EXPONENTIAL APPROXIMATION FOR HITTING TIMES IN MIXING PROCESSES

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Abstract

We present bounds of the error of the exponential approximation of the first occurrence time of a rare event in a stationary stochastic process with finite alphabet with α-mixing property with a summable function α or with general ϕ-mixing property. We prove a lower bound for this error in terms of the measure of the cylinder and an upper bound as a function of the measure of the cylinder plus the decay of correlation of the process.

Keywords. *Mixing stochastic processes, occurrence time of a rare event, exponential law.*

1 Introduction.

This article presents lower and upper bounds for the exponential approximation to the law of the first occurrence of a long string of symbols in a stochastic source with a finite alphabet. Our result holds for α-mixing processes with summable function α and for ϕ-mixing processes with general decreasing function ϕ. We prove that for any cylinder set A, the law of its hitting time, suitably rescaled, can be uniformly approximated by a mean one exponential law. The error in the approximation of this law is bounded

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from below by $C_1 I\!P(A)$. It is bounded above by $\inf_x\{x I\!P(A) + g(x)\}$ where $I\!P(A)$ is the measure of the cylinder and g is the function of decay of correlation of the process.

Moreover, we show that the scaling factor can be written as $\xi_{A,\mu(t)} I\!P(A)$, where $\xi_{A,\mu(t)}$ is bounded below and above by two strictly positive constants Ξ_1 and Ξ_2, respectively, independent of A, n and t.

Important recent papers on this subject are Galves-Schmitt (1997), Hirata-Saussol-Vaienti (1998) and Collet-Galves-Schmitt (1999).

Galves-Schmitt's approach introduces a correction factor in the approximating law. This approach works for all kind of cylinders but only in the case of ϕ-mixing processes with a summable function ϕ.

The paper by Hirata, Saussol and Vaienti presents a very elegant straightforward proof. However their result is interesting for cylinders that doesn't allows an immediate return to themselves.

Collet, Galves and Schmitt (1999) show how this correction factor depends on the return of the rare event onto itself in the firsts steps.

For a brief review on the field before 1997 we refer the reader to Galves and Schmitt (1997).

In this article we present a modification of Galves-Schmitt's approach. This new technique provides sharper upper bound for this type of approximation. The improvement is obtained in part by changing the scaling factor. Our technique works for all type of cylinders and for the mentioned enlarged family of processes. This answer in part one of the questions pointed out in the paper by Hirata, Saussol and Vaienti: What is the largest class of processes for which the exponential approximation hols? We refer to Doukhan (1995) for examples and references of α-mixing processes that are not ϕ-mixing and for ϕ-mixing decaying at any rate. We should say that an irreducible and aperiodic finite state Markov chain is ϕ-mixing with exponential decay. Moreover, Gibbs states with a potential with summable variations are exponentialy ϕ-mixing. See Bowen 1975 for definitions and properties.

Kac's lemma (see Cornfeld, Fomin, Sinai 1982) says that the right scaling factor for return times is $I\!P(A)$. This suggests to use it as scaling factor for hitting times. which in our notation means $\xi_{A,\mu(t)} = 1$. We show that it this is not always the case for hitting times. We study how $\xi_{A,\mu(t)}$ depends on A. In this sense, the use of this scaling factor provides a good approximation to the exponential law but could be difficult to calculate explicitly. To overcome this real difficulty we present in theorem 4 a computable approximation for this parameter in terms of the short times recurrence of each cylinder. Its corollary presents conditions, on the process and on the cylinders, to get the parameter λ_A equal to one; and also prove that the total mass of this type of cylinders is high. This behavior was indicated in Hirata, Saussol, Vaienti 1998; Collet, Galves, Schmitt 1999 and Saussol 1999. We also present a counterexample in which the parameter is not just $\lambda_A \neq 1$ but also so small as we want, according to the process.

This paper is organized as follows. In section 3 we state the theorems. In sections 4, 5 and 6 we prove the lemmata that is needed in the proofs of the theorems and in sections 7, 8 and 9 we prove the theorems.

2 The framework.

Let $\mathcal{E}$ be a finite set. Put $\Omega = \mathcal{E}^{\mathbb{Z}}$. For each $n \in \mathbb{Z}$ define $X_n : \Omega \to \Re$ the n-th coordinate projection. We denote by $T : \Omega \to \Omega$ the one-step-left shift operator.

We denote by $\mathcal{F}$ the σ-algebra over Ω generated by cylinders. Moreover we denote with $\mathcal{F}_I$ the σ-algebra generated by cylinders with coordinates in I, $I \subseteq \mathbb{Z}$.

For a subset $A \subseteq \Omega$ we say that $A \in \mathcal{C}_n$ if and only if

$$A = \{X_0 = a_0, \ldots, X_{n-1} = a_{n-1}\} \ ,$$

with $a_i \in \mathcal{E},\ i = 1, \ldots, n$.

We consider a stationary probability measure $\mathbb{P}$ over $\mathcal{F}$. We shall assume that in $\mathcal{E}$ there are no element of probability 0 or 1.

Let $\alpha = (\alpha(l))_{l \geq 0}$ and $\phi = (\phi(l))_{l \geq 0}$ be two going to zero sequences of positive real numbers. We shall say that $(\Omega, \mathcal{F}, \mathbb{P}, T)$ is *α-mixing* if, for all integers $n \geq 1$ and $l \geq 0$, the following inequality holds

$$\sup_{B \in \mathcal{F}_{\{0,..,n\}}, C \in \mathcal{F}_{\{n \geq 0\}}} \frac{\left| \mathbb{P}\left(B \cap T^{-(n+l+1)} C\right) - \mathbb{P}(B)\mathbb{P}(C) \right|}{\mathbb{P}(B)} = \alpha(l) \ ,$$

and *ϕ-mixing* if

$$\sup_{B \in \mathcal{F}_{\{0,..,n\}}, C \in \mathcal{F}_{\{n \geq 0\}}} \frac{\left| \mathbb{P}\left(B \cap T^{-(n+l+1)} C\right) - \mathbb{P}(B)\mathbb{P}(C) \right|}{\mathbb{P}(B)\mathbb{P}(C)} = \phi(l) \ ,$$

where in the above expressions the supremum is taken over the sets B and C, such that $\mathbb{P}(B) > 0$ in the first case and such that $\mathbb{P}(B)\mathbb{P}(C) > 0$ in the second one. Given $A \in \mathcal{C}_n$, we define the *entrance time* $\tau_A : \Omega \to \mathbb{N} \cup \{\infty\}$

as the following random variable defined on the probability space $(\Omega, \mathcal{F}, I\!P)$. For any $\omega \in \Omega$

$$\tau_A(\omega) = \inf\{k \geq 1 : T^k(\omega) \in A\}$$

Clearly, ϕ-mixing implies α-mixing. We recall that mixing (at any rate) implies ergodicity, and ergodicity ensures that τ_A is $I\!P$-almost surely finite (see Cornfeld, Fomin, Sinai 1982).

We shall use the classic probabilistic shorthand notation for events defined through random variables. We shall write $\{\tau_A = m\}$ instead of $\{\omega \in \Omega : \tau_A(\omega) = m\}$, $T^{-k}(A) = \{\omega \in \Omega : T^k(\omega) \in A\}$ and $\{X_r^s = x_r^s\} = \{X_r = x_r, \ldots, X_s = x_s\}$. As usual, the mean of a random variable X will be denoted by $I\!E(X)$. Wherever it is not ambiguous we will write C and c for different positive constants even in the same sequence of equalities/inequalities. Where a property holds for α and ϕ processes we shall write $*$-mixing.

3 Statement of the results.

We now state our main result.

Theorem 1. *Let $(\Omega, \mathcal{F}, I\!P, T)$ be ϕ-mixing or α-mixing with α summable. Then, there exists strictly positive constants Ξ_1, Ξ_2, C_1 and C_2 such that for any $n, A \in \mathcal{C}_n$ and $t > 0$ there exists $\xi_{A,\mu(t)} \in [\Xi_1, \Xi_2]$, for which the following inequality holds*

$$\sup_{t>0} \left| I\!P\left\{\tau_A > \frac{t}{\xi_{A,\mu(t)} I\!P(A)}\right\} - e^{-t}\right| \leq C_2 \inf_{\Delta \geq n} \left[2\Delta I\!P(A) + *(\Delta - n)\right] \ .$$

Moreover $e^{-\xi_{A,\mu(t)} I\!P(A)t}$ is a left-continuous non-increasing funtion of t over each $[k/I\!P(A), (k+1)/I\!P(A)]$ and for all $k \in I\!N$

$$\left| \lim_{t \to \left(\frac{k+1}{I\!P(A)}\right)^-} e^{-\xi_{A,\mu(t)}} - e^{-\xi_{A,\mu(k+1/I\!P(A))}} \right| \leq 2\Delta I\!P(A) + *(\Delta - n) \ .$$

We need first to prove the following results.

Theorem 2. *Let $(\Omega, \mathcal{F}, I\!P, T)$ be ϕ-mixing. Then, there exists strictly positive constants $\Lambda_1, \Lambda_2, C_1$ and C_2 such that for any n and any $A \in \mathcal{C}_n$, there exists $\lambda_A \in [\Lambda_1, \Lambda_2]$, for which the following inequality holds*

$$C_1 I\!P(A) \leq \sup_{t>0} \left| I\!P\left\{\tau_A > \frac{t}{\lambda_A I\!P(A)}\right\} - e^{-t}\right| \leq C_2\, b(A)$$

where $b(A) = I\!P(A)^\eta$, $0 < \eta < 1$ if $\phi(n) \leq C/n^\epsilon$ for all $n > n_o$ with $\epsilon > 0$ and $b(A) = \phi^{1/2}(n)$ otherwise.

In Galves-Schmitt, 1997 it was prove an upper bound of the type $I\!P(A)^\beta$ with $0 < \beta < 1$ for ϕ summable processes. Therefore, theorem 2 extends that result for *any* type of decreasing function ϕ.

Theorem 3. *Let $(\Omega, \mathcal{F}, I\!P, T)$ be α-mixing and let us suppose that the function $\alpha : I\!N \to I\!R_+$ is summable. Then, there exists strictly positive constants $\Lambda_1, \Lambda_2, C_1$, C_2 and κ such that for any n and any $A \in \mathcal{C}_n$, there exists $\lambda_A \in [\Lambda_1, \Lambda_2]$, for which the following inequality holds*

$$C_1 I\!P(A) \leq \sup_{t>0} \left| I\!P\left\{\tau_A > \frac{t}{\lambda_A I\!P(A)}\right\} - e^{-t}\right| \leq C_2 I\!P(A)^\kappa$$

We emphasize that in both theorems the constants $\Lambda_1, \Lambda_2, C_1, C_2, \eta$ and κ are independent of n and A. Notation $\Lambda_1, \Lambda_2, C_1$, and C_2 stand for (possibly) different constants in each theorem.

The difference between theorem 1 and theorems 2 and 3 is that the first one provides a more accurate upper bound given that we use $\xi_{A,\mu(t)}$ instead of λ_A as a correction factor. We first prove Theorems 2 and 3 following Galves-Schmitt 97 approach. After this is done, we use the estimate of $I\!P\{\tau_A > 1/I\!P(A)\}$ to prove that $1/I\!P(A)$ is a suitable scaling factor for the process and provides a sharper upper bound for the approximation of the exponential law.

We now present the theorem that gives an estimation of $\xi_{A,\mu(t)}$. Let s be a positive integer. Denoting by $I\!P_A$ the conditional probability on A we define

$$\zeta_{A,s} = I\!P_A\left\{\tau_A > \frac{n}{s}\right\} = \frac{I\!P\left\{\tau_A > \frac{n}{s} \cap A\right\}}{I\!P(A)} .$$

Theorem 4. *Let $(\Omega, \mathcal{F}, I\!P, T)$ be exponentially α-mixing. Let s be a positive integer. Then, there exist strictly positive constants Ψ_1, Ψ_2, C_1, C_2 and c such that for any n and any $A \in \mathcal{C}_n$, there exists $\zeta_{A,s} \in [\Psi_1, \Psi_2]$, for which the following inequality holds*

$$C_1 I\!P(A) \leq \sup_{t>0} \left| I\!P\left\{\tau_A > t\right\} - e^{-t\zeta_{A,s} I\!P(A)} \right| \leq C_2 e^{-cn} \ .$$

4 Estimates of the probability of a cylinder.

Lemma 1. *Let the process be ϕ-mixing with any decreasing function ϕ or α-mixing with α summable. There exists strictly positive constants C, $\bar{C}$, and Γ such that for any fixed positive integer n and any $A \in \mathcal{C}_n$ the following inequalities holds*

$$I\!P(A) \leq Ce^{-\Gamma n} \tag{1}$$

and

$$\sum_{k=1}^{n} I\!P\left(A \cap T^{-k} A\right) \leq \bar{C} I\!P(A) \ . \tag{2}$$

Proof. Take $A = \{X_1^n = a_1^n\}$. Using the α-mixing property we have

$$I\!P(A) \leq [\alpha(n_0 - 1) + \rho]^{[\frac{n}{n_0}]+1} \ ,$$

where

$$\rho = \sup\{I\!P(a_i) : a_i \in \mathcal{E}\}.$$

Since, by hypothesis, $\rho < 1$ and the function $\alpha(l) \to 0$, there exists an integer n_0 such that

$$(\rho + \alpha(n_0 - 1)) < 1 \ .$$

This proves (1). To prove (2), in α-mixing case we remark that

$$I\!P(A \cap T^{-k} A) \leq \alpha\left([\frac{k}{2}]\right) I\!P(A) + I\!P\left(A^{(k-[\frac{k}{2}])}\right) I\!P(A)$$

where $A^{(l)} = \{X_{n+1}^{n+l} = a_{n-l+1}^n\}$. Then, since α is summable and using (1)

$$\sum_{k=1}^{n} I\!P\left(A \cap T^{-k} A\right) \leq 2I\!P(A) \sum_{k=0}^{[\frac{n}{2}]} \alpha\left(l\right) + 2I\!P(A) \sum_{k=1}^{n-[\frac{n}{2}]} I\!P(A^{(l)}) \leq CI\!P(A) \ .$$

For the general ϕ-mixing case we refer to lemma 1 on Galves-Schmitt, 97.

5 Bounds for the parameter $\lambda_{A,f}$.

For any positive integer k let us define

$$N_k = \sum_{l=1}^{k} 1\!\!1_{T^{-l}(A)} \ , \tag{3}$$

where $1\!\!1_A$ is the indicator function of the set A. For any $\omega \in \Omega$, $N_k(\omega)$ is the number of times the process visits A, during the first k steps. We remark that

$$\{\tau_A \le k\} = \{N_k \ge 1\} \ .$$

Lemma 2. *For any real number $t \ge 1$ the following inequality holds*

$$I\!P\{\tau_A \le t\} \le t\, I\!P(A) \ . \tag{4}$$

Proof.

$$I\!P\{\tau_A \le t\} = \sum_{l=1}^{[t]} I\!P\{\tau_A = l\} \le \sum_{l=1}^{[t]} I\!P\left\{T^{-l}A\right\} = [t] I\!P(A) \tag{5}$$

where in the last equality we use the invariance of the measure $I\!P$ with respect to the transformation T.

Lemma 3. *Let the process be ϕ-mixing or α-mixing with α summable. For any positive integer n, any cylinder $A \in \mathcal{C}_n$ and $t > 0$, the following holds*

$$I\!P\{\tau_A \le t\} \ge \frac{[t]^2 I\!P(A)^2}{t\, I\!P(A) + C' t\, I\!P(A) + C(t\, I\!P(A))^2 + 2Kt\, I\!P(A)}, \tag{6}$$

where $C' > 0, C > 0$ and $K > 0$ are constants independent of n and A.

Proof. Let $N = N_{[t]}$. We first remark that

$$(I\!E(N))^2 = \left(I\!E(N 1\!\!1_{\{N \ge 1\}})\right)^2 \le I\!E(N^2) I\!P\{N \ge 1\}$$

where the last inequality follows from the Schwarz inequality. Therefore

$$I\!P\{\tau_A \le t\} = I\!P\{N \ge 1\} \ge \frac{(I\!E(N))^2}{I\!E(N^2)} \ .$$

By definition $I\!E(N) = [t] I\!P(A)$.

To obtain an upper bound for the denominator $I\!\!E(N^2)$, we decompose it as follows. Assume first $[t] > n$, then the definition of N gives

$$I\!\!E(N^2) = \sum_{l=1}^{[t]} I\!\!E\left(1\!\!1_{T^{-l}(A)}\right) \quad + \quad 2\sum_{l=1}^{n}([t]-l+1)\, I\!\!E\left(1\!\!1_A\, 1\!\!1_{T^{-l}(A)}\right)$$
$$+ \quad 2\sum_{l=n+1}^{[t]}([t]-l+1)\, I\!\!E\left(1\!\!1_A\, 1\!\!1_{T^{-l}(A)}\right). \quad (7)$$

The first term of this decomposition is $I\!\!E(N)$.
Using lemma 1 in the second term, we get

$$\sum_{l=1}^{n}([t]-l+1)I\!\!E\left(1\!\!1_A\, 1\!\!1_{T^{-l}(A)}\right) \leq [t]\sum_{l=1}^{n} P\{A\cap T^{-l}(A)\} \leq C' t\, I\!\!P(A)\ .$$

The α-mixing property provides upper bounds for the expectations inside the third term

$$I\!\!E\left(\mathbf{1}_A\, \mathbf{1}_A(T^l)\right) \leq I\!\!P(A)^2 + \alpha(l-n-1) I\!\!P(A).$$

Therefore the third term is bounded above by

$$2I\!\!P(A)\left(I\!\!P(A)\sum_{l=1}^{[t]-1} l + t\sum_{l=1}^{+\infty}\alpha(l)\right)\ .$$

Now we remark that

$$\sum_{l=1}^{[t]-1} l = \frac{1}{2}[t]([t]-1)\ .$$

Finally we use the hypothesis that the series $\sum_{l=1}^{+\infty}\alpha(l)$ is convergent to get the upper bound

$$2\sum_{l=n+1}^{[t]}([t]-l+1)\, I\!\!E\left(1\!\!1_A\, 1\!\!1_{T^{-l}(A)}\right) \leq (tI\!\!P(A))^2 + 2Kt\, I\!\!P(A),$$

where $K = \sum_{l=1}^{+\infty}\alpha(l) < +\infty$.

In the case of ϕ-mixing we simply observe that

$$\mathbb{E}\left(\mathbb{1}_A\ \mathbb{1}_{T^{-l}(A)}\right) \leq (1+\phi(l-n))\ \mathbb{P}(A)^2.$$

Therefore the third term is bounded above by

$$2\frac{1}{2}[t]([t]-1)\,(1+\phi(0))\,\mathbb{P}(A)^2 \leq C\ (t\,\mathbb{P}(A))^2\ .$$

For $t \leq n$ the proof looks the same with the third term absent. This concludes the proof of Lemma 3.

Let f_A be a positive integer such that

$$f_A\mathbb{P}(A) \leq \frac{1}{2}\ . \tag{8}$$

and let us define

$$\lambda_{A,f} = \frac{-\log \mathbb{P}\left\{\tau_A > f_A\right\}}{f_A\mathbb{P}(A)}\ .$$

Lemma 4. *There exists a positive integer n_0, such that, for all $n \geq n_0$ and $A \in \mathcal{C}_n$, such that the following inequalities hold*

$$\Lambda_1 \leq \lambda_{A,f} \leq \Lambda_2\ ,$$

where Λ_1 and Λ_2 are two positive constants independent of n, A and f_A.

Proof. To obtain the lower bound, we first remark that

$$\frac{\theta}{2} \leq 1-e^{-\theta} \leq \theta\ , \tag{9}$$

where the right inequality holds for all $\theta \geq 0$ and the left hold for all $\theta \in [0,1]$. Let us take $\theta = \theta(A) = -\log \mathbb{P}\left\{\tau_A > f_A\right\}$ and remark that $\mathbb{P}\left\{\tau_A \leq f_A\right\} = 1-e^{-\theta}$. Using (9) and lemma 3, we get

$$\frac{\theta(A)}{f_A\mathbb{P}(A)} \geq \frac{1}{1+C'+Cf_A\mathbb{P}(A)+2K} \geq \frac{1}{1+C'+C+2K} = \Lambda_1\ . \tag{10}$$

where C', C and K are the same constants that appear in lemma 3.

On the other hand, it follows from (9), (8) and lemma 2 that

$$-\log \mathbb{P}\left\{\tau_A > f_A\right\} \leq 2\mathbb{P}\left\{\tau_A \leq f_A\right\} \leq 2f_A\mathbb{P}(A)\ ,$$

for all $n \geq n_0$ and we can take $\Lambda_2 = 2$. This ends the proof of lemma 4.

6 The asympthotic independence property.

Lemma 5. *Suppose that the process is $*$-mixing. Let $A \in \mathcal{C}_n$ and $t, s \in I\!N$ such that $\max\{t, s\} \geq n$. For all $\Delta \in I\!N$ with $n \leq \Delta \leq \max\{t, s\}$, the following inequality holds*

$$\sup_{\Delta} |I\!P\{\tau_A > t + s\} - I\!P\{\tau_A > t\}\, I\!P\{\tau_A > s\}| \leq [2\Delta I\!P(A) + *(\Delta - n)] \ .$$

Proof. *We use Galves-Schmitt approach. We introduce a gap Δ from $t+1$ to $t+\Delta$. The idea is that this gap should be large enough for the process to loose memory but small enough in order to keep the probability of the event close to the original one. Without loose of generality we assume that $n \leq \Delta \leq s$.*

$$\begin{aligned}
& |I\!P\{\tau_A > t + s\} - I\!P\{\tau_A > t\}\ I\!P\{\tau_A > s\}| \\
\leq\ & \left|I\!P\{\tau_A > t + s\} - I\!P\left\{\tau_A > t \cap \tau_A \circ T^{-(t+\Delta)} > s - \Delta\right\}\right| \\
+\ & \left|I\!P\left\{\tau_A > t \cap \tau_A \circ T^{-(t+\Delta)} > s - \Delta\right\} - I\!P\{\tau_A > t\}\, I\!P\left\{\tau_A \circ T^{-(t+\Delta)} > s - \Delta\right\}\right| \\
+\ & |I\!P\{\tau_A > t\}\, I\!P\{\tau_A > s - \Delta\} - I\!P\{\tau_A > t\}\, I\!P\{\tau_A > s\}| \ . \qquad (11)
\end{aligned}$$

We use the stationarity of the measure and lemma 2 to get upper bound for the first and third terms. We use the mixing property on the second. Therefore equation (11) is bounded above by

$$I\!P\{\tau_A \leq \Delta\} + *(\Delta - n) + I\!P\{\tau_A \leq \Delta\} \leq 2\Delta I\!P(A) + *(\Delta - n) \ .$$

We shall need the following iterated version of lemma 5.

Lemma 6. *For all $A \in \mathcal{C}_n$ let us define $\theta = \theta(A) = -\log I\!P\{\tau_A > s\}$. Then, for any integer $k \geq 1$ and any $s > n$, the following inequality holds*

$$\left|I\!P\{\tau_A > k\, s\} - e^{-\theta k}\right| \leq \inf_{\Delta \geq n} \left\{ \frac{2\Delta I\!P(A) + *(\Delta - n)}{I\!P\{\tau_A \leq s\}} \right\} \ .$$

Proof. It is enough to prove, by induction, that

$$\left|I\!P\{\tau_A > k\, s\} - e^{-\theta k}\right| \leq [2\Delta I\!P(A) + *(\Delta - n)] \left[1 + e^{-\theta} + \ldots + e^{-\theta(k-2)}\right] \ ,$$

holds for any integer $k \geq 1$.

The result is trivially true for $k = 1$. Let us assume that it holds for $k \geq 1$. By the triangle inequality

$$\begin{aligned}\left|I\!P\{\tau_A > (k+1)\ s\} - e^{-\theta(k+1)}\right| &\leq \left|I\!P\{\tau_A > (k+1)\ s\} - I\!P\{\tau_A > k\ s\}e^{-\theta}\right| \\ &+ e^{-\theta}\left|I\!P\{\tau_A > k\ s\} - e^{-\theta k}\right| . \qquad (12)\end{aligned}$$

By lemma 5

$$\left|I\!P\{\tau_A > (k+1)\, s\} - I\!P\{\tau_A > k\, s\}e^{-\theta}\right| \leq 2\Delta I\!P(A) + *(\Delta - n)\ . \qquad (13)$$

Bounding the first term on the right hand side of (12) as in (13) and using the hypothesis that the result holds for k, we obtain the corresponding inequality for $k+1$. This concludes the proof of lemma 6.

7 Proof of Theorems 2 and 3.

It is enough to prove that the theorems hold for any cylinder $A \in \mathcal{C}_n$ with $n \geq n_0$. Let us use the shorthand notation $\lambda_A = \lambda_{A,f}$.

Proof of the lower bound.

The proof of the lower bound follows the proof of proposition 2.3 in Hirata-Saussol-Vaienti, 1998.

Since $I\!P\left\{\tau_A > t\right\}$ is constant in $[[t],[t]+1) = [k, k+1)$ we get that

$$\begin{aligned}\sup_{t\in[k,k+1)}\left|I\!P\left\{\tau_A > t\right\} - e^{-t\lambda_A I\!P(A)}\right| &\geq \frac{1}{2}\left|e^{-k\lambda_A I\!P(A)} - e^{-(k+1)\lambda_A I\!P(A)}\right| \\ &\geq e^{-k\lambda_A I\!P(A)}\frac{1 - e^{-\Lambda_1 I\!P(A)}}{2} \\ &\geq C\ \Lambda_1 I\!P(A)\ e^{-k\Lambda_2 I\!P(A)}.\end{aligned}$$

In particular

$$\sup_{t>0}\left|I\!P\left\{\tau_A > t\right\} - e^{-t\lambda_A I\!P(A)}\right| \geq C I\!P(A).$$

Proof of the upper bound.

Fix $t > 0$ and write $t = k\ [f_A] + r$ where $k = k(A) \geq 0$ is the integer part of $t/[f_A]$ and $0 \leq r < [f_A]$. Let us write t_A for $k\ [f_A]$

$$\begin{aligned}
\left|I\!\!P(\tau_A > t) - e^{-\lambda_A I\!\!P(A)t}\right| &\leq |I\!\!P(\tau_A > t) - I\!\!P(\tau_A > t_A)| \\
&+ \left|I\!\!P(\tau_A > t_A) - e^{-\lambda_A I\!\!P(A)t_A}\right| \\
&+ \left|e^{-\lambda_A I\!\!P(A)t_A} - e^{-\lambda_A I\!\!P(A)t}\right| . \qquad (14)
\end{aligned}$$

By stationarity and lemma 1

$$|I\!\!P(\tau_A > t) - I\!\!P(\tau_A > t_A)| \leq r\ I\!\!P(A) \leq f_A I\!\!P(A)\ . \qquad (15)$$

Lemma 6 and (10) give an upper bound for the second term in (14).

$$\left|I\!\!P(\tau_A > t_A) - e^{-\lambda_A I\!\!P(A)t_A}\right| \leq C \inf_{\Delta \geq n} \frac{[2\Delta I\!\!P(A) + *(\Delta - n)]}{f_A I\!\!P(A)}\ . \qquad (16)$$

Elementary calculus provides a first upper bound for the third term in (14)

$$\left|e^{-\lambda_A I\!\!P(A)t_A} - e^{-\lambda_A I\!\!P(A)t}\right| \leq \lambda_A I\!\!P(A)\ r \leq \Lambda_2 f_A I\!\!P(A)\ . \qquad (17)$$

For Theorem 3 we note that the summability of α implies that $\alpha(x) \leq C/x$. Choosing $f_A = \left(\frac{1}{I\!\!P(A)}\right)^{3/4}$ and $\Delta = \left(\frac{1}{I\!\!P(A)}\right)^{1/2} + n$ in the sum of the three terms (15), (16) and (17), we get the desired upper bound.

For Theorem 2, if $\phi(x) \leq C/x^\epsilon$ with $\epsilon > 0$, choose $f_A = \frac{1}{I\!\!P(A)^\gamma}$, $1 > \gamma > 0$, $\Delta = \frac{1}{I\!\!P(A)^\beta} + n$, $\gamma > \beta > 0$, and get

$$\begin{aligned}
\frac{\Delta I\!\!P(A) + \phi(\Delta - n)}{f_A I\!\!P(A)} + f_A I\!\!P(A) &\leq C I\!\!P(A)^{\gamma-\beta} + I\!\!P(A)^{\gamma+\epsilon\beta-1} + I\!\!P(A)^{1-\gamma} \\
&\leq C I\!\!P(A)^{1/2(\epsilon+1)}\ .
\end{aligned}$$

Otherwise, there is m such that $\phi(m) \geq m I\!\!P(A)$ with $m \geq n$ and $\phi(m) < 1$. Then taking $f_A = \frac{\phi^h(n)}{I\!\!P(A)}$, $\Delta = \frac{\phi^d(m)}{I\!\!P(A)} + n$ with $0 < h < d < 1$ we have

$$\frac{\phi(\Delta - n)}{f_A I\!\!P(A)} = \frac{\phi(\frac{\phi^d(m)}{I\!\!P(A)})}{\phi^h(n)} \leq \phi^{1-h}(n)\ ,$$

so, the speed of convergence is bounded by

$$\phi^{d-h}(n) + \phi^{1-h}(n) + \phi^{h}(n) \leq \phi^{(1-\varepsilon)/2}(n) \ .$$

for every $\varepsilon > 0$, and by continuity the theorem follows.

This, together with lemma 4, concludes the proof of Theorems 2 and 3.

8 Proof of the main theorem.

Let us write $\xi_{A,\mu} = \lambda_{A,f}$ with $f_A = \mu/I\!P(A)$ and $1 \leq \mu \leq 2$ and $\xi_A = \xi_{A,1}$. For $t \leq \Delta$ we have

$$\begin{aligned} \left| I\!P\left\{\tau_A > t\right\} - e^{-t\xi_{A,\mu(t)} I\!P(A)} \right| &= \left| I\!P\left\{\tau_A \leq t\right\} - \left(1 - e^{-t\xi_{A,\mu(t)} I\!P(A)}\right) \right| \\ &\leq I\!P\left\{\tau_A \leq \Delta\right\} + 1 - e^{-\Delta \xi_A I\!P(A)} \\ &\leq \Delta I\!P(A) + \Xi_2 \Delta I\!P(A) \ . \end{aligned}$$

For $t > \frac{1}{I\!P(A)}$, let us write $t = k\ \mu \frac{1}{I\!P(A)}$, with $k \in I\!N$ and $1 \leq \mu \leq \frac{k+1}{k}$. By lemma 6 and lemma 4

$$\begin{aligned} \left| I\!P\left\{\tau_A > t\right\} - e^{-tI\!P(A)\xi_{A,\mu}} \right| &= \left| I\!P\left\{\tau_A > k\mu \frac{1}{I\!P(A)}\right\} - I\!P\left\{\tau_A > \mu \frac{1}{I\!P(A)}\right\}^k \right| \\ &\leq \frac{2\Delta I\!P(A) + *(\Delta - n)}{I\!P\left\{\tau_A \leq \mu \frac{1}{I\!P(A)}\right\}} \\ &\leq C\left(2\Delta I\!P(A) + *(\Delta - n)\right) \ . \end{aligned}$$

For $\Delta < t < \frac{1}{I\!P(A)}$, let us write $t = \delta \frac{1}{I\!P(A)}$, then $\Delta I\!P(A) < \delta < 1$ and there are $k \in I\!N$ and $1 \leq \mu \leq 2$ such that $k\delta = \mu$. We apply the mean value theorem to the function $f(x) = x^{\frac{1}{k}}$. After this, we apply lemma 6 with gap of length Δ (since $t > \Delta$) and scale $\delta \frac{1}{I\!P(A)}$.

$$\begin{aligned} & \left| I\!P\left\{\tau_A > t\right\} - e^{-t\xi_{A,\mu} I\!P(A)} \right| \\ = & \left| I\!P\left\{\tau_A > \frac{\mu}{k} \frac{1}{I\!P(A)}\right\} - I\!P\left\{\tau_A > \mu \frac{1}{I\!P(A)}\right\}^{1/k} \right| \\ \leq & \left| I\!P\left\{\tau_A > \frac{\mu}{k} \frac{1}{I\!P(A)}\right\}^k - I\!P\left\{\tau_A > \mu \frac{1}{I\!P(A)}\right\} \right| \end{aligned}$$

$$
\begin{aligned}
& \times \frac{1}{k} \max \left\{ I\!P \left\{ \tau_A > \frac{\mu}{k} \frac{1}{I\!P(A)} \right\}^k ; I\!P \left\{ \tau_A > \mu \frac{1}{I\!P(A)} \right\} \right\}^{\frac{1}{k} - 1} \\
\leq \; & \frac{2\Delta I\!P(A) + \alpha(\Delta)}{I\!P \left\{ \tau_A \leq \frac{\mu}{k} \frac{1}{I\!P(A)} \right\}} \frac{1}{k} C \\
\leq \; & C \left(2\Delta I\!P(A) + \alpha(\Delta) \right) \ . \qquad (18)
\end{aligned}
$$

where the last inequality holds since $I\!P \left\{ \tau_A \leq \frac{\mu}{k} \frac{1}{I\!P(A)} \right\} \geq C \frac{\mu}{k}$ and the maximum is bounded (above and below) for $\Delta I\!P(A) \leq \delta \leq 1$ by lemma 4.

Moreover if $t \in [k/I\!P(A), (k+1)/I\!P(A)]$ then $t = k\mu \frac{1}{I\!P(A)}$ and $e^{-\xi_{A,\mu(t)} I\!P(A)t}$ is a left-continuous non-increasing function of t.

$$
\begin{aligned}
& \left| \lim_{t \to (k+1)/I\!P(A)} e^{-\xi_{A,\mu(t)}} - e^{-\xi_{A,\mu((k+1)/I\!P(A))}} \right| \\
= \; & \left| I\!P \left(\tau_A > \frac{k+1}{k} \frac{1}{I\!P(A)} \right)^k - I\!P \left(\tau_A > \frac{1}{I\!P(A)} \right)^{k+1} \right| \\
\leq \; & \left| I\!P \left(\tau_A > \frac{k+1}{k} \frac{1}{I\!P(A)} \right)^k - I\!P \left(\tau_A > (k+1) \frac{1}{I\!P(A)} \right) \right| \\
+ \; & \left| I\!P \left(\tau_A > (k+1) \frac{1}{I\!P(A)} \right) - I\!P \left(\tau_A > \frac{1}{I\!P(A)} \right)^{k+1} \right| \\
\leq \; & 2 \left(2\Delta I\!P(A) + *(\Delta - n) \right) \ .
\end{aligned}
$$

This together with lemma 4 concludes the proof.

Corollary. *For any exponentially decreasing α-mixing process, the rate of convergence in Theorem 1 is $nI\!P(A)$.*

Proof. It is enough to compute the minimum on the upper bound in theorem 2 and appliy lemma 1

9 Behavior of $\xi_{A,\mu(t)}$.

Proof of theorem 4. Define $\sigma_{A,f}$ as in (19)

$$
\begin{aligned}
\left| e^{-\lambda_{A,f} I\!P(A)t} - e^{-\zeta_{A,s} I\!P(A)t} \right| \; \leq \; & \left| e^{-\lambda_{A,f} I\!P(A)t} - e^{-\sigma_{A,f} I\!P(A)t} \right| \\
+ \; & \left| e^{-\sigma_{A,f} I\!P(A)t} - e^{-\zeta_{A,s} I\!P(A)t} \right| \ .
\end{aligned}
$$

First note that $\theta \leq 1 - e^{-\theta} + \theta^2/2$ for all $\theta \geq 0$, then

$$\sigma_{A,k} \quad := \quad \frac{I\!P\left\{\tau_A \leq k\right\}}{kI\!P(A)} \leq \frac{-\log I\!P\left\{\tau_A > k\right\}}{kI\!P(A)} \leq \sigma_{A,k} + 2kI\!P(A) \ , \quad (19)$$

and

$$|\sigma_{A,k} - \lambda_{A,k}| \leq CkI\!P(A) \ .$$

The second term is bounded by decomposing it in terms with $j \leq n/s$ and $j > n/s$, and using mixing properties for terms in the second block.

$$\begin{aligned}
& \left| e^{-\sigma_{A,f} I\!P(A)t} - e^{-\zeta_{A,s} I\!P(A)t} \right| \\
\leq \ & |\sigma_{A,f} - \zeta_{A,s}| \\
= \ & \frac{1}{f_A I\!P(A)} \left| \sum_{j=1}^{f_A} \left[I\!P\left\{\tau_A = n/s\right\} - I\!P\left\{\tau_A = j\right\} \right] \right| \\
\leq \ & \frac{1}{f_A I\!P(A)} \left[\sum_{j=n/s}^{f_A} I\!P\left\{ \tau_A \leq j - \frac{n}{s} \cap \tau_A \circ T^{-(j-\frac{n}{s})} = \frac{n}{s} \right\} + 2\frac{n}{s} I\!P(A) \right] \\
\leq \ & \frac{1}{f_A I\!P(A)} \sum_{j=n/s}^{f_A} I\!P\left\{ \tau_A \leq j - \frac{n}{s} \cap T^{-j} A^{(\frac{n}{2s})} \right\} + 2\frac{n}{s f_A} \\
\leq \ & \frac{1}{f_A I\!P(A)} \sum_{j=n/s}^{f_A} \left[I\!P\left\{ \tau_A \leq j - \frac{n}{s} \right\} I\!P(A^{(\frac{n}{2s})}) + I\!P\left\{ \tau_A \leq j - \frac{n}{s} \right\} \alpha(\frac{n}{2s}) \right] \\
+ \ & 2\frac{n}{s f_A} \\
\leq \ & \frac{s^2 f_A I\!P(A^{(\frac{n}{2s})})}{n^2} + f_A \alpha(\frac{n}{2s}) + 2\frac{n}{s f_A} \\
\leq \ & Ce^{-cn} \ .
\end{aligned}$$

where the last inequality follows choosing

$$f_A = \min\left\{ \frac{n^2}{s^2 I\!P(A^{(\frac{n}{4s})})} ; \frac{1}{\alpha(\frac{n}{4s})} \right\} \ ,$$

and using lemma 1. Moreover $\zeta_{A,s} \leq 1$ and since $|\xi_A - \zeta_{A,s}| \leq Ce^{-cn}$ and $\xi_A \geq \Xi_1 > 0$, there exist $\Psi_1 \leq \zeta_{A,s}$. This concludes the proof.

Let $s \in I\!N$. Define $\mathcal{B}_n = \mathcal{B}_n(s)$ as the set of $A \in \mathcal{C}_n$ which recur before time n/s, namely, $A \in \mathcal{B}_n$ if and only if there is an integer $1 \leq j \leq n/s$ such that $A \cap T^{-j} A \neq \emptyset$ for some $1 \leq j \leq n/s$.

Corollary. *Assume that the process is stationary and exponentially α-mixing and let $A \in \mathcal{E}^n \backslash \mathcal{B}_n$. Then in Theorem 3 we get $\lambda_A = 1$, namely, there are positive numbers c, C_1 and C_2 such that the following inequality holds*

$$C_1 \, I\!P\,(A) \le \sup_{t \ge 0} \left| I\!P \left\{ \tau_A > \frac{t}{I\!P(A)} \right\} - e^{-t} \right| \le C_2 \, e^{-cn}$$

Proof. If $A \in \mathcal{B}_n$ then $\zeta_{A,s} = 1$.

Now we prove that the cylinders for which the last corollary holds, are typical in the following sense

Lemma 7. *Let the process be α-mixing (at any rate). There exist $s \in I\!N$ and two positive constants C and c such that*

$$I\!P\{\mathcal{B}_n(s)\} \le C e^{-cn} \, .$$

Proof. Let s be a positive integer. Take $A = (X_i = a_i, \, i = 1, \ldots, n)$. We first observe that for any k, with $2 \le k \le n/s$

$$a_k^n = a_1^{n-k+1}$$

implies that

$$a_{rk+1}^{(r+1)k} = a_1^k \, ,$$

for $0 \le r < n/k$. Let $m(k) = [n/k]$, therefore

$$\begin{aligned}
& \sum_{a_1^n \in \mathcal{B}_n} I\!P\{X_1^n = a_1^n\} \\
\le \; & \sum_{k=2}^{n/s} \sum_{b_1^k \in \mathcal{E}^k} I\!P\{X_1^k = X_{k+1}^{2k} = \cdots = X_{(m(k)-1)k+1}^{m(k)k} = b_1^k, X_{m(k)k+1}^{m(k)k+r} = b_1^r\} \\
\le \; & \sum_{k=2}^{n/s} \sum_{b_1^k \in \mathcal{E}^k} \Big(\alpha(0) + I\!P\{X_1^k = b_1^k\}\Big) \, I\!P\{X_{k+1}^{2k} = \cdots = X_{(m(k)-1)k+1}^{m(k)k} = b_1^k\} \\
\le \; & \sum_{k=2}^{n/s} \sum_{b_1^k \in \mathcal{E}^k} \Big(\alpha(0) + I\!P\{X_1^k = b_1^k\}\Big) \, C e^{-c\, n(s-1)/s} \\
\le \; & C\alpha(0) e^{n\left[\frac{\ln|\mathcal{E}| - c\,(s-1)}{s}\right]} + C e^{-c\,n} \, ,
\end{aligned}$$

where we use the α-mixing property and lemma 1 together with the fact that $n-k \geq n-n/s$. Taking s large enough we get the exponential decay.

Counterexample. We now provide an example of a process and a sequence of cylinders A_n such that $\lambda_{A_n} < 1$.

Let $\mathcal{E} = \{0,1\}$ and $A = A_n = \{X_0 = 1, \ldots, X_{n-1} = 1\}$ $\forall n \in I\!N$. Suppose that X_n are i.i.d. with $I\!P(X_n = 1) = p$. Then $\zeta_{A,s} = 1-p$.

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