SYMBOLIC CALCULUS AND COMPOSITION FOR PSEUDODIFFERENTIAL OPERATORS OF POSITIVE AND NON-ORTHOGONAL TYPE

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Abstract. We consider symbolic calculus and composition of h pseudodifferential operators. We define a general class of symbols and formulate natural conditions under which an h pseudodifferential operator (or h P.D.O. for short) may be represented in terms of other quantisations, and under which the composition of two h P.D.O. is an h P.D.O. The first condition corresponds, in some sense, to the h P.D.O. being of *positive type*, and the second to the pair of h P.D.O. being of *non-orthogonal* type.

1. Introduction.

The theory of pseudodifferential operators (P.D.O. for short) and h admissible P.D.O. has been developed with operators taking symbols from various different classes. In the h P.D.O. case the theory developed in [HR] for example requires that, for large enough N, an h P.D.O. may be expressed as the sum of Weyl quantisations (of some functions) modulo a bounded operator with L^2 norm equal to $O(h^{N+1})$. It is therefore necessary, in the h P.D.O. case, to impose additional conditions on symbols, to force boundedness of any remainder operators, when considering symbolic calculus and composition. This is in contrast with the general (non h) P.D.O. theory where no L^2 boundedness conditions are usually required.

In this paper we define classes of symbols for which natural, sufficient conditions exist ensuring symbolic calculus and composition of h P.D.O. are valid. As demonstrated, the conditions correspond, in an intuitive sense, to the h P.D.O. being of positive type (not in the operator theoretic sense) and of non-orthogonal type. Though the results here follow essentially from those appearing in [H], chapter 18, the original input is the idea of h P.D.O. being of positive or non-orthogonal type.

2. Symbol Classes and h P.D.O.

For $x, \xi \in \mathbb{R}^n$ we denote by $\langle x, \xi\rangle$ and $\langle x\rangle$ the positive quantities $\langle x, \xi\rangle = (1 + |x|^2 + |\xi|^2)^{\frac{1}{2}}$ and $\langle x\rangle = \langle x, 0\rangle$. We recall the definition of a σ temperate function in the case when g_0 is the standard Euclidean metric on $\mathbb{R}^n \times \mathbb{R}^n$, see [H].

Definition 1. A positive, continuous function $m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is said to be σ temperate if there exist positive constants $C, N > 0$ such that,

$$m(x, \xi) \leq Cm(y, \eta)\langle x - y, \xi - \eta\rangle^N, \tag{1}$$

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for all $(x,\xi),(y,\eta)\in\mathbb{R}^n\times\mathbb{R}^n$.

In view of (1) it makes sense to define the following order functions.

Definition 2. For any σ temperate function m, the upper (respectively lower) order of m, denoted $\mathrm{ord}\,(m)$ (resp. $\mathrm{ord}\,'(m)$) is defined to be the least (greatest) $N\in\mathbb{R}$ such that the quantity $m(x,\xi)\langle x,\xi\rangle^{-N}$ is bounded above (below) for all $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$.

We record some basic facts about σ temperate functions and the order functions ord and $\mathrm{ord}\,'$. In what follows for any real-valued functions f and g we use the standard notation $(f\wedge g)(x)=\min\{f(x),g(x)\}$ and $(f\vee g)(x)=\max\{f(x),g(x)\}$. All the following statements follow by elementary manipulation of inequalities and the proofs (see [B], chapter 2) are therefore omitted.

Proposition 3. *Let m, m_1 and m_2 denote σ temperate functions. When m is positive m^{-1} denotes the function $(m(x,\xi))^{-1}$.*

i) *If m is positive, then m^{-1} is σ temperate.*
ii) *The pointwise product m_1m_2 is σ temperate.*
iii) *For all $r\geq 0$, the function $m^r=(m(x,\xi))^r$ is σ temperate. Hence, by i), if $m>0$, then m^r is σ temperate for all $r\in\mathbb{R}$.*
iv) *For all $r\geq 0$ the following (in)equalities hold,*

$$\begin{aligned}\mathrm{ord}\,(m_1m_2)&\leq \mathrm{ord}\,(m_1)+\mathrm{ord}\,(m_2)\\ \mathrm{ord}\,'(m_1m_2)&\geq \mathrm{ord}\,'(m_1)+\mathrm{ord}\,'(m_2)\\ \mathrm{ord}\,(m^r)&=r\ \mathrm{ord}\,(m)\\ \mathrm{ord}\,'(m^r)&=r\ \mathrm{ord}\,'(m).\end{aligned}$$

v) *If $m>0$, then* $\mathrm{ord}\,(m^{-1})=-\mathrm{ord}\,'(m)$ *and* $\mathrm{ord}\,'(m^{-1})=-\mathrm{ord}\,(m)$.
vi) *If $r\in\mathbb{R}$, then $\langle x,\xi\rangle^r$ is σ temperate and* $\mathrm{ord}\,(\langle x,\xi\rangle^r)=\mathrm{ord}\,'(\langle x,\xi\rangle^r)=r$.
vii) *The functions $m_1\wedge m_2$ and $m_1\vee m_2$ are σ temperate and,*

$$\begin{aligned}\mathrm{ord}\,(m_1\wedge m_2)&\leq \min(\mathrm{ord}\,(m_1),\mathrm{ord}\,(m_2))\\ \mathrm{ord}\,'(m_1\wedge m_2)&\geq \min(\mathrm{ord}\,'(m_1),\mathrm{ord}\,'(m_2))\\ \mathrm{ord}\,(m_1\vee m_2)&\leq \max(\mathrm{ord}\,(m_1),\mathrm{ord}\,(m_2))\\ \mathrm{ord}\,'(m_1\vee m_2)&\geq \max(\mathrm{ord}\,'(m_1),\mathrm{ord}\,'(m_2)).\end{aligned}$$

We now define the symbol classes we will work with.

Definition 4. Let m,ϕ,ψ denote σ temperate functions with $\phi(x,\xi)>0$ and $\psi(x,\xi)>0$ for all $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$. A function $a\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ is said to be a symbol of type ϕ,ψ and weight m if, for each pair of multi-indices α,β, there exists a constant $C_{\alpha,\beta}>0$ such that,

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\leq C_{\alpha,\beta}\ m(x,\xi)(\phi(x,\xi))^{-|\alpha|}(\psi(x,\xi))^{-|\beta|}$$

for all $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$. The space of such symbols will be denoted $S^{\phi,\psi}(m)$.

We note that the symbol spaces used in [HR] correspond, in our notation, to $S^{\mathbf{1},\mathbf{1}}(m)$ where $\mathbf{1}$ denotes the function equal to 1 everywhere. The following facts may be deduced from Proposition 3.

Proposition 5. *Let m, m_1, m_2 and $\phi, \phi_1, \phi_2, \psi, \psi_1, \psi_2 > 0$ denote σ temperate functions.*

i) *Suppose there exist constants $C_1, C_2 > 0$ and C_3 such that $\phi_1(x,\xi) \geq C_1\phi_2(x,\xi)$, $\psi_1(x,\xi) \geq C_2\psi_2(x,\xi)$ and $m_1(x,\xi) \leq C_3 m_2(x,\xi)$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Then,*

$$S^{\phi_1,\psi_1}(m_1) \subseteq S^{\phi_2,\psi_2}(m_2).$$

ii) *Suppose $a \in S^{\phi_1,\psi_1}(m_1)$ and $b \in S^{\phi_2,\psi_2}(m_2)$ then the pointwise product ab is such that,*

$$ab \in S^{\phi_1\wedge\phi_2,\psi_1\wedge\psi_2}(m_1 m_2).$$

iii) *Let $a \in S^{\phi,\psi}(m)$ then for any multi-indices α, β,*

$$\partial_x^\alpha \partial_\xi^\beta a \in S^{\phi,\psi}(m\phi^{-|\alpha|}\psi^{-|\beta|}).$$

iv) *Suppose ϕ and ψ have* $\operatorname{ord}'(\phi) \geq 0$ *and* $\operatorname{ord}'(\psi) \geq 0$ *then*

$$S^{\phi,\psi}(m) \subseteq S^{\mathbf{1},\mathbf{1}}(m).$$

Similarly to [HR] we define a class of h P.D.O. In what follows, and later on, for a suitable function $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $\tau \in [0,1]$, we denote by $a^\tau(x, hD_x)$ the τ quantisation of the function a. The operator $a^\tau(x, hD_x)$ is defined, acting on a Schwartz function $u \in \mathcal{S}(\mathbb{R}^n)$ as,

$$a^\tau(x, hD_x)u(x) = (2\pi h)^{-n} \int a(x + \tau(y-x), \xi) e^{ih^{-1}\xi\cdot(x-y)} u(y)\, dy\, d\xi.$$

The particular case when $\tau = \frac{1}{2}$ will be referred to as the Weyl quantisation and hence, we define $a^w(x, hD_x) := a^{\frac{1}{2}}(x, hD_x)$.

Definition 6. Let m and $\phi, \psi > 0$ denote σ temperate functions. An operator valued map $A(h) : (0, h_0] \to \mathcal{L}(\mathcal{S}(\mathbb{R}^n), L^2(\mathbb{R}^n))$, $h_0 > 0$, is said to be an h P.D.O. of type ϕ, ψ and weight m if there exist a sequence of functions $a_j \in S^{\phi,\psi}(m(\phi\psi)^{-j})$ and an integer $N_0 > 0$ such that, for all $N \geq N_0$,

$$\left\| A(h) - \sum_{0 \leq j \leq N} h^j a_j^w(x, hD_x) \right\| = O(h^{N+1}), \quad h \to 0, \tag{2}$$

where a_j^w denotes the Weyl quantisation of a_j and $\|\cdot\|$ denotes $L^2(\mathbb{R}^n)$ operator norm. We denote by $\mathcal{L}_{\text{ad}}^{\phi,\psi}(m)$ the space of such h P.D.O., and the formal series,

$$a(x,\xi;h) = \sum_{j\geq 0} h^j a_j(x,\xi), \tag{3}$$

is said to be the Weyl symbol of $A(h)$.

3. Symbolic Calculus for h P.D.O. of Positive Type.

We now discuss symbolic calculus in $\mathcal{L}_{\mathrm{ad}}^{\phi,\psi}(m)$.

Definition 7. A symbol $a \in S^{\phi,\psi}(m)$ or an h P.D.O. $A(h) \in \mathcal{L}_{\mathrm{ad}}^{\phi,\psi}(m)$ is said to be of positive type if $\mathrm{ord}\,'(\phi) \geq 0$, $\mathrm{ord}\,'(\psi) \geq 0$ and $\mathrm{ord}\,'(\phi\psi) > 0$.

As $\mathrm{ord}\,'(\mathbf{1}) = 0$, we note that an arbitrary symbol from the class $S^{\mathbf{1},\mathbf{1}}(m)$ is not necessarily of positive type.

Theorem 8. *Let $A(h) : (0, h_0] \to \mathcal{L}(\mathcal{S}(\mathbb{R}^n), L^2(\mathbb{R}^n))$ denote an operator valued map which satisfies the hypotheses of Definition 6 with the $a_j^w(x, hD_x)$ replaced by $a_j^{\tau_1}(x, hD_x)$ for some $\tau_1 \in [0, 1]$. If ϕ and ψ are such that $\mathrm{ord}\,'(\phi) \geq 0$, $\mathrm{ord}\,'(\psi) \geq 0$ and $\mathrm{ord}\,'(\phi\psi) > 0$ then $A(h) \in \mathcal{L}_{\mathrm{ad}}^{\phi,\psi}(m)$ and $A(h)$ is an h P.D.O. of positive type. Furthermore, for any $\tau_2 \in [0, 1]$, there exist a sequence of functions $b_j \in S^{\phi,\psi}(m(\phi\psi)^{-j})$ and an integer $N_1 > 0$ such that (2) holds for any $N \geq N_1$ and with the a_j^w replaced by $b_j^{\tau_2}$. We note that choosing $(N_1 + 1) \geq \frac{\mathrm{ord}\,(m)}{\mathrm{ord}\,'(\phi\psi)}$ (for any τ_1 and τ_2) will always suffice.*

If the hypotheses of Theorem 8 are satisfied then it makes sense to call the series $a = \sum_{j\geq 0} h^j a_j$ the τ_1 symbol of $A(h)$ and $b = \sum_{j\geq 0} h^j b_j$ the τ_2 symbol of $A(h)$. As in [R], the two symbols are then related by the formal expression,

$$b = e^{ih(\tau_1 - \tau_2) D_x \cdot D_\xi} a, \tag{4}$$

where the operator $e^{i\kappa D_x \cdot D_\xi}$ is defined as in [H], chapter 18 for example.

Proof. Under the hypotheses of the theorem, $S^{\phi,\psi}(m) \subseteq S^{\mathbf{1},\mathbf{1}}(m)$. Then, according to the definition of $e^{i\kappa D_x \cdot D_\xi}$ in [H] and using Taylor's theorem,

$$h^{-(N+1)} \left(e^{ih(\tau_1 - \tau_2) D_x \cdot D_\xi} a - \sum_{j \leq N} h^j b_j \right) \in S^{\mathbf{1},\mathbf{1}}(m(\phi\psi)^{-(N+1)}), \tag{5}$$

uniformly with respect to $h \in (0, h_0]$ and for all $N \geq 0$. According to Proposition 3, choosing $(N + 1) \geq \frac{\mathrm{ord}\,(m)}{\mathrm{ord}\,'(\phi\psi)}$ always ensures that the function $m(\phi\psi)^{-(N+1)}$ is bounded. Thus, using a standard L^2 boundedness result ([H], Theorem 18.6.3 for example), if $(N + 1) \geq \frac{\mathrm{ord}\,(m)}{\mathrm{ord}\,'(\phi\psi)}$, then the τ_2 quantisation of the symbol in (5) is bounded with norm $O(1)$ as $h \to 0$. The statement in the theorem then follows from these observations and [H], Theorem 18.5.10. $\square$

4. Composition for h P.D.O. of Non-Orthogonal Type.

Definition 9. Two symbols $a \in S^{\phi_1,\psi_1}(m_1)$ and $b \in S^{\phi_2,\psi_2}(m_2)$ or two h P.D.O. $A(h) \in \mathcal{L}_{\mathrm{ad}}^{\phi_1,\psi_1}(m_1)$ and $B(h) \in \mathcal{L}_{\mathrm{ad}}^{\phi_2,\psi_2}(m_2)$ are said to be of non-orthogonal type if $\mathrm{ord}\,'((\phi_1 \wedge \phi_2)(\psi_1 \wedge \psi_2)) > 0$, $\mathrm{ord}\,'(\phi_1 \wedge \phi_2) \geq 0$ and $\mathrm{ord}\,'(\psi_1 \wedge \psi_2) \geq 0$.

Again, as $\mathrm{ord}\,'((\mathbf{1} \wedge \mathbf{1})(\mathbf{1} \wedge \mathbf{1})) = \mathrm{ord}\,'(\mathbf{1}) = 0$, we note that two arbitrary symbols from the class $S^{\mathbf{1},\mathbf{1}}(m)$ are not necessarily non-orthogonal.

Theorem 10. *Let $A(h) \in \mathcal{L}_{\mathrm{ad}}^{\phi_1,\psi_1}(m_1)$ and $B(h) \in \mathcal{L}_{\mathrm{ad}}^{\phi_2,\psi_2}(m_2)$ denote two h P.D.O. of non-orthogonal type with Weyl symbols a and b. Then the composition $A(h)B(h) \in \mathcal{L}_{\mathrm{ad}}^{\phi_1\wedge\phi_2,\psi_1\wedge\psi_2}(m_1m_2)$ is an h P.D.O. of positive type with Weyl symbol,*

$$(a\sharp b)(x,\xi;h) = e^{\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)}a(x,\xi;h)\ b(y,\eta;h)\Big|_{(y,\eta)=(x,\xi)},$$

where $\sigma(D_x,D_\xi;D_y,D_\eta) = D_y\cdot D_\xi - D_x\cdot D_\eta$.

Proof. Define,

$$c_{j,k,l}(x,\xi) = \frac{(i)^l}{l!2^l}(\sigma(D_x,D_\xi;D_y,D_\eta))^l a_j(x,\xi)b_k(y,\eta)\Big|_{(y,\eta)=(x,\xi)}.$$

Then by definition and according to Proposition 5 each,

$$c_{j,k,l} \in S^{\phi_1\wedge\phi_2,\psi_1\wedge\psi_2}(m_1m_2(\phi_1\psi_1)^{-j}(\phi_2\psi_2)^{-k}(\phi_1\psi_2)^{-|\alpha|}(\psi_1\phi_2)^{-|\beta|}),$$

where $|\alpha+\beta| = l$. However, as $|\alpha+\beta| = l$,

$$(\phi_1\psi_2)^{-|\alpha|}(\phi_2\psi_1)^{-|\beta|} \le ((\phi_1\wedge\phi_2)(\psi_1\wedge\psi_2))^{-l},$$

and for all j and k,

$$(\phi_1\psi_1)^{-j}(\phi_2\psi_2)^{-k} \le ((\phi_1\wedge\phi_2)(\psi_1\wedge\psi_2))^{-(j+k)}.$$

Hence,

$$\sum_{k+l+r=j} c_{k,l,r} \in S^{\phi_1\wedge\phi_2,\psi_1\wedge\psi_2}(m_1m_2((\phi_1\wedge\phi_2)(\psi_1\wedge\psi_2))^{-j}), \tag{6}$$

for all $j \ge 0$. By hypothesis $S^{\phi_1\wedge\phi_2,\psi_1\wedge\psi_2}(m) \subseteq S^{\mathbf{1},\mathbf{1}}(m)$ and, similarly to before, the results of [H], chapter 18 and (6) imply,

(7)
$$h^{-(N+1)}\left(e^{\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)}ab\Big|_{(y,\eta)=(x,\xi)} - \sum_{j+k+l\le N} h^{j+k+l}c_{j,k,l}\right) \in S^{\mathbf{1},\mathbf{1}}(m_1m_2((\phi_1\wedge\phi_2)(\psi_1\wedge\psi_2))^{-(N+1)}),$$

uniformly with respect to $h \in (0,h_0]$ and for all $N \ge 0$. According to Proposition 3, choosing $(N+1) \ge \frac{\mathrm{ord}\ (m_1)+\mathrm{ord}\ (m_2)}{\mathrm{ord}\ '((\phi_1\wedge\phi_2)(\psi_1\wedge\psi_2))}$ ensures that the function $m_1m_2((\phi_1\wedge\phi_2)(\psi_1\wedge\psi_2))^{-(N+1)}$ is bounded. In this case, applying the same L^2 boundedness result as in the proof of Theorem 8, the Weyl quantisation of the symbol in (7) is bounded with norm $O(1)$ as $h \to 0$. The statement in the theorem then follows from these observations and [H], Theorem 18.5.4. □

Remark. In the case when $a = a_0(x,\xi)$ and $b = b_0(x,\xi)$ the above result is valid under the weaker condition that $\mathrm{ord}\ '(\phi_1\psi_2) > 0$ and $\mathrm{ord}\ '(\phi_2\psi_1) > 0$. Then choosing $(N+1) \ge \frac{\mathrm{ord}\ (m_1)+\mathrm{ord}\ (m_2)}{\min(\mathrm{ord}\ '(\phi_1\psi_2),\mathrm{ord}\ '(\phi_2\psi_1))}$ ensures that the function,

$$h^{-(N+1)}\left(e^{\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)}a_0b_0\Big|_{(y,\eta)=(x,\xi)} - \sum_{l\le N} h^l c_{0,0,l}\right),$$

is bounded, uniformly with respect to $h \in (0,h_0]$, and the required result is achieved. We note, however, that in the general case the stronger condition of non-orthogonality is necessary.

Finally we consider an example which explains the nomenclature used.

Example 11. Let $\rho_1, \rho_2, \delta_1, \delta_2 \geq 0$ and set $\phi_j = \langle x, \xi \rangle^{\rho_j}$ and $\psi_j = \langle x, \xi \rangle^{\delta_j}$. According to Proposition 3, the ϕ_j and ψ_j are σ temperate. Furthermore, $\operatorname{ord}'((\phi_1 \wedge \phi_2)(\psi_1 \wedge \psi_2)) = \min(\rho_1, \rho_2) + \min(\delta_1, \delta_2)$ and the condition $\operatorname{ord}'((\phi_1 \wedge \phi_2)(\psi_1 \wedge \psi_2)) > 0$ is equivalent to $\rho_1\rho_2 + \delta_1\delta_2 > 0$. The non-orthogonality condition, in this case, is therefore equivalent to requiring that the vectors (ρ_1, ρ_2) and (δ_1, δ_2) are non-orthogonal in $\mathbb{R}^2$. The positivity condition is equivalent to $\rho_j\delta_j > 0$.

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