SISSA 158/99/FM
DSM–QM464
April 2000

Some Properties of Non-linear σ-Models in Noncommutative Geometry

Ludwik Dabrowski [1], Thomas Krajewski [1,2], Giovanni Landi [2,3]

[1] *Scuola Internazionale Superiore di Studi Avanzati, Via Beirut 2-4, I-34014, Trieste, Italy*

[2] *Dipartimento di Scienze Matematiche, Università di Trieste P.le Europa 1, I-34127, Trieste, Italy*

[3] *INFN, Sezione di Napoli, Napoli, Italy.*

Abstract

We introduce non-linear σ-models in the framework of noncommutative geometry with special emphasis on models defined on the noncommutative torus. We choose as target spaces the two point space and the circle and illustrate some characteristic features of the corresponding σ-models. In particular we construct a σ-model instanton with topological charge equal to 1. We also define and investigate some properties of a noncommutative analogue of the Wess-Zumino-Witten model.

1 Introduction

One could say that the aim of noncommutative geometry is to generalize geometrical tools to "spaces whose coordinates fail to commute" [6], (see also [17], [16], [26]). One way to implement this program is to start with a given geometrical theory involving sets X endowed with additional structures and formulate them algebraically by using suitable subalgebras of the algebra of complex valued functions over X. Then one extends parts of the theory to noncommutative algebras, which are thought of as functions over "noncommutative spaces". Although much of the construction takes place at the algebraic level, it is necessary, in order to use the powerful machinery of functional analysis, to represent these algebras as operators on a Hilbert space. Accordingly, this can be seen from a physicist's point of view, as analogous to quantum mechanics: one trades the commutative algebra of functions over phase space for a noncommutative algebra of operators acting on a Hilbert space. Most of the geometrical ideas of classical mathematics can be "quantized". For example, topology can be formulated in terms of C*-algebras, commutative C*-algebras corresponding to locally compact spaces. Thus, noncommutative ones are referred to as "continuous functions over noncommutative locally compact spaces". In the compact case (i.e. when the algebra is unital), one can further define noncommutative vector bundles as finitely generated projective modules over a given unital C*-algebra which plays the role of functions over the base space. When this algebra is commutative, Serre-Swan's theorem asserts that these modules correspond to module of sections of vector bundles. Furthermore, methods of differential topology are also available within the realm of cyclic and Hochschild cohomology and this leads, via the coupling of the former with K-theory, to quantities that are stable under deformation and that generalize topological invariants, like, for instance, winding numbers and topological charges.

Noncommutative geometry has already proved to be useful in understanding various physical phenomena, like the integral quantum Hall effect [3] or the classical aspect of the Higgs sector of the standard model (see [22] for a review). Recent developments [8] and [23] also indicate that it is helpful in string theory. These last developments involve Yang-Mills fields defined on noncommutative spaces that fit into a broad formalism for gauge fields in noncommutative geometry which allows one to define connections, their curvature or the associated Yang-Mills action while preserving most of their classical aspects. For instance, one can prove a topological bound for the Yang-Mills action in dimension 4 [6]. Also, one can construct a Chern-Simons type theory and interpret its behavior under large gauge transformations as a coupling between cyclic cohomology and K-theory [13].

In this report, we will be interested in constructing analogues of two dimensional nonlinear σ-models within the noncommutative world. Since these models usually exhibit a very rich and easily accessible geometrical structure, we expect their noncommutative counterparts to be an ideal playground for a probe into the interplay between noncommutative geometry and field theory. This we shall try to exemplify by means of three different models: a continuous analogue of the Ising model which admits instantonic solutions, the analogue of the principal chiral model together with its infinite number of conserved currents and the noncommutative Wess-Zumino-Witten model together with its modified conformal invariance.

All ideas will be presented in a rather sketchy form and we refer to [10] (for fields with values in finite spaces) and [11] (for S^1-valued fields) for a detailed account.

2 General Aspects

In ordinary field theory, non-linear σ-models (see [27] for a review) are field theories whose configuration space consists of maps X from a Riemannian manifold Σ with metric g, which we assume to be compact and orientable, to an other Riemannian manifold $\mathcal{M}$ whose metric we denoted by G. In the physics literature, these manifolds are called source and target space respectively. By using local coordinates, the action functional is defined as

$$S[X] = \frac{1}{2\pi} \int_\Sigma \sqrt{g}\, g^{\mu\nu}\, G_{ij}(X) \partial_\mu X^i\, \partial_\nu X^j, \tag{1}$$

where as usual $g = \det g_{\mu\nu}$ and $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. When Σ is two dimensional, the action S is conformally invariant, since a rescaling of the metric $g \rightarrow ge^\sigma$, with σ being any map from Σ to $\mathbb{R}$, leaves it invariant. Accordingly, the action only depends on the conformal class of the metric and may be rewritten using a complex structure on Σ as

$$S[X] = \frac{i}{\pi} \int_\Sigma G_{ij}(X)\, \partial X^i \wedge \overline{\partial} X^j \tag{2}$$

where $\partial = \partial_z dz$ and $\overline{\partial} = \partial_{\overline{z}} d\overline{z}$, z being a suitable local complex coordinate.

Different choices of the source and target spaces lead to different field theories, some of them playing a major role in physics. Especially interesting are their applications to statistical field theory, and (supplemented by some supersymmetries) they are the basic building blocks of superstring theories.

From the mathematical point of view, the stationary points of the action functional (1) are harmonic maps from Σ to $\mathcal{M}$ and describe the extremal surfaces embedded in $\mathcal{M}$. Thus, a noncommutative generalization of the action functional of the non-linear σ-model should yield, as stationary points, noncommutative analogues of harmonic maps.

To generalize such a construction to the noncommutative case, we must dualize the previous picture and reformulate it in terms of the $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ of complex valued smooth functions defined respectively on Σ and $\mathcal{M}$. Embeddings X of Σ into $\mathcal{M}$ are in one to one correspondence with $*$-algebra morphisms π from $\mathcal{B}$ to $\mathcal{A}$, the correspondence being simply $\pi_X(f) = f \circ X$. Since this makes perfectly sense in the noncommutative case, we define our configuration space, for fixed, not necessarily commutative algebras $\mathcal{A}$ and $\mathcal{B}$, as the space of all $*$-algebra morphisms from $\mathcal{B}$ to $\mathcal{A}$.

The construction of the action functional is more tricky since it involves noncommutative generalizations of the conformal and Riemannian geometries. Following an idea of Connes [6] and [7], the former can be understood within the framework of positive Hochschild cohomology. Without entering into details, one observes that, in the commutative case, the trilinear map on $\mathcal{A}^3$ defined by

$$\phi(f_0, f_1, f_2) = \frac{i}{\pi} \int_\Sigma f_0 \partial f_1 \wedge \overline{\partial} f_2 \tag{3}$$

is an extremal element of the space of positive Hochschild cocycles that belongs to the cohomology class of the cyclic cocycle ψ defined by

$$\psi(f_0, f_1, f_2) = \frac{i}{2\pi} \int_\Sigma f_0 df_1 \wedge df_2. \tag{4}$$

Again, we refer to [6] for the general definitions and we simply notice that (3)-(4) still make sense for a general noncommutative algebra $\mathcal{A}$.

Roughly speaking, one can say that ψ allows to integrate 2-forms in dimension 2,

$$\frac{i}{2\pi}\int a_0 da_1 da_2 = \psi(a_0, a_1, a_2) \tag{5}$$

so that it is a metric independent object, whereas ϕ defines a suitable scalar product

$$\langle a_0 da_1, b_0 db_1\rangle = \phi(b_0^* a_0, a_1, b_1^*) \tag{6}$$

on the space of 1-forms and thus depends on the conformal class of the metric. Furthermore, this scalar product is positive and invariant with respect to the action of the unitary elements of $\mathcal{A}$ on 1-forms, and its relation to the cyclic cocylic ψ allows to prove various inequalities involving topological quantities.

Having such a cocycle ϕ, it is natural to compose it with a morphism $\pi:\ \mathcal{B}\to\mathcal{A}$ in order to obtain a positive cocycle on $\mathcal{B}$ defined by $\phi_\pi = \phi\circ(\pi\otimes\pi\otimes\pi)$. Since our goal is to build an action functional, which assigns a number to any morphism π, we have to evaluate the previous cocycle on a suitably chosen element of $\mathcal{B}^{\otimes 3}$. Such an element is provided by the noncommutative analogue of the metric on the target, which we take simply as a positive element $G = \sum_i b_0^i \delta b_1^i \delta b_2^i$ of the space of universal 2-forms $\Omega^2(\mathcal{B})$. Thus

$$S[\pi] = \phi_\pi(G) \tag{7}$$

is well defined and positive and we take it as a noncommutative analogue of the action functional of the non linear σ-model. Of course we consider π as the dynamical variable (the embedding) whereas ϕ (the conformal structure on the source) and G (the metric on the target) are background structures that have been fixed.

As an alternative, one could consider that only the metric G on the target is a background field, since the morphism $\pi:\ \mathcal{B}\to\mathcal{A}$ allows to define the induced metric $\pi_* G$ on the source as

$$\pi_* G = \sum_i \pi(b_0^i)\delta\pi(b_1^i)\delta\pi(b_2^i), \tag{8}$$

which is obviously a positive universal 2-form on $\mathcal{A}$. To such an object one can associate, by means of a variational problem (see [6] and [7]), a positive Hochschild cocycle that stands for the conformal class of the induced metric. As a result, the critical points of the corresponding σ-model describe "minimally embedded surfaces" in the noncommutative space associated with $\mathcal{B}$.

A scrupulous reader may be puzzled by such a formal and sketchy construction. However, in what follows we will mainly work out examples involving the noncommutative torus and only consider a fixed ϕ and fixed metrics on the two target space we will consider (the circle and the two point space). Accordingly, ϕ and G could be replaced by their explicit expressions. Nevertheless, we think that it may be useful to have a general setting. In particular, one easily reconstructs ordinary σ-models with suitable choices of ϕ and G.

3 Two points as a target space

3.1 A General Construction

The simplest example of a target space one can think of is that of a finite space made of two points, like in the Ising model. Of course, any continuous map from a connected surface to a discrete space is constant and the resulting (commutative) theory would be trivial. However, this is no longer true if the source is a noncommutative space and one has, in general, lots of such maps (i.e. algebra morphisms).

Let us first notice that the algebra $\mathcal{B} = \mathbb{C}^2$ of functions over a two point space is the unital algebra generated by a hermitian projection $e, e^2 = e^* = e$. Thus, any $*$-algebra morphism π from $\mathcal{B}$ to $\mathcal{A}$ is given by a hermitian projection $p = \pi(e)$ in $\mathcal{A}$. Choosing the metric $G = \delta e \delta e$ on the space of two points, the action functional simply reads

$$S[p] = \phi(1, p, p), \tag{9}$$

where ϕ is a given Hochschild cocycle standing for the conformal structure. Of course, one could choose other metrics on the two points space, but G is more interesting since it will lead to a topological bound for the corresponding action. We shall prove this fact for the noncommutative torus, but the procedure is general and only uses the idea of positivity in Hochschild cohomology.

3.2 The noncommutative two torus as a source space

For the sake of completeness we recall the very basic aspects of the noncommutative torus and refer the reader to [21] for a thorough survey. The algebra $\mathcal{A}_\theta$ of smooth functions on the noncommutative torus is the unital $*$-algebra made of power series of the form

$$a = \sum_{m,n \in \mathbb{Z}^2} a_{mn} \; U_1^m U_2^n \; , \tag{10}$$

with a_{mn} a complex-valued Schwarz function on $\mathbb{Z}^2$ that is, the sequence of complex numbers $\{a_{mn} \in \mathbb{C} \; , \; (m, n) \in \mathbb{Z}^2\}$ decreases rapidly at 'infinity'. The two unitary elements U_1, U_2 have commutation relations

$$U_2 \; U_1 = e^{2\pi i \theta} U_1 \; U_2 \; . \tag{11}$$

On $\mathcal{A}_\theta$ there is a unique normalized positive definite trace which we shall unusually denote by an integral symbol $\int : \mathcal{A}_\theta \to \mathbb{C}$ and which is given by

$$\int \Big(\sum_{(m,n) \in \mathbb{Z}^2} a_{mn} \; U_1^m U_2^n \Big) =: a_{00} \; . \tag{12}$$

This trace is invariant under the action of the commutative torus T^2 on $\mathcal{A}_\theta$ whose infinitesimal form is determined by two commuting derivations ∂_1, ∂_2 acting by

$$\partial_\mu(U_\nu) = 2\pi i \; \delta_\mu^\nu U_\nu \; , \quad \mu, \nu = 1, 2 \; . \tag{13}$$

The invariance just being the statement that $\int \partial_\mu(a) = 0$, $\mu = 1, 2$ for any $a \in \mathcal{A}_\theta$.

All the previous properties, even if elementary, turn out to be important for our construction since they allow us to use all tools of elementary calculus on a commutative torus. However one must bear in mind that, in order to develop a more general setting, one should work only with the corresponding cyclic and Hochschild cocycles that we shall now describe.

The cyclic 2-cocycle associated to the integration of 2-forms is simply given by

$$\psi(a_0, a_1, a_2) = \frac{i}{2\pi} \int \epsilon_{\mu\nu} a_0 \partial_\mu a_1 \partial_\nu a_2, \tag{14}$$

where $\epsilon_{\mu\nu}$ is the standard antisymmetric tensor. Its normalization ensures that for any hermitian projector $p \in \mathcal{A}_\theta$, the quantity $\psi(p, p, p)$ is an integer: it is indeed the index of a Fredholm operator.

Working with the standard Euclidean metric on the torus, the Hochschild cocycle ϕ is

$$\phi(a_0, a_1, a_2) = \frac{2}{\pi} \int a_0 \partial a_1 \overline{\partial} a_2 \tag{15}$$

where the complex derivations $\partial = 1/2(\partial_1 - i\partial_2)$ and $\overline{\partial} = 1/2(\partial_1 + i\partial_2)$ are combination of the previous derivations. Note that we consider ∂ and $\overline{\partial}$ as maps with values in $\mathcal{A}_\theta$ and not in the bimodule of 1-forms. A construction of the cocycle (15) as the conformal class of the Euclidean metric on the torus can be found in [6] and [7]. We remark that one can also work with a general constant metric whose conformal class is parametrized by a complex number τ that belongs to the upper half plane.

Accordingly, the action functional for our non-linear σ-model reads

$$\phi(1, p, p) = \frac{1}{2\pi} \int \partial_\mu p \partial_\mu p = \frac{1}{\pi} \int p \partial_\mu p \partial_\mu p \ , \tag{16}$$

the contraction with the Euclidean metric being understood.

As a subset of a topological vector space, the space $\mathcal{P}_\theta$ of all hermitian projectors of $\mathcal{A}_\theta$ comes equipped with a natural topology (in fact it is an infinite dimensional manifold) and we are interested in the study of the critical points of the action in a given connected component of $\mathcal{P}_\theta$. By carefully taking into account the non linear structure of the space $\mathcal{P}_\theta$, we get the field equations

$$p \ \Delta(p) \ - \ \Delta(p) \ p = 0 \ . \tag{17}$$

where $\Delta = \partial_\mu \partial_\mu$ is the laplacian.

The previous equation is a non linear equation of the second order and it is rather difficult to explicit its solutions in closed form. Following a standard route, we shall show that the absolute minima of (3.2) in a given connected component of $\mathcal{P}_\theta$ actually fulfill a first order equation which is easily solved.

Given a projector $p \in \mathcal{P}_\theta$, there is a 'topological charge' (the first Chern number) defined by [5]

$$Q(p) =: \frac{1}{2\pi i} \int p \Big[\partial_1(p) \partial_2(p) - \partial_2(p) \partial_1(p) \Big] \ \in \ \mathbb{Z} \ . \tag{18}$$

As in four dimensional Yang-Mills theory, this topological quantity yields a bound for the action functional.

Due to positivity of the trace $\int$ and its cyclic property, we have

$$\int \Big[\partial_\mu(p)\; p \pm i\epsilon_{\mu\alpha}\partial_\alpha(p)\; p\Big]^* \Big[\partial_\mu(p)\; p \pm i\epsilon_{\mu\beta}\partial_\beta(p)\; p\Big] \geq 0 \;, \tag{19}$$

from which we obtain the inequality

$$S(p) \geq \pm 2Q(p) \;. \tag{20}$$

The inequality (20), which gives a lower bound for the action, is the analogue of the one for ordinary σ-models [2]. Also, it is a two dimensional analogue of the inequality that occurs in four dimensional Yang-Mills theory. A similar bound for a model on the fuzzy sphere has been obtained in [1].

From (19) it is clear that the equality in (20) occurs exactly when the projector p satisfies the following *self-duality* or *anti-self duality* equations

$$\Big[\partial_\mu p \pm i\epsilon_{\mu\alpha}\partial_\alpha p\Big]\; p = 0 \;. \tag{21}$$

By using the derivations $\partial, \bar{\partial}$, the self duality equation (21) reduce to

$$\bar{\partial}(p)\; p = 0 \;, \tag{22}$$

while the anti-self duality one is

$$\partial(p)\; p = 0 \;. \tag{23}$$

Simple manipulations show directly that each of the equations (22) and (23) implies the field equations (17), as it should be.

In the next section, we will partially solve these equations.

3.3 The instantons of charge 1.

Before we proceed further, let us be more precise about the connected components of $\mathcal{P}_\theta$ [20]. The latter are parametrized by two integers m and n such that $m + n\theta > 0$. When $\theta \in]0, 1[$ is irrational, the corresponding projectors are exactly the projectors of trace $m + n\theta$ and the topological charge $Q(p)$ appearing in (18) is just n. We shall construct our solutions for $m = 0$ and $n = 1$ and postpone the general discussion to [10]. Thus we have to find projectors that belongs to the previous homotopy class and satisfy the self-duality equation $(\overline{\partial}p)p = 0$ or, equivalently, $p\partial p = 0$.

Although these equations look very simple, they are far from being easy to solve because of their non linear nature. To reduce them to a linear problem, we shall introduce the following material and mimic the original construction of Rieffel [19] but with the constraint arising from the self-duality equation.

The space $\mathcal{E} = \mathcal{S}(\mathbb{R})$ of Schwarz functions of one variable is made into a *right* module over $\mathcal{A}_{-1/\theta}$ by defining

$$\begin{aligned}(\xi\; V_1)(s) &=: \xi(s - 1/\theta) \;, \\ (\xi\; V_2)(s) &=: e^{2\pi i s}\xi(s) \;,\end{aligned} \tag{24}$$

for any $\xi \in \mathcal{E}$. It is easily checked that this defines an action on the right of the algebra generated by V_1 and V_2 and that the latter is isomorphic to $\mathcal{A}_{-1/\theta}$.

Furthermore, $\mathcal{E}$ admits also a left action of $\mathcal{A}_\theta$ given by

$$
\begin{aligned}
(U_1\xi)(s) &=: \xi(s-1) \ , \\
(U_2\xi)(s) &=: e^{2\pi i s\theta}\xi(s) \ .
\end{aligned} \tag{25}
$$

and one easily proves that the latter commutes with the right action of $\mathcal{A}_{-1/\theta}$. Besides, the elements of $\mathcal{A}_\theta$ acting on the left are exactly all linear operators from $\mathcal{E}$ to itself that commute with the right action of $\mathcal{A}_{-1/\theta}$, namely $\mathcal{A}_\theta \simeq \mathrm{End}_{\mathcal{A}_{-1/\theta}}(\mathcal{E})$.

On the module $\mathcal{E}$ there is also a $\mathcal{A}_{-1/\theta}$-valued hermitian structure, namely a sesquilinear map (antilinear in the first variable) $\langle \ , \ \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}_{-1/\theta}$ which is compatible with the right $\mathcal{A}_{-1/\theta}$-module structure of $\mathcal{E}$ (see [6] for explicit formulae). As a consequence, if $\xi \in \mathcal{E}$ is such that $\langle \xi, \xi \rangle$ is an invertible element of $\mathcal{A}_{-1/\theta}$, the endomorphism

$$
p = |\xi\rangle \, \frac{1}{\langle \xi, \xi \rangle} \, \langle \xi| \tag{26}
$$

is a self-adjoint idempotent (that is a projector) in the algebra $\mathcal{A}_\theta$ (due to the identification $\mathcal{A}_\theta \simeq \mathrm{End}_{\mathcal{A}_{-1/\theta}}(\mathcal{E})$). Here we are using a physicist's notation for an element $|\xi\rangle \in \mathcal{E}$ and the dual element $\langle \xi| \in \mathcal{E}^*$ is defined by means of the hermitian structure as $\langle \xi| \, (\eta) = \langle \xi, \eta \rangle \in \mathcal{A}_{-1/\theta}$ for any $\eta \in \mathcal{E}$.

In order to translate the self-duality equations for p to equations for ξ, we need to introduce a connection on $\mathcal{E}$. This is done [9] by means of two covariant derivatives explicitly given by $\nabla_1, \nabla_2 : \mathcal{E} \to \mathcal{E}$,

$$
(\nabla_1\xi)(s) =: 2\pi i\theta s \; \xi(s) \ , \quad \nabla_2\xi =: \frac{d\xi}{ds} \ , \tag{27}
$$

These two operators fulfill a Leibniz rule with respect to the right action

$$
\nabla_\mu(\xi a) = (\nabla_\mu\xi)a + \xi(\partial_\mu a), \quad \mu = 1, 2 \ . \tag{28}
$$

for any $\xi \in \mathcal{E}$ and $a \in \mathcal{A}_{-1/\theta}$ and $b \in \mathcal{A}_\theta$. Furthermore, they are compatible with the hermitian structure in the sense that

$$
\partial_\mu \langle \xi, \eta \rangle = \langle \nabla_\mu \xi, \eta \rangle + \langle \xi, \nabla_\mu \eta \rangle \ , \quad \mu = 1, 2 \ , \tag{29}
$$

for any $\xi, \eta \in \mathcal{E}$.

By introducing the operator $\overline{\nabla} = 1/2(\nabla_1 + i\nabla_2)$, it is easy to show that p satisfies the self-dual equations (22) if and only if there is an element $\rho \in \mathcal{A}_{-1/\theta}$ such that

$$
\overline{\nabla}\xi = \xi\rho. \tag{30}
$$

Thus, we manage to reduce the self-duality equation to a linear equation for ξ that can be easily solved in some simple cases.

When ρ is a constant element (i.e. it is proportional to the unit of $\mathcal{A}_{-1/\theta}$, $\rho = \lambda\,1$, with $\lambda \in \mathbb{C}$), equation (30) reduces to the simple differential equation

$$\frac{d\xi}{dt} + (2\pi\theta t + 2i\lambda)\,\xi = 0 \tag{31}$$

whose solutions are the gaussians

$$\xi_\lambda(t) = A\, e^{-\pi\theta\, t^2 - 2i\lambda\, t}, \tag{32}$$

and $A \in \mathbb{C}^*$ is an inessential normalization parameter.

We will show in [10] that, at least for θ small enough, the norms $\langle \xi_\lambda, \xi_\lambda \rangle$ are invertible. Accordingly, the gaussians (32) provide a two (real) parameter family of solutions $p_\lambda = |\xi_\lambda\rangle\,(\langle \xi_\lambda, \xi_\lambda \rangle)^{-1}\,\langle \xi_\lambda|$ of the self-duality equations (22), and one can show that the freedom we have in λ just corresponds to the action of the ordinary torus on $\mathcal{A}_\theta$ by translation. Thus, we interpret the solution as a two dimensional "instanton" in this simple " noncommutative Ising model" (remember that the target is just made of two points) and the freedom in λ in a sense corresponds to its location.

However, it is not obvious that different solutions of the self duality equations on $\mathcal{E}$, yield different projectors. In fact, ξ and $\xi^{'}$ provide different projectors if and only if they belong to different orbits of the action of the group of invertible elements of $\mathcal{A}_{-1/\theta}$ that acts on the right on $\mathcal{E}$. Obviously, this action preserves the invertibility of $\langle \xi, \xi \rangle$ while the structure of the self-duality equation (30) is preserved provided ρ is modified according to

$$\rho \to g^{-1}\rho g + g^{-1}\overline{\partial} g. \tag{33}$$

In a more physical language, this means that in trading p for ξ we have introduced spurious gauge degrees of freedom that we must get rid of. In the case of the Gaussians ξ_λ, it is easy to show that ξ_λ and $\xi_{\lambda'}$ are gauge equivalent if and only if $\xi_{\lambda'} = \xi_\lambda U_1^{n_1} U_2^{n_2}$ where n_1 and n_2 are integers. More generally, given a solution ξ of the self-dual equation

$$\overline{\nabla}\xi - \xi\rho = 0 \tag{34}$$

with $\rho \in \mathcal{A}_{-1/\theta}$, it is not clear that we can find a complex gauge transformation g (i.e. an invertible element of $\mathcal{A}_{-1/\theta}$) that allows to gauge transform ξ into one of the gaussians. If this were the case, it would mean that we had indeed constructed all self-dual solutions belonging to the corresponding homotopy class. This problem is tantamount to solve the following equation in g and λ,

$$\rho = \lambda + g\overline{\partial} g^{-1}. \tag{35}$$

Again, this can be done if θ is small enough [10]. The corresponding idea is simple: we first notice that the problem is trivial when $\theta = 1/n$, with $n \in \mathbb{N}^*$ because $\mathcal{A}_{-1/\theta}$ is commutative in this case. Indeed, the existence of the gauge transformation results from the Hodge decomposition of 1-forms. Then, we use the implicit function theorem in order to find how to deform the commutative solution, considered as functions of θ [10].

A few additional remarks are in order. Even if many of the methods we have used are similar to the ones used in the CP^N model rather than to the ones pertaining to the Ising model, we refrain from calling these "noncommutative CP^N models" since we want to emphasize the fact that our target space in made of two points and is not the manifold CP^N (or more general grassmanian manifolds). But obviously the ordinary grassmanian models can also be considered as "noncommutative Ising models" with a source described by matrix valued functions over an ordinary Riemann surface.

It is also worth remarking that we have been working with the Euclidean metric, but all constructions are readily extended to constant metrics whose conformal class are parametrized by a complex number τ in the upper half-plane. Then, the corresponding moduli space turns out to be a complex torus.

4 An analogue of the principal chiral model

Apart from finite spaces, the simplest possible target spaces are circles. Ordinary two dimensional field theories compactified on a circle have been extensively studied (see for instance [14]) and they essentially behave like free fields (with minor deviations). As we shall show in our next example, this is not the case for noncommutative models, the interaction arising from the noncommutative nature of the source.

To proceed, let us first recall that the algebra of function over the circle S^1 is generated by a unitary element U. Thus, specifying a $*$-algebra morphism π from the algebra of functions on the circle to a another $*$-algebra $\mathcal{A}$ is tantamount to select a unitary element $g = \pi(U)$ in $\mathcal{A}$. Accordingly, our configuration space is made of all unitary elements of $\mathcal{A}$. For the metric on the circle we shall take the most natural one, $G = \delta U \delta U^*$, while for the target space we take the Euclidean noncommutative torus, extension to other constant metrics being straightforward. Then, the Hochschild cocycle is the one in (15) and our action functional simply reads $S[g] = \phi(1, g, g^{-1})$, which reduces to

$$S[g] = \frac{1}{2\pi} \int \partial_\mu g \partial_\mu g^{-1} \ , \tag{36}$$

the variables being unitary elements in the algebra of the noncommutative torus $\mathcal{A}_\theta$.

Our model is analogous to a principal chiral model, with values in a unitary group of matrices with which it shares lots of properties, apart from non-locality. For the time being we shall limit our study to the existence of infinitely many conserved currents.

From the action functional (36), one readily obtain the equations of motion by varying $g \to g + \delta g$. As in the commutative case, they are equivalent to a current conservation

$$\partial_\mu \left(g^{-1} \partial_\mu g \right) = 0, \tag{37}$$

which expresses the invariance under the global $U(1)$ symmetry.

To construct infinitely many such currents, we use a standard induction that relies on the Hodge decomposition of differential forms. Since the latter reduce to a simple problem in linear algebra on the noncommutative torus we shall use it without further discussion and write any differential form as a unique sum of a harmonic one (i.e. a constant one), an exact one and a coexact one.

Let us assume that we have constructed the conserved current $J_\mu^{(n)}$ and let us build $J_\mu^{(n+1)}$. Since $J_\mu^{(n)}$ is conserved, the Hodge decomposition just tells that it is a sum of a constant form and a co-exact one. After an incorporation of the possible constant term into $J_\mu^{(n)}$, one can find $\chi \in \mathcal{A}_\theta$ such that

$$J_\mu^{(n)} = \epsilon_{\mu\nu}\partial_\nu \chi, \tag{38}$$

where $\epsilon_{\mu\nu}$ is the standard antisymmetric tensor. Then, let us introduce the gauge field $A_\mu = g^{-1}\partial_\mu g$ and the covariant derivative $D_\mu = \partial_\mu + A_\mu$. We define the next current as

$$J_\mu^{(n+1)} = D_\mu \chi \ . \tag{39}$$

It is easy to check that it is conserved, owing to the easily verified commutation rules $[\partial_\mu, D_\mu] = 0$ and $[D_\mu, D_\nu] = 0$. Starting with $J_\mu^{(1)} = g^{-1}\partial_\mu g$, by repeating the construction we can construct an infinite number of non local conserved currents.

Of course the series of new currents would stop whenever there appears a constant current. With some more work one can show that this does not happen unless one starts with a trivial solution of the equations of motion which is a product of the generators. One could also object that non trivial solutions of the equations of motion may not exist. Again this is not the case, since one can take $g = 2p - 1$, where p is one of the instantonic solutions we constructed in the previous section.

All previous construction is very elementary and follows directly from the ordinary field theoretical construction of the currents. The only point we want to emphasize is that the latter still works in noncommutative geometry. A more thorough survey of our theory along the classical lines [25] will be given in [11], including a generalization of unitons.

5 Addition of the Wess-Zumino term

Although the previous considerations are purely classical, the models can be quantized. This amounts to define and compute the partition function

$$\mathcal{Z} = \int [\mathcal{D}g] e^{-S[g]} \ , \tag{40}$$

together with the correlation functions

$$\langle g \otimes g \otimes \cdots \otimes g \rangle = \frac{1}{\mathcal{Z}} \int [\mathcal{D}g] (g \otimes g \otimes \cdots \otimes g) e^{-S[g]}, \tag{41}$$

as well as possible insertions of composite operators. Of course, none of these functional integrals are well defined and to give a precise meaning to them, one has to set up the renormalization procedure which yields some non trivial problems even in dimension two, these models being power counting renormalizable only in that dimension.

However, as far as the one-loop level is considered, this is easily achieved within the background field method. As its non-abelian cousins, our model exhibits a negative β function so that one may say that it is asymptotically free. Accordingly, one can definitely exclude the possibility of having a free field theory.

We shall see that, after addition of the so called "Wess-Zumino term", the model behaves almost like a free field (see, for instance, [15] and [24] for recent pedagogical reviews of the ordinary WZW model). Once again, we will only be sketchy because of lack of space, and refer to [11] for a detailed account.

To construct the Wess-Zumino term, let us start with a given unitary element g of $\mathcal{A}_\theta$. It is known from K-theory [20] that there always exist a curve g_t , $t \in [0,1]$ in the group of unitary elements of $\mathcal{A}_\theta$ that fulfills $g_1 = g$ and $g_0 = (U_1)^{n_1}(U_2)^{n_2}$, where (n_1, n_2) denotes the class of g in $\mathrm{K}_1(\mathcal{A}_\theta)$. Therefore, we can define the Wess-Zumino term as

$$S_{WZ}[g] = \frac{ik}{4\pi}\epsilon^{\mu\nu}\int_0^1 dt \int g_t^{-1}\frac{dg_t}{dt}\partial_\mu g_t^{-1}\partial_\nu g_t, \tag{42}$$

where k is an *a priori* arbitrary real number. As in the classical case, this term can be expressed as the integral over a solid noncommutative torus, but the latter depends on the class of g in K-theory, different classes yielding isomorphic and cobordant solid tori.

Although the model (42) depends on the curve g_t and not only on g one can show that, given any other curve $\tilde{g}_t$ connecting the same boundaries, the difference of the two Wess-Zumino terms can be expressed as an integral over a loop in the group of unitary elements of $\mathcal{A}_\theta$. Such a quantity may be easily identified with a coupling of a 3-cyclic cocycle with a unitary element of $C^\infty(S^1) \otimes \mathcal{A}_\theta$ and it can be shown to be proportional to an integer, as follows from a straightforward application of the index theorem (see [12] for a very elementary treatment). It turns out that if $k \in \mathbb{Z}$, the Wess-Zumino term is defined up to integral multiples of $2i\pi$.

Accordingly, we construct the Wess-Zumino-Witten action just by adding the previous term to the non-linear σ-model and we get,

$$S_{WZW}[g] = \frac{k}{8\pi}\int \partial_\mu g \partial_\mu g^{-1} + \frac{ik}{4\pi}\epsilon^{\mu\nu}\int_0^1 dt \int g_t^{-1}\frac{dg_t}{dt}\partial_\mu g_t^{-1}\partial_\nu g_t, \tag{43}$$

for positive k.

By introducing the usual operators ∂ and $\overline{\partial}$, algebraic manipulations involving integrations by part show that a Polyakov-Wiegman identity holds, namely

$$S_{WZ}[gh] = S_{WZ}[g] + S_{WZ}[h] + \frac{1}{4i\pi}\int g^{-1}\overline{\partial} g\, h\partial h^{-1}. \tag{44}$$

When $h = 1 + g^{-1}\delta g$, this identity allows one to write the variation of $S_{WZW}[g]$ as

$$\delta S_{WZW}[g] = -\frac{k}{2i\pi}\int g^{-1}\delta g \partial\left(g^{-1}\overline{\partial} g\right). \tag{45}$$

Then, the equations of motion can be written equivalently as

$$\partial \overline{J} = 0\ , \quad \text{or} \quad \overline{\partial} J = 0\ , \tag{46}$$

with $\overline{J} = g^{-1}\overline{\partial} g$ and $J = g\partial g^{-1}$.

One readily sees that there are very few solutions of the previous equation, since any holomorphic function on the noncommutative torus, defined as an element of the algebra in the kernel of $\overline{\partial}$, is constant.

In order to get non trivial solutions, we equip the torus with the Minkowski metric. Then the equations of motion are

$$\partial_+ \left(g^{-1}\partial_- g\right) = 0, \tag{47}$$

with $\partial_\pm = \partial_1 \mp \partial_2$. Apart from products of the generators U_1 and U_2, we will show that the general solution of equations (47) can be factorized as

$$g = g_+ g_-, \tag{48}$$

where $g_\pm$ are unitary elements of $\mathcal{A}_\theta$ satisfying the equations $\partial_\mp g_\pm = 0$.

To proceed, let us first assume that g belongs to the connected component of the identity. If this is not the case, we multiply it by a suitable product of the generators, given by the class of g in K-theory (the result is still a solution of the equation of motion). Now, it follows from the equation (47) that $J_- = g^{-1}\partial_- g$ belongs to the kernel of ∂_+ so that it can be expanded as a Laurent series in $U_1 U_2^{-1}$. Besides, since we are assuming that g belongs to the connected component of the identity, the constant mode of the expansion vanishes, since it is invariant under deformation of g. Therefore, the primitive

$$\int_- J_- \tag{49}$$

is well defined (one simply has to divide the coefficient in front of any monomial by the corresponding non vanishing power). As a consequence, the solutions of the remaining equation

$$\partial_- g = g J_- \tag{50}$$

are easily expressed as

$$g = g_+ e^{\int_- J_-} \tag{51}$$

with g_+ an arbitrary unitary element of the algebra of Laurent series in $U_1 U_2$. Then the factorization (47) is achieved by taking $g_- = e^{\int_- J_-}$. Thus, the unitary elements $g_\pm$ can be expanded as

$$g_\pm = \sum_{n\in\mathbb{Z}} g_\pm^{(n)} \left(U_1 U_2^{\pm 1}\right)^n, \tag{52}$$

and both can be interpreted as maps from circles $S^1_\pm$, which are the spaces of characters of the commutative algebras generated by $U_1 U_2^{\pm 1}$, to $U(1)$. Note however that the coordinates on S^1_+ do not commute with the ones on S^1_-.

Although this model almost looks like a free field theory, with commutative left and right movers, the standard parity symmetry that exchanges left and right has been broken and the theory, due to noncommutativity, always remembers that left movers must appear on the left. Alternatively, this may be understood as a lack of invariance of the Wess-Zumino term under the inversion $g \to g^{-1}$, while the kinetic term obviously enjoys this symmetry. From the strict point of view of solving the equation of motion, this is the only remainder of the noncommutative nature of the source space.

Obviously, the space of solutions of the equations of motion is invariant under gauge symmetry (respective multiplication on the left and the right by left and right moving unitaries) and under conformal symmetry (reparametrisation of $S^\pm$). However, general conformal transformations are not symmetries of the noncommutative torus in the sense that they do not correspond to automorphisms of the algebra $\mathcal{A}_\theta$; it is only the translations that can be lifted to automorphisms. Therefore, there is no *a priori* satisfactory way to define conformal transformations of g when it is not a solution of the equations of motion. Note that gauge transformation do not create any trouble since they correspond to inner automorphisms of the algebra.

Fortunately, one can construct analogues of conformal transformations that do leave the action invariant and reduce both on-shell and in the commutative case to ordinary conformal transformations. To proceed, let us introduce the (not necessarily conserved) left and right currents $J_\pm$, analogous of J and $\overline{J}$. Furthermore, let us introduce infinitesimal multiplets $\epsilon_\pm = (\epsilon_\pm^{(1)}, \ldots, \epsilon_\pm^{(n)})$ of left and right moving elements of $\mathcal{A}_\theta$. then, we define the infinitesimal transformations $\delta_{\epsilon_\pm}(g)$ as

$$\delta_{\epsilon_-}(g) = g \left(\sum_{\text{permutations}} \epsilon_-^{(i_1)} J_- \epsilon_-^{(i_2)} J_- \ldots \epsilon_-^{(i_{n-1})} J_- \epsilon_-^{(i_n)} \right) \tag{53}$$

and

$$\delta_{\epsilon_+}(g) = \left(\sum_{\text{permutations}} \epsilon_+^{(i_1)} J_+ \epsilon_+^{(i_2)} J_+ \ldots \epsilon_+^{(i_{n-1})} J_+ \epsilon_+^{(i_n)} \right) g \ , \tag{54}$$

with the sums running over all permutations of the indices $i_1, \ldots, i_n$.

One readily sees that, by replacing δg with $\delta_{\epsilon_\pm}(g)$ in the variation of the Wess-Zumino-Witten action, one gets the integral of a total derivative. Thus the variation vanishes even if g is not a solution of the equation of motion.

For the particular $n = 1$ case, the previous transformations reduce to gauge transformations. On shell or in the commutative case, the $n = 2$ transformations are just the conformal ones. The case $n > 2$ is more exotic, since two such transformations acting on the left and on the right do not in general commute. Furthermore, it is not clear whether these transformations close off-shell or not. Probably the closure of this algebra requires more general transformations. For instance, $[\delta_{\epsilon_+}, \delta_{\epsilon_-}]$ is a new symmetry that one has to introduce into the algebra. In the same vein, one also introduces transformations involving the derivatives of the currents. All these transformations are of the form

$$\delta_{\epsilon_-,\epsilon_+}(g) = g\, K_-(\epsilon_-, J_-) + K_+(\epsilon_+, J_+)\, g, \tag{55}$$

where $K_\pm(\epsilon_\pm, J_\pm)$ are suitable products of the currents, their derivatives and the corresponding parameters $\epsilon_\pm$.

Whether this procedure ends or not is not so clear since any computation is rather intricate due to the transformation of the currents themselves. One can also note that the $n > 2$ case will not yield symmetries of the ordinary $SU(N)$ non-abelian Wess-Zumino-Witten theory (apart from the $SU(2)$ case), since it does not preserve the unimodular condition of $SU(N)$, but they are bona fide transformations for fields with values in $U(N)$.

As a final remark, we mention that we have constructed the Wess-Zumino term associated with a particular cyclic cocycle of the noncommutative torus. But the procedure is general and given any $2n$-cyclic cocycle on an algebra $\mathcal{A}$ one can construct the associated

Wess-Zumino term in a similar way. Furthermore, the ambiguity in the definition will still be measured by the coupling of a $2n+1$-cyclic cocycle with an element of $K_1(C(S^1)\otimes\mathcal{A})$. This Wess-Zumino term may be added to the action of a principal chiral field, constructed with a Hochschild cocycle. In two dimensions, an analogue of the Polyakov-Wiegman identity still holds provided one uses a suitable scalar product in the LHS.

As a simple example, one can take the matrix algebra $\mathcal{A} = M_n(\mathbb{C})$ and the cyclic cocycle given by the trace. Then, for any unitary matrix g, it is easy to show that $S_{WZ}[g] = k \log\det g$. In this very simple case, the ambiguity in defining the Wess-Zumino term is nothing but the ambiguity one encounters when defining the argument of a complex number of modulus one, which is arbitrary up to $2i\pi\mathbb{Z}$. Furthermore, the Polyakov-Wiegman identity (we drop the kinetic term in this example) reduces to the statement that the argument of a product is the sum of the argument of its factors, up to $2i\pi\mathbb{Z}$.

We end here our sketchy discussion of the classical aspects of the noncommutative Wess-Zumino-Witten model. We are aware that many interesting questions have been left aside. In our opinion the main question is to understand how far from a free field theory our model stands.

Finally, we mention that actions analogous to ours have been obtained in [18] and [4].

References

[1] Baez S., A. P. Balachandran A.P., Vaidya S., Ydri B. *Monopoles and Solitons in Fuzzy Physics*, Commun. Math. Phys. **208** (2000) 787-798, `hep-th/9811169`.

[2] Belavin A.A., Polyakov A.M. *Metastable states of two-dimensional isotropic ferromagnets*, JETP Lett. **22** (1975) 245-247.

[3] Bellissard J. *K-theory of C^*-algebras in solid state physics*, Lecture notes in physics **237** (1986) 99-156.

[4] Chu C.-S. *Induced Chern-Simons and WZW action in Noncommutative Spacetime*, hep-th/0003007.

[5] Connes A. *C^*-algèbres et géométrie différentielle*, C.R. Acad. Sci. Paris Sér. A **290** (1980) 599-604.

[6] Connes A. *Noncommutative Geometry*, Academic Press, 1994.

[7] Connes A. *A short survey of noncommutative geometry*, `hep-th/0003006`.

[8] Connes A., Douglas M., Schwarz A. *Matrix theory compactification on tori*, J. High Energy Phys. **02** (1998) 003, `hep-th/9711162`.

[9] Connes A., Rieffel M. *Yang-Mills for Non-commutative Two-Tori*, in *Operator Algebras and Mathematical Physics*, Contemp. Math. **62** (1987) 237-266.

[10] Dabrowski L., Krajewski T., Landi G. *Non-linear σ models in noncommutative geometry: fields with values in finite spaces*, work in progress.

[11] Dabrowski L., Krajewski T., Landi G. *Non-linear σ models in noncommutative geometry: fields with values in S^1*, work in progress.

[12] Krajewski T. *Géométrie non commutative et interactions fondamentales*, PhD thesis, `math-ph/9903047`.

[13] Krajewski T. *Gauge invariance of the Chern-Simons action in noncommutative geometry*, ISI GUCCIA Conference "Quantum Groups, Noncommutative Geometry and Fundamental Physical Interactions", Palermo December 1997, `math-ph/9810015`.

[14] Gawedzki K. *Lectures on conformal field theory*, 1996-97 Quantum Field Theory Program at IAS (1997) available at `www.math.ias.edu/QFT`.

[15] Gawedzki K. *Conformal field theory: a case study*, `hep-th/9904145`.

[16] Landi G. *An introduction to noncommutative spaces and their geometries*, Lecture Notes in Physics, Springer-Verlag, 1997, `hep-th/97801078`.

[17] Madore J. *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, Cambridge University Press 2nd edition, 1999.

[18] Moreno E.F., Schaposnik F.A. *The Wess-Zumino-Witten term in noncommutative two-dimensional fermion model*, `hep-th/0002236`.

[19] Rieffel M. *C^*-algebras associated with irrational rotations*, Pacific J. Math. **93** (1981) 415-429.

[20] Rieffel M. *Projective modules over higher-dimensional noncommutative tori*, Can. J. Math. **40** (1988) 257-338.

[21] Rieffel M. *Non-commutative Tori -A case study of Non-commutative Differentiable Manifolds* Contemp. Math. **105** (1991) 191-211.

[22] Schücker T. *Geometries and forces*, Summer School "Noncommutative geometry and applications", Lisbon September 1997, `hep-th/9712095`.

[23] Seiberg N., Witten E. *String Theory and Noncommutative Geometry* JHEP **09** (1999) 032, `hep-th/9908142`.

[24] Shifman M. *ITEP lectures on particle physics and field theory*, Lecture notes in physics **62**, World Scientific, 1999.

[25] Uehlenbeck K. *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J. Differ. Geom. **30** (1989) 1-50.

[26] Várilly J. *An introduction to noncommutative geometry*, Summer School "Noncommutative geometry and applications", Lisbon September 1997, `physics/9709045`.

[27] Zakrzewski, W.J. *Low dimensional sigma models*, Adam Hilger, Bristol 1989.