{\ell^2}}\rangle < \infty.$$

So $\alpha_E(\hat{j}) \geq a\sqrt{\gamma}$ for any $a \in (0, c_d a_0)$. □

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