CUBE–ROOT BOUNDARY FLUCTUATIONS FOR DROPLETS IN RANDOM CLUSTER MODELS

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Abstract. For a family of bond percolation models on $\mathbb{Z}^2$ that includes the Fortuin-Kasteleyn random cluster model, we consider properties of the "droplet" that results, in the percolating regime, from conditioning on the existence of an open dual circuit surrounding the origin and enclosing at least (or exactly) a given large area A. This droplet is a close surrogate for the one obtained by Dobrushin, Kotecký and Shlosman by conditioning the Ising model; it approximates an area-A Wulff shape. The local part of the deviation from the Wulff shape (the "local roughness") is the inward deviation of the droplet boundary from the boundary of its own convex hull; the remaining part of the deviation, that of the convex hull of the droplet from the Wulff shape, is inherently long-range. We show that the local roughness is described by at most the exponent 1/3 predicted by nonrigorous theory; this same prediction has been made for a wide class of interfaces in two dimensions. Specifically, the average of the local roughness over the droplet surface is shown to be $O(l^{1/3}(\log l)^{2/3})$ in probability, where $l = \sqrt{A}$ is the linear scale of the droplet. We also bound the maximum of the local roughness over the droplet surface and bound the long-range part of the deviation from a Wulff shape, and we establish the absense of "bottlenecks," which are a form of self-approach by the droplet boundary, down to scale $\log l$. Finally, if we condition instead on the event that the total area of all large droplets inside a finite box exceeds A, we show that with probability near 1 for large A, only a single large droplet is present.

1. Introduction

Consider an Ising model in a finite box (or other "nice" region) Λ in $\mathbb{Z}^2$ at a supercritical inverse temperature β, with minus boundary condition. There is a positive magnetization $m(\beta)$, and if Λ is large, the expected and actual fraction of plus spins observed in Λ will each be approximately $(1 - m(\beta))/2$, with high probability.

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If, however, one conditions on the observed number of plus spins being sufficiently greater than the expected number, then, as first explicated by Dobrushin, Kotecký and Shlosman [11], the typical configuration contains a single macroscopic droplet of the plus phase, that is, a droplet in which the usual proportions of plus and minus spins are reversed. Further, the droplet will have a characteristic equilibrium crystal shape, the solution of an isoperimetric problem, given by the Wulff construction. If, for example, the temperature is such that the usual proportion of plus to minus spins is 20/80, and one conditions the fraction of plus spins to be 30%, then the typical configuration will show a 20/80 mix except inside a Wulff droplet in which the proportion is approximately 80/20. This droplet will cover approximately 1/6 of the box Λ, so as to account for nearly all of the excess plus spins.

This in an example of the general phenomenon of phase separation. The work of Dobrushin, Kotecký and Shlosman in [11] provides the first rigorous derivation of phase separation beginning from a local interaction.

In the joint construction [12] of the Ising model and the corresponding Fortuin-Kasteleyn random cluster model ([13]; see [14]), abbreviated "FK model," the droplet boundary appears as a circuit of open dual bonds. If a particular site, say the origin, is inside the droplet, one expects that the outermost open dual circuit Γ_0 containing the origin will closely approximate the droplet boundary. Since the Ising droplet has approximately a fixed area, we can gain information useful in studying the droplet by studying the FK model conditioned to have Γ_0 enclose at least, or exactly, a given area A. This is our main aim in this paper.

Since it is of interest to study phase separation beyond the context of the Ising model, we establish our results not just for the FK model but for general percolation models having properties known, or reasonably expected, to hold quite widely in the percolating regime. These include the FKG property, a special case of the Markov property, exponential decay of dual connectivity and certain mixing properties.

In [11] for very low temperatures, and in [16] for all subcritical temperatures, bounds are given for the boundary fluctuations of the Ising droplet, that is, for the deviation of the boundary of the observed droplet from the boundary of an appropriately translated and rescaled Wulff (that is, equilibrium crystal) shape. Let N denote the linear scale of the box. For a droplet also of linear scale N, the boundary fluctuations are shown to be of order at most $N^{3/4}\sqrt{\log N}$, and for a droplet of linear scale $l_N = N^\alpha$, with $2/3 < \alpha < 1$, the boundary fluctuations are shown to be of order less than $\sqrt{N^{2/3} l_N}$. There is typically never a droplet of linear scale N^α for $0 < \alpha < 2/3$; if the excess number of plusses is of order $N^{2\alpha}$ with $0 < \alpha < 2/3$, then these plusses are dispersed throughout the minus phase without the formation of a large droplet. Also, when phase separation does occur, other than the single large droplet there are typically no droplets of linear scale greater than $\log N$.

Heuristics suggest that the boundary fluctuation bounds of [11] and [16] are not sharp. To see what the correct fluctuation size should be, one must refine the analysis by considering three separate types of fluctuations. The first type is *shrinkage*—the actual droplet may be smaller than an ideal "full size" Wulff shape large enough to account for all excess plus spins, since some of the excess may be dispersed in the surrounding minus phase. So one should actually consider fluctuations about a shrunken Wulff shape enclosing the same area as the actual droplet. The second type is *local roughness*, defined as inward deviations of the droplet boundary from the boundary of its own convex hull. The third type is *long-wave fluctuations*, defined as deviations of the convex hull of the droplet from the shrunken Wulff shape. (More precise definitions will be made below.)

The local roughness is of particular interest, and is our main focus in this paper, because it is subject to the same type of interface-roughness heuristics as a wide variety of other dynamic and equilibrium systems, including first-passage percolation [21], [19], various deposition models [17], polymers in random environments [22], asymmetric exclusion processes [17] and longest increasing subsequences of random permutations [8], only the last of which is now well-understood rigorously. For a two-dimensional object of linear scale l, these heuristics predict fluctuations of order $l^{1/3}$ and a transverse correlation length of order $l^{2/3}$. For the local roughness, this transverse correlation length should appear as the typical separation between adjacent extreme points, where the droplet boundary touches the boundary of its convex hull (see Section 2.) This is what makes local roughness local—one expects distinct inward excursions of the droplet boundary from the convex hull boundary to interact only minimally. The main result of this paper is that in the random cluster model context, with probability approaching 1 as $l \to \infty$, the average local roughness is $O(l^{1/3}(\log l)^{2/3})$.

Results of Dobrushin and Hryniv [10] and Hryniv [15] (at very low temperatures) strongly suggest that the the fluctuations of the droplet boundary about the shrunken Wulff shape should be Gaussian, heuristically resembling roughly a rescaled Brownian bridge added radially to the Wulff shape. In particular, the long-wave fluctuations should be of order $l^{1/2}$. We are only able to bound these by $l^{2/3}(\log l)^{1/3}$, however, in the random cluster model context.

2. Definitions, Heuristics and Statement of Main Results

The results in this paper make use of only a few basic properties of the FK or other percolation model, so we will state our results for general bond percolation models satisfying these properties. A *bond*, denoted $\langle xy \rangle$, is an unordered pair of nearest neighbor sites of $\mathbb{Z}^2$. When convenient we view bonds as being open line segments in the plane; this should be clear from the context. In particular for $R \subset \mathbb{R}^2$, $\mathcal{B}(R)$ denotes the set of all bonds for which the corresponding open line segments are contained in R, and when we refer to distances between sets of bonds, we mean distances between the corresponding sets of line segments. The exception is for $\Lambda \subset \mathbb{Z}^2$, for which we set $\mathcal{B}(\Lambda) = \{\langle xy \rangle : x, y \in \Lambda\}$. (Again, this should be clear from the context.) For a set $\mathcal{D}$ of bonds we let $V(\mathcal{D})$ denote the set of all endpoints of bonds in $\mathcal{D}$, and

$$\partial \mathcal{D} = \{\langle xy \rangle : x \in V(\mathcal{D}), y \notin V(\mathcal{D})\}, \quad \overline{\mathcal{D}} = \mathcal{D} \cup \partial \mathcal{D}.$$

We write $\overline{\mathcal{B}}(\Lambda)$ for $\overline{\mathcal{B}(\Lambda)}$. A *bond configuration* is an element $\omega \in \{0,1\}^{\mathcal{B}(\mathbb{Z}^2)}$.

The *dual lattice* is the translation of the integer lattice by (1/2,1/2); we write x^* for $x + (1/2, 1/2)$. To each (regular) bond e of the lattice there corresponds a *dual bond* e^* which is its perpendicular bisector; the dual bond is defined to be open in a configuration ω precisely when the regular bond is closed, and the corresponding configuration of dual bonds is denoted ω^*. We write $(\mathbb{Z}^2)^*$ for $\{x^* : x \in \mathbb{Z}^2\}$. A *cluster* in a given configuration is a connected component of the graph with site set $\mathbb{Z}^2$ and all open bonds; *dual clusters* are defined analogously for open dual bonds. (In contexts where there is a boundary condition consisting of a configuration on the complement $\mathcal{D}^c$ for some set $\mathcal{D}$ of bonds, a cluster may include bonds in $\mathcal{D}^c$.) C_x and C_{x^*} denote the regular and dual clusters containing sites x and x^*, respectively. Given a set $\mathcal{D}$ of bonds, we write $\mathcal{D}^*$ for $\{e^* : e \in \mathcal{D}\}$. The set of all endpoints of bonds in $\mathcal{D}^*$ is denoted $V^*(\mathcal{D})$ or $V^*(\mathcal{D}^*)$.

For $\Lambda \subset \mathbb{Z}^2$ or $\Lambda \subset (\mathbb{Z}^2)^*$ we define

$$\partial \Lambda = \{x \notin \Lambda : x \text{ adjacent to } \Lambda\}, \quad \partial_{in} \Lambda = \{x \in \Lambda : x \text{ adjacent to } \Lambda^c\}$$

where adjacency is in the appropriate lattice $\mathbb{Z}^2$ or $(\mathbb{Z}^2)^*$.

A *(dual) path* is a sequence $\gamma = (x_0, \langle x_0 x_1 \rangle, x_1, \dots x_{n-1}, \langle x_{n-1} x_n \rangle, x_n)$ of alternating (dual) sites and bonds. γ is *self-avoiding* if all sites are distinct. We write $x \leftrightarrow y$ (in ω) if there is a path of open bonds (or open dual bonds, if x and y are dual sites) from x to y in ω. A *circuit* is a path with $x_n = x_0$ which has all bonds distinct and which does not "cross itself" (in the obvious sense.) Note we do allow $x_i = x_j$ for any $i \neq j$ here, i.e. a circuit may touch itself without crossing. A path or circuit is *open* in a bond configuration ω if all its bonds are open. The *exterior* of a circuit γ, denoted $\mathrm{Ext}(\gamma)$, is the unique unbounded component of the complement of γ in $\mathbb{R}^2$, and the *interior* $\mathrm{Int}(\gamma)$ is the union of the bounded components. An open circuit γ is called an *exterior circuit* in a configuration ω if $\gamma \cup \mathrm{Int}(\gamma)$ is maximal among all open circuits in ω. (These definitions differ slightly from what is common in the literature.) Similar definitions apply to dual circuits. A site x is surrounded by at most one exterior circuit; when this circuit exists we denote it Γ_x. For u, v points in a path or circuit ζ, let $\zeta^{[u,v]}$ and $\zeta^{(u,v)}$ denote the closed and open segments, respectively, of ζ from u to v (in the direction of positive orientation, for circuits.) $|\cdot|$ denotes the Euclidean norm for vectors, Euclidean length for curves, cardinality for finite sets, and Lebesgue measure for regions in $\mathbb{R}^2$ (which one should be clear from the context.) Euclidean distance is denoted $d(\cdot,\cdot)$. Define $d(A,B) = \inf\{d(x,y) : x \in A, y \in b\}$ for $A, B \subset \mathbb{R}^2$ and $d(x,A) = d(\{x\},A)$. We define the *average local roughness* of a circuit γ by

$$ALR(\gamma) = \frac{|\operatorname{Co}(\gamma) \backslash \operatorname{Int}(\gamma)|}{|\partial \operatorname{Co}(\gamma)|},$$

where $\operatorname{Co}(\cdot)$ denotes the convex hull. The *maximum local roughness* is

$$MLR(\gamma) = sup\{d(x, \partial \operatorname{Co}(\gamma)) : x \in \gamma\}.$$

By a *bond percolation model* we mean a probability measure P on $\{0,1\}^{\mathcal{B}(\mathbb{Z}^2)}$. The conditional distributions for the model P are

$$P_{\mathcal{D},\rho} = P(\cdot \mid \omega_e = \rho_e \text{ for all } e \in \mathcal{D}^c),$$

where $\mathcal{D} \subset \mathcal{B}(\mathbb{Z}^2)$. We say a bond percolation model P has *bounded energy* if there exists $p_0 > 0$ such that

$$p_0 < P(\omega_e = 1 \mid \omega_b, b \neq e) < 1 - p_0 \quad \text{for all } \{\omega_b, b \neq e\}. \tag{2.1}$$

From [9], bounded energy and translation invariance imply that there is at most one infinite cluster P-a.s. Write $\omega_{\mathcal{D}}$ for $\{\omega_e : e \in \mathcal{D}\}$ and let $\mathcal{G}_{\mathcal{D}}$ denote the σ-algebra generated by $\omega_{\mathcal{D}}$. P has the *weak mixing property* if for some $C, \lambda > 0$, for all finite sets $\mathcal{D}, \mathcal{E}$ with $\mathcal{D} \subset \mathcal{E}$,

$$\begin{aligned}\sup\{\operatorname{Var}(P_{\mathcal{E},\rho}(\omega_{\mathcal{D}} \in \cdot), P_{\mathcal{E},\rho'}(\omega_{\mathcal{D}} \in \cdot)) : \rho, \rho' \in \{0,1\}^{\mathcal{E}^c}\} \\ \leq C \sum_{x \in V(\mathcal{D}), y \in V(\mathcal{E}^c)} e^{-\lambda|x-y|},\end{aligned}$$

where $\operatorname{Var}(\cdot,\cdot)$ denotes total variation distance between measures. Roughly, the influence of the boundary condition on a finite region decays exponentially with distance from that region. Equivalently, for some $C, \lambda > 0$, for all sets $\mathcal{D}, \mathcal{F} \subset \mathcal{B}(\mathbb{Z}^2)$,

$$\begin{aligned}\sup\{|P(E \mid F) - P(E)| : E \in \mathcal{G}_{\mathcal{D}}, F \in \mathcal{G}_{\mathcal{F}}, P(F) > 0\} \\ \leq C \sum_{x \in V(\mathcal{D}), y \in V(\mathcal{F})} e^{-\lambda|x-y|}.\end{aligned} \tag{2.2}$$

P has the *ratio weak mixing property* if for some $C, \lambda > 0$, for all sets $\mathcal{D}, \mathcal{F} \subset \mathcal{B}(\mathbb{Z}^2)$,

$$\sup \left\{ \left| \frac{P(E \cap F)}{P(E)P(F)} - 1 \right| : E \in \mathcal{G}_{\mathcal{D}}, F \in \mathcal{G}_{\mathcal{F}}, P(E)P(F) > 0 \right\} \tag{2.3}$$
$$\leq C \sum_{x \in V(\mathcal{D}), y \in V(\mathcal{F})} e^{-\lambda |x-y|},$$

whenever the right side of (2.3) is less than 1.

Let $\mathrm{Open}(\mathcal{D})$ denote the event that all bonds in $\mathcal{D}$ are open.

The FK model [13] with parameters (p, q), $p \in [0,1], q > 0$ on a finite $\mathcal{D} \subset \mathcal{B}(\mathbb{Z}^2)$ is described by a weight attached to each bond configuration $\omega \in \{0,1\}^{\mathcal{B}(\mathcal{D})}$, which is

$$W(\omega) = p^{|\omega|}(1-p)^{|\mathcal{D}|-|\omega|} q^{C(\omega)},$$

where $|\omega|$ denotes the number of open bonds in ω and $C(\omega)$ denotes the number of open clusters in ω, counted in accordance with the boundary condition, if any; see [14] for details and further information. For integer $q \geq 1$ the FK model is a random cluster representation of the q-state Potts model at inverse temperature β given by $p = 1 - e^{-\beta}$. For the study of phase separation involving more than two species, for example in the Potts model, it is useful to be able to "tilt" the distribution with one or more external fields before calculating various probabilities, as well as quantities such as surface tension and magnetization. For the q-state Potts model with external fields h_i on species i, $i = 0, 1, \ldots, q-1$, we need only consider $0 = h_0 \geq h_1 \geq \ldots \geq h_{q-1}$ and then the factor $q^{C(\omega)}$ in the weight $W(\omega)$ is replaced by

$$\prod_{C \in \mathcal{C}(\omega)} \left(1 + (1-p)^{h_1|C|} + \ldots + (1-p)^{h_{q-1}|C|}\right),$$

where $\mathcal{C}(\omega)$ is the set of clusters in the configuration ω and $|C|$ denotes the number of sites in the cluster C. We call species i *viably dominant* if h_i is maximal, i.e. $h_i = h_0$. For each species i and for finite $\Lambda \subset \mathbb{Z}^2$, corresponding to the species-i boundary condition for the Potts model on Λ there is the *i-wired* boundary condition on $\overline{\mathcal{B}}(\Lambda)$ for the FK model, in which sites in Λ connected to $\partial\Lambda$ are considered a single cluster C_∂ and assigned weight

$$(1-p)^{h_i|C_\partial|}.$$

Given a circuit γ and a configuration $\rho \in \{0,1\}^{\mathcal{B}(\mathrm{Ext}(\gamma))}$, conditioning on ρ and on $\mathrm{Open}(\gamma)$ induces a boundary condition on $\mathcal{B}(\mathrm{Int}(\gamma))$ which is a mixture over i of the different i-wired boundary conditions. The weight assigned to i-wiring in this mixture is proportional to $(1-p)^{h_i N(\rho)}$, where $N(\rho)$ is the number of sites in γ plus the number of sites in $\mathrm{Ext}(\gamma)$ connected to γ in ρ. In the absense of external fields, i-wiring is the same for all i and the choice of ρ does not affect the boundary condition induced on $\mathcal{B}(\mathrm{Int}(\gamma))$, which is a form of Markov property, but if an external field is present this property fails. However, the weight assigned to i-wiring in the mixture for non-viably-dominant i is exponentially small in $|\gamma|$, uniformly in ρ, so for large γ the effect of ρ on the boundary condition is uniformly small.

Motivated by the preceding, we say a bond percolation model P has the *Markov property for open circuits* if for every circuit γ (of regular bonds), the bond configurations inside and outside γ are independent given the event $\mathrm{Open}(\gamma)$. We have seen that the FK model has this property if and only if there are no external fields. If P is the infinite-volume k-wired FK model for some viably dominant k, then letting ω_{int} and ω_{ext} denote the bond configurations inside and outside γ, respectively, we have from the preceding discussion for some $C, a > 0$

$$\sup\left\{\left|\frac{P(\omega_{int} \in A \mid \mathrm{Open}(\gamma), \omega_{ext} \in B)}{P(\omega_{int} \in A \mid \mathrm{Open}(\gamma))} - 1\right| : A \in \mathcal{G}_{\mathcal{B}(\mathrm{Int}(\gamma))}, B \in \mathcal{G}_{\mathcal{B}(\mathrm{Ext}(\gamma))}\right\} \leq Ce^{-a|\gamma|} \qquad \text{for all } \gamma. \tag{2.4}$$

When (2.4) holds we say P has the *near-Markov property for open circuits.* It is easy to see that one can interchange the roles of interior and exterior in (2.4). Further, if $\gamma_1, .., \gamma_k$ are circuits with disjoint interiors, $B_i \in \mathcal{G}_{\mathcal{B}(\mathrm{Int}(\gamma_i))}, A \in \mathcal{G}_{\mathcal{B}(\cap_i \mathrm{Ext}(\gamma_i))}$, then by easy induction on k,

$$\frac{P(A \mid \mathrm{Open}(\gamma_i) \cap B_i \text{ for all } i \leq k)}{P(A \mid \mathrm{Open}(\gamma_i) \text{ for all } i \leq k)} \leq \prod_{i \leq k} \frac{1 + Ce^{-a|\gamma_i|}}{1 - Ce^{-a|\gamma_i|}} \tag{2.5}$$

and

$$\frac{P(A \mid \mathrm{Open}(\gamma_i) \text{ for all } i \leq k)}{P(A \mid \mathrm{Open}(\gamma_i) \cap B_i \text{ for all } i \leq k)} \leq \prod_{i \leq k} \frac{1 + Ce^{-a|\gamma_i|}}{1 - Ce^{-a|\gamma_i|}}. \tag{2.6}$$

An event $A \subset \mathcal{B}(\mathbb{Z}^2)$ is called *increasing* if $\omega \in A$ and $\omega \leq \omega'$ imply $\omega' \in A$. Here $\omega \leq \omega'$ refers to the natural coordinatewise partial ordering. A bond percolation model P has the *FKG property* if A, B increasing implies $P(A \cap B) \geq P(A)P(B)$.

Throughout the paper, $\epsilon_1, \epsilon_2, \ldots,$ $c_1, c_2, \ldots$ and $K_1, K_2, \ldots$ are constants which depend only on P. We reserve ϵ_i for constants which are "sufficiently small," K_i for constants which are "sufficiently large," and c_i for those which fall in neither category.

Our basic assumptions will be that

(2.7) P is translation-invariant, invariant under 90° rotation, and has the FKG property, bounded energy and exponential decay of dual connectivity, and $P_{\Lambda,\rho}$ has the FKG property for all Λ, ρ.

When necessary we will also assume weak mixing, ratio weak mixing and/or the near-Markov property for open circuits.

Since P has the FKG property, $-\log P(0^* \leftrightarrow x^*)$ is a subadditive function of x, and therefore the limit

$$\tau(x) = \lim_{n\to\infty} -\frac{1}{n} \log P(0^* \leftrightarrow (nx)^*), \tag{2.8}$$

exists for $x \in \mathbb{Q}^2$, provided we take the limit through values of n for which $nx \in \mathbb{Z}^2$. This definition extends to $\mathbb{R}^2$ by continuity (see [2]); the resulting τ is a norm on $\mathbb{R}^2$, when the dual connectivity decays exponentially (i.e. $\tau(x)$ is positive for all $x \neq 0$, or equivalently by lattice symmetry, $\tau(x)$ is positive for some $x \neq 0$; we abbreviate this by saying *τ is positive.*) By standard subadditivity results,

$$P(0^* \leftrightarrow x^*) \leq e^{-\tau(x)} \quad \text{for all } x. \tag{2.9}$$

In the opposite direction, it is known [4] that if τ is positive, ratio weak mixing holds and some milder assumptions hold then for some ϵ_1 and K_1,

$$P(0^* \leftrightarrow x^*) \geq \epsilon_1 |x|^{-K_1} e^{-\tau(x)} \quad \text{for all } x \neq 0. \tag{2.10}$$

It follows from the fact that the surface tension τ is a norm on $\mathcal{R}^2$ with axis symmetry that, letting e_i denote the ith unit coordinate vector, for $\kappa_\tau = \tau(e_1)$ we have

$$\frac{1}{\sqrt{2}}\kappa_\tau \leq \frac{\tau(x)}{|x|} \leq \sqrt{2}\kappa_\tau \quad \text{for all } \; x \neq 0. \tag{2.11}$$

For a curve γ tracing the boundary of a convex region we define the τ-length of γ as the line integral

$$\mathcal{W}(\gamma) = \int_\gamma \tau(v_x)\, dx,$$

where v_x is the unit forward tangent vector at x and dx is arc length. The *Wulff shape* is the convex set $\mathcal{K}_1 = \mathcal{K}_1(\tau)$ which minimizes $\mathcal{W}(\partial V)$ subject to the constraint $|V| = 1$. (We also refer to multiples of $\mathcal{K}_1$ as Wulff shapes, when confusion is unlikely.) The Wulff shape actually minimizes $\mathcal{W}$ over a much larger class of γ than just boundaries of convex sets ([23],[24]) but that fact will not concern us here.

Let $d_\tau(\cdot,\cdot)$ denote τ-distance; $\mathrm{diam}(\cdot)$ and $\mathrm{diam}_\tau(\cdot)$ denote Euclidean diameter and τ-diameter, respectively. $B(x,r)$ and $B_\tau(x,r)$ denote the closed Euclidean and τ-balls, respectively, of radius r about x. We write $x + A$ for the translation of the set A by the vector x. d_H denotes Hausdorff distance.

The deviation of a closed curve γ from the boundary of an area-A Wulff shape is given by

$$\Delta_A(\gamma) = \inf_z d_H(\gamma, z + \partial(\sqrt{A}\mathcal{K}_1)).$$

As a convention, whenever we refer to the object in a finite class which maximizes or minimizes something, we implicitly assume there is a deterministic algorithm for breaking ties.

Our description of heuristics for the local roughness is nonrigorous, so we permit ourselves the following partly vague assumptions:

(i) The Wulff shape boundary has curvature bounded away from 0 and ∞.

(ii) For a droplet of any linear scale l, there is a characteristic length scale $\xi = \xi(l)$ representing the typical spacing between adjacent extreme points where the droplet touches the boundary of its convex hull.

(iii) On any length scale $n \leq \xi$ the fluctuations of the droplet boundary are of order $\sqrt{n}$.

Here (iii) is reasonable because within each inward excursion between extreme points, the droplet boundary is nearly unconstrained, except by surface tension. (i) is known for the Ising case from the exact solution (see [7], [20].) Under (i), an arc of $\partial(l\mathcal{K}_1)$ of length n deviates from the corresponding secant line by a distance of order n^2/l, so we call n^2/l the *curvature deviation* (on scale n.)

On the characteristic scale ξ the fluctuations and the curvature deviation should be of the same order, that is,

$$\frac{\xi^2}{l} \approx \sqrt{\xi}. \tag{2.12}$$

To see this, consider two adjacent extreme points x and y of the droplet boundary separated by a distance of order ξ. If the curvature deviation $\xi^2/l \gg \sqrt{\xi}$ this means the boundary between x and y is following the straight segment $\overline{xy}$ much more closely than it follows the arc of $\partial(l\mathcal{K}_1)$ from x to y, that is, the droplet has an approximate facet from x to y. But such facets are isoperimetrically disadvantageous since the Wulff shape lacks them under (i), so this is not a probable picture. Therefore we expect $\xi^2/l \leq \sqrt{\xi}$. On the other hand, if the curvature deviation $\xi^2/l \ll \sqrt{\xi}$ then even on length scales $n \gg \xi$ an arc of the Wulff shape boundary looks nearly flat compared to the droplet boundary fluctuations, so along such an arc the roughly n/ξ extreme points appear as a large number of local maxima of the droplet boundary above the Wulff shape boundary, all approximately collinear. This too is an unlikely picture, so we expect $\xi^2/l \geq \sqrt{\xi}$.

From (2.12) we get $\xi \approx l^{2/3}$, and then from (iii), we expect local roughness of order $l^{1/3}$. The same relation (2.12) occurs in the assorted systems mentioned in the introduction.

For $r > q > 0$, an (q,r)-*bottleneck* in an exterior dual circuit γ is an ordered pair (u,v) of sites in γ such that there exists a path of length at most q from u to v in $\mathrm{Int}(\gamma)$, and the segments $\gamma^{[u,v]}$ and $\gamma^{[v,u]}$ each have diameter at least r. When r is not very large (as in our main theorem, where r can be of order $\log l$) the absense of (q,r)-bottlenecks reflects a high degree of regularity in the structure of the boundary.

Our main theorem is the following.

Theorem 2.1. *Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7), the near-Markov property for open circuits, and the ratio weak mixing property. There exist K_i, ϵ_i such that for $A > K_2$ and $l = \sqrt{A}$, under the measure $P(\cdot \mid |\operatorname{Int}(\Gamma_0)| \geq A)$ with probability approaching 1 as $A \to \infty$ we have*

$$ALR(\Gamma_0) \leq K_3 l^{1/3} (\log l)^{2/3}, \tag{2.13}$$

$$\Delta_A(\partial \operatorname{Co}(\Gamma_0)) \leq K_4 l^{2/3} (\log l)^{1/3}, \tag{2.14}$$

$$MLR(\Gamma_0) \leq K_5 l^{2/3} (\log l)^{1/3}, \tag{2.15}$$

and, for $\epsilon_2 A \geq r \geq 15q \geq K_6 \log A$,

$$\Gamma_0 \text{ is } (q,r) - \text{bottleneck-free}. \tag{2.16}$$

It is easy to see that if $MLR(\gamma)$ and $\Delta_A(\partial \operatorname{Co}(\gamma))$ are each a sufficiently small fraction of $\sqrt{A}$, then $MLR(\gamma) = d_H(\gamma, \partial \operatorname{Co}(\gamma))$ and hence

$$\Delta_A(\gamma) \leq \Delta_A(\partial \operatorname{Co}(\gamma)) + MLR(\gamma). \tag{2.17}$$

(Here "sufficiently small" does not depend on A.) Hence provided A is large, (2.14) and (2.15) imply

$$\Delta_A(\Gamma_0) \leq (K_4 + K_5) l^{2/3} (\log l)^{1/3}. \tag{2.18}$$

Theorem 7.4 below shows that one may condition on $|\operatorname{Int}(\Gamma_0)| = A$ instead of on $|\operatorname{Int}(\Gamma_0)| \geq A$ in Theorem 2.1.

The basic strategy of the proof of Theorem 2.1 is like that of [6], [11] and [16]: one establishes a lower bound for $P(|\operatorname{Int}(\Gamma_0)| \geq A)$ and upper bounds for the (unconditional) probability that $|\operatorname{Int}(\Gamma_0)| \geq A$ but Γ_0 does not have the desired behavior, these upper bounds being much smaller than the lower bound. Both the upper and lower bounds involve coarse-graining to create a skeleton $(y_0, .., y_n, y_0)$ for Γ_0, and both require showing that the various connections $y_j \leftrightarrow y_{j+1}$ occur approximately independently, that is,

$$P(y_0 \leftrightarrow ... \leftrightarrow y_n \leftrightarrow y_0)$$

can be approximated in an appropriate sense by

$$\prod_{j \leq n} P(y_j \leftrightarrow y_{j+1})$$

and hence by

$$\exp(-\sum_{j\leq n}\tau(y_{j+1}-y_j))$$

(setting $y_{n+1}=y_0$.) For the lower bound this is a straightforward application of the FKG inequality, but for the upper bound the mixing properties established in Section 3 are needed, and our methods must handle "pathological" forms of Γ_0 having many near-self-intersections. The difficulties are of two types. First, near-independence generally requires large spatial separation, or, under the near-Markov property, separation by a circuit of open bonds, neither of which need be present in our context. Second, direct application of standard mixing properties such as weak mixing requires specifying in advance some *deterministic* spatially separated regions on which the near-independent events will occur, but in our context one does not know *a priori* where the paths $y_j \leftrightarrow y_{j+1}$ may go. This is where Lemma 3.2 below is important.

We consider now the special case of the FK model on $\mathcal{B}(\mathbb{Z}^2)$. For each (p,q) there is a value p^* dual to p at level q given by

$$\frac{(1-p^*)}{p^*}=\frac{p}{q(1-p)};$$

the dual configuration to the infinite-volume wired-boundary FK model at (p,q) is the infinite-volume free-boundary FK model at (p^*,q) (see [14].) The model has a percolation critical point $p_c(q)$ which for $q=1, q=2$ and $q\geq 25.72$ is known to coincide with the self-dual point $p_{sd}(q)=\sqrt{q}/(1+\sqrt{q})$ [18]; positivity of τ is known to hold for $p>p_{sd}(q)$ for these same values of q. For $2<q<25.72$, it is known that positivity of τ holds for $p>p_{sd}(q-1)^*$, where the $*$ refers to duality at level q [5]. The FK model without external field has the Markov property for open circuits; assuming positivity of τ it satisfies (2.7) (see [14]) and has the ratio weak mixing property [3]. In a forthcoming paper we will show that positivity of τ also implies ratio weak mixing for the FK model with external fields. For now, we can conclude the following from Theorem 2.1.

Theorem 2.2. *Let P be the FK model at (p,q) on $\mathbb{B}(\mathbb{Z}^2)$ with $q\geq 1$ (without external fields) and suppose the surface tension τ is positive. There exists K_2 such that for $A>K_2$ and $l=\sqrt{A}$, under the measure $P(\cdot \mid |\mathrm{Int}(\Gamma_0)|\geq A)$ with probability approaching 1 as $A\to\infty$, (2.13) – (2.16) hold.*

Following [16] we call a dual circuit γ *r-large* if $\mathrm{diam}_\tau(\gamma)>r$ and *r-small* if $\mathrm{diam}_\tau(\gamma)\leq r$.

When τ is positive, in the box $\Lambda_N=[-N,N]^2$ the largest open dual circuit typically has diameter of order $\log N$. Let $\mathfrak{C}_N$ denote the collection of all $(K\log N)$-large exterior open dual circuits contained in Λ_N; here and throughout this section, K is a fixed "sufficiently large" constant. To avoid any ambiguity, we impose a wired boundary condition on Λ_N. Rather than conditioning on $|\operatorname{Int}(\Gamma_0)| \geq A$ as in Theorem 2.1, it is sometimes of interest to condition on the event

$$\sum_{\gamma \in \mathfrak{C}_N} |\operatorname{Int}(\gamma)| \geq A.$$

In particular, it is natural to ask whether under such conditioning one has $|\mathfrak{C}_N| = 1$ with high probability. The alternative most difficult to rule out, as suggested by the form of the error term in Theorem 4.1, is the presence of one or more "small" $(K \log N)$-large open dual circuits, of diameter of order $l^{1/3}(\log l)^{2/3}$ or less, outside a single large open dual circuit enclosing nearly area A. In the context of phase separation in the Ising model, single-droplet theorems have been proved in [11] and [16]. However, the method there for ruling out "small" $(K \log N)$-large droplets uses not purely surface tension but rather an argument that it is energentically preferable to dissolve such "small" droplets by spreading the spins they contain throughout the bulk of the system, which is allowed because there is no constraint on the total volume contained in $(K \log N)$-large droplets. Thus what we seek here is different, essentially a single-droplet theorem based purely on surface tension considerations. Let $P_{N,w}$ denote the measure P conditioned on all bonds outside $\operatorname{Int}(\Lambda_N)$ being open, that is, the measure under a wired boundary condition on $\mathcal{B}(\operatorname{Int}(\Lambda_N))$.

Theorem 2.3. *Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7), the near-Markov property for open circuits, and the ratio weak mixing property. There exist ϵ_i, K_i such that for $N \geq 1, K_7(\log N)^2 \leq A \leq c_2 N^2$ and $l = \sqrt{A}$, under the measure $P_{N,w}(\cdot \mid \sum_{\gamma \in \mathfrak{C}_N} |\operatorname{Int}(\gamma)| \geq A)$, with probability approaching 1 (uniformly in A) as $N \to \infty$ we have*

$$|\mathfrak{C}_N| = 1, \tag{2.19}$$

and, for the unique open dual circuit γ in $\mathfrak{C}_N$,

$$ALR(\gamma) \leq K_3 l^{1/3}(\log l)^{2/3}, \tag{2.20}$$

$$MLR(\gamma) \leq K_8 l^{2/3}(\log l)^{1/3}, \tag{2.21}$$

$$\Delta_A(\partial \operatorname{Co}(\gamma)) \leq K_4 l^{2/3}(\log l)^{1/3} \tag{2.22}$$

and, for $\epsilon_3 A \geq r \geq 15q \geq K_9 \log A$,

$$\gamma \text{ is } (q,r) - \textit{bottleneck-free.} \tag{2.23}$$

Here c_2 is any constant less than

$$\sup\{c > 0 : \sqrt{c}\mathcal{K}_1 \subset [-1,1]^2\},$$

and K_3, K_4 are from Theorem 2.1.

As noted after Theorem 2.1, the FK model satisfies the assumptions of Theorem 2.3, provided that τ is positive.

3. Preliminaries—Coarse Graining and Mixing Properties

We first define our coarse-graining concepts. Our definition of the *s-hull skeleton* follows [1], with some added refinements. For a contour γ let E_γ denote the set of extreme points of $\mathrm{Co}(\gamma)$ and let $\gamma_{co} : [0,1] \to \mathbb{R}^2$ be a curve which traces $\partial\,\mathrm{Co}(\gamma)$ in the direction of positive orientation, beginning at the leftmost lattice site u_0 having minimal second coordinate. When confusion is unlikely we also use γ_{co} to denote the image of this curve. Note that

$$u_0 \in E_\gamma \subset \gamma_{co} \cap \gamma \cap (\mathbb{Z}^2)^*.$$

To define the s-hull skeleton we require that the τ-diameter of γ be at least $2s$. We traverse γ in the direction of positive orientation, beginning at, say, the leftmost lattice site u_0 having minimal second coordinate. Given $u_0, .., u_j \in E_\gamma$, go forward from u_j along γ_{co} until either u_0 or $\partial B_\tau(u_j, s)$ is reached, at some point u'_{j+1}. Then backtrack along γ_{co} until a point of E_γ is reached (possibly u'_{j+1}, meaning we backtrack zero distance.) If this backtracking does not require going all the way back to u_j, then this new point of E_γ is labeled u_{j+1}. If instead the backtracking does require going all the way back to u_j, then from u'_{j+1} continue forward along γ_{co}, necessarily in a straight line, to the next point of E_γ, which is then labeled u_{j+1}. Stop the process when $u_{m+1} = u_0$ for some m. The *s-hull pre-skeleton* is then $(u_0, .., u_{m+1})$. (A similar definition, under the name "s-hull skeleton," may be found in [1].) The sites $u_0, .., u_{m+1}$ are sites of E_γ which appear in order in γ_{co} (and in γ.) Therefore $\mathrm{Co}(\{u_0, .., u_{m+1}\})$ is a convex polygon bounded by the polygonal path $u_0 \to .. \to u_{m+1}$. Now

$$\mathcal{W}(\gamma_{co}) \geq \sum_{j=0}^{m} \tau(u_{j+1} - u_j),$$

and as noted in [1], we have

$$\tau(u_{j+2} - u_j) > s \quad \text{for all} \quad 0 \leq j \leq m-2.$$

Therefore

$$m \leq 1 + 2\mathcal{W}(\gamma_{co})/s. \tag{3.1}$$

To obtain the s-hull skeleton we refine the s-hull pre-skeleton. This is necessary because if γ_{co} has a sharp corner, then the polygonal path $u_0 \to .. \to u_{m+1}$ may clip this corner excessively, meaning part of γ may be too far outside the polygon for our needs. We must add vertices so that, whenever possible, the angular change between successive segments of the polygonal path does not exceed $s/\text{diam}(\gamma)$, which is of the order of the angular change we would obtain if γ were a circle. By convexity, for each $x \in \gamma_{co}$ there exists a forward tangent vector v_x and a corresponding forward tangent line; for $y \neq x \in \gamma_{co}$ let $\alpha(x, y)$ denote the angle measured counterclockwise from v_x to v_y. Fix $0 \leq j \leq m$ and let $u_{j0} = u_j$. Note that $\alpha(u_j, \cdot)$ is a nondecreasing function as one traces γ_{co} from u_j to u_{j+1}. Having defined $u_{j0}, .., u_{jk} \in E_\gamma \cap \gamma_{co}^{[u_{j-1}, u_{j+1}]}$, let $u_{j,k+1}$ be the first point of γ_{co} after $u_{j,k}$ for which $\alpha(u_{j,k}, u_{j,k+1}) \geq s/\operatorname{diam}_\gamma$. If there is no such point, then set $u_{j,k+1} = u_{j+1}$ and stop the process. Necessarily $u_{j,k+1}$ is a lattice site in E_γ.

We call the sites u_{jk} strictly between u_j and u_{j+1} *refinement sites*. Let $(w_0, .., w_n, w_{n+1})$ be a relabeling of all sites $u_{jk}, 0 \leq j \leq m, k \geq 1$, in order on γ_{co}, with $w_0 = w_{n+1} = u_0$; we call $(w_0, .., w_n, w_{n+1})$ the *s-hull skeleton* of γ and denote it $\text{HSkel}_s(\gamma)$. The polygonal path $w_0 \to .. \to w_{n+1}$ is denoted $\text{HPath}_s(\gamma)$. It is easy to see that the angle between $u_{jk} - u_{j,k-1}$ and $u_{j,k+2} - u_{jk}$ cannot be less than $s/\operatorname{diam}(\gamma)$, for $k \geq 1$. It follows that the number of refinement sites satisfies

$$n - m \leq 4\pi \text{diam}(\gamma)/s$$

With (3.1) this shows that the number of sites in the s-hull skeleton satisfies

$$n + 1 \leq K_{10} \text{diam}(\gamma)/s. \tag{3.2}$$

As with the pre-skeleton, the sites $w_0, .., w_{n+1}$ appear in order in γ and in γ_{co}, and $\text{Co}(\{w_0, .., w_{n+1}\})$ is a convex polygon bounded by $\text{HPath}_s(\gamma)$.

A key property of the s-hull skeleton involves the extent to which γ can go outside $\text{Co}(\{w_0, .., w_{n+1}\})$. Let T_j denote the triangle formed by the segment $\overline{w_j w_{j+1}}$, the forward tangent line at w_j and the backward tangent line at w_{j+1}. The angle between these two tangent lines is at most $s/\operatorname{diam}(\gamma)$. Now every point of γ outside $\text{Co}(\{w_0, .., w_{n+1}\})$ is in some T_j, and $\gamma_{co} \cap T_j \subset \gamma^{[w_j, w_{j+1}]}$. Let

$$J = \{j \leq n : \tau(w_{j+1} - w_j) \leq 2s\}.$$

For distinct points $x, y \in \mathbb{R}^2$ let H_{xy} ($\overline{H}_{xy}$) denote the open (closed) halfspace which is to the right of the line from x to y. From the construction of the s-hull pre-skeleton, if $\gamma^{(u_j, u_{j+1})} \not\subset B_\tau(u_j, s)$ for some j, then $\gamma \cap H_{u_j u_{j+1}} = \phi$. It follows that if $\tau(w_{j+1} - w_j) > 2s$ for some j, then w_j and w_{j+1} are sites of the s-hull pre-skeleton and $\gamma \cap H_{w_j w_{j+1}} = \phi$. Thus we have

$$\operatorname{Int}(\gamma) \backslash \operatorname{Int}(\operatorname{HPath}_s(\gamma)) \subset \cup_{j \in J} T_j. \tag{3.3}$$

For $j \in J$ we have $|T_j| \leq K_{11} s^3 / \operatorname{diam}(\gamma)$ and

$$d\big(x, \operatorname{Int}(\operatorname{HPath}_s(\gamma))\big) \leq K_{12} s^2 / \operatorname{diam}(\gamma) \quad \text{for all } x \in T_j.$$

With (3.3) this shows that

$$|\operatorname{Int}(\gamma) \backslash \operatorname{Int}(\operatorname{HPath}_s(\gamma))| \leq K_{13} s^2 \tag{3.4}$$

and

$$\sup_{x \in \operatorname{Co}(\gamma)} d\big(x, \operatorname{Int}(\operatorname{HPath}_s(\gamma))\big) \leq K_{12} s^2 / \operatorname{diam}(\gamma). \tag{3.5}$$

From (3.5) and convexity it follows that

$$\mathcal{W}(\gamma_{co}) \leq \mathcal{W}(\operatorname{HPath}_s(\gamma)) + K_{14} s^2 / \operatorname{diam}(\gamma). \tag{3.6}$$

We turn now to mixing properties. The following is an immediate consequence of the definition of ratio weak mixing.

Lemma 3.1. *([4]) Suppose P has the ratio weak mixing property. There exists a constant K_{15} as follows. Suppose $r > 3$ and $\mathcal{D}, \mathcal{E} \subset \mathbb{Z}^2$ with $\operatorname{diam}(\mathcal{E}) \leq r$ and $d(\mathcal{D}, \mathcal{E}) \geq K_{15} \log r$. Then for all $A \in \mathcal{G}_{\mathcal{D}}$ and $B \in \mathcal{G}_{\mathcal{E}}$, we have*

$$\frac{1}{2} P(A) P(B) \leq P(A \cap B) \leq 2 P(A) P(B).$$

A weakness of Lemma 3.1 is that the locations $\mathcal{D}, \mathcal{E}$ of the two events must be deterministic. The next lemma applies only to a limited class of events but allows the locations to be partially random. For $\mathcal{C} \subset \mathcal{D} \subset \mathcal{B}(\mathbb{Z}^2)$ we say an event $A \subset \{0,1\}^{\mathcal{D}}$ *occurs on* $\mathcal{C}$ (or on $\mathcal{C}^*$) in $\omega \in \{0,1\}^{\mathcal{D}}$ if $\omega' \in A$ for every $\omega' \in \{0,1\}^{\mathcal{D}}$ satisfying $\omega'_e = \omega_e$ for all $e \in \mathcal{C}$. For a possibly random set $\mathcal{F}(\omega)$ we say A *occurs only on* $\mathcal{F}$ (or equivalently, on $\mathcal{F}^*$) if $\omega \in A$ implies A occurs on $\mathcal{F}(\omega)$ in ω. We say events A and B *occur at separation* r in ω if there exist $\mathcal{C}, \mathcal{E} \subset \mathcal{D}$ with $d(\mathcal{C}, \mathcal{E}) \geq r$ such that A occurs on $\mathcal{C}$ and B occurs on $\mathcal{E}$ in ω. Let $A \circ_r B$ denote the event that A and B occur at separation r. Let $\mathcal{D}^r = \{e \in \mathcal{B}(\mathbb{Z}^2) : d(e, \mathcal{D}) \leq r\}$.

In the next lemma we give two alternate hypotheses, in the interest of wider applicability, though either hypothesis alone suffices for our purposes in this paper.

Lemma 3.2. *Assume (2.7) and either (i) the ratio weak mixing property or (ii) both the weak mixing property and the near-Markov property for open circuits. There exist constants K_i, ϵ_i as follows. Let $\mathcal{D} \subset \mathcal{B}(\mathbb{Z}^2)$, $x^* \in (\mathbb{Z}^2)^*$ and $r > K_{16} \log |\mathcal{D}|$, and let A, B be events such that A occurs only on C_{x^*} and $B \in \mathcal{G}_{\mathcal{D}}$. Then*

$$P(A \circ_r B) \leq (1 + K_{17} e^{-\epsilon_5 r}) P(A) P(B). \tag{3.7}$$

Proof. First suppose P has the ratio weak mixing property. For $y^* \in V^*(\mathcal{D}^r)$, let $B^{y^*} = B(y^*, r/12)$, $B^{y^*}_{\mathbb{Z}} = B^{y^*} \cap (\mathbb{Z}^2)^*$ and $D^{y^*} = B(y^*, r/6)$, $D^{y^*}_{\mathbb{Z}} = D^{y^*} \cap (\mathbb{Z}^2)^*$. Define events

$$G_{y^*} = [y^* \leftrightarrow \partial_{in} B^{y^*}_{\mathbb{Z}} \text{ by an open dual path}],$$

$$Z_{y^*} = [A \text{ and } B \text{ occur at separation } r \text{ on } \overline{\mathcal{B}}(D^{y^*})^c],$$

$$Q = \cup_{y^* \in V^*(\mathcal{D}^*)} (Z_{y^*} \cap G_{y^*}).$$

Then by (2.9),

$$P(G_{y^*}) \leq K_{18} e^{-\epsilon_6 r}. \tag{3.8}$$

Let C and λ be as in (2.2). Then for some K_i, ϵ_i depending on C, λ, provided K_{16} is chosen large enough

$$\begin{aligned} &P((A \circ_r B) \cap Q) \\ &\quad \leq \sum_{y^* \in V^*(\mathcal{D}^r)} P(Z_{y^*}) P(G_{y^*} \mid Z_{y^*}) \\ &\quad \leq \sum_{y^* \in V^*(\mathcal{D}^r)} P(Z_{y^*}) (P(G_{y^*}) + K_{19} r^2 e^{-\epsilon_7 r}) \\ &\quad \leq K_{20} r^2 |\mathcal{D}| P(A \circ_r B)(K_{18} e^{-\epsilon_6 r} + K_{19} r^2 e^{-\epsilon_7 r}) \\ &\quad \leq K_{21} e^{-\epsilon_8 r} P(A \circ_r B). \end{aligned} \tag{3.9}$$

Therefore

$$P(A \circ_r B) \leq (1 - K_{21} e^{-\epsilon_8 r})^{-1} P((A \circ_r B) \cap Q^c). \tag{3.10}$$

Suppose $\omega \in (A \circ_r B) \cap Q^c$, and consider $\mathcal{C}, \mathcal{E}$ with $\mathcal{C}^* \subset C_{x^*}(\omega)$, $\mathcal{E} \subset \mathcal{D}$ and $d(\mathcal{C}, \mathcal{E}) \geq r$ for which A occurs on $\mathcal{C}^*$ and B occurs on $\mathcal{E}$ in ω. If $r/2 < d(y^*, \mathcal{C}) \leq r/2 + 1$ for some $y^* \in V^*(\mathcal{D}^r)$, then $y^* \notin C_{x^*}(\omega)$ since $\omega \notin Z_{y^*} \cap G_{y^*}$. Therefore $C_{x^*}(\omega) \subset \mathcal{C}^{r/2}$ and hence $d(\mathcal{E}, C_{x^*}(\omega)) > r/2$. For $\mathcal{F} \subset \mathcal{B}(\mathbb{Z}^2)$ let $B_{\mathcal{F},r}$ be the event that B occurs on some set $\mathcal{E} \subset \mathcal{D}$ with $d(\mathcal{E}, \mathcal{F}) > r/2$. We call $\mathcal{F}$ A-*sufficient* if $\omega_e = 0$ for all $e \in \mathcal{F}$ implies $\omega \in A$. Let $\mathfrak{A}$ denote the set of all finite A-sufficient subsets of $\mathcal{B}(\mathbb{Z}^2)$. Since the event $[C_{x^*} = \mathcal{F}^*]$ occurs on $\overline{\mathcal{B}}(V(\mathcal{F}))$ for all $\mathcal{F} \subset \mathcal{B}(\mathbb{Z}^2)$, we have by ratio weak mixing

$$
\begin{aligned}
(3.11) \qquad & P((A \circ_r B) \cap Q^c) \\
& \leq \sum_{\mathcal{F} \in \mathfrak{A}} P([C_{x^*} = \mathcal{F}^*] \cap B_{\mathcal{F},r}) \\
& \leq \sum_{\mathcal{F} \in \mathfrak{A}} (1 + K_{22}|\mathcal{D}|e^{-\epsilon_9 r}) P(C_{x^*} = \mathcal{F}^*) P(B_{\mathcal{F},r}) \\
& \leq (1 + K_{23} e^{-\epsilon_{10} r}) P(A) P(B).
\end{aligned}
$$

Combining (3.10) and (3.11) proves (3.7), provided K_{16} is sufficiently large.

Under hypothesis (ii) of weak mixing and the near-Markov property for open circuits, the proof through the first inequality of (3.11) is still valid, but we need to modify the rest of (3.11) as follows. Fix $\mathcal{F} \in \mathfrak{A}$ and let $F = [C_{x^*} = \mathcal{F}^*]$, $\mathcal{D}' = \mathcal{D} \backslash \mathcal{F}^r$, $\mathcal{A} = \{e \in \mathcal{B}(\mathbb{Z}^2) : r/6 \leq d(e, \mathcal{D}') \leq r/3\}$. Let $\mathfrak{C}$ be the set of all circuits (of regular bonds) in $\mathcal{A}$ which surround $\mathcal{D}'$ and let $O_{\mathcal{A}}$ be the event that some circuit $\alpha \in \mathfrak{C}$ is open; for $\omega \in O_{\mathcal{A}}$ there is a unique outermost open circuit in $\mathfrak{C}$, which we denote $\Gamma = \Gamma(\omega)$. Note that

$$(3.12) \qquad O_{\mathcal{A}}^c \subset \cup_{y^* \in V^*(\mathcal{A})} G_{y^*},$$

$$(3.13) \qquad |\alpha| \geq r \quad \text{for all } \alpha \in \mathfrak{C},$$

and

$$(3.14) \qquad [\Gamma = \alpha] = \mathrm{Open}(\alpha) \cap G_\alpha \quad \text{for some event } G_\alpha \in \mathcal{G}_{\mathcal{B}(\mathrm{Ext}(\alpha))}.$$

Let

$$p_\alpha = P(\Gamma = \alpha), \qquad p_{F\alpha} = P(\Gamma = \alpha \mid F), \qquad p_{FB\alpha} = P(\Gamma = \alpha \mid F \cap B_{\mathcal{F},r}).$$

By weak mixing we have for $\delta = K_{24} e^{-\epsilon_{11} r}$:

$$(3.15) \qquad \sum_{\alpha \in \mathfrak{C}} |p_{F\alpha} - p_\alpha| \leq \delta, \qquad \sum_{\alpha \in \mathfrak{C}} |p_{FB\alpha} - p_\alpha| \leq \delta.$$

Define the set of “good” circuits

$$\mathfrak{R} = \{\alpha \in \mathfrak{C} : p_{F\alpha} \leq p_\alpha (1 + \sqrt{\delta})\}$$

and let

$$h(\alpha) = \left(\frac{p_{F\alpha}}{p_\alpha} - 1 \right) 1_{[\alpha \in \mathfrak{C} \backslash \mathfrak{R}]}.$$

From (3.13), (3.14) and the near-Markov property for open circuits, if r is sufficiently large we obtain

$$P(B_{\mathcal{F},r} \mid F \cap [\Gamma = \alpha]) \le (1+\delta')P(B_{\mathcal{F},r} \mid \Gamma = \alpha), \tag{3.16}$$

where $\delta' = 3Ce^{-ar} < 1$ for C, a as in (2.4).

We need to bound $P(F \cap B_{\mathcal{F},r})$. To do this, we decompose $F \cap B_{\mathcal{F},r}$ into 3 pieces by intersecting it with $[\Gamma \in \mathfrak{R}], [\Gamma \in \mathfrak{C}\backslash\mathfrak{R}]$ and $O_{\mathcal{A}}^c$ (the latter meaning there is no Γ.) We then show that the first piece is approximately bounded by $P(F)P(B_{\mathcal{F},r})$, and the other two pieces are negligible relative to the size of the full event. Specifically, from (3.16),

$$\begin{aligned} &P(F \cap B_{\mathcal{F},r} \cap [\Gamma \in \mathfrak{R}]) \\ &\quad = \sum_{\alpha \in \mathfrak{R}} p_{F\alpha} P(F) P(B_{\mathcal{F},r} \mid F \cap [\Gamma = \alpha]) \\ &\quad \le (1+\sqrt{\delta})(1+\delta')P(F) \sum_{\alpha \in \mathfrak{R}} p_\alpha P(B_{\mathcal{F},r} \mid \Gamma = \alpha) \\ &\quad \le (1 + 2\sqrt{\delta} + \delta')P(F)P(B_{\mathcal{F},r}). \end{aligned} \tag{3.17}$$

Next, one application of (3.15) yields

$$E(h(\Gamma)1_{O_{\mathcal{A}}}) = \sum_{\alpha \in \mathfrak{C}\backslash\mathfrak{R}} h(\alpha)p_\alpha \le \delta;$$

this and a second application of (3.15), with Markov's inequality, yield

$$\begin{aligned} &P(F \cap B_{\mathcal{F},r} \cap [\Gamma \in \mathfrak{C}\backslash\mathfrak{R}]) \\ &\quad = P(F \cap B_{\mathcal{F},r})P(h(\Gamma) > \sqrt{\delta} \mid F \cap B_{\mathcal{F},r}) \\ &\quad \le P(F \cap B_{\mathcal{F},r})(P(h(\Gamma) > \sqrt{\delta}) + \delta) \\ &\quad \le P(F \cap B_{\mathcal{F},r})(\sqrt{\delta} + \delta). \end{aligned} \tag{3.18}$$

Finally, similarly to (3.9) we have using (3.12)

$$\begin{aligned} P(F \cap B_{\mathcal{F},r} \cap O_{\mathcal{A}}^c) &\le \sum_{y^* \in V^*(\mathcal{A})} P(F \cap B_{\mathcal{F},r} \cap G_{y^*}) \\ &\le \delta'' P(F \cap B_{\mathcal{F},r}), \end{aligned} \tag{3.19}$$

where $\delta'' = K_{25}e^{-\epsilon_{12}r}$. Combining (3.17), (3.18) and (3.19) we obtain

$$\begin{aligned} P(F \cap B_{\mathcal{F},r}) &\le (1 - \delta - \sqrt{\delta} - \delta'')^{-1} P(F \cap B_{\mathcal{F},r} \cap [\Gamma \in \mathfrak{R}]) \\ &\le (1 + K_{26}e^{-\epsilon_{13}r})P(F)P(B_{\mathcal{F},r}). \end{aligned} \tag{3.20}$$

Summing over $\mathcal{F}$ yields

$$\sum_{\mathcal{F}\in\mathfrak{A}} P([C_{x^*} = \mathcal{F}^*] \cap B_{\mathcal{F},r}) \leq (1 + K_{26}e^{-\epsilon_{13}r})P(A)P(B)$$

which substitutes for (3.11). □

4. Lower Bounds for Open Dual Circuit Probabilities

In this section we prove the following result.

Theorem 4.1. *Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7), the near-Markov property for open circuits, positivity of τ and the ratio weak mixing property. There exist K_i such that for $A > K_{27}$ and $l = \sqrt{A}$,*

$$P(|\operatorname{Int}(\Gamma_0)| \geq A) \geq \exp(-w_1\sqrt{A} - K_{28}l^{1/3}(\log l)^{2/3}).$$

The size of the error term $K_{28}l^{1/3}(\log l)^{2/3}$ in Theorem 4.1 is important because it determines what "bad" behaviors can be ruled out as unlikely—in particular, those which have probability at most $\exp(-w_1 l - cl^{1/3}(\log l)^{2/3}))$ for some $c > K_{28}$. Though our error term is likely not optimal—according to [15] the optimal error term may be of order $\log l$—it is enough of an improvement over the corresponding results in [11] and [16] to enable us to establish an apparently near-optimal bound on the local roughness.

The proof of our Theorem 4.1 relies on the following result, the halfspace version of (2.10).

Theorem 4.2. *([4]) Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7), positivity of τ and the ratio weak mixing property. There exist ϵ_{14}, K_{29} such that for all $x \neq y \in \mathbb{R}^2$ and all dual sites $u, v \in H_{xy}$,*

$$P(u \leftrightarrow v \text{ via an open dual path in } H_{xy}) \geq \frac{\epsilon_{14}}{|x|^{K_{29}}} e^{-\tau(v-u)}$$

Proof of Theorem 4.1. Let $s = l^{2/3}(\log l)^{1/3}$ and $\delta = K_{30}s^2/l$, with K_{30} to be specified. Let $(y_0, .., y_n, y_0)$ be the s-hull skeleton of $\partial(l + \delta)\mathcal{K}_1$. For each i let y_i' be a dual site with $y_i' \in H_{y_{i-1}y_i} \cap H_{y_iy_{i+1}}$ and $|y_i' - y_i| \leq 2\sqrt{2}$. By (3.5), provided K_{30} is large enough we have

$$\operatorname{Co}(\{y_0, .., y_n\}) \supset l\mathcal{K}_1 \quad \text{and hence} \quad |\operatorname{Co}(\{y_0, .., y_n\})| \geq A.$$

Further,

$$\sum_{j=0}^{n} \tau(y_{j+1}' - y_j') \leq \mathcal{W}(\partial(l + \delta)\mathcal{K}_1) + 4\kappa_\tau n \leq w_1 l + K_{31}l^{1/3}(\log l)^{2/3} \tag{4.1}$$

(with $y_{n+1} = y_0$.) Therefore using the FKG property, Theorem 4.2, (4.1) and (3.2),

$$
\begin{aligned}
(4.2) \qquad P(|\operatorname{Int}(\Gamma_0)| \geq A) &\geq P(\Gamma_0 \text{ encloses } \operatorname{Co}(\{y_0, .., y_n\})) \\
&\geq P(y_j' \leftrightarrow y_{j+1}' \text{ via a path in } H_{y_j y_{j+1}} \text{ for all } j \leq n) \\
&\geq \prod_{j=0}^{n} P(y_j' \leftrightarrow y_{j+1}' \text{ via a path in } H_{y_j y_{j+1}}) \\
&\geq \left(\frac{\epsilon_{14}}{l^{K_{29}}}\right)^{(n+1)} \exp\left(-\sum_{j=0}^{n} \tau(y_{j+1}' - y_j')\right) \\
&\geq \exp(-w_1 l - K_{32} l^{1/3} (\log l)^{2/3}).
\end{aligned}
$$

□

5. Upper Bounds for Open Dual Circuit Probabilities

We need to develop a method of cutting a dual circuit across a bottleneck, modifying the bond configuration to create two dual circuits. The cutting procedure is simplified if the bottleneck is clean, in the following sense. The *canonical path* from dual site $u = (x_1, y_1)$ to dual site $v = (x_2, y_2)$ is the path, denoted ζ_{uv}, which goes horizontally from (x_1, y_1) to (x_2, y_1), then vertically to (x_2, y_2). We call a bottleneck (u, v) *clean* if $\zeta_{uv} \subset \operatorname{Int}(\gamma)$ (except for the endpoints u, v.) The next lemma will enable us to restrict our cutting to clean bottlenecks.

Lemma 5.1. *If a dual circuit γ contains a (q, r)-bottleneck for some $r > 3q > 0$, then γ contains a clean $(q, r/3)$-bottleneck.*

Proof. Suppose γ contains a (q, r)-bottleneck (u, v). We have two disjoint paths from u to v: $\gamma^{[u,v]}$ and $\gamma^{[v,u]}$ (traversed backwards.) Each of these may intersect ζ_{uv} a number of times. Accordingly, ζ_{uv} contains a finite sequence of sites $u = x_0, x_1, .., x_m = v$ such that the segment ζ_i of ζ_{uv} between x_{i-1} and x_i satisfies $\zeta_i \subset \operatorname{Int}(\gamma)$ for all $i \in I$ and $\zeta_i \subset \operatorname{Ext}(\gamma) \cup \gamma$ for all $i \notin I$, where I consists either of all odd i or of all even i. For $i \in I$, we call the segment of ζ_{uv} with endpoints x_{i-1} and x_i an *interior gap*. Let ψ be a dual path from u to v in $\operatorname{Int}(\gamma)$ with $|\psi| \leq q$. We can extend ψ to a doubly infinite path ψ^+ by adding on (possibly non-lattice) paths ψ_1 from v to ∞ and ψ_2 from ∞ to u, both in $\operatorname{Ext}(\gamma)$. The path ψ^+ divides the plane into two regions, $A_L \supset \gamma^{[v,u]}$ to the left of ψ^+ and $A_R \supset \gamma^{[u,v]}$ to the right. Replacing ψ with ζ_{uv} in the definition of ψ^+, we obtain another doubly infinite path ζ^+. The path ζ^+ is not necessarily self-avoiding, but $\mathbb{R}^2 \backslash \zeta^+$ has exactly two unbounded components B_L and B_R, to the left and right of ζ^+, respectively. Since $\operatorname{diam}(\zeta_{uv}) \leq q$, there exist sites $z_1 \in \gamma^{[u,v]}$ and $z_2 \in \gamma^{[v,u]}$ for which $d(z_j, \zeta_{uv}) \geq (r - q)/2 > q$ $(j = 1, 2)$

and hence $z_1 \in B_R, z_2 \in B_L$. Let θ be a (possibly non-lattice) path from z_1 to z_2 in $\mathrm{Int}(\gamma)$. Then θ must intersect ζ^+, and hence must intersect ζ_{uv}, necessarily in some interior gap. Thus every θ from z_1 to z_2 in $\mathrm{Int}(\gamma)$ must cross at least one interior gap, so there exists an interior gap ζ_i which separates z_1 and z_2, that is, exactly one of z_1, z_2 is in $\gamma^{[x_{i-1},x_i]}$. It follows that (x_{i-1}, x_i) is a clean $(q,(r-q)/2)$-bottleneck. Since $(r-q)/2 > r/3$, the proof is complete. □

Define $R_x = x + [-1/2, 1/2]^2$ and $R_x^+ = x + [-1,1]^2$. Let $Q_1(u,v) = \cup_{x \in \zeta_{uv}} R_x$ and $Q_2(u,v) = \cup_{x\in\zeta_{uv}} R_x^+$. Note that

$$|\partial Q_2(u,v)| \leq 4q + 8.$$

Let $J_1(u,v), J_2(u,v), ...$ be an enumeration of the subsets of $\partial Q_2(u,v)$. We say a clean (q,r)-bottleneck (u,v) in a dual circuit γ is of *type* η if the set of bonds in $\partial Q_2(u,v)$ which are contained (except possibly for endpoints) in $\mathrm{Int}(\gamma)$ is precisely $J_\eta(u,v)$.

We assume we have a fixed but arbitrary algorithm for choosing a particular (q,r)-bottleneck, which we then call *primary*, from any circuit containing one or more (q,r)-bottlenecks. When a configuration ω includes an exterior dual circuit γ for which (u,v) is a primary (q,r)-bottleneck of type η, we can apply a procedure, which we term *bottleneck surgery* (on γ, at (u,v)) to create a new configuration denoted $Y_{uv\eta}(\omega)$. Bottleneck surgery consists of replacing the configuration ω with the configuration given by

$$Y_{uv\eta}(\omega)_e = \begin{cases} 1, & \text{if } e \in \partial Q_1(u,v); \\ 0, & \text{if } e^* \in J_\eta(u,v); \\ \omega_e, & \text{otherwise,} \end{cases} \tag{5.1}$$

for each bond e. The configuration $Y_{uv\eta}(\omega)$ then contains two or more disjoint open dual circuits α_i, each consisting of some dual bonds of γ and some dual bonds of $J_\eta(u,v)$, with no open dual path connecting α_i to α_j for $i \neq j$, and with

$$\cup_i \mathrm{Int}(\alpha_i) = \mathrm{Int}(\gamma) \backslash Q_2(u,v). \tag{5.2}$$

Further,

$$|Q_2(u,v)| + \sum_{i:\alpha_i (\kappa_\tau r/3)-\mathrm{small}} |\mathrm{Int}(\alpha_i)| \leq K_{33} r^2, \tag{5.3}$$

and, since γ is exterior, there is no open dual path connecting α_i to α_j for $i \neq j$. We call each α_i an (q,r)-*offspring* or an (q,r)-*descendant* of γ. A (q,r)-offspring of an (q,r)-descendant is also an (q,r)-descendant, iteratively. We may perform bottleneck surgery on each (q,r)-offspring of γ which contains a clean (q,r)-bottleneck, and iterate this process until no descendant of γ contains such a clean (q,r)-bottleneck (necessarily after a finite number of surgeries.) The bottleneck-free (q,r)-descendants are called *final* (q,r)*-descendants*. Among final (q,r)-descendants, the one enclosing maximal area is called the *maximal* (q,r)*-descendant* of γ and denoted $\alpha_{\max,\gamma}$. The set of all $(\kappa_\tau r/3)$-large final (q,r)-descendants of γ is denoted $\mathfrak{F}_{(q,r)}(\gamma)$; the non-maximal among these form the set $\mathfrak{F}'_{(q,r)}(\gamma) = \mathfrak{F}_{(q,r)}(\gamma)\backslash\{\alpha_{\max,\gamma}\}$. Note that since γ is exterior, so is each offspring of γ.

It is useful to note the following general fact about norms on $\mathbb{R}^2$, which can be verified by a simple geometric argument. Let C be a convex set; then

$$\mathcal{W}(\partial C) \leq 6\,\mathrm{diam}_\tau(C). \tag{5.4}$$

Define

$$u(c,A) = \max(w_1 A^{1/2} - cA^{1/6}, 0)$$

and

$$D_{(q,r)}(\gamma) = \sum_{\alpha\in\mathfrak{F}_{(q,r)}(\gamma)} \mathrm{diam}_\tau(\alpha), \qquad D'_{(q,r)}(\gamma) = \sum_{\alpha\in\mathfrak{F}'_{(q,r)}(\gamma)} \mathrm{diam}_\tau(\alpha).$$

The following lemma is related to (5.4).

Lemma 5.2. *Let γ be a circuit, let $A = |\,\mathrm{Int}(\gamma)|$, and let $q \geq 1, r \geq 15q$. Then*

$$D_{(q,r)}(\gamma) \geq \frac{1}{6} w_1 \sqrt{A}. \tag{5.5}$$

Proof. We may assume γ contains a clean (q,r)-bottleneck (u,v), for otherwise (5.5) is immediate from (5.4). We have

$$q + 2\sqrt{2} \leq \frac{1}{4} r.$$

Let S denote the union of $Q_2(u,v)$ and all $(\kappa_\tau r/3)$-small offspring of γ, and let

$$R = \{z \in \mathbb{R}^2 : d_\tau(z, Q_2(u,v)) \leq \kappa_\tau r/3\}.$$

Then

$$S \subset R, \qquad \mathrm{diam}(R) \leq q + 2\sqrt{2} + \frac{2\sqrt{2}r}{3}$$

and

$$|R| \leq \left|\left\{z \in \mathbb{R}^2 : d(z, Q_2(u,v)) \leq \frac{\sqrt{2}r}{3}\right\}\right| \leq \pi\left(\frac{q + 2\sqrt{2} + \frac{2\sqrt{2}r}{3}}{2}\right)^2 \leq r^2. \tag{5.6}$$

Note that the set $\{\alpha_1, \alpha_2, ...\}$ of $(\kappa_\tau r/3)$-large offspring of γ can be divided into two disjoint classes: *right offspring*, which intersect $\gamma^{[u,v]}$, and *left offspring*, which intersect $\gamma^{[v,u]}$. Also, every point of γ is either in a left offspring, in a right offspring, or in S. The diameter of $Q_2(u,v)$ is at most $q + 2\sqrt{2} \le r/6$, while the diameters of $\gamma^{[u,v]}$ and $\gamma^{[v,u]}$ are at least r, so the right and left classes each include at least one $(5\kappa_\tau r/6\sqrt{2})$-large offspring,. Further, if

$$D_{(q,r)}(\gamma) \ge \operatorname{diam}_\tau(\gamma) \tag{5.7}$$

then (5.5) follows from (5.4). Let w and x be sites of γ with $d_\tau(w,x) = \operatorname{diam}_\tau(\gamma)$. At least one of these points is not in $Q_2(u,v)$, so we may assume w is in some α_i. There are now four possibilities. First, if also $x \in \alpha_i$, then (5.7) holds. Second, if instead $x \in S$, then there exists a $(5\kappa_\tau r/6\sqrt{2})$-large offspring $\alpha_j \ne \alpha_i$, and we have

$$\begin{aligned} D_{(q,r)}(\gamma) &\ge \operatorname{diam}_\tau(\alpha_i) + \operatorname{diam}_\tau(\alpha_j) \\ &\ge \operatorname{diam}_\tau(\gamma) - \sqrt{2}\kappa_\tau(q + 2\sqrt{2}) - \frac{\kappa_\tau r}{3} + \frac{5\kappa_\tau r}{6\sqrt{2}} \\ &\ge \operatorname{diam}_\tau(\gamma) \end{aligned}$$

and again (5.7) holds. Third, suppose $x \in \alpha_k$ for some $k \ne i$ and there exists a third $(\kappa_\tau r/3)$-large offspring α_l with $l \ne i, k$. Let $d_m = \operatorname{diam}_\tau(\alpha_m)$. Then

$$d_i + d_k \ge \operatorname{diam}_\tau(\gamma) - \sqrt{2}\kappa_\tau(q + 2\sqrt{2}) \ge \operatorname{diam}_\tau(\gamma) - d_l$$

so once more, (5.7) holds.

The fourth possibility is that $x \in \alpha_k$ for some $k \ne i$ and α_i, α_k are the only $(\kappa_\tau r/3)$-large offspring; each is necessarily actually $(5\kappa_\tau r/6\sqrt{2})$-large. From (5.4) we have

$$A \le |R| + |\operatorname{Int}(\alpha_i)| + |\operatorname{Int}(\alpha_k)| \le r^2 + \frac{36}{w_1^2}(d_i^2 + d_k^2).$$

Using this and the fact that $w_1 \le 4\kappa_\tau$ (since the unit square encloses unit area) we obtain

$$\frac{w_1^2}{36}A \le \frac{4}{9}\kappa_\tau^2 r^2 + d_i^2 + d_k^2 \le 2d_i d_k + d_i^2 + d_k^2.$$

Taking square roots yields (5.5). □

For $k \ge 0$ define the events

$$\begin{aligned} M_y(k,q,r,A,A',d',t) = [|\mathfrak{F}'_{(q,r)}(\Gamma_0)| = k] \cap [|\operatorname{Int}(\Gamma_0)| = A] \cap [|\operatorname{Int}(\alpha_{\max,\Gamma_0})| = A'] \\ \cap [D'_{(q,r)}(\Gamma_0) \in [d', d'+1)] \cap [\mathcal{W}(\partial \operatorname{Co}(\alpha_{\max,\Gamma_0})) \ge t]. \end{aligned}$$

We first consider $k = 0$, which means $\alpha_{\max,\gamma} = \gamma$ and $D_{(q,r)}(\gamma) = 0$; larger values will be handled later by induction.

Proposition 5.3. *Assume (2.7) and either (i) the ratio weak mixing property or (ii) both the weak mixing property and the near-Markov property for open circuits. Then there exist constants ϵ_i, K_i as follows. Let $A \geq 1, t_+ \geq 0, t = w_1\sqrt{A} + t_+ \geq 2$, and $\epsilon_{16}t > r > 15q > K_{34}\log t$. Then*

$$P(M_0(0,q,r,A,A,0,t)) \leq e^{-u(K_{35}r^{2/3},A)-\frac{1}{2}t_+}. \tag{5.8}$$

Proof. From the definition of w_1 we may assume $t_+ \geq 0$. It follows easily from (2.9) that for some K_{36}, K_{37},

$$P(\mathrm{diam}_\tau(\Gamma_0) \geq t) \leq K_{36}t^4 e^{-t} \leq e^{-u(K_{37}r^{2/3},A)-t_+} \tag{5.9}$$

so by (5.4) it suffices to consider

$$M_0'(q,r,A,t) = M_0(0,q,r,A,A,0,t) \cap [t/6 \leq \mathrm{diam}_\tau(\Gamma_0) \leq t].$$

Suppose $\omega \in M_0'(q,r,A,t)$. Fix $\alpha > 0$ to be specified, let $s = \alpha t^{2/3}r^{1/3}$ and suppose $\mathrm{HSkel}_s(\Gamma_0) = (y_0, .., y_{m+1})$. By (3.1),

$$m \leq K_{38}\alpha^{-1}t^{1/3}r^{-1/3}. \tag{5.10}$$

Let $B_i = B(y_i, 4r) \cap (\mathbb{Z}^2)^*$. Let $I = \{i \leq m : |y_{i+1} - y_i| > 8r\}$. For each $i \in I$ there is a segment $\Gamma_0^{[w_i,x_i]} \subset \Gamma_0^{[y_i,y_{i+1}]}$ entirely outside $B_i \cup B_{i+1}$, with $w_i \in \partial B_i$ and $x_i \in \partial B_{i+1}$. We next show that

$$d(\Gamma_0^{[w_i,x_i]}, \Gamma_0^{[w_j,x_j]}) > q/2 \quad \text{for all } i \neq j \text{ in } I. \tag{5.11}$$

If not, there exist $u \in \Gamma_0^{[w_i,x_i]}$, $v \in \Gamma_0^{[w_j,x_j]}$ and a dual path ψ from u to v in $\mathrm{Co}(\Gamma_0)$ with $|\psi| \leq q$. Let a be the last site of ψ in $\Gamma_0^{[y_i,y_{i+1}]}$, and b the first site of ψ after a which is in some segment $\Gamma_0^{[y_k,y_{k+1}]}$ with $k \neq i$. Since all sites y_l are extreme points, we must have $\psi^{(a,b)} \subset \mathrm{Int}(\Gamma_0)$. We claim that (a,b) is a $(q,3r)$-bottleneck. By Lemma 5.1 this is a contradiction, so our claim will establish (5.11). Suppose $i < k$; the proof if $i > k$ is similar. We have $\psi \subset B(u,q)$ and $u \notin B_{i+1}$ so $\psi \cap B(y_{i+1}, 3r+1) = \phi$. Therefore $\Gamma_0^{[a,b]}$ contains a segment in $\overline{\mathcal{B}}(B_{i+1})$ which includes y_{i+1} and has diameter at least $3r$. Similarly since $v \notin B_i, \Gamma_0^{[b,a]}$ contains a segment in $\mathcal{B}(B_i)$ which includes y_i and has diameter at least $3r$. This proves the claim and thus (5.11).

From (3.6) and (5.10) we have

$$\begin{aligned}\sum_{i\in I} \tau(x_i - w_i) &\geq \mathcal{W}(\mathrm{HPath}_s(\Gamma_0)) - K_{39}mr \\ &\geq \mathcal{W}(\partial\,\mathrm{Co}(\Gamma_0)) - K_{40}\alpha^2 t^{1/3}r^{2/3} - K_{39}mr \\ &\geq t - K_{41}(\alpha^2 + \alpha^{-1})t^{1/3}r^{2/3}.\end{aligned} \tag{5.12}$$

Equation (5.12) shows that it is optimal to take α of order 1 in our choice of s, so we now set $\alpha = 1$.

For $w_i \in \partial(B_i \cap \mathbb{Z}^2)$ and $x_i \in \partial(B_{i+1} \cap \mathbb{Z}^2)$ for each $i \leq m$, let $A(w_0, x_0, .., w_m, x_m)$ be the event that for each $i \in I$ there is an open dual path α_i from w_i to x_i in $B_\tau(w_i, t)$, with $d(\alpha_i, \alpha_j) > q/2$ for all $i \neq j$. Then we have shown

$$\begin{aligned} (5.13) \qquad & P(M_0'(q, r, A, t)) \cap [\mathrm{HSkel}_s(\Gamma_0) = (y_0, .., y_{m+1})]) \\ & \leq \sum_{w_0} \sum_{x_0} \cdots \sum_{w_m} \sum_{x_m} P(A(w_0, x_0, .., w_m, x_m)) \\ & \leq (K_{42} r^2)^{m+1} \max_{w_0, x_0, .., w_m, x_m} P(A(w_0, x_0, .., w_m, x_m)). \end{aligned}$$

Provided K_{34} is sufficiently large, Lemma 3.2, (5.12) and induction give

$$\begin{aligned} (5.14) \qquad P(A(w_0, x_0, .., w_m, x_m)) &\leq 2^m \prod_{i \in I} P(w_i \leftrightarrow x_i) \\ &\leq 2^m e^{-t + 2K_{41} t^{1/3} r^{2/3}} \end{aligned}$$

which with (5.10) and (5.13) yields

$$\begin{aligned} (5.15) \qquad & P(M_0'(q, r, A, t) \cap [\mathrm{HSkel}_s(\Gamma_0) = (y_0, .., y_{m+1})]) \\ & \leq e^{-t + K_{43} t^{1/3} r^{2/3}}. \end{aligned}$$

The number of possible $(y_0, .., y_{m+1})$ in (5.15) is at most $(K_{44} t^2)^{m+1}$, which with (5.15) yields

$$(5.16) \qquad P(M_0'(q, r, A, t) \leq e^{-t + K_{45} t^{1/3} r^{2/3}}$$

provided K_{34}, and hence r, is large enough. It is easily verified that, provided ϵ_{16} is small enough,

$$K_{45}(t_+ + w_1 A^{1/2})^{1/3} r^{2/3} \leq 2K_{45}(w_1 A^{1/2})^{1/3} r^{2/3} + \frac{1}{2} t_+,$$

by considering two cases according to which of t_+ and $w_1 A^{1/2}$ is larger. This and (5.16) establish (5.8) for M_0'; as we have noted, this and (2.9) establish (5.8) as given. □

Remark 5.4. Let I be any increasing event. Since the event on the left side of (5.14) is decreasing, its probability is not increased by conditioning on I. It follows easily that Proposition 5.3 remains true if the probability on the left side of (5.8) is conditioned on I, even though $M_0(0, q, r, A, A, 0, t)$ is not itself a decreasing event.

Under (2.7), open dual bonds do not percolate, so for every bounded set A there is a.s. an innermost open circuit surrounding A; we denote this circuit by $\Theta(A)$.

An *enclosure event* is an event of form

$$\cap_{i\leq n}(\mathrm{Open}(\alpha_i)\cap[\alpha_i\leftrightarrow\infty])$$

where $\alpha_1,..,\alpha_n$ are circuits (of regular bonds.) This includes the degenerate case of the full space $\{0,1\}^{\mathcal{B}(\mathbb{Z}^2)}$. Clearly any such event is increasing.

Proposition 5.5. *Assume (2.7), the weak mixing property and the near-Markov property for open circuits. There exist constants K_i,ϵ_i as follows. Let $A\geq A'\geq 3, k\geq 0, t_+\geq 0, t=w_1\sqrt{A'}+t_+, d'\geq 0$, and $\epsilon_{17}(w_1\sqrt{A}+d'+t_+)\geq r\geq 15q\geq K_{46}\log A$. Then*

$$P(M_0(k,q,r,A,A',d',t))\leq \exp\left(-u(K_{47}r^{2/3},A)-\frac{1}{60}t_+-\frac{1}{10}d'\right) \tag{5.17}$$

and

$$P(M_0(k,q,r,A,A',d',t))\leq \exp\left(-\frac{1}{2}d'\right)P(M_0(0,q,r,A',A',0,t)). \tag{5.18}$$

Proof. We will refer to the requirement $\epsilon_{17}(w_1\sqrt{A}+d'+t_+)\geq r$ as the *size condition*, and to all other assumptions of the Proposition collectively as the *basic assumptions.*

We first prove (5.17). We proceed by induction on k, using Proposition 5.3 for $k=0$. Fix q,r and define

$$L_y(k,A,A',d,d',t)=M_y(k,q,r,A,A',d',t)\cap[d\leq D_{(q,r)}(\Gamma_y)<d+1],$$

where

$$\frac{1}{6}w_1\sqrt{A}-1\leq d\leq K_{48}A,\quad d\geq\frac{1}{6}t_+-1. \tag{5.19}$$

If K_{48} is large enough then, from Lemma 5.2 and the lattice nature of Γ_y, $L_y(k,A,A',d,d',t)$ is empty if any of the inequalities in (5.19) fails. Note that for some K_{49},

$$M_y(k,q,r,A,A',d',t)\subset[\mathrm{diam}(\Gamma_y)\leq K_{49}A].$$

Our induction hypothesis is that for some constants c_i, for all $j<k$, all A,A',t,d' satisfying the basic assumptions, and all d satisfying (5.19), for any enclosure event E, we have

$$P(L_0(j,A,A',d,d',t)\mid E)\leq\exp(-\frac{9}{10}d+\frac{\kappa_\tau}{40}jr); \tag{5.20}$$

if the size condition is also satisfied, then in addition

$$\begin{aligned}P(&L_0(j,A,A',d,d',t)\mid E)\\&\leq\exp(-u(K_{35}r^{2/3},A)-\frac{1}{60}t_+-\frac{1}{10}d'),\end{aligned} \tag{5.21}$$

with K_{35} from (5.8). We wish to verify these hypotheses for $j = k$.

For $j = 0$ it suffices to consider $d' = 0$ and (5.21) is Proposition 5.3 (together with Remark 5.4), while (5.20) follows easily from the first inequality in (5.9), if K_{46} is large. Hence we may assume $k \geq 1$ and fix A, A', d, d'. Let $\Psi(u, v, \eta)$ denote the event that $L_0(k, A, A', d, d', t)$ occurs with (u, v) a primary (q, r)-bottleneck in Γ_0 of type η, and let $\Phi(u, v, \eta) = \{Y_{uv\eta}(\omega) : \omega \in \Psi(u, v, \eta)\}$. Let E be an enclosure event; it is easy to see that bottleneck surgery cannot destroy E, that is,

$$\Psi(u, v, \eta) \cap E \subset \Phi(u, v, \eta) \cap E.$$

(This is the reason for considering only enclosure events, not general increasing events.) Since $|\partial Q_1(u, v)| + |J_\eta(u, v)| \leq K_{48} q$, we then have from the bounded energy property:

$$P(\Psi(u, v, \eta) \mid E) \leq e^{K_{49} q} P(\Phi(u, v, \eta) \mid E). \tag{5.22}$$

Fix u, v, η and for $y_1, .., y_l \in \mathbb{Z}^2$ and $l, m, A_i, k_i, d_i, d_i' \geq 0$ in $\mathbb{Z}$, let

$$Z = Z(l, m, y_1, .., y_l, A_1, .., A_l, d_1, .., d_l, d_1', .., d_l', k_1, .., k_l)$$

denote the event that there exist disjoint exterior open dual circuits $\alpha_1, .., \alpha_l$ such that:

(i) α_i surrounds y_i, $|\operatorname{Int}(\alpha_i)| = A_i$, $\operatorname{diam}_\tau(\alpha_i) \geq \kappa_\tau r/3$, $|\mathfrak{F}'_{(q,r)}(\alpha_i)| = k_i$, $d_i \leq D_{(q,r)}(\alpha_i) < d_i + 1$ and $d_i' \leq D'_{(q,r)}(\alpha_i) < d_i' + 1$ for all $i \leq l$;

(ii) letting α_{main} denote the open dual circuit enclosing maximal area among all offspring of all α_i, we have α_{main} an offspring of α_m satisfying $|\operatorname{Int}(\alpha_{\text{main}})| = A'$ and $\mathcal{W}(\partial \operatorname{Co}(\alpha_{\text{main}})) \geq w_1\sqrt{A'} + t_+$;

(iii) there is no open dual path connecting α_i to α_n for $i \neq n$.

We supress the parameters in Z when confusion is unlikely. Then considering only $(\kappa_\tau r/3)$-large offspring we see that

$$\Phi(u, v, \eta) \subset \cup\, Z(l, m, y_1, .., y_l, A_1, .., A_l, d_1, .., d_l, d_1', .., d_l', k_1, .., k_l) \tag{5.23}$$

where the union is over all parameters satisfying

$$2 \leq l \leq \min(4q, k+1), \quad m \leq l, \quad y_i \in (\mathbb{Z}^2)^* \text{ with } d(y_i, \zeta_{uv}) \leq 2, \quad A_i \geq \frac{\kappa_\tau r}{6},$$

$$A_m \geq A', \quad A - K_{33} r^2 \leq \sum_{i \leq l} A_i \leq A, \quad \sum_{i \leq l} k_i = k + 1 - l, \tag{5.24}$$

$$\frac{1}{6} w_1 \sqrt{A_i} - 1 \leq d_i \leq K_{48} A_i, \quad d_i' \leq d_i, \tag{5.25}$$

$$d' - l \le d'_m + \sum_{i \ne m} d_i \le d', \quad d - l \le \sum_{i \le l} d_i \le d, \tag{5.26}$$

$$d_m - d'_m + 1 \ge \frac{1}{6}(w_1\sqrt{A'} + t_+). \tag{5.27}$$

Here (5.27) and the first inequality in (5.25) follow from (ii) above and (5.4),and K_{33} is from (5.3). Temporarily fix such a set of parameters and let $\nu_1, .., \nu_l$ be circuits with

$$\mathrm{diam}_\tau(\nu_i) \ge \frac{\kappa_\tau r}{3} \quad \text{for all } i, \text{ and } \mathrm{Int}(\nu_i) \cap \mathrm{Int}(\nu_j) = \phi \text{ for } i \ne j.$$

Define events

$$\tilde{L}_i = \cup_{3 \le B \le A_i} L_{y_i}(k_i, A_i, B, d_i, d'_i, w_1\sqrt{B}), \quad i \ne m,$$

$$\tilde{L}_m = L_{y_m}(k_m, A_i, A', d_m, d'_m, t),$$

$$T_i = [\Theta(\Gamma_{y_i}) = \nu_i], \quad Y_i = \tilde{L}_i \cap T_i \quad \text{for } i \le m, \quad T = \cap_{i \le l} T_i.$$

Then

$$Z \cap T \subset \cap_{i \le l} Y_i. \tag{5.28}$$

Note that as in (3.14),

$$Y_i = \mathrm{Open}(\nu_i) \cap [\nu_i \leftrightarrow \infty] \cap G_i \text{ for some } G_i \in \mathcal{G}_{\mathcal{B}(\mathrm{Int}(\nu_i))}, \quad \text{for every } i \le m. \tag{5.29}$$

Define events

$$R = \cup_{i \le l-1} \mathrm{Int}(\nu_i), \qquad F = \cap_{i \le l-1} \mathrm{Ext}(\nu_i),$$

$$\tilde{G} = \cap_{i \le l-1} G_i, \qquad H = \cap_{i \le l-1} \mathrm{Open}(\nu_i), \qquad N = \cap_{i \le l-1}[\nu_i \leftrightarrow \infty],$$

and let L_l denote the event that $\tilde{L}_l$ occurs on $\mathcal{B}(F)$. Then

$$N \cap E \cap H = E_R \cap E_F \cap H \quad \text{for some } E_R \in \mathcal{G}_{\mathcal{B}(R)},\ E_F \in \mathcal{G}_{\mathcal{B}(F)}, \tag{5.30}$$

$$\cap_{i \le l-1} Y_i = H \cap N \cap \tilde{G} \tag{5.31}$$

and

$$\cap_{i \le l} Y_i \subset L_l. \tag{5.32}$$

The relation between area A_i and diameter d_i tells us roughly whether the circuit α_i (or its collection of descendants) is regular or irregular; we thereby subdivide the circuits into "large regular," "small regular" and "irregular" categories as follows:

$$I_1 = \{i \le l : d_i < 4w_1\sqrt{A_i}, A_i \ge c_1 r^2\},$$

$$I_2 = \{i \leq l : d_i < 4w_1\sqrt{A_i}, A_i < c_1 r^2\},$$

$$I_3 = \{i \leq l : d_i \geq 4w_1\sqrt{A_i}\}$$

where $c_1 = \max(1/\epsilon_{17}^2, (3K_{35}/w_1)^3)$ is chosen so that

$$u(K_{35}r^{2/3}, A_i) \geq \frac{2}{3}w_1\sqrt{A_i}, \quad i \in I_1. \tag{5.33}$$

Let

$$\mu_i = \max\left[u(K_{35}r^{2/3}, A_i) + \frac{1}{10}d_i, \frac{9}{10}d_i - \frac{\kappa_\tau}{40}k_i r\right], \quad i \in I_1\backslash\{m\},$$

$$\mu_i = \frac{9}{10}d_i - \frac{\kappa_\tau}{40}k_i r, \quad i \in I_2 \cup I_3,$$

$$\mu_m = \max\left[u(K_{35}r^{2/3}, A_m) + \frac{1}{60}t_+ + \frac{1}{10}d_m', \frac{9}{10}d_m - \frac{\kappa_\tau}{40}k_m r\right] \quad \text{if } m \in I_1$$

(cf. (5.20).) Now $H \cap E_F$ is an enclosure event so by the induction hypotheses (5.20) and (5.21), summing over $B \leq A_l$ gives

$$P(L_l \mid H \cap E_F) \leq A_l e^{-\mu_l}. \tag{5.34}$$

(Note that the size condition can be used here for $i \in I_1$.) Since $|\nu_i| \geq \epsilon_{18} r$ for all i, from (2.5) and (2.6), provided K_{46} is sufficiently large we get

$$P(L_l \cap E_F \mid H \cap \tilde{G} \cap E_R) \leq (1 + e^{-\epsilon_{18}r/2})P(L_l \cap E_F \mid H) \tag{5.35}$$

and

$$P(E_F \mid H) \leq (1 + e^{-\epsilon_{18}r/2})P(E_F \mid H \cap \tilde{G} \cap E_R). \tag{5.36}$$

Combining (5.29) – (5.36) we obtain

$$
\begin{aligned}
(5.37)\qquad & P((\cap_{i\leq l}Y_i)\cap E) \\
&\leq P(L_l\cap(\cap_{i\leq l-1}Y_i)\cap E)P(T_l\mid L_l\cap(\cap_{i\leq l-1}Y_i)\cap E) \\
&= P(L_l\cap H\cap\tilde{G}\cap E_R\cap E_F)P(T_l\mid L_l\cap(\cap_{i\leq l-1}Y_i)\cap E) \\
&\leq (1+e^{-\epsilon_{18}r/2})P(L_l\cap E_F\mid H)P(H\cap\tilde{G}\cap E_R) \\
&\qquad\cdot P(T_l\mid L_l\cap(\cap_{i\leq l-1}Y_i)\cap E) \\
&= (1+e^{-\epsilon_{18}r/2})P(L_l\mid E_F\cap H)P(E_F\mid H)P(H\cap\tilde{G}\cap E_R) \\
&\qquad\cdot P(T_l\mid L_l\cap(\cap_{i\leq l-1}Y_i)\cap E) \\
&\leq (1+e^{-\epsilon_{18}r/2})^2A_le^{-\mu_l}P(E_F\mid H\cap\tilde{G}\cap E_R)P(H\cap\tilde{G}\cap E_R) \\
&\qquad\cdot P(T_l\mid L_l\cap(\cap_{i\leq l-1}Y_i)\cap E) \\
&= (1+e^{-\epsilon_{18}r/2})^2A_le^{-\mu_l}P((\cap_{i\leq l-1}Y_i)\cap E)P(T_l\mid L_l\cap(\cap_{i\leq l-1}Y_i)\cap E).
\end{aligned}
$$

Summing over ν_l (which appears via T_l), dividing by $P(I)$ and iterating this (taking H and N to be the full space and $R=\phi, F=\mathbb{R}^2$, at the last iteration step) we obtain using (5.28)

$$
(5.38)\qquad P(Z\mid E)\leq(1+e^{-\epsilon_{18}r/4})^{2l}\prod_{i\leq l}A_ie^{-\mu_i}\leq 2A^l\exp(-\sum_{i\leq l}\mu_i).
$$

We now want to sum (5.38) over all parameters of Z allowed in (5.23). We first view l as fixed and allow the other parameters to vary. Note that the number of parameter choices is at most $(K_{50}A)^{5l+1}$, and the number of possible (u,v,η) is at most $(K_{50}A)^3$, for some K_{50}. Suppose we can show that, under the basic assumptions,

$$
(5.39)\qquad \sum_{i\leq l}\mu_i\geq\frac{9}{10}d-\frac{\kappa_\tau}{40}kr+\frac{\kappa_\tau}{100}(l-1)r
$$

and, if the size condition is also satisfied,

$$
(5.40)\qquad \sum_{i\leq l}\mu_i\geq u(K_{35}r^{2/3},A)+\frac{1}{10}d'+\frac{1}{60}t_++\frac{\kappa_\tau}{100}(l-1)r.
$$

Then from (5.23) and (5.38),

$$
P(\Phi(u,v,\eta)\mid I)\leq 2(K_{50}A)^{5l+2}\exp\left(-\frac{9}{10}d+\frac{\kappa_\tau}{40}kr-\frac{\kappa_\tau}{40}(l-1)r\right),
$$

and if the size condition is satisfied,

$$P(\Phi(u,v,\eta)\mid I)\le (K_{50}A)^{5l+2}\exp\Bigg(-\max\Bigg[u(K_{35}r^{2/3},A)+\frac{1}{10}d'+\frac{1}{60}t_+,\\ \frac{9}{10}d-\frac{\kappa_\tau}{40}kr\Bigg]-\frac{\kappa_\tau}{100}(l-1)r\Bigg).$$

Thus, summing over u,v,η, then over l, and using $r>K_{46}\log A$ and (5.22), we obtain (5.20) and (5.21) for $j=k$.

Now (5.39) is a direct consequence of (5.24) and (5.26), so we turn to (5.40) and assume that the size condition holds. Let

$$\delta_i=\begin{cases}\frac{1}{10}d_i, & i\ne m\\ \frac{1}{10}d'_m+\frac{1}{60}t_+, & i=m\end{cases}$$

and set $\beta_1=\beta_3=1/2,\beta_2=1/10$. We claim that

$$\mu_i\ge\beta_j w_1\sqrt{A_i}+\delta_i+\frac{1}{10}d_i-1\quad\text{for every } i\in I_j, j=1,2,3. \tag{5.41}$$

Observe that

$$d_i\ge(k_i+1)\frac{\kappa_\tau r}{3}-1\ge(k_i+1)\frac{\kappa_\tau r}{4},\quad d'_i\ge k_i\frac{\kappa_\tau r}{3}-1\ge k_i\frac{\kappa_\tau r}{4} \tag{5.42}$$

and hence

$$\mu_i\ge\frac{4}{5}d_i,\quad i\le l. \tag{5.43}$$

This yields

$$\mu_i\ge\frac{4}{5}d_i\ge w_1\sqrt{A_i}+\frac{11}{20}d_i,\quad i\in I_3, \tag{5.44}$$

which proves (5.41) for $i\in I_3\backslash\{m\}$. For $i\in I_1$ we can use (5.33), (5.42) and a convex combination of the lower bounds (5.43) and $u(K_{35}r^{2/3},A_i)$ for μ_i to obtain

$$\mu_i\ge\frac{3}{4}u(K_{35}r^{2/3},A_i)+\frac{1}{5}d_i\ge\frac{1}{2}w_1\sqrt{A_i}+\frac{1}{5}d_i,\quad i\in I_1, \tag{5.45}$$

which proves (5.41) for $i\in I_1\backslash\{m\}$. From (5.25),

$$d_i\ge\frac{1}{8}w_1\sqrt{A_i}+\frac{1}{4}d_i-\frac{3}{4},\quad i\le l,$$

and hence by (5.43)

$$\mu_i\ge\frac{4}{5}d_i\ge\frac{1}{10}w_1\sqrt{A_i}+\frac{1}{5}d_i-\frac{3}{5},\quad i\le l, \tag{5.46}$$

which proves (5.41) for $i\in I_2\backslash\{m\}$.

We need slightly different estimates for $i = m$. If $m \in I_3$ then using (5.44) and (5.27) we obtain

$$\mu_m \geq w_1\sqrt{A_m} + \frac{1}{2}d_m \geq w_1\sqrt{A_m} + \frac{1}{4}d_m + \frac{1}{4}(d'_m - 1) + \frac{1}{24}t_+, \tag{5.47}$$

which proves (5.41) for $i = m$. If $m \in I_1$ then by (5.45) and (5.27),

$$\mu_m \geq \frac{1}{2}w_1\sqrt{A_m} + \frac{1}{5}d_m \geq \frac{1}{2}w_1\sqrt{A_m} + \frac{1}{10}d_m + \frac{1}{10}(d'_m - 1) + \frac{1}{60}t_+, \tag{5.48}$$

which again proves (5.41) for $i = m$. Finally if $m \in I_2$ then by (5.46), (5.48) remains valid with 1/10 in place of 1/2, and 3/5 subtracted from the right side, once again proving (5.41) for $i = m$. Thus (5.41) holds in all cases.

The next step is to sum (5.41) over i. There are 2 cases.

Case 1. $d \geq 20w_1\sqrt{A}$. Then using (5.41), (5.26) and (5.42),

$$\begin{aligned}\sum_{i\leq l}\mu_i &\geq \sum_{i\leq l}(\delta_i + \frac{1}{10}d_i + 1) \\ &\geq \frac{1}{10}(d' - l) + \frac{1}{60}t_+ + \frac{1}{20}(d - l) + \frac{1}{20}\sum_{i\leq l}d_i + l \\ &\geq \frac{1}{10}d' + \frac{1}{60}t_+ + w_1\sqrt{A} + \frac{\kappa_\tau}{80}lr\end{aligned} \tag{5.49}$$

which proves (5.40).

Case 2. $d < 20w_1\sqrt{A}$. By (5.26), (5.27) and (5.42),

$$6d' + t_+ \leq 6(d + l + 1) \leq 7d \leq 140w_1\sqrt{A}.$$

This and the size condition imply

$$r < 141\epsilon_{17}w_1\sqrt{A} \leq \frac{\kappa_\tau\sqrt{A}}{80K_{33}w_1}$$

if ϵ_{17} is small enough, with K_{33} as in (5.3), and then

$$\sqrt{A - K_{33}r^2} \geq \sqrt{A}\left(1 - \frac{K_{33}r^2}{A}\right) \geq \sqrt{A} - \frac{\kappa_\tau r}{80w_1}. \tag{5.50}$$

Let

$$S_j = \sum_{i \in I_j} A_i, \quad j = 1, 2, 3, \quad \text{and} \quad S = S_1 + S_2 + S_3.$$

Provided ϵ_{17} is small enough, we have by (5.3) and (5.50)

$$w_1 \sqrt{S} \geq w_1 \sqrt{A} - \frac{\kappa_\tau r}{80}. \tag{5.51}$$

It is easily checked that

$$\sqrt{a+b} \leq \sqrt{a} + \theta \sqrt{b} \quad \text{for } 0 \leq b \leq 4\theta^2 a. \tag{5.52}$$

Choose i_{13}, i_2 satisfying

$$A_{i_{13}} = \max_{i \in I_1 \cup I_3} A_i, \qquad A_{i_2} = \max_{i \in I_2} A_i.$$

(If $I_1 \cup I_3$ or I_2 is empty then the corresponding i_{13} or i_2 is undefined.) We now consider two subcases.

Case 2a. $I_2 = \phi$ or $A_{i_2} \leq \frac{1}{25}(S_1 + S_3)$. From the definition of μ_i if $i_{13} \in I_1$, and from (5.41) if $i_{13} \in I_3$, we have

$$\mu_{i_{13}} \geq w_1 \sqrt{A_{i_{13}}} - \lambda + \delta_{i_{13}}, \tag{5.53}$$

where $\lambda = \min(K_{35} r^{2/3} A_{i_{13}}^{1/6}, w_1 \sqrt{A_{i_{13}}})$. Therefore by (5.52) with $\theta = 1/2$,

$$\sum_{i \in I_1 \cup I_3} \mu_i \geq w_1 \sqrt{S_1 + S_3} - \lambda + \sum_{i \in I_1 \cup I_3} \delta_i + \sum_{i \in I_1 \cup I_3, i \neq i_{13}} \frac{1}{10} d_i - l. \tag{5.54}$$

Hence by (5.52) with $\theta = 1/25$, and (5.51), (5.42), (5.54) and (5.41),

$$\begin{aligned} \sum_{i \leq l} \mu_i &\geq w_1 \sqrt{S} - \lambda + \sum_{i \leq l} \delta_i + \frac{\kappa_\tau}{40}(l-1)r - l \\ &\geq w_1 \sqrt{A} - \frac{\kappa_\tau r}{80} - \lambda + \frac{1}{10}(d' - l) + \frac{1}{60} t_+ + \frac{\kappa_\tau}{40}(l-1)r - l \\ &\geq u(K_{35} r^{2/3}, A) + \frac{1}{10} d' + \frac{1}{60} t_+ + \frac{\kappa_\tau}{100}(l-1)r \end{aligned} \tag{5.55}$$

which gives (5.40).

Case 2b. $A_{i_2} > \frac{1}{25}(S_1 + S_3)$. Let us relabel $(A_i, i \in J_2)$ as $B_1 \geq .. \geq B_n$. We have

$$S_1 + S_3 \leq 25 c_1 r^2,$$

while from (5.3), provided ϵ_{17} is small enough,

$$S \geq A - K_{33} r^2 \geq 32,525 c_1 r^2,$$

so

$$S_2 \geq 32,500 c_1 r^2 \geq 325 \sum_{m=1}^{100} B_i,$$

which implies

$$\frac{1}{20}\left(\sum_{m=101}^{n} B_m\right)^{1/2} \geq \frac{9}{10}\left(\sum_{m=1}^{100} B_m\right)^{1/2}.$$

Using this, and using (5.52) twice (with $\theta = 1$ and with $\theta = 1/20$), we get

$$\begin{aligned}
(5.56)\quad & \frac{1}{10}\sum_{m=1}^{n}\sqrt{B_i} \\
&\geq \frac{1}{10}\left(\sum_{m=1}^{100} B_m\right)^{1/2} + \frac{1}{10}\sum_{m=101}^{n}\sqrt{B_i} \\
&\geq \left(\sum_{m=1}^{100} B_m\right)^{1/2} + \frac{1}{20}\sum_{m=101}^{n}\sqrt{B_i} + \frac{1}{20}\left(\sum_{m=101}^{n} B_i\right)^{1/2} - \frac{9}{10}\left(\sum_{m=1}^{100} B_m\right)^{1/2} \\
&\geq \sqrt{S_2}.
\end{aligned}$$

If $I_1 \cup I_3 \neq \phi$ then (5.54) remains valid. This, with (5.41), (5.42) and (5.56), shows that, whether $I_1 \cup I_3 = \phi$ or not, (5.55) (with $\lambda = 0$ if $I_1 \cup I_3 = \phi$) still holds.

The proof of (5.40), and thus of (5.21), is now complete. Taking $y = 0$ and E the full configuration space in (5.21), and summing over d satisfying (5.19) shows that

$$(5.57)\qquad P(M_0(k,q,r,A,A',d',t)) \leq K_{48} A \exp\left(-u(K_{35} r^{2/3}, A) - \frac{1}{60}t_+ - \frac{1}{10}d'\right).$$

This proves (5.17), with $K_{47} = 2K_{35}$.

It remains to prove (5.18). This is similar to the proof of (5.21), so we will only describe the changes. Again fix q, r. We make the same induction hypothesis, except that (5.21) is replaced by

$$(5.58)\qquad P(L_0(j,A,A',d,d',t) \mid E) \leq \exp\left(-\frac{4}{5}d'\right) P(M_0(0,q,r,A',A',0,t) \mid E),$$

and the requirement that the size condition be satisfied is removed. This hypothesis is true for $j = 0$, where only $d' = 0$ is relevant; hence we fix $k \geq 1$ and A, A', d, d'. Let

$$f(A',t) = -\log P(M_0(0,q,r,A',A',0,t) \mid E).$$

In place of μ_i we use

$$\hat{\mu}_i = \frac{9}{10}d_i - \frac{\kappa_\tau}{40}k_i r, \quad i \neq m,$$

$$\hat{\mu}_m = \max\left[\frac{4}{5}d'_m + f(A',t), \frac{9}{10}d_m - \frac{\kappa_\tau}{40}k_i r\right].$$

In place of (5.39), (5.40) and their multi-case proofs, we have simply, using the first half of (5.42),

$$\sum_{i\neq m}\left(\frac{1}{10}d_i - \frac{\kappa_\tau}{40}k_i r\right) \geq \frac{\kappa_\tau r}{40}(l-1) \geq l + \frac{\kappa_\tau r}{50}(l-1)$$

and hence using (5.26)

$$\begin{aligned}\sum_{i\leq l}\hat{\mu}_i &\geq f(A',t) + \frac{4}{5}(d'-l) + \sum_{i\neq m}\left(\frac{1}{10}d_i - \frac{\kappa_\tau}{40}k_i r\right)\\ &\geq f(A',t) + \frac{4}{5}d' + \frac{\kappa_\tau r}{50}(l-1).\end{aligned}$$

This leads directly to (5.58), as in the proof of (5.21). In place of (5.57) we have

$$P(M_0(k,q,r,A,A',d',t)) \leq K_{48}A\exp\left(-\frac{4}{5}d'\right)P(M_0(0,q,r,A',A',0,t)).$$

For $k \geq 1$ we have $d' \geq \kappa_\tau r/3$, so provided K_{46} is large enough, this proves (5.18). ☐

Lemma 5.6. *Let $q \geq 1, r \geq 15q$, let γ be a dual circuit and let $\alpha_{\max,\gamma}$ be its maximal (q,r)-descendant. Then*

$$\mathcal{W}(\partial\,\mathrm{Co}(\gamma)) \leq \mathcal{W}(\partial\,\mathrm{Co}(\alpha_{\max,\gamma})) + 19D'_{(q,r)}(\gamma).$$

Proof. We may assume γ has at least one bottleneck. If (u,v) is a primary bottleneck in γ, and the $(\kappa_\tau r/3)$-large offspring of γ are $\alpha_1,..,\alpha_k$, then

$$\mathrm{Int}(\gamma) \subset B_\tau(u, \frac{\kappa_\tau r}{3} + 2\kappa_\tau q) \cup \cup_{i\leq k}\mathrm{Int}(\alpha_i)$$

and therefore from (5.4), γ can be surrounded by a (non-lattice) loop of τ-length at most

$$12\left(\frac{\kappa_\tau r}{3} + 2\kappa_\tau q\right) + \sum_{i\leq k}\mathcal{W}(\partial\,\mathrm{Co}(\alpha_i)).$$

Since $\partial\,\mathrm{Co}(\gamma)$ minimizes the τ-length over all such loops, it follows that

$$\mathcal{W}(\partial\,\mathrm{Co}(\gamma)) \leq 6\kappa_\tau r + \sum_{i\leq k}\mathcal{W}(\partial\,\mathrm{Co}(\alpha_i)).$$

Iterating this, and using (5.4) and $D'_{(q,r)}(\gamma) \geq \frac{\kappa_\tau r}{3}|\mathfrak{F}'_{(q,r)}(\gamma)|$, we obtain

$$\begin{aligned}\mathcal{W}(\partial\operatorname{Co}(\gamma)) &\leq 6\kappa_\tau r|\mathfrak{F}'_{(q,r)}(\gamma)| + \sum_{\alpha\in\mathfrak{F}_{(q,r)}(\gamma)} \mathcal{W}(\partial\operatorname{Co}(\alpha)) \\ &\leq \mathcal{W}(\partial\operatorname{Co}(\alpha_{\max,\gamma})) + 19D'_{(q,r)}(\gamma).\end{aligned}$$

□

The next theorem, together with Theorem 4.1, shows roughly that for a droplet of size A, there is a cost for the convex hull boundary τ-length exceeding the minimum $w_1\sqrt{A}$ by an amount s_+, this cost being exponential in s_+, and there is an exponential cost for positive $D'_{(q,r)}(\Gamma_0)$.

Theorem 5.7. *Assume (2.7) and either (i) the ratio weak mixing property or (ii) both the weak mixing property and the near-Markov property for open circuits. There exist constants c_i as follows. Let $A > K_{51}, s_+ \geq 0, s = w_1\sqrt{A} + s_+$ and $d' \geq 0$. Then*

$$\begin{aligned}P(|\operatorname{Int}(\Gamma_0)| \geq A, \mathcal{W}(\partial\operatorname{Co}(\Gamma_0)) \geq s, D'_{(q,r)}(\Gamma_0) \geq d') \\ \leq \exp\left(-u(K_{52}(\log A)^{2/3}, A) - \frac{1}{1520}s_+ - \frac{1}{20}d'\right).\end{aligned} \tag{5.59}$$

Proof. Let $K_{53} \geq K_{46}$ (of Proposition 5.5) and

$$q_B = \frac{1}{15}K_{53}\log B, \qquad r_B = K_{53}\log B,$$

$$t_+(n, A') = \max(s - 19n - w_1\sqrt{A'}, 0),$$

$$I_1(n) = \{A' \in \mathbb{Z}^+ : t_+(n, A') \geq \frac{s_+}{2}\},$$

$$I_2(n) = \{A' \in \mathbb{Z}^+ : t_+(n, A') < \frac{s_+}{2}, w_1\sqrt{A'} \leq w_1\sqrt{A} + 19n\},$$

and

$$I_3(n) = \{A' \in \mathbb{Z}^+ : t_+(n, A') < \frac{s_+}{2}, w_1\sqrt{A'} > w_1\sqrt{A} + 19n\}.$$

Then using Lemma 5.6,

(5.60)

$$
\begin{aligned}
&P(|\operatorname{Int}(\Gamma_0)| \geq A, \mathcal{W}(\partial \operatorname{Co}(\Gamma_0)) \geq s, D'_{(q,r)}(\Gamma_0) \geq d') \\
&\quad \leq \sum_{B \geq A} \sum_{n \geq d'} P(|\operatorname{Int}(\Gamma_0)| = B, D'_{(q,r)}(\Gamma_0) \in [n, n+1), \\
&\qquad\qquad\qquad \mathcal{W}(\partial \operatorname{Co}(\alpha_{\max,\Gamma_0})) \geq s - 19n) \\
&\quad \leq \sum_{B \geq A} \sum_{n \geq d'} \Bigg[\sum_{A' \leq B, A' \in I_1(n)} \sum_{k \geq 0} P\big(M_0(k, q_B, r_B, B, A', n, w_1\sqrt{A'} + t_+(n, A'))\big) \\
&\qquad\qquad + \sum_{A' \leq B, A' \in I_2(n)} \sum_{k \geq 0} P(M_0(k, q_B, r_B, B, A', n, w_1\sqrt{A'})) \\
&\qquad\qquad + \sum_{A' \leq B, A' \in I_3(n)} \sum_{k \geq 0} P(M_0(k, q_B, r_B, B, A', n, w_1\sqrt{A'})) \Bigg].
\end{aligned}
$$

The events $M_0(k, q_B, r_B, B, A', n, \cdot)$ are empty unless $n+1 > (k+1)\kappa_\tau r/4$ (cf. (5.42)); if K_{53}, and hence r, is large enough, this implies $k \leq n$, so we may restrict the sums in (5.60) to such k. Presuming A is large enough, $u(K_{47}(\log B)^{2/3}, B)$ is strictly positive for all $B \geq A$. For $A' \in I_1(n)$ we apply Proposition 5.5 to get

$$
\begin{aligned}
(5.61)\qquad &\sum_{A' \leq B, A' \in I_1(n)} \sum_{k \leq n} P(M_0(k, q_B, r_B, B, A', n, w_1\sqrt{A'} + t_+(n, A')) \\
&\qquad \leq (n+1)B \exp\left(-w_1\sqrt{B} + K_{47}B^{1/6}(\log B)^{2/3} - \frac{s_+}{120} - \frac{1}{10}n \right).
\end{aligned}
$$

Note that if $I_2(n)$ or $I_3(n)$ is nonempty we must have $s_+ = s - w_1\sqrt{A} > 0$. If $A' \in I_2(n)$ we have

$$
\frac{1}{2}(s - w_1\sqrt{A}) > s - 19n - w_1\sqrt{A'} \geq s - w_1\sqrt{A} - 38n
$$

and hence $n \geq s_+/64$. Therefore

$$\sum_{A' \leq B, A' \in I_2(n)} \sum_{k \leq n} P(M_0(k, q_B, r_B, B, A', n, w_1\sqrt{A'})) \tag{5.62}$$
$$\leq (n+1)B \exp\left(-w_1\sqrt{B} + K_{47}B^{1/6}(\log B)^{2/3} - \frac{1}{10}n\right)$$
$$\leq (n+1)B \exp\left(-w_1\sqrt{B} + K_{47}B^{1/6}(\log B)^{2/3} - \frac{1}{20}n - \frac{1}{1520}s_+\right).$$

If $A' \in I_3(n)$ we have

$$s_+ + w_1\sqrt{A} - w_1\sqrt{A'} - 19n \leq t_+(n, A') \leq \frac{s_+}{2}$$

so that

$$2(w_1\sqrt{A'} - w_1\sqrt{A}) > w_1\sqrt{A'} - w_1\sqrt{A} + 19n \geq \frac{s_+}{2}$$

which implies

$$w_1\sqrt{B} \geq w_1\sqrt{A'} > w_1\sqrt{A} + \frac{s_+}{4}.$$

Therefore

(5.63)
$$\sum_{A' \leq B, A' \in I_3(n)} \sum_{k \leq n} P(M_0(k, q_B, r_B, B, A', n, w_1\sqrt{A'}))$$
$$\leq (n+1)B \exp\left(-w_1\sqrt{B} + K_{47}B^{1/6}(\log B)^{2/3} - \frac{1}{10}n\right)$$
$$\leq (n+1)B \exp\left(-\frac{1}{2}w_1\sqrt{B} - \frac{1}{2}w_1\sqrt{A} - \frac{1}{8}s_+ + K_{47}B^{1/6}(\log B)^{2/3} - \frac{1}{10}n\right).$$

We can now use (5.61),(5.62) and (5.63) to sum over n and B in (5.60), obtaining (5.59). ∎

Part of our main result is an easy consequence of Theorem 5.7.

Proof of Theorem 2.1,(2.13) and (2.14). From the definition of w_1 and Theorem 5.7, for any c, if A is sufficiently large,

$$
\begin{aligned}
(5.64) \qquad & P(|\operatorname{Int}(\Gamma_0)| \geq A, ALR(\Gamma_0) > cl^{1/3}(\log l)^{2/3}) \\
&\leq P(|\operatorname{Int}(\Gamma_0)| \geq A, |\operatorname{Co}(\Gamma_0)| \geq A + cw_1 l^{4/3}(\log l)^{2/3}) \\
&\leq P\left(|\operatorname{Int}(\Gamma_0)| \geq A, \mathcal{W}(\partial \operatorname{Co}(\Gamma_0)) \geq w_1\sqrt{A} + \frac{cw_1^2 l^{1/3}(\log l)^{2/3}}{3} \right) \\
&\leq \exp(-u(K_{52}(\log A)^{2/3}, A) - \epsilon_{19} cl^{1/3}(\log l)^{2/3}).
\end{aligned}
$$

If we take c sufficiently large, this and Theorem 4.1 prove that (2.13) holds with conditional probability approaching 1 as $A \to \infty$.

Next, from the quadratic nature of the Wulff variational minimum (see [1], [11]), for any a, b, if A is sufficiently large,

$$
\begin{aligned}
(5.65) \qquad & P(A \leq |\operatorname{Int}(\Gamma_0)| \leq A + aw_1 l^{4/3}(\log l)^{2/3}, \Delta_A(\partial \operatorname{Co}(\Gamma_0)) > bl^{2/3}(\log l)^{1/3}) \\
&\leq \sum_B P(|\operatorname{Int}(\Gamma_0)| = B, \Delta_B(\partial \operatorname{Co}(\Gamma_0)) > \frac{b}{2} l^{2/3}(\log l)^{1/3}) \\
&\leq \sum_B P(|\operatorname{Int}(\Gamma_0)| = B, \mathcal{W}(\partial \operatorname{Co}(\Gamma_0)) \geq w_1\sqrt{A} + \epsilon_{20} bl^{1/3}(\log l)^{2/3}) \\
&\leq \exp(-u(K_{52}(\log A)^{2/3}, A) - \epsilon_{21} bl^{1/3}(\log l)^{2/3}),
\end{aligned}
$$

where the sums are over $A \leq B \leq A + aw_1 l^{4/3}(\log l)^{2/3}$. Now

$$P(|\operatorname{Int}(\Gamma_0)| > A + aw_1 l^{4/3}(\log l)^{2/3})$$

can be bounded as in (5.64), so if we take a, b sufficiently large, (5.65) and Theorem 4.1 prove that (2.14) holds with conditional probability approaching 1 as $A \to \infty$. $\square$

6. Proof of the Single Droplet Theorem

Let Φ_N denote the open dual circuit in Λ_N enclosing maximal area. Let

$$T_N = \sum_{\gamma \in \mathfrak{C}_N} \operatorname{diam}_\tau(\gamma), \quad T_N' = \sum_{\gamma \in \mathfrak{C}_N \setminus \{\Phi_N\}} \operatorname{diam}_\tau(\gamma).$$

Let $G_N(k, A, A', d, d')$ denote the event that there are exactly $k+1$ $(K \log N)$-large exterior open dual circuits in Λ_N, with

$$\sum_{\gamma \in \mathfrak{C}_N} |\operatorname{Int}(\gamma)| = A, \quad |\operatorname{Int}(\Phi_N)| = A', \quad d \leq T_N < d+1, \quad d' \leq T_N < d'+1.$$

As we will see, by mimicking the proof of (5.18) it is easy to obtain

$$
\begin{aligned}
(6.1)\qquad & P_{N,w}(G_N(k,A,A',d',t)) \\
&\leq \exp\left(-\frac{4}{5}d'\right) P_{N,w}(|\operatorname{Int}(\Gamma_y)| = A' \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2),
\end{aligned}
$$

and then, summing as in Theorem 5.7, roughly

$$
\begin{aligned}
(6.2)\qquad & P_{N,w}(\sum_{\gamma \in \mathfrak{C}_N} |\operatorname{Int}(\gamma)| \geq A, T'_N \geq d') \\
&\leq \exp\left(-\frac{1}{2}d'\right) P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A - v \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2).
\end{aligned}
$$

with $v \leq \sqrt{A}$. (Statement (6.2) is for motivation only—the actual statement we prove is (6.62).) Note that $|\mathfrak{C}_N| > 1$ implies $T'_N \geq K \log N$; hence to prove (2.19) we would like a result somewhat like (6.2) but with A on the right side in place of $A - v$. To replace $A - v$ with A we need to know something of how the probability on the right side of (6.2) behaves as a function of v, which is obtainable from our next two results. For $N > 0$ and $0 < A < B$, we say that a lattice site y is (Λ_N, A, B)-*compatible* if there exists z such that $y \in z + \sqrt{A}\mathcal{K}_1$ and $z + \sqrt{B}\mathcal{K}_1 \subset \Lambda_N$.

Proposition 6.1. *Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7), the near-Markov property for open circuits, and the ratio weak mixing property. There exist K_i, ϵ_i such that for $A \geq K_{54}$ and $\delta \geq K_{55} \log A$ we have*

$$
(6.3)\qquad P(|\operatorname{Int}(\Gamma_y)| \geq A + \delta\sqrt{A}) \geq e^{-\epsilon_{22}\delta} P(|\operatorname{Int}(\Gamma_y)| \geq A).
$$

From this proposition we will obtain its analog for $P_{N,w}$, which is as follows.

Proposition 6.2. *Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7), the near-Markov property for open circuits, and the ratio weak mixing property. There exist K_i, ϵ_i such that for $N \geq 1, K_{56} \leq A \leq c_2 N^2, K_{57} \log A < \delta \leq \epsilon_{23}\sqrt{A}$ and y $(\Lambda_N, A/2, (1+\epsilon_{24})A)$-compatible we have*

$$
(6.4)\qquad P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A + \delta\sqrt{A}) \geq e^{-\epsilon_{25}\delta} P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A).
$$

Here c_2 is from Theorem 2.3.

These propositions will require some preliminary results.

Lemma 6.3. *Suppose τ is positive. There exists ϵ_{26} such that if $q \geq 1, r \geq 15q, A > A' > 0$ and γ is a dual circuit with $|\operatorname{Int}(\gamma)| = A, |\operatorname{Int}(\alpha_{\max,\gamma})| = A'$ then*

$$
D'_{(q,r)}(\gamma) \geq \epsilon_{26}\sqrt{A - A'}.
$$

Proof. Let $\alpha_1,..,\alpha_k$ be the non-maximal final (q,r)-descendants of γ, and $A_i = |\operatorname{Int}(\alpha_i)|$. Then

$$D'_{(q,r)}(\gamma) \geq \sum_{i\leq k} \max\left(w_1\sqrt{A_i}, \frac{\kappa_\tau r}{3}\right) \geq \frac{1}{2}w_1\sqrt{\sum_{i\leq k} A_i} + \frac{1}{6}k\kappa_\tau r$$

while (cf. (5.6))

$$A - A' \leq kr^2 + \sum_{i\leq k} A_i.$$

The lemma follows easily. □

Remark 6.4. The proof of Proposition 5.5 shows that the bounds (5.17) and (5.18) are valid conditionally on any enclosure event E, with the following modification: on the right side of (5.18) one must replace $P(M_0(\cdots))$ with $\sup_{y\in\Lambda_N\cap\mathbb{Z}^2} P_{N,w}(M_y(\cdots) \mid E)$. Let E_N denote the enclosure event $\operatorname{Open}(\partial\Lambda_N) \cap [\partial\Lambda_N \nleftrightarrow \infty]$. Assuming P has the near-Markov property for open circuits, probabilities under $P_{N,w}$ and under $P(\cdot \mid E_N)$ differ by a factor of at most $1+Ce^{-aN}$ for some C, a. Therefore Proposition 5.5 is valid with (5.17) and (5.18) replaced by

$$P_{N,w}(M_y(k,q,r,A,A',d',t)) \leq \exp(-u(K_{47}r^{2/3}, A) - \frac{1}{60}t_+ - \frac{1}{10}d') \tag{6.5}$$

and

$$P_{N,w}(M_y(k,q,r,A,A',d',t)) \leq \exp\left(-\frac{1}{2}d'\right) P_{N,w}(\cup_{x\in B(y,K_{58}A)} M_x(0,q,r,A',A',0,t)) \tag{6.6}$$

for all $y \in \Lambda_N \cap \mathbb{Z}^2$, for some K_{58}. It follows easily from (6.5) that Theorem 5.7, and then (2.13) and (2.14) of Theorem 2.1, extend similarly. Further, because the constraint $A \leq c_2N^2$ in Theorem 2.3 and Proposition 6.1 allows the appropriate size of Wulff shape to fit inside Λ_N, and weak mixing adequately eliminates boundary effects, Theorem 4.1 is also valid for $P_{N,w}$ when $A \leq c_2N^2$.

For $x, y \in (\mathbb{Z}^2)^*, r > 0$ and $G \subset \mathbb{R}^2$, we say there is an *r-near dual connection* from x to y in G if for some $u, v \in (\mathbb{Z}^2)^*$ with $d(u,v) \leq r$, there are open dual paths from x to u and from y to v in G. Let $N(x,y,r,G)$ denote the event that such a connection exists. The following result is from [4].

Lemma 6.5. *Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7) and the ratio weak mixing property. There exist K_i such that if $|x| > 1$ and $r \geq K_{59}\log|x|$ then*

$$P(N(0,x,r,\mathbb{R}^2)) \leq e^{-\tau(x)+K_{60}r}.$$

The next lemma proves (2.15) in Theorem 2.1.

Lemma 6.6. *Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7), the near-Markov property for open circuits, and the ratio weak mixing property. There exist ϵ_i, K_i such that for $A > K_{61}, l = \sqrt{A}$ and $\epsilon_{27}A \geq r \geq 15q \geq K_{62}\log A$, under the measure $P(\cdot \mid |\mathrm{Int}(\Gamma_0)| \geq A)$, with probability approaching 1 as $A \to \infty$ we have*

$$MLR(\Gamma_0) \leq K_{63}l^{2/3}(\log l)^{1/3}. \tag{6.7}$$

We may and henceforth do assume that $K_{63} \geq K_4$ of Theorem 2.1.

Proof of Lemma 6.6. The proof is partly a modification of that of Proposition 5.3, so we use the notation of that proof. The basic idea is that a large inward deviation of $\Gamma_0^{[y_k,y_{k+1}]}$ from $\partial\,\mathrm{Co}(\Gamma_0)$ for some k reduces the factor $P(w_k \leftrightarrow x_k)$ in (5.14).

First observe that the proof of (5.11) actually shows that

$$d(\Gamma_0^{[y_i,y_{i+1}]}, \Gamma_0^{[w_j,x_j]}) > q/2 \quad \text{for all } i \neq j \text{ with } j \in I. \tag{6.8}$$

Let Z be the site in Γ_0 most distant from $\partial\,\mathrm{Co}(\Gamma_0)$. Let $K_{64} > \sqrt{4\pi^{-1}w_1K_3}$ be a constant to be specified, and suppose that Γ_0 satisfies (2.13) and (2.14), but for some $k \leq m$ and some z we have $Z = z \in \Gamma_0^{[y_k,y_{k+1}]}$ and $d(z, \partial\,\mathrm{Co}(\{y_0, \dots, y_{m+1}\})) > 51K_{64}l^{2/3}(\log l)^{1/3}$. By (3.5) we have $z \in \mathrm{Co}(\{y_0, \dots, y_{m+1}\})$. Let R_k denote the region bounded by $\Gamma_0^{[y_k,y_{k+1}]}$ and by the segment of $\partial\,\mathrm{Co}(\Gamma_0)$ from y_k to y_{k+1}. Let ℓ_k denote the line through y_k and y_{k+1}. There exists a site $z' \in \ell_k$ with the following property: a line tangent to $\partial B_\tau(y_k, \tau(z'-y_k))$ at z' passes through z. This z' is a projection of z onto ℓ_k and satisfies

$$\tau(y_k - z) \geq \tau(y_k - z'), \quad \tau(y_{k+1} - z) \geq \tau(y_{k+1} - z'). \tag{6.9}$$

Let

$$D = B(z, K_{64}l^{2/3}(\log l)^{1/3}), \qquad E = B(z, 2K_{64}l^{2/3}(\log l)^{1/3}),$$

$$D' = B(z', K_{64}l^{2/3}(\log l)^{1/3}), \qquad E' = B(z', 13K_{64}l^{2/3}(\log l)^{1/3}),$$

$$F' = B(z', 14K_{64}l^{2/3}(\log l)^{1/3}).$$

Write $D_{\mathbb{Z}}, D'_{\mathbb{Z}}$, etc. for the intersection of D, D', etc. with $(\mathbb{Z}^2)^*$. We have several cases.

Case 1. $\Gamma_0^{[y_k,y_{k+1}]}$ contains a $(K_{59}\log A)$-near dual connection from y_k to y_{k+1} outside E. Here K_{59} is from Lemma 6.5. Then this event and the event $[z \leftrightarrow \partial_{in}D_{\mathbb{Z}}]$ occur at separation $K_{64}l^{2/3}(\log l)^{1/3}$, so by Lemmas 3.1 and 6.5,

$$\begin{aligned} P([z \leftrightarrow \partial_{in} D_{\mathbb{Z}}] \cap N(y_k, y_{k+1}, K_{59} \log A, E^c)) \\ \leq 2P(z \leftrightarrow \partial_{in} E_{\mathbb{Z}}) P(N(y_k, y_{k+1}, K_{59} \log A, E^c) \\ \leq \exp\left(-\tau(y_{k+1} - y_k) - \frac{1}{2} K_{64} l^{2/3} (\log l)^{1/3}\right). \end{aligned} \tag{6.10}$$

Case 2. $\Gamma_0^{[y_k, y_{k+1}]}$ contains no $(K_{59} \log A)$-near connection from y_k to y_{k+1} outside E, and $D' \cap \ell_k \subset \overline{y_k y_{k+1}}$. In this case, if $\Gamma_0^{[y_k, y_{k+1}]}$ did not intersect D', then at least half of D' would be contained in R_k. (This requires (3.5), which shows that we cannot have $D' \subset \mathrm{Int}(\Gamma_0)$.) Then from (2.13),

$$\frac{1}{2}\pi K_{64}^2 l^{4/3} (\log l)^{2/3} \leq |R_k| \leq |\partial \,\mathrm{Co}(\Gamma_0)| ALR(\Gamma_0) \leq 2w_1 K_3 l^{4/3} (\log l)^{2/3},$$

contradicting the definition of K_{64}. Thus $\Gamma_0^{[y_k, y_{k+1}]}$ must intersect D'.

Let h_1 be the first site of $\Gamma_0^{[y_k, v]}$ in $\partial E_{\mathbb{Z}}$, and let h_2 be the last site of $\Gamma_0^{[v, y_{k+1}]}$ in $\partial E_{\mathbb{Z}}$. Note that by the definition of Case 2, we have

$$d(\Gamma_0^{[y_k, h_1]}, \Gamma_0^{[h_2, y_{k+1}]}) > K_{59} \log A.$$

We have two subcases within Case 2.

Case 2a. $\Gamma_0^{[y_k, h_1]}$ and $\Gamma_0^{[h_2, y_{k+1}]}$ contain $(K_{59} \log A)$-near connections from y_k to h_1 and from h_2 to y_{k+1}, respectively, both outside F'. Then these near-connections occur at separation $K_{59} \log A$ from each other and from the event $[\partial_{in} D'_{\mathbb{Z}} \leftrightarrow \partial E'_{\mathbb{Z}}]$, so by Lemmas 3.2 and 6.5, since $\tau(h_2 - h_1) \leq \kappa_\tau \sqrt{2}(4K_{64} l^{2/3} (\log l)^{1/3} + 2)$,

(6.11)

$$\begin{aligned} &P(N(y_k, h_1, K_{59} \log A, (F')^c), N(h_2, y_{k+1}, K_{59} \log A, (F')^c), \text{ and } [\partial_{in} D'_{\mathbb{Z}} \leftrightarrow \partial E'_{\mathbb{Z}}] \\ &\qquad \text{occur at mutual separation } K_{59} \log A \text{ for some } h_1, h_2 \in \partial E_{\mathbb{Z}}) \\ &\leq 4 \sum_{h_1, h_2 \in \partial E_{\mathbb{Z}}} P(N(y_k, h_1, K_{59} \log A, (F')^c) \\ &\qquad \cdot P(N(h_2, y_{k+1}, K_{59} \log A, (F')^c)) P(\partial_{in} D'_{\mathbb{Z}} \leftrightarrow \partial E'_{\mathbb{Z}}) \\ &\leq 4 |\partial_{in} D'_{\mathbb{Z}}| |\partial E'_{\mathbb{Z}}| \sum_{h_1, h_2 \in \partial E_{\mathbb{Z}}} \exp\Big(-\tau(h_1 - y_k) - \tau(y_{k+1} - h_2) \\ &\qquad - 6\sqrt{2} K_{64} \kappa_\tau l^{2/3} (\log l)^{1/3} + K_{65} \log A\Big) \\ &\leq \exp(-\tau(y_{k+1} - y_k) - K_{64} \kappa_\tau l^{2/3} (\log l)^{1/3}). \end{aligned}$$

(Here we have actually used a trivial extension of Lemma 3.2, since the near-connection events occur not necessarily on the dual cluster of a single x^* but rather on the union of two such clusters.)

Case 2b. $\Gamma_0^{[y_k,h_1]}$ does not contain a $(K_{59}\log A)$-near connection from y_k to h_1 outside F'. (The other alternative to Case 2a within Case 2, symmetric to this one, is that $\Gamma_0^{[h_2,y_{k+1}]}$ does not contain a $(K_{59}\log A)$-near connection from h_2 to y_{k+1} outside F'; the proof is similar.) In this case the events $[y_k \leftrightarrow \partial F'_{\mathbb{Z}}$ outside $F' \cup E]$, $[\partial F'_{\mathbb{Z}} \leftrightarrow \partial E_{\mathbb{Z}}$ outside $F' \cup E]$ and $[\partial E_{\mathbb{Z}} \leftrightarrow y_{k+1}$ outside $E]$ occur at separation $K_{59}\log A$ from each other. Further, we have for $f, g \in \partial F'_{\mathbb{Z}}$, using (7.3),

$$
\begin{aligned}
(6.12)\qquad & \tau(f-y_k)+\tau(y_{k+1}-h_2)+\tau(h_1-g)\\
&\geq \tau(z'-y_k)+\tau(y_{k+1}-z)-\sqrt{2}\kappa_\tau(|z'-f|+|z-h_2|)\\
&\qquad + \frac{\kappa_\tau}{\sqrt{2}}(|z-z'|-|z-h_1|-|z'-g|)\\
&\geq \tau(z'-y_k)+\tau(y_{k+1}-z')+2K_{64}\kappa_\tau l^{2/3}(\log l)^{1/3}\\
&\geq \tau(y_{k+1}-y_k)+2K_{64}\kappa_\tau l^{2/3}(\log l)^{1/3}.
\end{aligned}
$$

Hence by Lemma 3.2,

$$
\begin{aligned}
(6.13)\quad & P([y_k \leftrightarrow \partial F'_{\mathbb{Z}} \text{ outside } F' \cup E], [\partial F'_{\mathbb{Z}} \leftrightarrow \partial E_{\mathbb{Z}} \text{ outside } F' \cup E] \text{ and}\\
&\qquad [\partial E_{\mathbb{Z}} \leftrightarrow y_{k+1} \text{ outside } F' \cup E] \text{ occur at mutual separation } K_{59}\log A)\\
&\leq 4P(y_k \leftrightarrow \partial F'_{\mathbb{Z}})P(\partial F'_{\mathbb{Z}} \leftrightarrow \partial E_{\mathbb{Z}})P(\partial E_{\mathbb{Z}} \leftrightarrow y_{k+1})\\
&\leq 4 \sum_{h_1,h_2 \in \partial F'_{\mathbb{Z}}} \sum_{f,g \in \partial E_{\mathbb{Z}}} \exp(-\tau(f-y_k)-\tau(y_{k+1}-h_2)-\tau(h_1-g))\\
&\leq \exp(-\tau(y_{k+1}-y_k)-K_{64}\kappa_\tau l^{2/3}(\log l)^{1/3}).
\end{aligned}
$$

Case 3. $\Gamma_0^{[y_k,y_{k+1}]}$ contains no $(K_{59}\log A)$-near connection from y_k to y_{k+1} outside E, and $D' \cap \ell_k \not\subset \overline{y_k y_{k+1}}$. As in Case 2b there are two symmetric alternatives within this, and we need only consider one, so we assume $d(z, y_{k+1}) \leq d(z, y_k)$. Then the events $[y_k \leftrightarrow \partial E_{\mathbb{Z}}]$ and $[\partial E_{\mathbb{Z}} \leftrightarrow y_{k+1}]$ occur at separation $K_{59}\log A$ and we have

$$\tau(z-y_k) \geq \tau(z'-y_k) \geq \tau(y_{k+1}-y_k) - K_{64}\sqrt{2}\kappa_\tau l^{2/3}(\log l)^{1/3}.$$

Hence

$$
\begin{aligned}
(6.14)\qquad & P([y_k \leftrightarrow \partial E_{\mathbb{Z}}] \text{ and } [\partial E_{\mathbb{Z}} \leftrightarrow y_{k+1}] \text{ occur at separation } K_{59} \log A) \\
&\le 2P(y_k \leftrightarrow \partial E) P(\partial E \leftrightarrow y_{k+1}) \\
&\le 2 \sum_{h_1, h_2 \in \partial E} \exp(-\tau(h_1 - y_k) - \tau(y_{k+1} - h_2)) \\
&\le 2 \exp\left(-\tau(z - y_k) + \sqrt{2}\kappa_\tau |z - h_1|) - \frac{\kappa_\tau}{\sqrt{2}}(|y_{k+1} - z| - |z - h_2|\right) \\
&\le 2 \exp(-\tau(y_{k+1} - y_k) - 30K_{64}\kappa_\tau l^{2/3}(\log l)^{1/3}).
\end{aligned}
$$

Now let $J(y_k, y_{k+1})$ denote the event that there is an open dual path from y_k to y_{k+1} containing a site z with $d(z, \ell_k) \ge 51K_{64}l^{2/3}(\log l)^{1/3}$. Combining the three cases we obtain

$$
P(J(y_k, y_{k+1})) \le 3 \exp(-\tau(y_{k+1} - y_k) - \frac{1}{2}K_{64}\kappa_\tau l^{2/3}(\log l)^{1/3}),
$$

and then analogously to (5.14),

$$
P(A(w_0, x_0, \dots, w_m, x_m) \cap J(y_k, y_{k+1})) \le 2^m P(J(y_k, y_{k+1})) \prod_{i \in I \setminus \{k\}} P(w_i \leftrightarrow x_i).
$$

Analogously to (5.14) – (5.16), this leads to

(6.15)

$$
\begin{aligned}
&P(|\operatorname{Int}(\Gamma_0)| \ge A, MLR(\Gamma_0) \ge 51K_{64}l^{2/3}(\log l)^{1/3}, \Gamma_0 \text{ is } (q,r) - \text{bottleneck-free}) \\
&\quad \le \exp(-w_1\sqrt{A} - \frac{1}{3}K_{64}\kappa_\tau l^{2/3}(\log l)^{1/3}).
\end{aligned}
$$

We will need the following straightforward extension of (5.18), under the conditions of Proposition 5.5:

(6.16)

$$
\begin{aligned}
&P(M_0(k, q, r, A, A', d', t) \cap [MLR(\alpha_{\max,\Gamma_0}) \ge 51K_{64}l^{2/3}(\log l)^{1/3}]) \\
&\qquad \le \exp\left(-\frac{1}{2}d'\right) P\big(M_0(0, q, r, A', A', 0, t) \cap [MLR(\Gamma_0) \ge 51K_{64}l^{2/3}(\log l)^{1/3}]\big).
\end{aligned}
$$

It is easy to see (cf. the proof of Lemma 5.6) that

$$
(6.17)\qquad MLR(\alpha_{\max,\Gamma_0}) \ge MLR(\Gamma_0) - 3\kappa_\tau^{-1} D'_{(q,r)}(\Gamma_0).
$$

Let $g(A) = (3\epsilon_{26})^{-2} K_{64}^2 \kappa_\tau^2 l^{4/3} (\log l)^{2/3}$, with ϵ_{26} from Lemma 6.3. From Theorem 5.7, Lemma 6.3, (6.15), (6.17) and (6.16),

(6.18)

$$
\begin{aligned}
&P(|\operatorname{Int}(\Gamma_0)| \geq A, MLR(\Gamma_0) \geq 52K_{64} l^{2/3} (\log l)^{1/3}) \\
&\quad \leq P(|\operatorname{Int}(\Gamma_0)| \geq A, D'_{(q,r)}(\Gamma_0) \geq \frac{1}{3} K_{64}\kappa_\tau l^{2/3} (\log l)^{1/3}) \\
&\qquad + P(|\operatorname{Int}(\Gamma_0)| \geq A, MLR(\alpha_{\max,\Gamma_0}) \geq 51K_{64} l^{2/3} (\log l)^{1/3}, \\
&\qquad\qquad\qquad |\operatorname{Int}(\Gamma_0)| - |\operatorname{Int}(\alpha_{\max,\Gamma_0})| < g(A)) \\
&\quad \leq \exp(-w_1\sqrt{A} - \frac{1}{60} K_{64}\kappa_\tau l^{2/3} (\log l)^{1/3}) \\
&\qquad + \sum_{B \geq Q} \sum_{d' \geq 0} \sum_{B - g(A) < A' \leq B} \sum_{k \leq d'} P\Big(M_0(k, q, r, B, A', d', w_1\sqrt{A'}) \\
&\qquad\qquad\qquad\qquad\qquad \cap [MLR(\alpha_{\max,\Gamma_0}) \geq 51K_{64} l^{2/3} (\log l)^{1/3}]\Big) \\
&\quad \leq \exp(-w_1\sqrt{A} - \frac{1}{60} K_{64}\kappa_\tau l^{2/3} (\log l)^{1/3}) \\
&\qquad + \sum_{A' \geq A - g(A)} \sum_{A' \leq B < A' + g(A)} \sum_{d' \geq 0} (d'+1) e^{-d'/2} P\Big(M_0(0, q, r, A', A', 0, w_1\sqrt{A'}) \\
&\qquad\qquad\qquad\qquad\qquad \cap [MLR(\Gamma_0) \geq 51K_{64} l^{2/3} (\log l)^{1/3}]\Big) \\
&\quad \leq \exp(-w_1\sqrt{A} - \frac{1}{60} K_{64}\kappa_\tau l^{2/3} (\log l)^{1/3}) \\
&\qquad + 10g(A) P\Big(|\operatorname{Int}(\Gamma_0)| \geq A - g(A), \Gamma_0 \text{ is } (q,r)-\text{bottleneck-free}, \\
&\qquad\qquad\qquad MLR(\Gamma_0) \geq 51K_{64} l^{2/3} (\log l)^{1/3}\Big) \\
&\quad \leq \exp(-w_1\sqrt{A} - \frac{1}{70} K_{64}\kappa_\tau l^{2/3} (\log l)^{1/3}).
\end{aligned}
$$

Here the restriction to $k \leq d'$ is permissible as in the proof of Theorem 5.7. Provided we take K_{64} large enough, with Theorem 4.1 this completes the proof. □

Proof of Proposition 6.1. Let $l = \sqrt{A}$ and $r = 15q = K_{46} \log A$, where K_{46} is from Proposition 5.5. Let $\operatorname{Mid}(\Gamma_0)$ denote the Wulff shape of area $\frac{3}{4}|\Gamma_0|$ centered at the center of mass of $\operatorname{Int}(\Gamma_0)$. From translation invariance, Theorems 2.1 and 4.1, (2.17) and (6.18) we have (recalling $K_{63} = 52K_{64}$)

$$\begin{aligned}
&P(|\operatorname{Int}(\Gamma_0)| \geq A, D'_{(q,r)}(\Gamma_0) \geq \frac{1}{3}K_{64}\kappa_\tau l^{2/3}(\log l)^{1/3}) \\
&+ P(|\operatorname{Int}(\Gamma_0)| \geq A, \Delta_A(\Gamma_0) > 2K_{63}l^{2/3}(\log l)^{1/3}) \\
&+ P(|\operatorname{Int}(\Gamma_0)| \geq A, \Delta_A(\Gamma_0) \leq 2K_{63}l^{2/3}(\log l)^{1/3}, 0 \notin \operatorname{Mid}(\Gamma_0)) \\
&\qquad \leq \frac{1}{2}P(|\operatorname{Int}(\Gamma_0)| \geq A).
\end{aligned}$$

With this fact, we can repeat the argument of (6.18), but excluding reference to $MLR(\cdot)$, to obtain (using again $g(A) = (3\epsilon_{26})^{-2}K_{64}^2\kappa_\tau^2 l^{4/3}(\log l)^{2/3}$)

$$\begin{aligned}
(6.19)\quad & P(|\operatorname{Int}(\Gamma_0)| \geq A) \\
&\leq \frac{1}{2}P(|\operatorname{Int}(\Gamma_0)| \geq A) \\
&\qquad + P\big(|\operatorname{Int}(\alpha_{\max,\Gamma_0})| \geq A - g(A), \\
&\qquad\qquad \Delta_A(\alpha_{\max,\Gamma_0}) \leq 2K_{63}l^{2/3}(\log l)^{1/3}, 0 \in \operatorname{Mid}(\Gamma_0)\big) \\
&\leq \frac{1}{2}P(|\operatorname{Int}(\Gamma_0)| \geq A) \\
&\qquad + 10g(A)P\big(|\operatorname{Int}(\Gamma_0)| \geq A - g(A), \Gamma_0 \text{ is } (q,r)-\text{bottleneck-free}, \\
&\qquad\qquad \Delta_A(\Gamma_0) \leq 2K_{63}l^{2/3}(\log l)^{1/3}, 0 \in \operatorname{Mid}(\Gamma_0)\big),
\end{aligned}$$

so that

$$\begin{aligned}
(6.20)\quad & P(|\operatorname{Int}(\Gamma_0)| \geq A) \\
&\leq 20g(A)P\big(|\operatorname{Int}(\Gamma_0)| \geq A - g(A), \Gamma_0 \text{ is } (q,r)-\text{bottleneck-free}, \\
&\qquad \Delta_A(\Gamma_0) \leq 2K_{63}l^{2/3}(\log l)^{1/3}, 0 \in \operatorname{Mid}(\Gamma_0)\big).
\end{aligned}$$

The idea now is to split Γ_0 into two halves and approximate the probability on the right side of (6.20) by the product of the probabilities of the two halves. With this independence the two halves can in effect be pulled apart from one another to increase the area enclosed by Γ_0 at only a small cost in increased boundary length. To accomplish this we first need some definitions. Let ρ be a path from $x_2 = (a_2, b_2)$ to $x_1 = (a_1, b_1)$ in the slab $S_{x_1x_2} = \{(x,y) \in \mathbb{R}^2 : b_2 \leq y \leq b_1\}$. Let $J_L(\rho)$ and $J_R(\rho)$ denote the regions to the left and right, respectively, of ρ in $S_{x_1x_2}$. The *right-side area* determined by ρ is

$$\mu_R(\rho) = |J_R(\rho) \cap \Lambda_N| - |J_R(\overline{x_1x_2}) \cap \Lambda_N|,$$

evaluated for N large enough that Λ_N contains ρ. (Note that for such N, the right-side area does not vary with N. Also, in our definition the path ρ must be oriented so that $b_1 \leq b_2$.) The *left-side area* $\mu_L(\rho)$ is defined similarly using the left side of ρ. Let X_1 and X_2 be the points of Γ_0 of maximum and minimum second coordinate, respectively, using the leftmost if there is more than one. Then

$$|\operatorname{Int}(\Gamma_0)| = \mu_L(\Gamma_0^{[X_2,X_1]}) + \mu_R(\hat{\Gamma_0}^{[X_2,X_1]}),$$

where $\hat{\Gamma_0}$ is Γ_0 traversed in the direction of negative orientation. Let

$$B_i = B(X_i, 4r), \qquad i = 1, 2,$$

and let U_i, V_i be the first and last lattice sites, respectively, of the segment of $\Gamma_0 \cap B_i$ containing X_i. Let W_1 be the first site in $\Gamma_0^{[U_1,X_1]}$ for which

$$d(W_1, \Gamma_0^{[X_1,U_2]}) \leq q,$$

and let Z_1 be the closest site to W_1 in $\Gamma_0^{[X_1,X_2]}$. W_2 and Z_2 are defined similarly with subscripts 1 and 2 interchanged.

Suppose now that Γ_0 is (q, r)-bottleneck-free and satisfies

$$\Delta_A(\Gamma_0) \leq 2K_{63} l^{2/3} (\log l)^{1/3} \tag{6.21}$$

for K_{63} of Lemma 6.6. Then B_1 and B_2 are disjoint, and since

$$\operatorname{diam}(\Gamma_0^{[U_1,X_1]}) > r \quad \text{and} \quad \operatorname{diam}(\Gamma_0^{[X_1,V_1]}) > r,$$

the absense of bottlenecks implies

$$d(\Gamma_0^{[X_2,U_1]}, \Gamma_0^{[X_1,U_2]}) > q \quad \text{and} \quad d(\Gamma_0^{[V_2,X_1]}, \Gamma_0^{[V_1,X_2]}) > q.$$

It follows that $W_i \neq U_i, Z_i \in \Gamma_0^{[X_i,V_i]}$ $(i = 1, 2)$ and

$$d(\Gamma_0^{[X_2,W_1]}, \Gamma_0^{[X_1,W_2]}) > q - 1 > \frac{q}{2}.$$

When ρ and σ are paths such that the endpoint of ρ is the initial point of σ, we let (ρ, σ) denote the path obtained by concatenating σ and ρ. Then

$$|\mu_L(\Gamma_0^{[X_2,X_1]}) - \mu_L((\Gamma_0^{[X_2,W_1]}, \zeta_{W_1X_1}))| \leq |B_1| = 16\pi r^2,$$

since the paths differ only inside B_1. Again we may interchange 1, 2 and L, R. Let $K_{66} > 0$ and let $D_x = B(x, K_{66} l^{2/3} (\log l)^{1/3})$ for $x \in \mathbb{R}^2$. Presuming K_{66} is large enough, (6.21) implies that

$$\Gamma_y^{[X_1,X_2]} \backslash (D_{X_1} \cup D_{X_2}) \subset J_L(\overline{X_1X_2}), \quad \Gamma_y^{[X_2,X_1]} \backslash (D_{X_1} \cup D_{X_2}) \subset J_R(\overline{X_1X_2}).$$

Hence using Lemma 3.2,

$$
\begin{aligned}
(6.22)\quad & P(|\operatorname{Int}(\Gamma_0)| \geq A - u, \Gamma_0 \text{ is } (q,r) - \text{bottleneck-free}, \\
&\qquad \Delta_A(\Gamma_0) \leq 2K_{63} l^{2/3} (\log l)^{1/3}, 0 \in \operatorname{Mid}(\Gamma_0)) \\
&\leq \sum_{A-u \leq A_L + A_R \leq 2A} \ \sum_{x_1, x_2 \in B(y, \sqrt{A})} \ \sum_{w_1, z_1 \in B(x_1, 4r)} \ \sum_{w_2, z_2 \in B(x_2, 4r)} \\
&\qquad P(X_1 = x_1, X_2 = x_2, W_1 = w_1, W_2 = w_2, Z_1 = z_1, Z_2 = z_2 \\
&\qquad \text{ and there exist paths } \rho_R \text{ from } x_2 \text{ to } w_1 \text{ and } \rho_L \text{ from } w_2 \text{ to } x_1 \\
&\qquad \text{ satisfying } \mu_L((\rho_R, \zeta_{w_1 x_1})) \geq A_R - 16\pi r^2, \\
&\qquad \mu_R((\zeta_{x_2 w_2}, \rho_L)) \geq A_L - 16\pi r^2, d(\rho_L, \rho_R) \geq \frac{q}{2}, z_1 \in \rho_L, z_2 \in \rho_R, \\
&\qquad \rho_L \backslash (D_{x_1} \cup D_{x_2}) \subset J_L(\overline{x_1 x_2}), \rho_R \backslash (D_{x_1} \cup D_{x_2}) \subset J_R(\overline{x_1 x_2}), \\
&\qquad 0 \in J_R((\zeta_{x_2 w_2}, \rho_L)) \cap J_L((\rho_R, \zeta_{w_1 x_1})) \backslash (D_{x_1} \cup D_{x_2})) \\
&\leq \sum_{A-u \leq A_L + A_R \leq 2A} \ \sum_{x_1, x_2 \in B(y, \sqrt{A})} \ \sum_{w_1, z_1 \in B(x_1, 4r)} \ \sum_{w_2, z_2 \in B(x_2, 4r)} \\
&\qquad 2P(x_2 \leftrightarrow w_1 \text{ via an open dual path } \rho_R \\
&\qquad\qquad \text{with } \mu_L((\rho_R, \zeta_{w_1 x_1})) \geq A_R - 16\pi r^2, z_2 \in \rho_R, \\
&\qquad\qquad \rho_R \backslash (D_{x_1} \cup D_{x_2}) \subset J_R(\overline{x_1 x_2}), \\
&\qquad\qquad 0 \in J_L((\rho_R, \zeta_{w_1 x_1})) \backslash (D_{x_1} \cup D_{x_2})) \\
&\qquad \cdot P(w_2 \leftrightarrow x_1 \text{ via an open dual path } \rho_L \\
&\qquad\qquad \text{with } \mu_R((\zeta_{x_2 w_2}, \rho_L)) \geq A_L - 16\pi r^2, z_1 \in \rho_L, \\
&\qquad\qquad \rho_L \backslash (D_{x_1} \cup D_{x_2}) \subset J_L(\overline{x_1 x_2}), \\
&\qquad\qquad 0 \in J_R((\zeta_{x_2 w_2}, \rho_L)) \backslash (D_{x_1} \cup D_{x_2}))
\end{aligned}
$$

Let us assume for convenience that δ is an integer (if not, the necessary modifications are simple), and let x_1', w_1', z_2' and x_2' be the lattice sites which are 2δ units to the right of x_1, w_1, z_2 and x_2, respectively. We now "pull apart" the two halves of Γ_0 by replacing each of these four sites by its right-shifted counterpart in the first probability on the right side of (6.22). Specifically, by the FKG property,

$$
\begin{aligned}
(6.23)\quad & P(x_2' \leftrightarrow w_1' \text{ via an open dual path } \rho_R \\
&\qquad \text{with } \mu_L((\rho_R, \zeta_{w_1'x_1'})) \geq A_R - 16\pi r^2, z_2' \in \rho_R, \\
&\qquad \rho_R \backslash (D_{x_1'} \cup D_{x_2'}) \subset J_R(\overline{x_1'x_2'}), \\
&\qquad 0 \in J_L((\rho_R, \zeta_{w_1'x_1'})) \backslash (D_{x_1'} \cup D_{x_2'})) \\
&\quad \cdot P(w_2 \leftrightarrow x_1 \text{ via an open dual path } \rho_L \\
&\qquad \text{with } \mu_R((\zeta_{x_2w_2}, \rho_L)) \geq A_L - 16\pi r^2, z_1 \in \rho_L, \\
&\qquad \rho_L \backslash (D_{x_1} \cup D_{x_2}) \subset J_L(\overline{x_1x_2}), \\
&\qquad 0 \in J_R((\zeta_{x_2w_2}, \rho_L)) \backslash (D_{x_1} \cup D_{x_2})) \\
&\quad \cdot P(\mathrm{Open}(\zeta_{z_1w_1'}))P(\mathrm{Open}(\zeta_{w_2z_2'})) \\
&\leq P(|\Gamma_0| \geq A_L + A_R + \frac{3}{2}\delta\sqrt{A} - 32\pi r^2 - |D_{x_1}| - |D_{x_2}| - |D_{x_1'}| - |D_{x_2'}|) \\
&\leq P(|\Gamma_0| \geq A + \delta\sqrt{A}).
\end{aligned}
$$

From (6.20), (6.22), (6.23) and the bounded energy property we obtain

$$
\begin{aligned}
(6.24)\qquad & P(|\operatorname{Int}(\Gamma_0)| \geq A) \\
&\leq K_{67}A^4r^4ue^{K_{68}\delta}P(|\operatorname{Int}(\Gamma_0)| \geq A + \delta\sqrt{A}) \\
&\leq e^{K_{69}\delta}P(|\operatorname{Int}(\Gamma_0)| \geq A + \delta\sqrt{A}),
\end{aligned}
$$

completing the proof. □

Proof of Propostion 6.2. Let $B = A + \delta\sqrt{A}$. Since $[|\operatorname{Int}(\Gamma_y)| \geq A]$ is a decreasing event, we have

$$(6.25)\qquad P(|\operatorname{Int}(\Gamma_y)| \geq A) \geq P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A),$$

so it is enough to show

$$(6.26)\qquad P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq B) \geq e^{-\epsilon_{25}\delta}P(|\operatorname{Int}(\Gamma_y)| \geq B).$$

Let

$$r_i^+ = \sqrt{B} + 2K_{63}il^{2/3}(\log l)^{1/3}), \quad r^- = \sqrt{B} - 2K_{63}l^{2/3}(\log l)^{1/3}).$$

Fix $y \in \mathbb{Z}^2$ and let $z \in \mathbb{Z}^2$ with $y \in z + r^-\mathcal{K}_1$ and $z + \sqrt{(1+\epsilon_{24})A}\mathcal{K}_1 \subset \Lambda_N$, where ϵ_{24} is to be specified. Then $\sqrt{A/2} < r^- < r_5^+ < \sqrt{B}$ and

$$(6.27)\qquad d(z + r_5^+\mathcal{K}_1, \partial\Lambda_N) \geq \tfrac{1}{8}\epsilon_{24}\sqrt{A},$$

provided ϵ_{23} and ϵ_{24} are small and A is large enough.

Let $\tilde{\Gamma}_y^i$ denote the outermost open dual circuit surrounding y in $z+r_i^+\mathcal{K}_1$ and define the event

$$F_i = [|\operatorname{Int}(\tilde{\Gamma}_y^i)| \geq B, d_H(\partial \operatorname{Co}(\tilde{\Gamma}_y^i), z + \partial\sqrt{B}\mathcal{K}_1) \leq K_4 l^{2/3}(\log l)^{1/3},$$
$$MLR(\tilde{\Gamma}_y^i) \leq K_{63} l^{2/3}(\log l)^{1/3}]$$

Note that $F_i \subset [\tilde{\Gamma}_y^i \subset z+r_1^+\mathcal{K}_1]$. Let E_i be the event that there exist $u \notin z+r_{i+1}^+\mathcal{K}_1, v \notin z+r_i^+\mathcal{K}_1$ for which $d(u,v) \geq \frac{1}{2}d(u, z+r_i^+\mathcal{K}_1)$ and $u \leftrightarrow v$ via an open dual path outside $z + r_i^+\mathcal{K}_1$. Then for some ϵ_{28},

$$P_{N,w}(E_i) \leq P(E_i) \leq \exp(-\epsilon_{28} l^{2/3}(\log l)^{1/3}), \tag{6.28}$$

$$F_5 \subset [\tilde{\Gamma}_y^5 \subset z + r_1^+\mathcal{K}_1] \subset [\tilde{\Gamma}_y^5 = \tilde{\Gamma}_y^3]$$

and

$$F_5 \cap [\tilde{\Gamma}_y^5 \neq \Gamma_y] \subset [\Gamma_y \not\subset z + r_5^+\mathcal{K}_1, \tilde{\Gamma}_y^5 \subset z + r_1^+\mathcal{K}_1] \subset E_4.$$

Here the last inclusion follows from the fact that if $\Gamma_y \not\subset z+r_5^+\mathcal{K}_1$ and $\tilde{\Gamma}_y^5 \subset z+r_1^+\mathcal{K}_1$ then Γ_y must surround or intersect $\tilde{\Gamma}_y^5$. The events E_4 and F_3 necessarily occur at separation $2K_{63}l^{2/3}(\log l)^{1/3}$, so by Lemma 3.1 we have

$$P_{N,w}(F_5 \cap [\tilde{\Gamma}_y^5 \neq \Gamma_y]) \leq P_{N,w}(E_4 \cap F_3) \leq 2P_{N,w}(E_4)P_{N,w}(F_3). \tag{6.29}$$

We want to replace $P(F_3)$ with $P(F_5)$ on the right side of (6.29). We have

$$F_3 \backslash F_5 \subset F_3 \cap [\tilde{\Gamma}_y^5 \not\subset z + r_3^+\mathcal{K}_1] \subset F_3 \cap E_2. \tag{6.30}$$

Let $\omega_1 \in F_1$ be a bond configuration on $\mathcal{B}(z + r_1^+\mathcal{K}_1)$. Conditionally on ω_1, F_3 is an increasing event (since F_3 requires $\tilde{\Gamma}_y^3 \subset z + r_1^+\mathcal{K}_1$, meaning $\tilde{\Gamma}_y^3$ is part of ω_1) and E_2 is a decreasing one, so using (6.28), (6.30) and Lemma 3.1 again,

$$\begin{aligned} &P_{N,w}(F_3\backslash F_5 \mid \omega_1) \\ &\quad \leq P_{N,w}(F_3 \mid \omega_1)P_{N,w}(E_2 \mid \omega_1) \\ &\quad \leq P_{N,w}(F_3 \mid \omega_1)P(E_2 \mid \omega_1) \\ &\quad \leq 2P_{N,w}(F_3 \mid \omega_1)P(E_2) \\ &\quad \leq \frac{1}{2}P_{N,w}(F_3 \mid \omega_1). \end{aligned} \tag{6.31}$$

Therefore $P_{N,w}(F_3) \leq 2P_{N,w}(F_5)$. With (6.29), (6.28), (6.27) and Lemma 3.1 this shows that

$$\begin{aligned}
(6.32) \qquad & P(|\operatorname{Int}(\Gamma_y)| \geq B, d_H(\partial \operatorname{Co}(\Gamma_y), z + \partial\sqrt{B}\mathcal{K}_1) \leq K_4 l^{2/3}(\log l)^{1/3}, \\
& \qquad MLR(\Gamma_y) \leq K_{63} l^{2/3}(\log l)^{1/3}) \\
& \leq P(F_5) \\
& \leq 2P_{N,w}(F_5) \\
& \leq 4P_{N,w}(F_5 \cap [\Gamma_y = \tilde{\Gamma}_y^5]) \\
& \leq 4P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq B, d_H(\partial \operatorname{Co}(\Gamma_y), z + \partial\sqrt{B}\mathcal{K}_1) \leq K_4 l^{2/3}(\log l)^{1/3}, \\
& \qquad MLR(\Gamma_y) \leq K_{63} l^{2/3}(\log l)^{1/3}).
\end{aligned}$$

By Theorem 2.1, Lemma 6.6 and translation invariance, there exists a site y' such that

$$\begin{aligned}
(6.33) \qquad & \frac{1}{2B} P(|\operatorname{Int}(\Gamma_y)| \geq B) \\
& = \frac{1}{2B} P(|\operatorname{Int}(\Gamma_{y'})| \geq B) \\
& \leq \frac{1}{B} P(|\operatorname{Int}(\Gamma_{y'})| \geq B, \Delta_B(\partial \operatorname{Co}(\Gamma_{y'})) \leq K_4 l^{2/3}(\log l)^{1/3}, \\
& \qquad MLR(\Gamma_{y'}) \leq K_{63} l^{2/3}(\log l)^{1/3}) \\
& \leq P(|\operatorname{Int}(\Gamma_{y'})| \geq B, d_H(\partial \operatorname{Co}(\Gamma_{y'}), z + \partial\sqrt{B}\mathcal{K}_1) \leq K_4 l^{2/3}(\log l)^{1/3}, \\
& \qquad MLR(\Gamma_{y'}) \leq K_{63} l^{2/3}(\log l)^{1/3}).
\end{aligned}$$

But the last event implies that $\Gamma_{y'}$ surrounds $z + r^- \mathcal{K}_1$, which contains y. Hence the last probability in (6.33) is bounded by the first probability in (6.32), so that

$$(6.34) \qquad \frac{1}{2B} P(|\operatorname{Int}(\Gamma_y)| \geq B) \leq 4P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq B).$$

Since $\delta \geq K_{57} \log A$, this completes the proof of (6.26). □

The next lemma includes the analog of Lemma 6.6 for $P_{N,w}$.

Lemma 6.7. *Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7), the near-Markov property for open circuits, and the ratio weak mixing property. There exist ϵ_i, K_i such that for $N \geq 1, K_{70}(\log N)^2 \leq A \leq c_2 N^2$ (with c_2 from Theorem 2.3) and $l = \sqrt{A}$,* *we have*

(6.35)
$$P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A \textit{ and } MLR(\Gamma_y) > K_{71} l^{2/3}(\log l)^{1/3} \textit{ for some } y \in \Lambda_N \cap \mathbb{Z}^2) \leq \exp(-\epsilon_{29} l^{2/3}(\log l)^{1/3}) P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A \textit{ for some } y \in \Lambda_N \cap \mathbb{Z}^2),$$

(6.36)
$$P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A \textit{ and } \Delta_A(\Gamma_y) > K_{72} l^{2/3}(\log l)^{1/3} \textit{ for some } y \in \Lambda_N \cap \mathbb{Z}^2) \leq \exp(-\epsilon_{30} l^{2/3}(\log l)^{1/3}) P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A \textit{ for some } y \in \Lambda_N \cap \mathbb{Z}^2),$$

(6.37)
$$P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A \textit{ and } ALR(\Gamma_y) > K_{73} l^{1/3}(\log l)^{2/3} \textit{ for some } y \in \Lambda_N \cap \mathbb{Z}^2) \leq \exp(-\epsilon_{31} l^{1/3}(\log l)^{2/3}) P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A \textit{ for some } y \in \Lambda_N \cap \mathbb{Z}^2),$$

and for $\epsilon_{32} A \geq r \geq 15q \geq K_{74} \log A$ *and* $\kappa_\tau r/3 \leq d' \leq \sqrt{A}$,

(6.38)
$$P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A, D'_{(q,r)}(\Gamma_y) > d' \textit{ and } |\operatorname{Int}(\Gamma_y)| - |\operatorname{Int}(\alpha_{\max,\Gamma_y})| < \epsilon_{33} d' \sqrt{A} \textit{ for some } y \in \Lambda_N \cap \mathbb{Z}^2) \leq \exp(-\epsilon_{34} d') P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A \textit{ for some } y \in \Lambda_N \cap \mathbb{Z}^2).$$

Proof. We begin with (6.35). The proof of Lemma 6.6 is valid for $P_{N,w}$ through (6.18) (see Remark 6.4), which gives that for c sufficiently large,

$$P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A, MLR(\Gamma_y) \geq 52 c l^{2/3}(\log l)^{1/3} \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2) \leq |\Lambda_N \cap \mathbb{Z}^2| \exp\left(-w_1 \sqrt{A} - \frac{1}{80} c \kappa_\tau l^{2/3}(\log l)^{1/3}\right). \tag{6.39}$$

If $l^{2/3}(\log l)^{1/3} \geq \log N$, we can complete the proof as we did in Lemma 6.6. Thus suppose

$$l^{2/3}(\log l)^{1/3} < \log N. \tag{6.40}$$

Since $A \leq c_2 N^2$, provided we choose ϵ_{35} small enough the set $J_A = \{y \in \Lambda_N \cap \mathbb{Z}^2 : y \text{ is } (\Lambda_N, A/2, (1+\epsilon_{35})A) - \text{compatible}\}$ satisfies $|J_A| \geq \epsilon_{36} N^2$. Let E_y denote the event that $A \leq |\operatorname{Int}(\Gamma_y)| \leq 2A$ and $\Delta_A(\partial \operatorname{Co}(\Gamma_y)) \leq K_4 l^{1/3}(\log l)^{2/3}$. Let F_y denote the event that Γ_y is the unique exterior dual circuit in Λ_N satisfying both $A \leq |\operatorname{Int}(\Gamma_x)| \leq 2A$ for some $x \in J_A$ and $\Delta_A(\partial \operatorname{Co}(\Gamma_x)) \leq K_4 l^{1/3}(\log l)^{2/3}$. From Theorem 2.1, Theorem 4.1 and Remark 6.4, we have for $y \in J_A$

$$P_{N,w}(E_y^c \mid |\operatorname{Int}(\Gamma_y)| \geq A) \leq \frac{1}{4}. \tag{6.41}$$

Also, from the near-Markov property for open circuits, the FKG property, and Theorem 5.7, provided K is large,

(6.42)

$$
\begin{aligned}
&P_{N,w}(F_y^c \mid E_y) \\
&\qquad = \sum_{\nu} P_{N,w}(F_y^c \mid E_y \cap [\Theta(\Gamma_y) = \nu]) P_{N,w}(\Theta(\Gamma_y) = \nu \mid E_y) \\
&\qquad \le \sum_{\nu} 2P_{N,w}(|\operatorname{Int}(\Gamma_x)| \ge A \text{ for some } x \in J_A \cap \operatorname{Ext}(\nu)) P_{N,w}(\Theta(\Gamma_y) = \nu \mid E_y) \\
&\qquad \le 2P_{N,w}(|\operatorname{Int}(\Gamma_x)| \ge A \text{ for some } x \in J_A) \\
&\qquad \le \frac{1}{4},
\end{aligned}
$$

so by Theorem 4.1 and Remark 6.4, for $y \in J_A$, using the $(\Lambda_N, A/2, (1+\epsilon_{35})A)$-compatibility of y,

$$
P_{N,w}(F_y) \ge \frac{1}{2} P_{N,w}(|\operatorname{Int}(\Gamma_y)| \ge A) \ge \exp(-w_1\sqrt{A} - K_{76} l^{1/3} (\log l)^{2/3}).
$$

Combining these facts we obtain

$$
\begin{aligned}
2AP_{N,w}&(|\operatorname{Int}(\Gamma_y)| \ge A \text{ for some } y \in J_A) \\
&\ge 2AP_{N,w}(\cup_{y \in J_A} F_y) \\
&\ge \sum_{y \in J_A} P_{N,w}(F_y) \\
&\ge \epsilon_{36} N^2 \exp(-w_1\sqrt{A} - K_{76} l^{1/3} (\log l)^{2/3}).
\end{aligned} \tag{6.43}
$$

Here the second inequality uses the fact that F_y can occur in each bond configuration for at most $2A$ sites $y \in J_A$. With (6.40) and (6.39), provided we take c large enough, (6.43) gives

$$
\begin{aligned}
&\exp(-l^{2/3} (\log l)^{1/3}) P_{N,w}(|\operatorname{Int}(\Gamma_y)| \ge A \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2) \\
&\qquad \ge |\Lambda_N \cap \mathbb{Z}^2| \exp(-w_1\sqrt{A} - K_{77} l^{2/3} (\log l)^{1/3}) \\
&\qquad \ge P_{N,w}(|\operatorname{Int}(\Gamma_y)| \ge A, MLR(\Gamma_y) \ge 52cl^{2/3} (\log l)^{1/3} \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2),
\end{aligned}
$$

which completes the proof of (6.35).

The proofs of (6.36) and (6.37) are essentially the same, except that for (6.36), in place of (6.39) we have the following. From Remark 6.4 and the proof of (2.14) of

Theorem 2.1, for some K_{78} and for c sufficiently large:

$$\begin{aligned}(6.44)\quad & P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A, \Delta_A(\partial \operatorname{Co}(\Gamma_y)) \geq cl^{1/3}(\log l)^{2/3} \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2) \\ & \leq |\Lambda_N \cap \mathbb{Z}^2| \exp(-w_1\sqrt{A} - K_{78}c\kappa_\tau l^{1/3}(\log l)^{2/3}).\end{aligned}$$

With (2.17) and (6.35) this proves (6.36). The modification is similar for (6.37).

For (6.38) we must do more, because d' may be much smaller than the order of the error term $l^{1/3}(\log l)^{2/3}$ in Theorem 4.1. Let $a = \epsilon_{33}d'\sqrt{A}$, with ϵ_{33} still to be specified. From Theorem 5.7, Remark 6.4 and (6.43) we have

$$\begin{aligned}(6.45)\quad & P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq 2A, D'_{(q,r)}(\Gamma_y) \geq d' \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2) \\ & \leq |\Lambda_N \cap \mathbb{Z}^2| \exp\left(-\frac{1}{20}d' - u(K_{52}(\log 2A)^{2/3}, 2A)\right) \\ & \leq \exp\left(-\frac{1}{20}d'\right) P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2).\end{aligned}$$

From Proposition 5.5 and Remark 6.4, similarly to (6.18) we have

$$\begin{aligned}(6.46)\quad & P_{N,w}\big(A \leq |\operatorname{Int}(\Gamma_y)| < 2A, D'_{(q,r)}(\Gamma_y) > d' \text{ and} \\ & \qquad |\operatorname{Int}(\Gamma_y)| - |\operatorname{Int}(\alpha_{\max,\Gamma_y})| < a \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2\big) \\ & \leq \sum_{y\in\Lambda_N\cap\mathbb{Z}^2} \sum_{A\leq B<2A} \sum_{n\geq d'} \sum_{B-a<A'\leq B} \sum_{k\leq n} P_{N,w}(M_y(k,q,r,B,A',n,w_1\sqrt{A'})) \\ & \leq \sum_{y\in\Lambda_N\cap\mathbb{Z}^2} \sum_{A-a\leq A'<2A} \sum_{A'\leq B<A'+a} \sum_{n\geq d'} \\ & \qquad (n+1)e^{-n/2} P_{N,w}(\cup_{x\in B(y,K_{58}A)} M_x(0,q,r,A',A',0,w_1\sqrt{A'})) \\ & \leq \sum_{y\in\Lambda_N\cap\mathbb{Z}^2} K_{79}a(d'+1)e^{-d'/2} \\ & \qquad \cdot P_{N,w}(A-a \leq |\operatorname{Int}(\Gamma_x)| < 2A \text{ for some } x \in B(y,K_{58}A)) \\ & \leq K_{80}a(d'+1)e^{-d'/2}A^2 \sum_{x\in\Lambda_N\cap\mathbb{Z}^2} P_{N,w}(A-a \leq |\operatorname{Int}(\Gamma_x)| < 2A).\end{aligned}$$

Let $\tilde{F}_y$ denote the event that $A - a \leq |\operatorname{Int}(\Gamma_y)| < 2A$ and Γ_y is the unique exterior dual circuit in Λ_N satisfying $|\operatorname{Int}(\Gamma_y)| \geq A - a$. Similarly to (6.42) and (6.43) we have

$$\begin{aligned}
(6.47) \qquad & \sum_{x \in \Lambda_N \cap \mathbb{Z}^2} P_{N,w}(A - a \leq |\operatorname{Int}(\Gamma_x)| < 2A) \\
&\leq \sum_{x \in \Lambda_N \cap \mathbb{Z}^2} 2P_{N,w}(\tilde{F}_x) \\
&\leq 4AP_{N,w}(A - a \leq |\operatorname{Int}(\Gamma_x)| < 2A \text{ for some } x \in \Lambda_N \cap \mathbb{Z}^2).
\end{aligned}$$

From (6.36) we have

$$\begin{aligned}
(6.48) \qquad & P_{N,w}(|\operatorname{Int}(\Gamma_x)| \geq A - a \text{ for some } x \in \Lambda_N \cap \mathbb{Z}^2) \\
&\leq 2 \sum_{x \in \Lambda_N \cap \mathbb{Z}^2} P_{N,w}(|\operatorname{Int}(\Gamma_x)| \geq A - a, \Delta_A(\Gamma_x) \leq K_{72} l^{2/3} (\log l)^{1/3}).
\end{aligned}$$

But the last event, which says roughly that Γ_x approximates the appropriate Wulff shape, implies that Γ_x surrounds some $(\Lambda_N, (A - a)/2, (1 + \epsilon_{24})(A - a))$-compatible site y. (Take y to be the closest site to 0 in $\operatorname{Int}(\Gamma_x)$.) Here ϵ_{24} is from Proposition 6.2. Thus provided ϵ_{33} is small, (6.48) and Proposition 6.2 give

(6.49)

$$\begin{aligned}
& P_{N,w}(|\operatorname{Int}(\Gamma_x)| \geq A - a \text{ for some } x \in \Lambda_N \cap \mathbb{Z}^2) \\
&\qquad \leq 2e^{\epsilon_{25}\epsilon_{33}d'} \sum_{x \in \Lambda_N \cap \mathbb{Z}^2 \cap (\mathbb{Z}^2)^*} P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A \text{ for some } y \in B(x, \sqrt{A}) \cap J_{A-a}) \\
&\qquad \leq 2K_{81} A e^{\epsilon_{25}\epsilon_{33}d'} \sum_{y \in J_{A-a}} P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A).
\end{aligned}$$

We can repeat the argument of (6.41) – (6.43) (excluding the last inequality of (6.43)) once more to get

$$(6.50) \qquad \sum_{y \in J_{A-a}} P_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A) \leq 4AP_{N,w}(|\operatorname{Int}(\Gamma_y)| \geq A \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2).$$

Provided ϵ_{33} is small, together, (6.45), (6.46), (6.47), (6.49) and (6.50) prove the lemma. $\square$

Proof of Theorem 2.3. We begin by obtaining an analog of (5.18), by mimicking its proof. We omit some details becuase of the similarity. First fix N and let K_{82} be a constant to be specified. Our induction hypothesis is that for every $j < k, K_{82} \leq$

$A' \leq A, d \geq d' \geq 0$ and enclosure event E, we have

$$\begin{aligned}(6.51)\qquad & P_{N,w}(G_N(j,A,A',d,d') \mid E) \\ & \leq e^{-\frac{4}{5}d'} P_{N,w}(|\operatorname{Int}(\Gamma_y)| = A' \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2 \mid E)\end{aligned}$$

and

$$(6.52)\qquad P_{N,w}(G_N(j,A,A',d,d') \mid E) \leq \exp\left(-\frac{9}{10}d + \frac{1}{20}jK\log N\right)$$

and

$$(6.53)\qquad P_{N,w}(G_N(j,A,A',d,d') \mid E) \leq \exp\left(-u(K_{52}(\log N)^{2/3}, A) - \frac{1}{20}d'\right).$$

We need only consider parameters satisfying

$$(6.54)\qquad d+1 \geq (j+1)K\log N, \qquad d'+1 \geq jK\log N, \quad d'+1 \geq \frac{1}{6}w_1\sqrt{A-A'},$$

the last inequality following from (5.7) and subadditivity of the square root. For $j = 0$, we need only consider $A = A', d' = 0$ and (6.51) is immediate from the definitions, while (6.52) follows easily from the first inequality in (5.9) and the fact that E is increasing, and (6.53) follows from Theorem 5.7. Hence we may assume $k \geq 1$ and fix A, A', d, d', E.

Let $\omega \in G_N(k,A,A',d,d')$. There may be open dual paths α in ω each connecting Φ_N to another open dual circuit $\gamma \in \mathfrak{C}_N$. Since Φ_N is exterior, for each such γ there is a unique bond $\langle xy\rangle$ with $x \in \Phi_N, y \in \operatorname{Ext}(\Phi_N)$ which is part of such an α. We denote the set of all such bonds by $\mathcal{A}_N$. Thus $|\mathcal{A}_N| \leq |\mathfrak{C}_N| - 1$. Let

$$\tilde{G}_N(k,A,A',d,d') = G_N(k,A,A',d,d') \cap [\mathcal{A}_N = \phi].$$

When this event occurs, the only $(K\log N)$-large open dual circuit in $\operatorname{Int}(\Theta(\Phi_N))$ is Φ_N. From the above and the bounded energy property we have

$$P_{N,w}(G_N(k,A,A',d,d') \mid E) \leq |\mathcal{B}(\Lambda_N)|^k p_0^{-k} P_{N,w}(\tilde{G}_N(k,A,A',d,d') \mid E),$$

where p_0 is from (2.1). Given a circuit $\nu \subset \Lambda_N$ define the event

$$\begin{aligned}(6.55)\qquad F(\nu,d_1) =& [|\operatorname{Int}(\Gamma_y)| = A', d_1 \leq \operatorname{diam}_\tau(\Gamma_y) < d_1+1, \Theta(\Gamma_y) = \nu \text{ and} \\ & \Gamma_y \text{ is the only } (K\log N)-\text{large open dual circuit in } \operatorname{Int}(\nu), \\ & \text{for some } y \in \operatorname{Int}(\nu)].\end{aligned}$$

Note that $F(\nu,d_1) \in \mathcal{G}_{\mathcal{B}(\operatorname{Int}(\nu)\cup\nu)}$. It follows easily from the near-Markov property and the first inequality in (5.9) that provided K is large, given $F(\nu,d_1) \cap E$ with high probability there are no $(K \log N)$-large open dual circuits outside ν; more precisely,

$$P_{N,w}\big(G_N(0, A', A', d_1, 0) \cap [\Theta(\Phi_N) = \nu] \mid E \cap F(\nu, d_1)\big) \geq \frac{1}{2}.$$

Hence as in the proof of Proposition 5.5, conditioning on $\Theta(\Phi_N)$ and using the near-Markov property and (6.52) for $j = k-1$ gives

$$\begin{aligned}
(6.56)\quad & P_{N,w}(G_N(k, A, A', d, d') \mid E) \\
&\leq |\mathcal{B}(\Lambda_N)|^k p^{-k} P_{N,w}(\tilde{G}_N(k, A, A', d, d') \mid E) \\
&\leq K_{83} p^{-k} N^{2k} \sum_{\nu} P_{N,w}(\tilde{G}_N(k, A, A', d, d') \cap [\Theta(\Phi_N) = \nu] \mid E) \\
&\leq 2K_{83} p^{-k} N^{2k} \sum_{d_1} \sum_{\nu} P_{N,w}(F(\nu, d_1) \mid E) \\
&\qquad \cdot P_{N,w}(\tilde{G}_N(k, A, A', d, d') \cap [\Theta(\Phi_N) = \nu] \mid E \cap F(\nu, d_1)) \\
&\leq 8K_{83} p^{-k} N^{2k} \sum_{d_1, d_2'} \sum_{0 < A_1 \leq A - A'} \sum_{\nu} \\
&\qquad P_{N,w}(G_N(0, A', A', d_1, 0) \cap [\Theta(\Phi_N) = \nu] \mid E) \\
&\qquad \cdot P_{N,w}(G_N(k-1, A - A', A_1, d', d_2') \mid E \cap \mathrm{Open}(\nu) \cap [\nu \nleftrightarrow \infty]) \\
&\leq 8K_{83} p^{-k} N^{2k+6} \sum_{d_1} \sum_{\nu} P_{N,w}(G_N(0, A', A', d_1, 0) \cap [\Theta(\Phi_N) = \nu] \mid E) \\
&\qquad\qquad \cdot \exp\left(-\frac{9}{10} d' + \frac{1}{20}(k-1) K \log N\right) \\
&\leq 8K_{83} p^{-k} N^{2k+6} \exp\left(-\frac{9}{10} d' + \frac{1}{20}(k-1) K \log N\right) \\
&\qquad \cdot P_{N,w}(|\operatorname{Int}(\Gamma_y)| = A', d - d' - 1 \leq \operatorname{diam}_\tau(\Gamma_y) \\
&\qquad\qquad < d - d' + 1 \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2 \mid E) \\
&\leq \exp\left(-\frac{4}{5} d'\right) P_{N,w}(|\operatorname{Int}(\Gamma_y)| = A', d - d' - 1 \leq \operatorname{diam}_\tau(\Gamma_y) \\
&\qquad\qquad < d - d' + 1 \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2 \mid E).
\end{aligned}$$

Here the sums include $d_1 \in \{d - d' - 1, d - d'\}$ and $0 \leq d_2' < d'$. Now (6.56) establishes (6.51) for $j = k$; we next establish (6.52). We have by (2.9)

$$\begin{aligned}
&P_{N,w}(d - d' - 1 \leq \operatorname{diam}_\tau(\Gamma_y) < d - d' + 1 \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2 \mid E) \\
&\leq |\Lambda_N \cap \mathbb{Z}^2|^2 e^{-(d - d' - 1)}.
\end{aligned}$$

Hence provided K is large enough, the right side of the next-to-last inequality in (6.56) is bounded by

$$\exp\left(-\frac{9}{10}d + \frac{1}{20}kK\log N\right),$$

which proves (6.52) for $j = k$. Turning to (6.53), from (6.52), (6.53) for $j = k-1$, and Theorem 2.1, the right side of the fourth inequality in (6.56) is bounded by

$$\begin{aligned}(6.57)\qquad & 8K_{84}p^{-k}N^{2k+4}\exp(-w_1\sqrt{A'} + K_{52}(\log N)^{2/3}(A')^{1/6})) \\ & \cdot\exp\left(-\max\left[w_1\sqrt{A-A'} - K_{52}(\log N)^{2/3}(A-A')^{1/6}),\right.\right. \\ & \left.\left.\frac{9}{10}d' - \frac{1}{20}(k-1)K\log N\right]\right).\end{aligned}$$

We now have two cases.

Case 1. $A' \geq \frac{1}{3}A$. From (6.54) we have

$$d' \geq \frac{1}{2}kK\log N + (K\log N)^{2/3}(\frac{1}{12}w_1\sqrt{A'})^{1/3}.$$

We can use this, (5.52) with $\theta = 1/\sqrt{2}$, (6.53) with $j = k-1$, and the fact that the maximum exceeds any convex combination to conclude that, provided K is sufficiently large, (6.57) is bounded above by

$$\begin{aligned}(6.58)\quad & 8K_{84}p^{-k}N^{2k+4}\exp\left(-w_1\sqrt{A'} - \frac{3}{4}w_1\sqrt{A-A'} - \frac{9}{40}d' + K_{52}(\log N)^{2/3}(A')^{1/6}\right. \\ & \left. + \frac{3}{4}K_{52}(\log N)^{2/3}(A-A')^{1/6} + \frac{1}{80}(k-1)K\log N\right) \\ & \leq \exp\left(-u(K_{52}(\log N)^{2/3}, A) - \frac{1}{20}d'\right).\end{aligned}$$

Case 2. $A' < \frac{1}{3}A$. In this case it is easily checked that we must have $k \geq 2$ and $d + 1 \geq 3w_1\sqrt{A/3}$ for $G_N(k, A, A', d, d')$ to be nonempty. But then

$$\frac{9}{10}d - \frac{1}{20}kK\log N \geq \frac{4}{5}(d+1) \geq w_1\sqrt{A} + \frac{1}{20}d',$$

so (6.53) follows from (6.52).

The proof of (6.53) for $j = k$ is now complete. Summing (6.53) as in (5.60),

$$
\begin{aligned}
(6.59)\quad & P_{N,w}\Big(\sum_{\gamma\in\mathfrak{C}_N} |\operatorname{Int}(\gamma)| \geq A, T'_N \geq d'\Big) \\
&\leq \sum_{A\leq B\leq|\Lambda_N|} \ \sum_{d\leq \operatorname{diam}_\tau(\Lambda_N)} \ \sum_{d'\leq n\leq d} \ \sum_{A'\leq B} \ \sum_{k\leq n} P_{N,w}(G_N(k,B,A',d,n)) \\
&\leq K_{85}N^4 \sum_{B\geq A} \ \sum_{n\geq d'} \exp\left(-\frac{1}{20}n - u(K_{52}(\log N)^{2/3}, B)\right) \\
&\leq \exp\left(-\frac{1}{20}d' - w_1\sqrt{A} + K_{86}(\log N)^{2/3} l^{1/3}\right).
\end{aligned}
$$

If K_{87} and K_7 are large enough, then (6.59), Theorem 4.1 and Remark 6.4 show that

$$
\begin{aligned}
(6.60)\qquad & P_{N,w}\left(\sum_{\gamma\in\mathfrak{C}_N} |\operatorname{Int}(\gamma)| \geq A, T'_N \geq K_{87} l^{1/3}(\log N)^{2/3}\right) \\
&\leq \exp\left(-\frac{1}{40}K_{87} l^{1/3}(\log N)^{2/3}\right) P_{N,w}(|\operatorname{Int}(\Gamma_0)| \geq A)
\end{aligned}
$$

and

$$
P_{N,w}\left(\sum_{\gamma\in\mathfrak{C}_N} |\operatorname{Int}(\gamma)| \geq 2A\right) \leq e^{-\epsilon_{37} l} P_{N,w}(|\operatorname{Int}(\Gamma_0)| \geq A).
$$

Therefore

$$
\begin{aligned}
(6.61)\quad & \\
& P_{N,w}\left(\sum_{\gamma\in\mathfrak{C}_N} |\operatorname{Int}(\gamma)| \geq A, |\mathfrak{C}_N| > 1\right) \\
&\leq P_{N,w}\left(\sum_{\gamma\in\mathfrak{C}_N} |\operatorname{Int}(\gamma)| \geq A, T'_N \geq K \log N\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le P_{N,w}\left(\sum_{\gamma\in\mathfrak{C}_N} |\operatorname{Int}(\gamma)| \ge A, T'_N \ge K_{87} l^{1/3}(\log N)^{2/3}\right)\\
&\qquad + P_{N,w}\left(\sum_{\gamma\in\mathfrak{C}_N} |\operatorname{Int}(\gamma)| \ge 2A\right)\\
&\qquad + P_{N,w}\left(A \le \sum_{\gamma\in\mathfrak{C}_N} |\operatorname{Int}(\gamma)| < 2A, K\log N \le T'_N < K_{87} l^{1/3}(\log N)^{2/3}\right)\\
&\le 2\exp\left(-\frac{1}{40}K_{87} l^{1/3}(\log N)^{2/3}\right) P_{N,w}\left(\sum_{\gamma\in\mathfrak{C}_N} |\operatorname{Int}(\gamma)| \ge A\right)\\
&\qquad + P_{N,w}\left(A \le \sum_{\gamma\in\mathfrak{C}_N} |\operatorname{Int}(\gamma)| < 2A, K\log N \le T'_N < K_{87} l^{1/3}(\log N)^{2/3}\right).
\end{aligned}
$$

To prove (2.19), then, we need to bound the last probability in (6.61). We will sum as in (6.59), but this time using (6.51) instead of (6.53). Let $a = 72w_1^{-2}K_{87}^2 l^{2/3}(\log l)^{4/3}$. By (6.54), we need only consider

$$A' \ge B - 36w_1^{-2}(d'+1)^2 \ge B - a.$$

Therefore using (6.52),

(6.62)

$$
\begin{aligned}
&P_{N,w}\left(A \le \sum_{\gamma\in\mathfrak{C}_N} |\operatorname{Int}(\gamma)| < 2A, K\log N \le T'_N < K_{87} l^{1/3}(\log N)^{2/3}\right)\\
&\quad\le \sum_{A\le B<2A}\ \sum_{B-a<A'\le B}\ \sum_{d\le \operatorname{diam}_\tau(\Lambda_N)}\ \sum_{d'\ge K\log N}\ \sum_{k\le d'} P_{\Lambda_N}(G_N(k,B,A',d,d'))\\
&\quad\le \sum_{A'\ge A-a}\ \sum_{A'\le B<A'+a}\ \sum_{d'\ge K\log N}\\
&\qquad\qquad 4\kappa_\tau N(d'+1)e^{-\frac{4}{5}d'} P_{N,w}(|\operatorname{Int}(\Gamma_y)| = A' \text{ for some } y\in\Lambda_N\cap\mathbb{Z}^2)\\
&\quad\le e^{-\frac{1}{2}K\log N} P_{N,w}(|\operatorname{Int}(\Gamma_y)| \ge A-a \text{ for some } y\in\Lambda_N\cap\mathbb{Z}^2).
\end{aligned}
$$

By Lemma 6.7 and (2.17),

$$
\begin{aligned}
(6.63)\quad &P_{N,w}(|\operatorname{Int}(\Gamma_y)| \ge A-a \text{ for some } y\in\Lambda_N\cap\mathbb{Z}^2)\\
&\quad\le 2P_{N,w}(|\operatorname{Int}(\Gamma_y)| \ge A-a,\\
&\qquad\qquad \Delta_{A-a}(\Gamma_y) < K_{63} l^{2/3}(\log l)^{1/3} \text{ for some } y\in\Lambda_N\cap\mathbb{Z}^2).
\end{aligned}
$$

With (6.62) and Proposition 6.2 (taking $\delta = K_{57} \log A$) this shows that, if K is sufficiently large,

$$
\begin{aligned}
(6.64) \quad & P_{N,w}(A \le \sum_{\gamma \in \mathfrak{C}_N} |\operatorname{Int}(\gamma)| < 2A, K \log N \le T'_N < K_{87} l^{1/3} (\log N)^{2/3}) \\
&\le 2e^{-\frac{1}{2} K \log N} P_{N,w}\big(|\operatorname{Int}(\Gamma_{y'})| \ge A - a \text{ for some} \\
&\qquad (\Lambda_N, (A-a)/2, (1+\epsilon_{24})(A-a)) - \text{compatible } y'\big) \\
&\le 2e^{-\frac{1}{2} K \log N + \epsilon_{25} \delta} \sum_{y' \in \Lambda_N \cap \mathbb{Z}^2} P_{N,w}(|\operatorname{Int}(\Gamma_{y'})| \ge A) \\
&\le e^{-\frac{1}{4} K \log N} P_{N,w}\left(\sum_{\gamma \in \mathfrak{C}_N} |\operatorname{Int}(\gamma)| \ge A \right).
\end{aligned}
$$

With (6.61) this proves (2.19). Then (2.19) and Lemma 6.7 prove (2.20)–(2.22).

It remains to establish (2.23). Let $\epsilon_{38} > 0$ to be specified, let $n_0 = \min\{n : 2^n \kappa_\tau r/3 > \epsilon_{38}\sqrt{A}\}$, and let $b_n = \epsilon_{26}^{-2} 2^{2n} (\kappa_\tau r/3)^2$. Then provided ϵ_{38} is small enough, we have

$$
b_n < \epsilon_{33} 2^{n-1} \frac{\kappa_\tau r}{3} \sqrt{A} \quad \text{for all } n \le n_0,
$$

with ϵ_{33} from Lemma 6.7. We have

(6.65)

$$
\begin{aligned}
& P_{N,w}(|\operatorname{Int}(\Gamma_y)| \ge A \text{ and } \Gamma_y \text{ is not } (q,r) - \text{bottleneck-free for some } y \in \Lambda_N \cap \mathbb{Z}^2) \\
&\quad \le P_{N,w}(|\operatorname{Int}(\Gamma_y)| \ge A \text{ and } D'_{(q,r)}(\Gamma_y) \ge \frac{\kappa_\tau r}{3} \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2) \\
&\quad \le P_{N,w}(|\operatorname{Int}(\Gamma_y)| \ge A \text{ and } D'_{(q,r)}(\Gamma_y) \ge \epsilon_{38}\sqrt{A} \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2) \\
&\qquad + \sum_{n=1}^{n_0} P_{N,w}\Big(|\operatorname{Int}(\Gamma_y)| \ge A, 2^{n-1}\frac{\kappa_\tau r}{3} \le D'_{(q,r)}(\Gamma_y) < 2^n \frac{\kappa_\tau r}{3}, \\
&\qquad\qquad |\operatorname{Int}(\Gamma_y)| - |\operatorname{Int}(\alpha_{\max,\Gamma_y})| < b_n \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\le |\Lambda_N \cap \mathbb{Z}^2| \exp\left(-\frac{1}{20}\epsilon_{38}\sqrt{A} - u(K_{52}(\log A)^{2/3}, A)\right) \\
&\quad + \sum_{n=1}^{n_0} \exp\left(-\epsilon_{34} 2^{n-1}\frac{\kappa_\tau r}{3}\right) P_{N,w}(|\operatorname{Int}(\Gamma_y)| \ge A \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2) \\
&\le \exp\left(-\frac{1}{40}\epsilon_{38}\sqrt{A}\right) P_{N,w}(|\operatorname{Int}(\Gamma_0)| \ge A) \\
&\quad + K_{88} \exp\left(-\epsilon_{34}\frac{\kappa_\tau r}{3}\right) P_{N,w}(|\operatorname{Int}(\Gamma_y)| > A \text{ for some } y \in \Lambda_N \cap \mathbb{Z}^2), \\
&\le K_{89} \exp(-\epsilon_{39} r) P_{N,w}\left(\sum_{\gamma \in \mathfrak{C}_N} |\operatorname{Int}(\gamma)| > A\right).
\end{aligned}
$$

Here the second inequality uses Lemma 6.3, the third inequality uses Theorem 5.7 and Lemma 6.7, and the fourth inequality follows from Theorem 4.1. With (2.19) this proves (2.23). □

Proof of Theorem 2.1, (2.15) and (2.16). For (2.15) we need only observe that Lemma 6.7 is valid for P in place of $P_{N,w}$ and only $y = 0$ considered. After these same changes, (2.16) follows from (6.65) with the last inequality omitted. □

7. Conditioning on the Exact Area

In Theorems 2.1—2.3, one could as well consider conditioning on $|\operatorname{Int}(\Gamma_0)| = A$ in place of $|\operatorname{Int}(\Gamma_0)| \ge A$. It is straightforward to alter the existing proof to prove these theorems under this conditioning once one has a lower bound like Theorem 4.1 for $P(|\operatorname{Int}(\Gamma_0)| = A)$. For this we need first some definitions and lemmas. In the interest of space we will not give full details in the proofs; these details involve many of the same technicalities we have encountered earlier. Consider distinct points $x, y \in \mathbb{R}^2$ with $|y - x| \ge 4\sqrt{2}$. We let $U(x, y)$ denote the open slab between the tangent line to $\partial B_\tau(x, \tau(y - x))$ at y and the parallel line through x; we call $U(x, y)$ the *natural slab* of x and y. (Note the tangent line is not necessarily unique; if it is not we make some arbitrary choice.) We have $U(x, y) = U(y, x)$. It follows from the definition of $U(x, y)$ that if u and v are on opposite sides of $U(x, y)$, then

$$
\tau(v - u) \ge \tau(y - x). \tag{7.1}
$$

The portion of $U(x, y)$ which is strictly to the right of the line from x to y is called the *natural half-slab* of x and y and denoted $U_R(x, y)$. For $x, y \in U(u, v)$ we let $\tilde{U}_{uv}(x, y)$ denote the open slab with sides parallel to those of $U(u, v)$ with x in one edge of $\tilde{U}_{uv}(x, y)$ and y in the other edge. We let $\tilde{U}^R_{uv}(x, y) = \tilde{U}_{uv}(x, y) \cap U_R(u, v)$.

One or two of the dual sites adjacent to x are in $U(x,y)$; we let x' denote such a site, making an arbitrary choice if there are two. We define y' analogously as a dual site in $U(x,y)$ adjacent to y.

By a *face of the dual lattice* we mean a square $z + [-\frac{1}{2}, \frac{1}{2}]^2$ with $z \in \mathbb{Z}^2$. Let $V(x,y)$ denote the interior of the union of all faces of the dual lattice whose interiors are contained in $\tilde{U}_{xy}(x',y')$, and let $T_x(x,y)$ and $T_y(x,y)$ denote the components of $V(x,y)^c$ containing x and y, respectively. (These components are distinct since $|x-y| \geq 4\sqrt{2}$.) Then $V(x,y) \subset \tilde{U}_{xy}(x',y') \subset U(x,y)$.

The following lemma says roughly that typical open dual paths from x to y are connected to the boundary of the natural slab only near x and y.

Lemma 7.1. *Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7). There exist constants $K_{90}, K_{91}, \epsilon_{40}$ as follows. For all $x, y \in (\mathbb{Z}^2)^*$ and $J \geq K_{90} \log|x-y|$,*

$$P(x \leftrightarrow z \textit{ for some } z \in \partial V(x,y) \backslash (x + \Lambda_J) \mid x \leftrightarrow y) \leq K_{91} e^{-\epsilon_{40} J}.$$

Proof. As mentioned above, we omit some details. Suppose $x \leftrightarrow y$ and $x \leftrightarrow z$ for some $z \in \partial T_x(x,y) \backslash (x + \Lambda_J)$, via open dual paths. If K_{92} is large, one can trivially dispose of the case in which $x \leftrightarrow B(x, K_{92}|y-x|)^c$, so we hence forth tacitly consider only connections occuring inside $B(x, K_{92}|y-x|)$; in particular this means $|z-x| \leq K_{92}|y-x|$. There are then two cases: either there is an $(\epsilon_{41} J)$-near connection from x to y in $B(z, J/10)^c$, or there is not; here ϵ_{41} is to be specified. In the first case, there is also an open dual path from z to $\partial(B(z, J/20) \cap (\mathbb{Z}^2)^*)$, so we can apply (2.10) and Lemmas 3.1 and 6.5 (assuming K_{90} is large) to obtain

$$\begin{aligned} &P(N(x, y, \epsilon_{41} J, B(z, J/10)^c) \cap [z \leftrightarrow \partial(B(z, J/20) \cap (\mathbb{Z}^2)^*)]) \\ &\quad \leq K_{93} J e^{-\tau(y-x) + K_{60}\epsilon_{41} J} e^{-\kappa_\tau J/40} \\ &\quad \leq K_{94} e^{-\epsilon_{42} J} P(x \leftrightarrow y), \end{aligned} \tag{7.2}$$

provided ϵ_{41} is chosen small enough and K_{90} large enough. In the second case, there exists dual sites v, w just outside $B(z, J/10)$ and open dual paths $x \leftrightarrow v$ and $w \leftrightarrow y$ occuring at separation $\epsilon_{41} J$. It follows easily from (7.1) and the fact that z is close to $\partial U(x,y)$ that

$$\tau(y-w) \geq \tau(y-x) - \tfrac{1}{5}\kappa_\tau J.$$

Also, since $|z-x| \geq J$,

$$\tau(v-x) \geq \tau(z-x) - \tfrac{1}{5}\kappa_\tau J \geq \tfrac{1}{2}\kappa_\tau J. \tag{7.3}$$

Therefore by Lemma 3.2 and (2.10),

$$
\begin{aligned}
(7.4) \qquad & P(N(x, y, \epsilon_{41} J, B(z, J/10)^c)^c \cap [x \leftrightarrow y] \cap [x \leftrightarrow z]) \\
& \leq \sum_{v,w} 2P(x \leftrightarrow v)P(w \leftrightarrow y) \\
& \leq K_{95} J^2 e^{-\tau(y-x) - \frac{3}{10}\kappa_\tau J} \\
& \leq K_{96} e^{-\epsilon_{43} J} P(x \leftrightarrow y).
\end{aligned}
$$

Now (7.2) and (7.4), summed over z with $|z-x| \leq K_{92}|y-x|$, prove the lemma. □

For dual sites x and y, we say that $x \leftrightarrow y$ *cylindrically* if there is an open dual path γ from x to y in $U(x, y)$ and every open dual path from γ to $U(x, y)^c$ passes through x or y.

For $\mathcal{D}$ a subgraph of the dual lattice (or just a set of dual bonds, which we may view as such a subgraph), and $A \subset \mathbb{R}^2$, we define the *bond boundary of* $\mathcal{D}$ *in* A to be the set of bonds contained in A having exactly one endpoint in $\mathcal{D}$. (As always, we view bonds as open intervals contained in the plane.)

Lemma 7.2. *Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7). There exist constants K_{97}, ϵ_{44} as follows. For all $x, y \in (\mathbb{Z}^2)^*$,*

$$P(x \leftrightarrow y \text{ cylindrically}) \geq \epsilon_{44}|y-x|^{-K_{97}} P(x \leftrightarrow y).$$

Proof. Fix x, y and let $J = K_{90} \log|y-x|$, with K_{90} from Lemma 7.1. Let $\mathcal{J}_x(x, y)$ denote the bond boundary of $\partial T_x(x, y) \cap \mathcal{B}(x+\Lambda_J)$ in $\mathbb{R}^2$, and define $\mathcal{J}_y(x, y)$ analogously. For $e \in \mathcal{J}_x(x, y)$ let A_e denote the event that all dual bonds in $\mathcal{B}(\partial T_x(x, y) \cap (x+\Lambda_J))$ are open, e and $\langle xx' \rangle$ are open and all other bonds in $\mathcal{J}_x(x, y)$ are closed; define A_e analogously for $e \in \mathcal{J}_y(x, y)$. Define the event

$$B = [x \leftrightarrow z \text{ for some } z \in \partial V(x, y) \backslash (x + \Lambda_J)]$$

(cf. Lemma 7.1.) Given a configuration $\omega \in [x \leftrightarrow y] \cap B^c$, there necessarily exists an open dual path from $\partial T_x(x, y) \cap (x + \Lambda_J)$ to $\partial T_y(x, y) \cap (y + \Lambda_J)$ in $V(x, y)$ which contains only one bond in $\mathcal{J}_x(x, y)$ and one bond in $\mathcal{J}_y(x, y)$; we denote these two bonds by $b_{xy}(x, \omega)$ and $b_{xy}(y, \omega)$, respectively, making an arbitrary choice if more than one choice is possible. If the configuration ω is in $[x \leftrightarrow y] \cap B^c \cap [b_{xy}(x) = e] \cap [b_{xy}(y) = f]$ for some e, f, then we can modify at most $16J$ bonds (those in $\partial T_x \cap \mathcal{B}(\partial T_x(x, y) \cap (x+\Lambda_J)))$ to obtain a configuration in $A_e \cap A_f$; in the resulting configuration we have $x \leftrightarrow y$ cylindrically. Therefore from the bounded energy property we have

$$P(x \leftrightarrow y \text{ cylindrically} \mid [x \leftrightarrow y] \cap B^c) \geq e^{-K_{98} J}$$

This and Lemma 7.1 prove the lemma. □

It is easy to check that Lemmas 7.1 and 7.2 are valid if we restrict to connections in the halfspace H_{xy}. More precisely, under the assumptions of Lemma 7.2 we have

$$P(x \leftrightarrow y \text{ cylindrically in } U_R(x,y)) \geq \epsilon_{44}|y-x|^{-K_{97}} P(x \leftrightarrow y). \tag{7.5}$$

Additionally, we can extend the idea of cylindrical connections as follows: for $u, v \in \mathbb{R}^2$ and x, y dual sites in $U(u,v)$, we say that $x \leftrightarrow y$ (u,v)*-cylindrically* if there is an open dual path from x to y in $\tilde{U}_{uv}(x,y)$ and the dual cluster of x and y intersects $\partial \tilde{U}_{uv}(x,y)$ only at x and y. Provided $x, y \in U_R(u,v)$, the proof of Lemma 7.2 shows that

$$P(x \leftrightarrow y \ (u,v)\text{-cylindrically in } \tilde{U}^R_{uv}(x,y)) \geq \epsilon_{44}|y-x|^{-K_{97}} P(x \leftrightarrow y \text{ in } H_{xy}). \tag{7.6}$$

Theorem 7.3. *Let P be a percolation model on $\mathcal{B}(\mathbb{Z}^2)$ satisfying (2.7), the near-Markov property for open circuits, positivity of τ and the ratio weak mixing property. There exist K_i such that for $A > K_{99}$ and $l = \sqrt{A}$,*

$$P(|\operatorname{Int}(\Gamma_0)| = A) \geq \exp(-w_1\sqrt{A} - K_{100} l^{1/3} (\log l)^{2/3}).$$

Proof. Fix A large and let $l = \sqrt{A}$. Let a_1 denote the vertical coordinate of the point where $\partial\mathcal{K}_1$ meets the positive vertical axis. Let $s = l^{2/3}(\log l)^{1/3}$ and $\delta = K_{30}s^2/l$, with K_{30} as in the proof of Theorem 4.1. Let $\alpha = \partial(l+\delta)\mathcal{K}_1$ and let $(z_0', .., z_n', z_0')$ be the s-hull skeleton of α. It is an easy exercise in geometry to see that the natural half-slabs $U_R(z_i', z_{i+1}'), i = 0, .., n,$ are disjoint. (Our labeling as usual is cyclical: $z_{n+1}' = z_0'$.) For some K_{101} to be specified, let us call a pair (z_i', z_{i+1}') from the skeleton *very short* if $|z_{i+1}' - z_i'| \leq 2\sqrt{2}$, *short* if $2\sqrt{2} < |z_{i+1}' - z_i'| \leq 2K_{101}\log l$ and *long* if $|z_{i+1}' - z_i'| > 2K_{101}\log l$. In what follows, very short pairs can be handled quite trivially but tediously, so for convenience we will assume there are no very short pairs. For long pairs we define x_i' and y_{i+1}' to be the points on the line segment $\overline{z_i' z_{i+1}'}$ at distance $K_{101}\log l$ from z_i' and from z_{i+1}', respectively, and let x_i, y_{i+1} be dual sites in $U_R(z_i', z_{i+1}')$ within distance $\sqrt{2}$ of x_i' and y_{i+1}', respectively. For short pairs we let $x_i = y_{i+1}$ be a dual site in $U_R(z_i', z_{i+1}')$ within distance $\sqrt{2}$ of the midpoint of $\overline{z_i' z_{i+1}'}$. With minor modification of the definition of the s-hull skeleton, we may assume the set $\{z_0', .., z_n'\}$ has lattice symmetry, that is, for each z_i' the reflection of z_i' across the horizontal or vertical axis is another z_j', and analogously for the sites x_i and y_i. For each i we let ϕ_i denote a dual lattice path of minimal length from y_i to x_i outside $\operatorname{Co}(\{z_0', .., z_n'\}) \cup \tilde{U}_{z_{i-1}' z_i'}(x_{i-1}, y_i) \cup \tilde{U}_{z_i' z_{i+1}'}(x_i, y_{i+1})$. We call such a ϕ_i a *short link.* Let $\mathcal{C}_i$ denote the bond boundary of ϕ_i in $(\operatorname{Co}(\{z_0', .., z_n'\}) \cup \tilde{U}_{z_{i-1}' z_i'}(x_{i-1}, y_i) \cup \tilde{U}_{z_i' z_{i+1}'}(x_i, y_{i+1}))^c$.

Let λ_a be the vertical line through $(a, 0)$. Let $H_L(x)$ and $H_R(x)$ denote the open half planes to the left and right, respectively, of the vertical line through x. Let $H_U(x)$ and $H_B(x)$ denote the open half planes above and below the horizontal line through x, respectively. (In general we use the convention that subscripts L, R, U, B

refer to left, right, upper and lower halfspaces, respectively, with combinations, such as LU, referring to quadrants.) Let $S(x,y)$ denote the open slab between the vertical lines through x and y. Let N be the integer part of $a_1 l/2$, M the integer part of $l^{2/3}(\log l)^{1/3}$ and D the integer part of $l^{1/3}(\log l)^{2/3}$. Let

$$u'_{RU} = H_R(0) \cap H_U(0) \cap \alpha \cap \lambda_{M+\frac{1}{2}}, \quad v'_{RU} = H_R(0) \cap H_U(0) \cap \alpha \cap \lambda_{M+D+\frac{1}{2}},$$

$$w'_{RU} = H_R(0) \cap H_U(0) \cap \alpha \cap \lambda_{N+\frac{1}{2}}, \quad x'_{RU} = H_R(0) \cap H_U(0) \cap \alpha \cap \lambda_{N+D+\frac{1}{2}}.$$

(Note each of these intersections is a single point.) We call these 4 points *determining points*. Lattice symmetry yields corresponding determining points with appropriate subscripts in the other three quadrants. We may assume that u'_{RU} is one of the sites z'_i of the s-hull skeleton of α (if not, we add u'_{RU} to the skeleton), and analogously for the other sites just defined. Let u_{RU} be the second closest dual site above u'_{RU} in $\lambda_{M+\frac{1}{2}}$, and analogously for v_{RU}, w_{RU}, x_{RU}. If $u'_{RU} = z'_i$, for some i, we define z_i to be u_{RU}, and again analogously for the other determining points. Loosely, the idea is to remove from Γ_0 its intersection with each of the width-D vertical slabs $S(x_{LU}, w_{LU}), S(v_{LU}, u_{LU}), S(u_{RU}, v_{RU}), S(w_{RU}, x_{RU})$, then raise or lower the segments of Γ_0 between these slabs to adjust the area as desired, then reconnect these segments to make a new circuit enclosing area A. To do this we must first ensure that Γ_0 intersects each vertical line bounding any of these four slabs only twice.

We refer to the 4 width-D vertical slabs above as *removal slabs*. We call the 5 regions $H_L(x_{LU}), S(w_{LU}, v_{LU}), S(u_{LU}, u_{RU}), S(v_{RU}, w_{RU}), H_R(x_{RU})$ (whose closures together form the complement of the 4 removal slabs) *retention regions*. By a *retained segment* we mean a connected component of the intersection of α with a retention region. Each retained segment has the form $\alpha^{(z'_j, z'_k)}$ for some j,k; we call z'_j an *initial determining point* and z'_k a *final determining point*, and call (j,k) a *retention pair*. We let J^{ret} denote the set of all 8 retention pairs. For each initial determining point z'_j, in the boundary of some retained region F, we let ψ_j be a dual lattice path from z_j to x_j in $F \backslash \tilde{U}_{z'_j z'_{j+1}}(x_j, y_{j+1})$, of minimal length, and let $\mathcal{D}_j$ be the bond boundary of ψ_j in $\overline{F} \backslash \tilde{U}_{z'_j z'_{j+1}}(x_j, y_{j+1})\}$. For each final determining point z'_k we let ψ_k be a dual lattice path from y_k to z_k in $F \backslash \tilde{U}_{z'_{k-1} z'_k}(x_{k-1}, y_k)$, of minimal length, and define $\mathcal{D}_k$ analogously to $\mathcal{D}_j$. We refer to ψ_j and ψ_k as the *endpaths* of the retention pair (j,k).

For each retention pair (j,k) let $I_{jk} = \{i : z'_i, z'_{i+1} \in \alpha^{(z'_j, z'_k)}\}$ and let Q_{jk} denote the event that (i) for each $i \in I_{jk} \cup \{j\}$, we have $x_i \leftrightarrow y_{i+1}$ (z'_i, z'_{i+1})-cylindrically in $\tilde{U}^R_{z'_i z'_{i+1}}(x_i, y_{i+1})$, (ii) for each $i \in I_{jk}$ we have ϕ_i open and all bonds in $\mathcal{C}_i$ closed, and (iii) both endpaths of (j,k) are open and all bonds in $\mathcal{D}_j \cup \mathcal{D}_k$ are closed. These 3 component events are denoted $Q_{(i)}(j,k)$, $Q_{(ii)}(j,k)$ and $Q_{(iii)}(j,k)$. For a configuration in Q_{jk}, the paths $x_i \leftrightarrow y_{i+1}$ together with the short links ϕ_i and the two endpaths form an open dual path from z_j to z_k outside $\mathrm{Co}(\{z'_0, .., z'_n\})$, and there is no open dual connection from this path to any point of the retention region boundary except z_j and z_k. By Lemma 3.1, (7.6) and Theorem 4.2, provided K_{101} is large we have

$$\begin{aligned}(7.7)\quad & P(Q_{(i)}(j,k)) \\ & \geq \left(\tfrac{1}{2}\right)^{|I_{jk}|} \prod_{i \in I_{jk} \cup \{j\}} P(x_i \leftrightarrow y_{i+1}\ (z'_i, z'_{i+1}) - \text{cylindrically in } \tilde{U}^R_{z'_i z'_{i+1}}(x_i, y_{i+1})) \\ & \geq \left(\frac{\epsilon_{45}}{l}\right)^{|I_{jk}|+1} \exp\left(-\sum_{i \in I_{jk} \cup \{j\}} \tau(y_{i+1} - x_i)\right) \\ & \geq \exp\left(-\sum_{i \in I_{jk} \cup \{j\}} \tau(z_{i+1} - z_i) - K_{102}|I_{jk}| \log l\right).\end{aligned}$$

From the bounded energy property,

$$P\big(Q_{(ii)}(j,k) \cap Q_{(iii)}(j,k) \mid Q_{(i)}(j,k)\big) \geq \exp(-K_{103}|I_{jk}| \log l)$$

which with (7.7) yields

$$P(Q_{jk}) \geq \exp\left(-\sum_{i \in I_{jk} \cup \{j\}} \tau(z_{i+1} - z_i) - K_{104}|I_{jk}| \log l\right). \tag{7.8}$$

For a configuration $\omega \in Q_{jk}$, and for F the retention region with $z'_j, z'_k \in \partial F$, we can associate an area $R_{jk}(\omega)$ as follows. There is a unique outermost open dual path $\Xi_{jk}(\omega)$ from z_j to z_k in F. If F is a halfspace ($H_L(x_{LU})$ or $H_R(x_{RU})$), then $R_{jk}(\omega)$ is the area of the region between $\Xi_{jk}(\omega)$ and $\overline{z_j z_k}$. If F is a slab and $z'_j, z'_k \in H_U(0)$, then $R_{jk}(\omega)$ is the area of the region in $F \cap H_U(0)$ below $\Xi_{jk}(\omega)$. If F is a slab and $z'_j, z'_k \in H_B(0)$, then $R_{jk}(\omega)$ is the area of the region in $F \cap H_B(0)$ above $\Xi_{jk}(\omega)$. We define corresponding nonrandom areas R^0_{jk} similarly but using $\alpha^{(z'_j z'_k)}$ in place of $\Xi_{jk}(\omega)$. Then $R_{jk}(\omega) \geq R^0_{jk}$.

It is not hard to see that for some K_{105} we have

$$\begin{aligned}(7.9)\qquad & P\big(R_{jk} \geq R^0_{jk} + K_{105} l^{4/3} (\log l)^{2/3} \text{ for some} \\ & \qquad \text{retention pair } (j,k) \mid \cap_{(j,k) \in J^{ret}} Q_{jk}\big) \leq \frac{1}{2}.\end{aligned}$$

In fact, if this were false we could obtain extra area $K_{105} l^{4/3} (\log l)^{2/3}$ almost "for free" in Theorem 4.1; more precisely, one could replace A with $A + K_{105} l^{4/3} (\log l)^{2/3}$ on the left side in the conclusion of that theorem. But, assuming K_{105} is large, this would contradict Theorem 5.7. It follows from (7.9) that for each retention pair (z'_j, z'_k) there exists $a_{jk} \in [R^0_{jk}, R^0_{jk} + K_{105} l^{4/3} (\log l)^{2/3})$ such that

$$\begin{aligned} P\left(\cap_{(j,k)\in J^{ret}}[R_{jk} = a_{jk}] \mid \cap_{(j,k)\in J^{ret}} Q_{jk}\right) \\ \geq \frac{1}{2}(K_{105} l^{4/3} (\log l)^{2/3})^{-8}. \end{aligned} \tag{7.10}$$

Let

$$R_1 = \sum_{(j,k)\in J^{ret}} a_{jk},$$

where the sum is over the 8 retention pairs.

We call (k,j) a *removal pair* if $\alpha^{(z'_k, z'_j)}$ is a connected component of the intersection of α with some removal slab. Let J^{rem} denote the set of all 8 removal pairs. For each removal pair (k,j) and corresponding removal slab F, let χ_{kj} be a dual path from z_k to z_j in $\overline{F} \backslash \mathrm{Co}(\{z'_0, .., z'_n\})$, and let G_{kj} denote the event that all bonds in χ_{kj} are open, while all bonds in the bond boundary of χ_{kj} in $(\psi_j \cup \psi_k)^c$ are closed. We call χ_{kj} a *long link*. There are 2 long links in each removal slab, one each in the upper and lower half planes. Let R_2 be the total area in the 4 removal slabs, between the upper and lower long link in each slab. Assuming the long links are chosen to have length of order D (say, at most $4D$), we have from the bounded energy property that

$$\begin{aligned} P\Big(\cap_{(k,j)\in J^{rem}} G_{z'_k, z'_j} \mid (\cap_{(j,k)\in J^{ret}} Q_{jk}) \cap (\cap_{(j,k)\in J^{ret}}[R_{jk} = a_{jk}])\Big) \\ \geq e^{-K_{106} D} \\ \geq e^{-K_{107} l^{1/3} (\log l)^{2/3}}. \end{aligned} \tag{7.11}$$

Define the event

$$E = \left(\cap_{(j,k)\in J^{ret}} Q_{jk}\right) \cap \left(\cap_{(j,k)\in J^{ret}}[R_{jk} = a_{jk}]\right) \cap \left(\cap_{(k,j)\in J^{rem}} G_{kj}\right).$$

For a configuration $\omega \in E$, there is an open dual circuit surrounding $\mathrm{Co}(\{z'_0, .., z'_n\})$ satsifying the constraint that it include all of the short links ϕ_i, long links χ_{kj} and endpaths ψ_j. There is a unique outermost such circuit subject to this constraint, obtained by taking the outermost (z'_i, z'_{i+1})-cylindrical connection from x_i to y_{i+1} for each i; we denote this circuit $\Gamma_1(\omega)$. Because of the cylindrical nature of these connections and the closed state of the bond boundaries of the short and long links, we have $\Gamma_1(\omega) = \Gamma_0(\omega)$, unless $\Gamma_0(\omega)$ and $\Gamma_1(\omega)$ are disjoint with $\Gamma_0(\omega)$ surrounding $\Gamma_1(\omega)$ and no open dual path connecting $\Gamma_0(\omega)$ to $\Gamma_1(\omega)$. It therefore follows from the near-Markov property that

$$P(\Gamma_0 \neq \Gamma_1 \mid E) \leq e^{-\epsilon_{46} l} \leq \frac{1}{2}.$$

By (3.2),

$$\sum_{(j,k)\in J^{ret}} |I_{jk}| \leq K_{108}\frac{l}{s} \leq K_{109} l^{1/3}(\log l)^{-1/3}.$$

Using these facts with (7.10), (7.11), Lemma 3.1, (7.8) and (4.1) (which is still valid here), we obtain

$$\begin{aligned} (7.12) \qquad & P(|\Gamma_0| = R_1 + R_2) \\ & \geq P(E \cap [\Gamma_0 = \Gamma_1]) \\ & \geq \frac{1}{2}P(E) \\ & \geq \frac{1}{2}(K_{105} l^{4/3}(\log l)^{2/3})^{-8} e^{-K_{107} l^{1/3}(\log l)^{2/3}} P(\cap_{(j,k)\in J^{ret}} Q_{jk}) \\ & \geq e^{-K_{110} l^{1/3}(\log l)^{2/3}} \prod_{(j,k)\in J^{ret}} P(Q_{jk}) \\ & \geq \exp\left(-\sum_{i=0}^{n} \tau(z_{i+1} - z_i) - K_{111} l^{1/3}(\log l)^{2/3} \right) \\ & \geq \exp\left(-w_1\sqrt{A} - K_{112} l^{1/3}(\log l)^{2/3} \right). \end{aligned}$$

Let $\theta\omega$ be the upward shift of a configuration ω by 1 unit, and for an event G let $\theta^m G = \{\omega : \theta^{-m}\omega \in G\}$. Let J_B^{ret} be the set consisting of the 3 retention pairs corresponding to segments of α in the lower half plane. Given a constant K_{113}, for $m \leq K_{113} l^{1/3}(\log l)^{2/3}$ we can replace Q_{jk} with $\theta^m Q_{jk}$ for all $(j,k) \in J_B^{ret}$ throughout the argument leading to (7.12) at the expense of only a possible increase in K_{111}, provided we alter the 4 long links χ_{kj} in the outer 2 removal slabs to connect to the appropriate shifted sites $z_i + (0,m)$ instead of to z_i. (The possible increase in K_{111} reflects a possible reduction in the probabilities of the events G_{kj}, resulting in an increase of K_{107} in (7.11).) We can readily keep the area R_2 fixed when we so alter the long links. We thereby obtain

$$P(|\Gamma_0| = R_1 + R_2 - (2N+1)m) \geq \exp(-w_1\sqrt{A} - K_{114} l^{1/3}(\log l)^{2/3}).$$

Provided K_{113} is large, we can choose $m \in \mathbb{Z}$ so that

$$|R_1 + R_2 - (2N+1)m - A| \leq N.$$

We can then repeat this entire argument, but shift upward (by some amount $q \leq K_{113} l^{1/3}(\log l)^{2/3}$) only the event Q_{jk} for the central of the 3 retention pairs in J_B^{ret}.

This gives

(7.13)

$$P(|\Gamma_0| = R_1 + R_2 - (2N+1)m - (2M+1)q) \geq \exp(-w_1\sqrt{A} - K_{115}l^{1/3}(\log l)^{2/3}).$$

We can choose k so that

$$|R_1 + R_2 - (2N+1)m - (2M+1)q - A| \leq M \leq l^{2/3}(\log l)^{1/3}.$$

But it is easy to see that one can alter the long links to change R_2 by any amount up to $l^{2/3}(\log l)^{1/3}$, so that

$$R_1 + R_2 - (2N+1)m - (2M+1)q = A,$$

at the expense of only a possible increase to K_{115} in (7.13). With (7.13) this completes the proof. □

Now that we can use Theorem 7.3 in place of Theorem 4.1, all proofs leading to Theorems 2.1— 2.3 remain valid under conditioning on $|\operatorname{Int}(\Gamma_0)| = A$ in place of $|\operatorname{Int}(\Gamma_0)| \geq A$. This establishes the following results.

Theorem 7.4. *Under the assumptions in Theorem 2.1 or Theorem 2.2, under the measure $P(\cdot \mid |\operatorname{Int}(\Gamma_0)| = A)$ the conclusions (2.13) – (2.16) hold with probability approaching 1 as $A \to \infty$.*

Theorem 7.5. *Under the assumptions in Theorem 2.3, under the measures $P_{N,w}(\cdot \mid \sum_{\gamma \in \mathfrak{C}_N} |\operatorname{Int}(\Gamma_0)| = A)$ the conclusions (2.19) – (2.23) hold with probability approaching 1 as $A \to \infty$.*

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