PIECEWISE ISOMETRIES HAVE ZERO TOPOLOGICAL ENTROPY

Jérôme Buzzi

Centre de Mathématique de l'Ecole Polytechnique / C.N.R.S.

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Introduction

We prove that piecewise isometries in arbitrary dimension have zero topological entropy. This was proven in dimension 2 by Gutkin and Haydn [4] for more general invertible maps, obtaining also a new proof of a result of Katok [5]. But in higher dimensions it was only conjectured while much weaker results were proven (see Goetz [3] and the references therein).

Definition 1. *A piecewise isometry is* $(X, \mathcal{Z}, T)$ *with:*

(1) $X \subset \mathbb{R}^d$, $d = 1, 2, \ldots$.
(2) $\mathcal{Z}$ *a finite collection of pairwise disjoint open polytopes, i.e., finite intersections of open half-spaces, such that* $Y = \bigcup_{Z \in \mathcal{Z}} Z$ *is a dense subset of* X.
(3) $T : Y \to X$ *is such that any restriction to an element of* $\mathcal{Z}$ *is an isometry.*

Definition 2. *The topological entropy of* $(X, \mathcal{Z}, T)$ *is:*

$$h_{\mathrm{top}}(T) = \limsup_{n\to\infty} \frac{1}{n} \log N(T^n)$$

where $N(T^n) = \#\mathcal{Z}^n$ *and* $\mathcal{Z}^n = \{[A_1 \ldots A_n] := A_1 \cap T^{-1}A_2 \cap \cdots \cap T^{-n+1}A_n : A_k \in \mathcal{Z}\} \setminus \{\emptyset\}$.

In other words, $h_{\mathrm{top}}(T)$ is the usual topological entropy (see [2]) of the symbolic dynamics of $(X, \mathcal{Z}, T)$, i.e., the subshift $\{A \in \mathcal{Z}^{\mathbb{N}} : \bigcap_{n\geq 0} \overline{[A_0 \ldots A_n]} \neq \emptyset\}$.

Remark that any ergodic invariant measure has zero exponents and therefore zero entropy, but T being only partially defined on X this does not tell anything about $h_{\mathrm{top}}(T)$.

Theorem 3. *The topological entropy of any piecewise isometry is zero.*

The proof is based on an induction on the dimension d. It uses ideas of Tsujii [6].

Localization

This notion is due to Tsujii [6] who calls it "differentiation". It extracts from a map its local behavior around one point.

Definition 4. *The* **localization** *of a piecewise isometry* $(X, \mathcal{Z}, T)$ *at the point* $x \in X$ *is the piecewise isometry* $(X_x, \mathcal{Z}_x, T_x)$ *defined as follows:*

(1) $X_x = \mathbb{R}^d$.

(2) $\mathcal{Z}_x = \{Z_x : Z \in \mathcal{Z}\} \setminus \{\emptyset\}$ *with:*
$$Z_x = \{z \in \mathbb{R}^d : \exists \epsilon_0 > 0 \text{ s.t. } \forall 0 < \epsilon < \epsilon_0 \; x + \epsilon \cdot z \in Z\}.$$
Set $Y_x = \bigcup_{Z \in \mathcal{Z}_x} Z$.

(3) $T_x : Y_x \to X_x$ *is defined by:*
$$T_x(y) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left(T(x + \epsilon \cdot y) - \lim_{\delta \to 0^+} T(x + \delta \cdot y) \right) \qquad \text{for all } y \in Y_x.$$

Remark 5. If $(X, \mathcal{Z}, T)$ is a localization of some map, then:

(1) $X = \mathbb{R}^d$.

(2) Each $Z \in \mathcal{Z}$ is a positive cone, i.e., is invariant by positive homotheties centered at $0 \in \mathbb{R}^d$.

Remark 6. What is happening for T around a point x coincides with what is happening for T_x around 0. More precisely, if $(X, \mathcal{Z}, T)$ is a piecewise isometry and if $(X_x, \mathcal{Z}_x, T_x)$ is its localization at some point $x \in X$ then there exists $\epsilon > 0$ such that:
$$\{Z \cap B(x, \epsilon) : Z \in \mathcal{Z}\} \setminus \{\emptyset\} = \tau_x \left(\{Z_x \cap B(0, \epsilon) : Z_x \in \mathcal{Z}_x\} \right)$$
$$T|B(x, \epsilon) = \tau_x \circ T_x|B(0, \epsilon) \circ \tau_{-x}$$
where τ_x is the translation by x on $\mathbb{R}^d$.

We shall iterate localizations, i.e., we shall consider piecewise isometries of the form:
$$(X_{x_1, x_2, \dots, x_n}, \mathcal{Z}_{x_1, x_2, \dots, x_n}, T_{x_1, x_2, \dots, x_n})$$
for $x_1 \in X$, $x_2, \dots, x_n \in \mathbb{R}^d$ where $T_{x_1, x_2, \dots, x_n} = (T_{x_1, x_2, \dots, x_{n-1}})_{x_n}$, etc. This is what Tsujii called "higher order differentials".

Remark 7 [6]. As Tsujii observed, all $Z \in \mathcal{Z}_{x_1, x_2, \dots, x_n}$ are invariant by translations along the vector space generated by $x_2, x_3, \dots, x_n$, i.e.,
$$Z + \operatorname{vect}(x_2, x_3, \dots, x_n) = Z$$
One says that $T_{x_2, x_3, \dots, x_n}$ is $\operatorname{vect}(x_2, x_3, \dots, x_n)$**-cylindrical.**

In particular, if $\dim \operatorname{vect}(x_2, x_3, \dots, x_n) = d$, then $\mathcal{Z}_{x_1, x_2, \dots, x_n} = \{\mathbb{R}^d\}$.

$\dim \operatorname{vect}(x_2, x_3, \dots, x_n)$ is called the **order** of the localization.

Lemma 8. *Starting from a given piecewise isometry and taking localizations of any order only finitely many maps are obtained.*

Proof. Observe that a piecewise isometry has only finitely many distinct localizations (of order 1). In fact their number is bounded by the number of simplexes (open faces of all dimensions) involved in the complexes $\bar{Z}$ for $Z \in \mathcal{Z}$. Conclude by induction on the order of the localizations using that if $x_n \in \operatorname{vect}(x_2 \dots, x_{n-1})$ then $T_{x_1, \dots, x_n} = T_{x_1, \dots, x_{n-1}}$. $\square$

Multiplicity

Definition 9. *The* **multiplicity** *of* $(X, \mathcal{Z}, T)$ *at* $x \in X$ *is:*

$$M(T, x) = \#\{Z \in \mathcal{Z} : x \in \overline{Z}\}.$$

Lemma 10. *For a piecewise isometry* T*, there is* $C < \infty$ *such that, for all* $n \geq 0$*,*

$$N(T^n) \leq C \cdot \sup_{x \in X} M(T^n, x).$$

Proof. See our paper [1, Proposition 4]. □

Proposition 11. *Let* T *be a piecewise isometry.*

There exists $C < \infty$ *such that for all* $r \geq 0$*, all* $n \geq 0$ *and all* $(x_1, \ldots, x_r) \in X \times (\mathbb{R}^d)^{r-1}$*:*

$$M((T^n)_{x_1,\ldots,x_r}) \leq C \cdot \sup_{y_1,\ldots,y_{r+1}} N(T_{y_1,\ldots,y_{r+1}})^n$$

$y_1, \ldots, y_{r+1}$ *ranging over* $X \times (\mathbb{R}^d)^r$ *under the condition, if* $r \geq 2$*, that:*

$$\dim(\mathrm{vect}(y_2, \ldots, y_{r+1})) = \dim(\mathrm{vect}(x_2, \ldots, x_r)) + 1.$$

This is the analogon in our setting of Tsujii's estimate [6, Theorem 12].

The Theorem follows easily from the Proposition and the fact that $N(S) = 1$ for any localization S of order d.

Proof of the proposition. Observe first that if $r = 0$ then:

$$M(T^n) = \sup_{x \in X} M(T^n, x) = \sup_{x \in X} N(T^n_x) \leq \left(\sup_{x \in X} N(T_x)\right)^n$$

hence the proposition is obvious in this case.

So we assume $r \geq 1$ and let $x_1, \ldots, x_r \in X \times (\mathbb{R}^d)^{r-1}$. Set $V = \mathrm{vect}(x_2, \ldots, x_r)$.

Clearly, $M((T^n)_{x_1,\ldots,x_r}) = M((T^n)_{x_1,\ldots,x_r}, 0)$.

Let $S = S(x_2, \ldots, x_r)$ be the unit sphere of $V^\perp$ centered at 0.

Remark that it is sufficient to bound the minimal cardinality $n(\mathcal{Z}^n_{x_1,\ldots,x_r}, \epsilon)$ of a cover of S by subsets included in an element of $\mathcal{Z}^n_{x_1,\ldots,x_r}$ and with diameter at most $\epsilon > 0$, for $\epsilon > 0$ small enough.

Indeed, any element of $\mathcal{Z}^n_{x_1,\ldots,x_r}$ touching 0 must intersect the cylinder $S + V$, hence must intersect S as it is V-cylindrical. Thus, $M((T^n)_{x_1,\ldots,x_r}, 0) \leq n(\mathcal{Z}^n_{x_1,\ldots,x_r}, \epsilon)$.

Observe that:

$$n(\mathcal{Z}^{n+1}_{x_1,\ldots,x_r}, \epsilon) \leq n(\mathcal{Z}^n_{x_1,\ldots,x_r}, \epsilon) \cdot \sup_{y_1,\ldots,y_r} m(\mathcal{Z}_{y_1,\ldots,y_r}, \epsilon)$$

where $m(\mathcal{Z}_{y_1,\ldots,y_r}, \epsilon)$ is the maximal number of elements of $\mathcal{Z}_{y_1,\ldots,y_r}$ that a subset of $S(y_2, \ldots, y_r)$ with diameter ϵ can meet. (This uses that T is an isometry on each one of its pieces and that it would be enough to take $(y_1, \ldots, y_r)$ among the images of $(x_1, \ldots, x_r)$ by the local branches at x_1 of T^n).

Clearly $n(\mathcal{Z}_{x_1,\dots,x_r},\epsilon) \leq \left(\frac{2\sqrt{d}}{\epsilon}\right)^d \cdot \#\mathcal{Z}$. Thus it is enough to show that there exists $\epsilon > 0$ depending only on T such that:

$$m(\mathcal{Z}_{y_1,\dots,y_r},\epsilon) \leq \sup_{y_{r+1}} N(T_{y_1,\dots,y_{r+1}}), \tag{1}$$

y_{r+1} ranging over $\mathbb{R}^d$ under condition $\dim(\{y_2,\dots,y_{r+1}\}) = \dim(\{y_2,\dots,y_r\})+1$.

Now, for any $\epsilon > 0$:

$$m(\mathcal{Z}_{y_1,\dots,y_r},\epsilon) \leq \sup_{y_{r+1}\in S(y_2,\dots,y_r)} \#\{Z \in \mathcal{Z}_{y_1,\dots,y_r} : Z \cap B(y_{r+1},\epsilon) \cap S \neq \emptyset\}.$$

To finish the proof, we show that there exists $\epsilon > 0$ depending only on T such that, for all $(y_1,\dots,y_r)$ in $X \times (\mathbb{R}^d)^{r-1}$, all $y \in S(y_2,\dots,y_r)$:

$$\#\{Z \in \mathcal{Z}_{y_1,\dots,y_r} : Z \cap B(y,\epsilon) \neq \emptyset\} \leq N(T_{y_1,\dots,y_r,y}). \tag{2}$$

By Remark 6, for each point $y \in S = S(y_2,\dots,y_r)$, there exists $\epsilon(y) > 0$ such that $T_{y_1,\dots,y_r}$ coincides with $T_{y_1,\dots,y_r,y}$ on $B(y,\epsilon(y))$.

By compactness of S, there exist finitely many points, say $y^1,\dots,y^p$, in S such that S is covered by the balls $B(y^i,\epsilon(y^i))$, $i = 1,\dots,p$. Take $\epsilon > 0$ a Lebesgue number for this cover (i.e., such that any ϵ-ball centered at some point of S is contained in some $B(y^i,\epsilon(y^i))$). Such an ϵ satisfies (2) for all $y \in S(y_2,\dots,y_r)$, $(y_1,\dots,y_r)$ being *fixed*.

As there are only finitely many localizations of T of any order (Lemma 8), one can find an $\epsilon > 0$ which is adequate for *all* $y_1,\dots,y_r$. The claim (1) and therefore the proposition follow immediately. □

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CMAT / Ecole Polytechnique / 91128 Palaiseau cedex / France

E-mail address: buzzi@math.polytechnique.fr