Mean-field Critical Behavior for Contact Processes

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Abstract

We prove the triangle condition for the basic contact process in sufficiently high dimensions and for the spatially spread-out contact process in spatial dimension $d > 4$. Combining with the result of [4], we prove the mean-field critical behavior for these processes: $\theta(\lambda) \approx \lambda - \lambda_c$ as $\lambda \downarrow \lambda_c$ and $\chi(\lambda) \approx (\lambda_c - \lambda)^{-1}$ as $\lambda \uparrow \lambda_c$, where $\theta(\lambda)$ and $\chi(\lambda)$ are the spread probability and the susceptibility of infection respectively, and λ_c is the critical infection rate. Our results imply that the upper critical dimension for the contact processes is four.

Contents

1 Introduction **2**

2 The models and the main results **2**
2.1 The models 2
2.2 The main results 5

3 Proof of Theorem 2.2 **6**
3.1 Discretization 6
3.2 Structure of the proof of Theorem 3.1 8
3.3 Bootstrapping argument 9
3.4 Bounds on the Gaussian quantities 12

4 The lace expansion **15**

5 Estimates **18**
5.1 Estimate of $\widehat{\Gamma}_N(K)$ for $N \geq 0$ 19
5.1.1 Estimate of $\widehat{g}_0(K)$ 19
5.1.2 Estimate of $\widehat{g}_n(K)$ for $n \geq 1$ 21
5.2 Estimate of $\widehat{R}_N(K)$ for $N \geq 1$ 30
5.3 Estimate of $\widehat{\Gamma}_N(O) - \widehat{\Gamma}_N(K)$ for $N \geq 0$ 30

1 Introduction

The (basic) contact process is a famous model of a spread of a disease, and exhibits a phase transition. This model has been studied for about twenty-five years, and many interesting results listed in Section 2.1 have been obtained.

Recently it has been shown in [4] that the critical behavior for the contact process becomes simple under a certain diagrammatic condition. This diagrammatic condition is called *the triangle condition* after a shape formed out of three routes of infection in space-time. This condition is expected to hold in spatial dimension $d > 4$. By using *the lace expansion*, it has been proved that a discrete analogue of the triangle condition holds for unoriented percolation in $d \gg 6$ in [14] and for oriented percolation in spatial dimension $d \gg 4$ in [20]. The contact process closely resembles an oriented percolation model. As in [6], we can construct an oriented percolation model which converges to the contact process in a limit of the temporal spacing $\varepsilon \downarrow 0$. Applying the lace expansion for discrete models to the discretized contact process and taking a continuum limit, we prove that the triangle condition for the contact process holds in spatial dimension $d \gg 4$. We also prove that the triangle condition holds in $d > 4$ for the spatially spread-out contact process defined in Section 2.1, which is expected to exhibit the same critical behavior as the basic contact process.

The lace expansion produces a convolution equation for *the connectivity function* for the discretized contact processes. If we simply estimate this convolution equation as in estimating that for discrete models, we can not take a limit $\varepsilon \downarrow 0$; even if we can take a continuum limit, the expansion does not converge (see the introduction of Section 5 for more detail). These potential difficulties are overcome by the following two points:

1. to extract a factor of ε from a point where at least two routes of infection in space-time intersect,

2. to select a parameter responsible for convergence of the lace expansion, even if ε is very small.

The first point, which is inspired by the idea of [4], enables us to take a continuum limit. Thanks to the second point, convergence of the lace expansion is realized.

This paper is organized as follows: in Section 2.1, we introduce two types of contact processes: the basic and the spatially spread-out contact processes. In order to arouse the readers' interest on the critical phenomena for these processes, we also list several interesting results which have been obtained so far. The main theorem is presented in Section 2.2. This theorem is proved by discretizing the temporal axis, and the discretization in time is performed in Section 3.1. In Section 3.2, we describe the structure of the proof of a discrete version of the main theorem. The difficult part of the proof is discussed in Section 3.3. It is here that the lace expansion is used. The expansion is derived through repeated applications of the inclusion-exclusion relation, and is proved in Section 4. Thanks to the above two major points of this paper, the expansion converges and has a meaningful continuum limit for sufficiently high dimensional basic contact process or for sufficiently spread-out contact process in $d > 4$. Detailed estimates for convergence of the lace expansion and existence of a limit in $\varepsilon \downarrow 0$ are presented in Section 5.

2 The models and the main results

2.1 The models

We define contact processes on $\mathbb{Z}^d$ through the Harris graphical representation (see [4] and references therein) to construct a percolation structure. Consider the graph $\mathbb{Z}^d \times \mathbb{R}$ in which $\mathbb{Z}^d$ denotes the spatial component and $\mathbb{R}$ denotes the temporal component. Along each time line $\{x\} \times \mathbb{R}$, Poisson points with intensity 1 are placed independently of the other point processes. These points are called *points of recovery from infection*. For each ordered pair of distinct time lines from $\{x\} \times \mathbb{R}$ to $\{y\} \times \mathbb{R}$, *arrows of infection* are drawn from x to y by a Poisson process with intensity $\lambda J_{x,y}$ independently of the other Poisson processes of arrows and points. We require the coupling constants $\{J_{x,y}\}_{x,y \in \mathbb{Z}^d}$ to be non-negative and $\mathbb{Z}^d$-invariant: invariant under translations, rotations by $\pi/2$ and reflections in coordinate hyperplanes. The infection rate λ is the only parameter in these models.

We consider the following cases of the coupling constant $J_{o,x}$:

$$J_{o,x} = \begin{cases} \mathbb{I}_{\{\|x\|_1 = 1\}}, & \text{for } \textit{the basic contact process}, \\ \dfrac{\mathbb{I}_{\{0<\|x\|_\infty \le L\}}}{\sum_y \mathbb{I}_{\{0<\|y\|_\infty \le L\}}}, & \text{for } \textit{the spatially spread-out contact process}, \end{cases}$$

where $\|x\|_1 = \sum_i |x_i|$ and $\|x\|_\infty = \sup_i |x_i|$ for $x = (x_1, x_2, \cdots, x_d)$, and $\mathbb{I}_E$ denotes the indicator function of the condition E which takes the value 1 if the condition E is satisfied and 0 otherwise.

Remark 2.1. We can consider a more general class for the spatially spread-out contact process. See for the references [13], [14] and [20].

From now on we use upper case letters for points in $\mathbb{Z}^d \times \mathbb{R}$. Particularly we use O to denote the space-time origin. For $X \in \mathbb{Z}^d \times \mathbb{R}$, we write $\sigma(X)$ and $\tau(X)$ for the spatial and the temporal components of X respectively. X is said to be *connected to* Y (equivalently Y is connected *from* X) if there exists a path in $\mathbb{Z}^d \times \mathbb{R}$ from X to Y using oriented spatial arrows and temporal line segments traversed in the increasing time direction without traversing points of recovery (see Figure 1). A site X is considered to

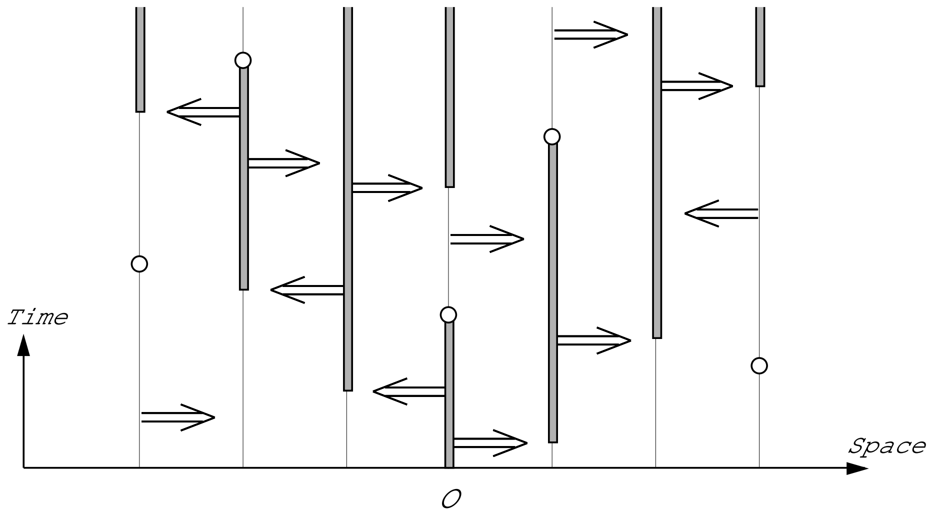

Figure 1: The graphical representation of the basic contact process on $\mathbb{Z}^1$. The arrows and the circles in this figure are the arrows of infection and the points of recovery from infection respectively. This figure shows a route of infection from the space-time origin O.

be connected to itself. '$X \to Y$' means that X is connected to Y. For $X \in \mathbb{Z}^d \times \mathbb{R}$, we define *the cluster* of X for the set of sites which are connected from X:

$$\mathbf{C}(X) = \{ Y \in \mathbb{Z}^d \times \mathbb{R} \,:\, X \to Y \}.$$

We write $\mathbb{P}^\lambda$ and $\mathbb{E}^\lambda$ for the associated probability measure and the expectation operator. We define *the spread probability* $\theta(\lambda)$ as

$$\theta(\lambda) = \mathbb{P}^\lambda\big(\mathbf{C}(O) \cap (\mathbb{Z}^d \times \{t\}) \neq \varnothing, \, \forall t \big),$$

and *the critical infection rate* $\lambda_{\rm c} = \sup\{ \lambda \,:\, \theta(\lambda) = 0 \}$. It has been proved that the critical infection rate is positive and finite (cf., [7], [15] and [17]). We are also interested in the expected total duration of the infection, called *the susceptibility*

$$\chi(\lambda) = \mathbb{E}^\lambda\big[\, |\mathbf{C}(O)| \, \big],$$

where

$$|\mathbf{C}(O)| = \sum_{\sigma(X)} \int_0^\infty d\tau(X) \, \mathbb{I}_{\{O \to X\}}.$$

If we define the connectivity function from O to X as $\phi^\lambda(O, X) \equiv \mathbb{P}^\lambda(O \to X)$, then by the monotone convergence theorem, we can rewrite the susceptibility as

$$\chi(\lambda) = \sum_{\sigma(X)} \int_0^\infty d\tau(X) \; \phi^\lambda(O, X).$$

We define the generating function of the cluster size distribution as

$$M^\lambda(h) = 1 - \int_0^\infty dl \; e^{-h\,l} \, \mathbb{P}^\lambda\big(|\mathbf{C}(O)| = l \big).$$

This quantity is analogous to the magnetization in ferromagnetic spin models.

We list here several interesting results which are related to the critical phenomena for the basic contact process. All these results can be extended to the spatially spread-out contact process.

1. (Bezuidenhout&Grimmett[5]) At the critical infection rate, $\theta(\lambda_{\rm c}) = 0$.
2. (Bezuidenhout&Grimmett[6]) If $\lambda < \lambda_{\rm c}$, then there exists a positive constant $m(\lambda)$ such that
$$\mathbb{P}^\lambda\big(|\mathbf{C}(O)| \geq l \big) \leq e^{-m(\lambda)\,l}, \quad \forall l \geq 0.$$
This implies that $\lambda_{\rm c}$ can also be characterized by $\sup\{ \lambda \,:\, \chi(\lambda) < \infty \}$.
3. (Barsky&Wu[4] and Bezuidenhout&Grimmett[6]) There exist positive constants C_i, C'_i, $i = 1, 2, 3$ such that
$$\begin{cases} \theta(\lambda) \geq C_1\,(\lambda - \lambda_{\rm c}), & \text{if } 0 \leq \lambda - \lambda_{\rm c} \leq C'_1, \\ \chi(\lambda) \geq C_2\,(\lambda_{\rm c} - \lambda)^{-1}, & \text{if } 0 < \lambda_{\rm c} - \lambda \leq C'_2, \\ M^{\lambda_{\rm c}}(h) \geq C_3\, h^{1/2}, & \text{if } 0 < h \leq C'_3. \end{cases}$$

4. (Barsky&Wu[4]) Under *the triangle condition*, there exist positive constants C_i, C'_i, $i = 1, 2, 3$ such that

$$\begin{cases} \theta(\lambda) \le C_1 \, (\lambda - \lambda_c), & \text{if } 0 \le \lambda - \lambda_c \le C'_1, \\ \chi(\lambda) \le C_2 \, (\lambda_c - \lambda)^{-1}, & \text{if } 0 < \lambda_c - \lambda \le C'_2, \\ M^{\lambda_c}(h) \le C_3 \, h^{1/2}, & \text{if } 0 < h \le C'_3. \end{cases}$$

The triangle condition is that

$$\sup_{X : \substack{\|\sigma(X)\|_1 \ge r, \\ \tau(X) \ge 0}} \nabla^{\lambda_c}(X) \to 0, \qquad \text{as } r \uparrow \infty, \tag{2.1}$$

where

$$\nabla^{\lambda}(X) = \sum_{\sigma(Y),\, \sigma(Z)} \int_{\tau(X)}^{\infty} d\tau(Y) \int_{\tau(Y)}^{\infty} d\tau(Z) \; \phi^{\lambda}(O, Z) \, \phi^{\lambda}(X, Y) \, \phi^{\lambda}(Y, Z).$$

It is expected that the observables behave in the following form of power laws near and at the critical point:

$$\begin{cases} \theta(\lambda) \approx (\lambda - \lambda_c)^{\beta}, & \text{as } \lambda \downarrow \lambda_c, \\ \chi(\lambda) \approx (\lambda_c - \lambda)^{-\gamma}, & \text{as } \lambda \uparrow \lambda_c, \\ M^{\lambda_c}(h) \approx h^{1/\delta}, & \text{as } h \downarrow 0, \end{cases}$$

where '$f(x) \approx g(x)$ as $x \to x_0$' means that there exist positive and finite constants c_1 and c_2 such that

$$c_1 \, g(x) \le f(x) \le c_2 \, g(x),$$

holds for all x close to x_0. Since $M^{\lambda}(h)$ can also be interpreted as the Laplace transform of the cluster size distribution, the above behavior for $M^{\lambda_c}(h)$ in the vicinity of $h = 0$ is equivalent to

$$\mathbb{P}^{\lambda_c}\big(|\mathbf{C}(O)| \ge l \big) \approx l^{-1/\delta}, \quad \text{as } l \uparrow \infty,$$

by the Tauberian theorem (cf., [18]). The exponents β, γ and δ are called *the critical exponents*, which generally depend on the spatial dimension. As in many other statistical mechanical models, it is generally expected (and is proved in this paper for some cases) that there exists d_c such that the critical exponents take on dimension-independent values with $\beta = 1$, $\gamma = 1$ and $\delta = 2$, as soon as d is greater than d_c. These dimension-independent values are called *the mean-field exponents*, and d_c is called *the upper critical dimension*, which is expected to be 4 for the contact processes. The above third and fourth results imply that the mean-field exponents appear under the triangle condition.

2.2 The main results

In this paper, we prove a certain condition that implies the triangle condition, in order to understand the mean-field critical behavior for the contact processes. We define the Fourier transform of $\phi^{\lambda}(O, X)$ as

$$\widehat{\phi}^{\lambda}(K) = \sum_{\sigma(X)} \int_0^{\infty} d\tau(X) \; \phi^{\lambda}(O, X) \, e^{i\, K \cdot X},$$

where $K = (k, \omega) \in [-\pi, \pi]^d \times \mathbb{R}$, and $K \cdot X = k \cdot x + \omega t$ for $X = (x, t)$. From now on we write Π_d for $[-\pi, \pi]^d$. Our main result can be stated as follows.

Theorem 2.1. *For the basic contact process in sufficiently high dimensions, or for the spatially spread-out contact process in $d > 4$ with L sufficiently large,*

1. $\sup_X \nabla^{\lambda_c}(X) < \infty$,
2. *the model exhibits the mean-field critical behavior: $\beta = 1$, $\gamma = 1$ and $\delta = 2$.*

The triangle condition (2.1) immediately follows from the first result through the Riemann-Lebesgue lemma or the Hausdorff-Young inequality. Combining with the result of [4], we can obtain the second result. The first result is an immediate consequence of the following theorem.

Theorem 2.2. *Under the same condition as in Theorem 2.1, the infrared bound*

$$|\,\hat{\phi}^\lambda(K)\,| \le \frac{1}{C_\sigma \, \|k\|^2 + C_\tau \, |\omega|}, \quad \text{uniformly in } \lambda < \lambda_c,$$

holds for some positive constants C_σ and C_τ, where $K = (k, \omega)$ and $\|k\|^2 \equiv \|k\|_2^2 = \sum_i k_i^2$.

The basic and the spatially spread-out contact processes are expected to exhibit the same critical behavior: these processes are expected to be in the same *universality class.* Therefore the upper critical dimension d_c is expected not to be above 4 by the above theorem. A necessary condition for $d_c = 4$ may be proved through an argument analogous to that of [24].

3 Proof of Theorem 2.2

We prove Theorem 2.2 along the following line:

1. we discretize the temporal axis and consider a kind of $d+1$-dimensional oriented bond percolation,
2. we apply the lace expansion to the connectivity function for this oriented percolation,
3. we estimate *the irreducible two-point functions* appearing in the above expansion in order to obtain a ε-uniform infrared bound.

We organize this section as follows: in Section 3.1, the discretization in time is performed. A discrete version of Theorem 2.2 is also presented. In Section 3.2, we describe the structure of the proof of this theorem, which is the same as that used in [14] and [20]. The difficult parts of the proof are explained in Section 3.3, in which the lace expansion is used.

3.1 Discretization

The discretized models are the following oriented bond percolation: fix $\varepsilon \in (0, 1)$. As in the case of continuous time, we use upper case letters for points in $\mathbb{Z}^d \times \varepsilon\,\mathbb{Z}$. We call an ordered pair of sites (X, Y) with $\tau(X) + \varepsilon = \tau(Y)$ a *bond* from X to Y. For a bond (X, Y), X and Y are called *the bottom* and *the top* of (X, Y) respectively. If $\sigma(Y) = \sigma(X)$, (X, Y) is called a *temporal* bond. If on the other hand $\sigma(Y) \neq \sigma(X)$, (X, Y) is called a *spatial* bond, and we define $\mathcal{J}_{X,Y}$ for $J_{\sigma(X),\sigma(Y)}$. When $\mathcal{J}_{X,Y}$ is the coupling constant of the basic contact process, we call this model *the nearest-neighbor model.* When $\mathcal{J}_{X,Y}$ is the coupling constant of the spatially spread-out contact process, we simply call this model *the spread-out model.* A bond (X, Y) is *open* with probability $p(X, Y)$ defined as

$$p(X, Y) = \lambda\, \varepsilon\, \mathcal{J}_{X,Y}\, \mathbb{I}_{\{\sigma(Y)\neq\sigma(X)\}} + (1-\varepsilon)\, \mathbb{I}_{\{\sigma(Y)=\sigma(X)\}},$$

and *closed* with probability $1 - p(X, Y)$, independently of the other bonds. The identity $p(X, Y) = p(X+Z, Y+Z)$ holds for any Z, because of translation invariance of the coupling constants. Connections for these oriented percolation are made along open bonds (see Figure 2).

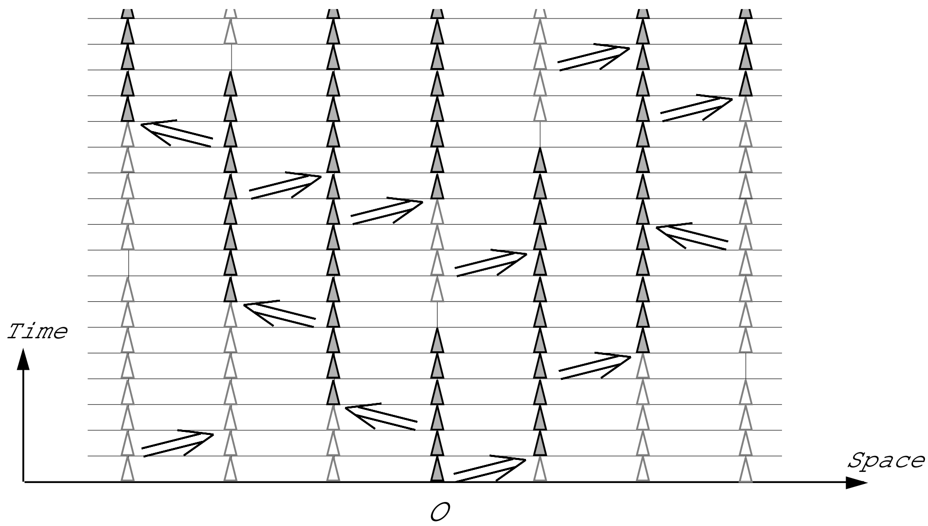

Figure 2: This figure is obtained from Figure 1 by discretizing the temporal axis. The triangles (resp., the arrows) in this figure are open temporal (resp., spatial) bonds. We shade the triangles which are connected from O.

We write $\mathbb{P}_\varepsilon^\lambda$ and $\mathbb{E}_\varepsilon^\lambda$ for the associated probability measure and the expectation operator. It can be proved as in [6] that $\mathbb{P}_\varepsilon^\lambda$ converges weakly to $\mathbb{P}^\lambda$ as $\varepsilon \downarrow 0$. Therefore the contact processes are well-approximated by the above discretized models.

As in the case of continuous time, we use the notation '$\rightarrow$' for the connection of these oriented percolation, and $\mathbf{C}(X)$ for the cluster of X. We define the susceptibility as

$$\chi_\varepsilon(\lambda) = \varepsilon \sum_X \varphi_\varepsilon^\lambda(O, X),$$

where $\varphi_\varepsilon^\lambda(O, X) \equiv \mathbb{P}_\varepsilon^\lambda(O \rightarrow X)$ is the connectivity function from O to X.

The oriented percolation considered also undergoes a phase transition: there exists λ_c^ε such that $\chi_\varepsilon(\lambda)$ is finite for all $\lambda < \lambda_c^\varepsilon$ and diverges as $\lambda \uparrow \lambda_c^\varepsilon$. For $\lambda < \lambda_c^\varepsilon$, if we define the Fourier transform of $\varphi_\varepsilon^\lambda(O, X)$ as

$$\widehat{\varphi}_\varepsilon^\lambda(K) = \varepsilon \sum_X \varphi_\varepsilon^\lambda(O, X)\, e^{i\,K\cdot X},$$

then we have $\chi_\varepsilon(\lambda) = \widehat{\varphi}_\varepsilon^\lambda(O)$.

Theorems 2.1 and 2.2 are immediate consequences of the following theorem.

Theorem 3.1. *1. For $\varepsilon_0 \in (0, 1)$, there exists $d_0 > 4$ such that, for the nearest-neighbor model with $d > d_0$ and $\varepsilon < \varepsilon_0$, the following infrared bound holds:*

$$|\,\widehat{\varphi}_\varepsilon^\lambda(K)\,| \le \frac{1}{C_\sigma\,\|k\|^2 + C_\tau\,|\omega|}, \quad \textit{uniformly in } \lambda < \lambda_c^\varepsilon,$$

for some ε-independent positive constants C_σ and C_τ.

2. For $d > 4$, there exists L_0 such that, for the spread-out model with $L > L_0$ and $\varepsilon < 1$, the above infrared bound holds.

3.2 Structure of the proof of Theorem 3.1

We omit the subscript ε and the superscript λ of various quantities in the rest of this paper. We also omit the superscript ε of λ_c^ε.

Before describing the structure of the proof, we define several quantities. *The triangle diagram $\mathcal{T}(X)$* is defined as

$$\mathcal{T}(X) = \varepsilon^2 \sum_{Y,\,Z} \{\,\varphi(O,\,Z)\,\varphi(X,\,Y)\,\varphi(Y,\,Z) - \varDelta_{O,\,Z}\,\varDelta_{X,\,Y}\,\varDelta_{Y,\,Z}\,\}, \tag{3.1}$$

where $\varDelta_{O,\,X}$ is the probability that O is connected to X by using only temporal bonds, defined as

$$\varDelta_{O,\,X} = \sum_{n=0}^{\infty} (\,1-\varepsilon\,)^n\,\mathbb{I}_{\{X=(o,\,n\,\varepsilon)\}}.$$

In [14] and [20], $\delta_{O,\,X} \equiv \mathbb{I}_{\{O=X\}}$ has been used instead of $\varDelta_{O,\,X}$ in defining the triangle diagram. However we must use $\varDelta_{O,\,X}$ for the models considered, in order to obtain a desirable bound on the triangle diagram in large d or large L (see Section 3.4). We also define the quantities

$$\mathcal{S} = \varepsilon \sum_{Z} \{\,\varphi(O,\,Z) - \varDelta_{O,\,Z}\,\}^2, \qquad \mathcal{W}(X) = \varepsilon \sum_{Z} \|\,\sigma(Z)\,\|^2\,\varphi(O,\,Z)\,\varphi(X,\,Z).$$

We consider the random walk on $\mathbb{Z}^d \times \varepsilon\,\mathbb{Z}$, whose transition probability $p_{\mathrm{G}}(O,\,X)$ is defined as

$$p_{\mathrm{G}}(O,\,X) = \frac{\varepsilon\,\mathcal{J}_{O,\,X}}{|\mathcal{J}|}\,\mathbb{I}_{\{(O,\,X)\text{ is a spatial bond}\}} + (\,1-\varepsilon\,)\,\mathbb{I}_{\{(O,\,X)\text{ is a temporal bond}\}},$$

where $|\mathcal{J}| = \sum_X \mathcal{J}_{O,\,X}$, which is equal to $2\,d$ for the nearest-neighbor model or is equal to 1 for the spread-out model. The two-point function (or the Green function) $G(O,\,X)$ of this random walk is given by the quantity satisfying the convolution equation

$$G(O,\,X) = \delta_{O,\,X} + \sum_{Z} p_{\mathrm{G}}(O,\,Z)\,G(Z,\,X). \tag{3.2}$$

We also define the Gaussian quantities $\mathcal{T}_{\mathrm{G}}(X)$, $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{W}_{\mathrm{G}}(X)$ corresponding to $\mathcal{T}(X)$, $\mathcal{S}$ and $\mathcal{W}(X)$ respectively, defined by replacing φ with G.

The proof of Theorem 3.1 is achieved by showing that the following three statements hold.

1. For each X, $\mathcal{T}(X)$, $\mathcal{S}$ and $\mathcal{W}(X)$ are continuous in $\lambda \le \lambda_{\mathrm{c}}$.

2. For $d > 4$ and $\lambda \le \lambda_{\mathrm{G}} \equiv |\mathcal{J}|^{-1}$, there exist constants $C_{\mathcal{T}}$, $C_{\mathcal{S}}$, $C_{\mathcal{W}}$ and $C'_{\mathcal{W}}$ such that

$$\mathcal{T}(X) \le \mathcal{T}_{\mathrm{G}}(X) \le C_{\mathcal{T}}\,\mu, \qquad \mathcal{S} \le \mathcal{S}_{\mathrm{G}} \le C_{\mathcal{S}}\,\nu, \qquad \mathcal{W}(X) \le \mathcal{W}_{\mathrm{G}}(X) \le \begin{cases} C_{\mathcal{W}}\,\rho, & \text{if } X = O, \\ C'_{\mathcal{W}}\,\varrho, & \text{if } X \ne O. \end{cases} \tag{3.3}$$

For the nearest-neighbor model, $C_{\mathcal{T}}$, $C_{\mathcal{S}}$, $C_{\mathcal{W}}$ and $C'_{\mathcal{W}}$ depend on ε but not on d, and μ, ν, ρ and ϱ are equal to d^{-1}. For the spread-out model, $C_{\mathcal{T}}$, $C_{\mathcal{S}}$, $C_{\mathcal{W}}$ and $C'_{\mathcal{W}}$ depend on d but not on L and ε, and

$$\mu = L^{-d/2}\,(\ln L)^{d/4}, \qquad \nu = \mu^2, \qquad \rho = L^{4-d}\,(\ln L)^d, \qquad \varrho = L^2\,\mu.$$

3. Let d or L be sufficiently large, and fix $\lambda \in [\,\lambda_{\mathrm{G}},\, \lambda_{\mathrm{c}})$ arbitrarily. We define $\mathfrak{P}_a$ for the statement that the following set of inequalities holds:

$$\mathcal{T}(X) \le a\,C_{\mathcal{T}}\,\mu, \qquad \mathcal{S} \le a\,C_{\mathcal{S}}\,\nu, \qquad \mathcal{W}(X) \le \begin{cases} a\,C_{\mathcal{W}}\,\rho, & \text{if } X = O, \\ a\,C'_{\mathcal{W}}\,\varrho, & \text{if } X \ne O, \end{cases} \qquad \lambda\,|\mathcal{J}| \le a. \tag{3.4}$$

Then $\mathfrak{P}_4$ implies $\mathfrak{P}_3$.

The first and the second statements imply that, for every X, $\mathcal{T}(X)$, $\mathcal{S}$ and $\mathcal{W}(X)$ are small satisfying $\mathfrak{P}_3$ for small λ, and continuous up to λ_{c}. The third statement implies that, for sufficiently large d or L, there is a forbidden region in the graph of $\mathcal{T}(X)$, $\mathcal{S}$ and $\mathcal{W}(X)$. Therefore we can see that the stronger statement $\mathfrak{P}_3$ indeed hold at λ_{c}. The infrared bound is obtained in the course of the proof of the third statement.

The first statement is a consequence of continuity of the connectivity function and the monotone convergence theorem. The second statement follows from the inequality $\varphi(O,\,X) \le G(O,\,X)$ which holds in $\lambda \le \lambda_{\mathrm{G}}$, and from bounds on the Gaussian quantities $\mathcal{T}_{\mathrm{G}}(X)$, $\mathcal{S}_{\mathrm{G}}$ and $\mathcal{W}_{\mathrm{G}}(X)$ proved in Section 3.4. The difficult part of the proof is to obtain the third statement, and here the lace expansion is used.

3.3 Bootstrapping argument

We describe the outline for deriving the stronger statement $\mathfrak{P}_3$ under the weaker inequalities in (3.4) with $a = 4$. We suppose $\varepsilon \le 1/3$ in the rest of this paper. We can prove the case of $\varepsilon > 1/3$ with slight modifications, and we omit this case in this paper.

In Section 4, we obtain the following lace expansion for the connectivity function for $\lambda < \lambda_{\mathrm{c}}$:

$$\varphi(O,\,X) = \mathit{\Gamma}_N(O,\,X) + \sum_{(U,\,V)} \mathit{\Gamma}_N(O,\,U)\,p(U,\,V)\,\varphi(V,\,X) + (-1)^{N+1}\,R_{N+1}(O,\,X),$$

where $\mathit{\Gamma}_N(O,\,X) = \sum_{n=0}^{N} (-1)^n\, g_n(O,\,X)$, $g_n(O,\,X)$ is the irreducible two-point function, and $R_N(O,\,X)$ is a remainder, which are explained in Section 4. Taking the Fourier transform of the above convolution equation, we obtain

$$\widehat{\varphi}(K) = \frac{\widehat{\mathit{\Gamma}}_N(K) + (-1)^{N+1}\,\widehat{R}_{N+1}(K)}{1 - \left\{\, 1 - \varepsilon + \lambda\,|\mathcal{J}|\,\varepsilon\,\widehat{D}(k) \,\right\} e^{i\,\omega\,\varepsilon}\,\widehat{\mathit{\Gamma}}_N(K)\,\varepsilon^{-1}}, \tag{3.5}$$

where

$$\widehat{D}(k) = \sum_X \frac{\mathcal{J}_{O,\,X}}{|\mathcal{J}|}\, e^{i\,k\cdot\sigma(X)}. \tag{3.6}$$

Especially $\widehat{D}(k) = d^{-1}\sum_i \cos k_i$ for the nearest-neighbor model. Under the inequalities in (3.4) with $a = 4$, we obtain in Section 5.1 the following bounds on the irreducible two-point functions:

$$\big|\,\widehat{g}_n(K)\,\big| \le \begin{cases} \varepsilon + \mathcal{O}(\mu)\,\varepsilon^2, & \text{for } n = 0, \\ \mathcal{O}(\mu)^n\,\varepsilon^2, & \text{for } n \ge 1. \end{cases} \tag{3.7}$$

In Section 5.2, we obtain the inequality

$$\big|\,\widehat{R}_{N+1}(K)\,\big| \le \varepsilon^{-1}\,\widehat{g}_N(O)\,\widehat{\varpi}(O)\,\chi(\lambda), \tag{3.8}$$

where $\widehat{\varpi}(K) = \sum_X p(O,\,X)\,e^{i\,K\cdot X}$. Therefore for sufficiently large d or L, we can take N to infinity to obtain $\widehat{\Gamma}_N(K) \to \widehat{\Gamma}(K) = \varepsilon + \mathcal{O}(\mu)\,\varepsilon^2$ and $\widehat{R}_{N+1}(K) \to 0$. These facts enable us to rewrite $\widehat{\varphi}(K)$ as

$$\begin{aligned}
\widehat{\varphi}(K) &= \frac{\widehat{\Gamma}(K)}{1 - \lambda\,|\mathcal{J}|\,\widehat{D}(k)\,e^{i\,\omega\,\varepsilon}\,\widehat{\Gamma}(K) - (\,1-\varepsilon\,)\,e^{i\,\omega\,\varepsilon}\,\widehat{\Gamma}(K)\,\varepsilon^{-1}} \\
&= \frac{\widehat{\Gamma}(K)}{\lambda\,|\mathcal{J}|\,\big\{\,\widehat{\Gamma}(O) - \widehat{D}(k)\,e^{i\,\omega\,\varepsilon}\,\widehat{\Gamma}(K)\,\big\} + (\,1-\varepsilon\,)\,\frac{\widehat{\Gamma}(O) - e^{i\omega\varepsilon}\,\widehat{\Gamma}(K)}{\varepsilon} + \frac{\widehat{\Gamma}(O)}{\chi(\lambda)}} \\
&= \frac{\widehat{\Gamma}(K)/\widehat{\Gamma}(O)}{\lambda\,|\mathcal{J}|\,\big\{\,1 - \widehat{D}(k)\,\big\} + \widehat{\varpi}(k,\,0)\,\frac{1-e^{i\omega\varepsilon}}{\varepsilon} + \widehat{\varpi}(K)\,\frac{\widehat{\Gamma}(O)-\widehat{\Gamma}(K)}{\varepsilon\,\widehat{\Gamma}(O)} + \chi(\lambda)^{-1}}.
\end{aligned} \tag{3.9}$$

In the second identity, we used the identity

$$\chi(\lambda) = \widehat{\varphi}(O) = \frac{\widehat{\Gamma}(O)}{1 - \lambda\,|\mathcal{J}|\,\widehat{\Gamma}(O) - (\,1-\varepsilon\,)\,\widehat{\Gamma}(O)\,\varepsilon^{-1}}.$$

The numerator is non-negative for sufficiently large d or L, and thus the denominator is also non-negative. Therefore

$$\lambda\,|\mathcal{J}| \le \frac{1 - (\,1-\varepsilon\,)\,\widehat{\Gamma}(O)\,\varepsilon^{-1}}{\widehat{\Gamma}(O)} = 1 + \frac{\varepsilon - \widehat{\Gamma}(O)}{\varepsilon\,\widehat{\Gamma}(O)} \le 1 + \mathcal{O}(\mu). \tag{3.10}$$

Next we estimate the denominator of the right hand side of (3.9). By the inequalities $\varepsilon \le 1/3$ and (3.10),

$$\widehat{\varpi}(k,\,0) \ge 1 - \varepsilon - \lambda\,|\mathcal{J}|\,\varepsilon \ge \frac{1}{3}\,\big\{\,1 - \mathcal{O}(\mu)\,\big\}.$$

Therefore $\widehat{\varpi}(k,\,0) \ge 0$ for sufficiently large d or L. Together with the trivial inequalities

$$\lambda\,|\mathcal{J}| \ge 1, \qquad \chi(\lambda)^{-1} \ge 0, \qquad 1 - \widehat{D}(k) \ge 0, \qquad \Re\big[\,1 - e^{i\,\omega\,\varepsilon}\,\big] \ge 0,$$

we obtain

$$
\begin{aligned}
&\left| \lambda\, |\mathcal{J}|\, \{\, 1 - \widehat{D}(k)\, \} + \widehat{\varpi}(k,\, 0)\, \frac{1 - e^{i\,\omega\,\varepsilon}}{\varepsilon} + \widehat{\varpi}(K)\, \frac{\widehat{\Gamma}(O) - \widehat{\Gamma}(K)}{\varepsilon\, \widehat{\Gamma}(O)} + \chi(\lambda)^{-1} \right|^2 \\
&\quad \geq \left| \lambda\, |\mathcal{J}|\, \{\, 1 - \widehat{D}(k)\, \} + \widehat{\varpi}(k,\, 0)\, \frac{1 - e^{i\,\omega\,\varepsilon}}{\varepsilon} + \chi(\lambda)^{-1} \right|^2 - \left| \widehat{\varpi}(K)\, \frac{\widehat{\Gamma}(O) - \widehat{\Gamma}(K)}{\varepsilon\, \widehat{\Gamma}(O)} \right|^2 \\
&\quad \geq \{\, 1 - \widehat{D}(k)\, \}^2 + \widehat{\varpi}(k,\, 0)^2 \left| \frac{1 - e^{i\,\omega\,\varepsilon}}{\varepsilon} \right|^2 - \left| \widehat{\varpi}(K)\, \frac{\widehat{\Gamma}(O) - \widehat{\Gamma}(K)}{\varepsilon\, \widehat{\Gamma}(O)} \right|^2 .
\end{aligned} \tag{3.11}
$$

In Section 5.3, we obtain

$$
|\, \widehat{\Gamma}(O) - \widehat{\Gamma}(K)\, | \leq \sum_n \left(\frac{\|k\|^2}{2\, d}\, \mathcal{G}_n^\sigma + |\omega|\, \mathcal{G}_n^\tau \right), \tag{3.12}
$$

where

$$
\mathcal{G}_n^\sigma = \varepsilon \sum_X g_n(O,\, X)\, \|\, \sigma(X)\, \|^2, \qquad\qquad \mathcal{G}_n^\tau = \varepsilon \sum_X g_n(O,\, X)\, \tau(X).
$$

Under the inequalities in (3.4) with $a = 4$, we also prove in Section 5.3 that

$$
\mathcal{G}_n^\sigma \leq \begin{cases} \mathcal{O}(\varrho)\, \varepsilon^2, & \text{for } n = 0, \\ n^2\, \mathcal{O}(\varrho)\, \mathcal{O}(\mu)^{n-1}\, \varepsilon^2, & \text{for } n \geq 1, \end{cases} \qquad \mathcal{G}_n^\tau \leq \begin{cases} \mathcal{O}(\mu)\, \varepsilon^2, & \text{for } n = 0, \\ n\, \mathcal{O}(\mu)^n\, \varepsilon^2, & \text{for } n \geq 1, \end{cases} \tag{3.13}
$$

and thus $\sum_n \mathcal{G}_n^\sigma \leq \mathcal{O}(\varrho)\, \varepsilon^2$ and $\sum_n \mathcal{G}_n^\tau \leq \mathcal{O}(\mu)\, \varepsilon^2$ for sufficiently large d or L. Using (3.12) and

$$
1 - \widehat{D}(k) \geq \frac{\|k\|^2}{2\, \pi^2\, d}, \qquad\qquad |\, 1 - e^{i\,\omega\,\varepsilon}\, | \geq \frac{2\,\varepsilon}{\pi}\, |\omega|,
$$

the right hand side of (3.11) is bounded from below by

$$
\begin{aligned}
&\{\, 1 - \widehat{D}(k)\, \}^2 + \left\{ \frac{2}{\pi}\, \widehat{\varpi}(k,\, 0)\, |\omega| \right\}^2 - \left| \frac{\widehat{\varpi}(K)}{\varepsilon\, \widehat{\Gamma}(O)} \right|^2 \left\{ \sum_n \left(\frac{\|k\|^2}{2\, d}\, \mathcal{G}_n^\sigma + |\omega|\, \mathcal{G}_n^\tau \right) \right\}^2 \\
&\quad \geq \frac{1}{2} \left\{ A_\sigma\, \{\, 1 - \widehat{D}(k)\, \} + \frac{2}{3\, \pi}\, A_\tau\, |\omega| \right\}^2,
\end{aligned} \tag{3.14}
$$

where

$$
A_\sigma = \sqrt{1 - 2 \left| \frac{\widehat{\varpi}(K)}{\varepsilon\, \widehat{\Gamma}(O)} \right|^2 \left(\pi^2 \sum_n \mathcal{G}_n^\sigma \right)^2}, \qquad A_\tau = \sqrt{\{\, 3\, \widehat{\varpi}(k,\, 0)\, \}^2 - 2 \left| \frac{\widehat{\varpi}(K)}{\varepsilon\, \widehat{\Gamma}(O)} \right|^2 \left(\frac{3}{2}\, \pi \sum_n \mathcal{G}_n^\tau \right)^2}.
$$

As stated above, $A_\sigma \geq 1 - \mathcal{O}(\varrho^2)$ and $A_\tau \geq 1 - \mathcal{O}(\mu)$ for sufficiently large d or L. Along the line of bounding the Gaussian triangle diagram, we use the inequalities (3.10) and (3.14) to obtain

$$
\overline{\mathcal{T}} \equiv \sup_X \mathcal{T}(X) \leq 2\, C_{\mathcal{T}}\, \mu + \mathcal{O}(\mu^2). \tag{3.15}
$$

We can also obtain the stronger inequality for $\mathcal{S}$ with some care for the spread-out model.

Finally we bound $\mathcal{W}(X)$. Integrating by parts, we have

$$\mathcal{W}(X) = \begin{cases} \displaystyle\sum_{j=1}^{d} \int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\pi)^{d+1}} \,|\, \partial_j \widehat{\varphi}(K) \,|^2, & \text{if } X = O, \\ \displaystyle -\sum_{j=1}^{d} \int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\pi)^{d+1}} \,\{\, \partial_j^2 \widehat{\varphi}(K) \,\}\, \widehat{\varphi}(-K)\, e^{-i\,K\cdot X}, & \text{if } X \neq O, \end{cases}$$

where $\partial_j = \partial/\partial k_j$. We differentiate the right hand side of (3.9) to obtain

$$\begin{aligned} \partial_j \widehat{\varphi}(K) &= \widehat{\varphi}(K)\, \frac{\partial_j \widehat{\Gamma}(K)}{\widehat{\Gamma}(K)} + \widehat{\varphi}(K)^2 \left\{ \lambda\, |\mathcal{J}|\, e^{i\,\omega\,\varepsilon}\, \partial_j \widehat{D}(k) + \frac{\widehat{\varpi}(K)\, \partial_j \widehat{\Gamma}(K)}{\varepsilon\, \widehat{\Gamma}(K)} \right\}, \\ \partial_j^2 \widehat{\varphi}(K) &= \widehat{\varphi}(K)\, \frac{\partial_j^2 \widehat{\Gamma}(K)}{\widehat{\Gamma}(K)} + 2\, \widehat{\varphi}(K)^2\, \frac{\partial_j \widehat{\Gamma}(K)}{\widehat{\Gamma}(K)} \left\{ \lambda\, |\mathcal{J}|\, e^{i\,\omega\,\varepsilon}\, \partial_j \widehat{D}(k) + \frac{\widehat{\varpi}(K)\, \partial_j \widehat{\Gamma}(K)}{\varepsilon\, \widehat{\Gamma}(K)} \right\} \\ &\quad + \widehat{\varphi}(K)^2 \left\{ \lambda\, |\mathcal{J}|\, e^{i\,\omega\,\varepsilon}\, \partial_j^2 \widehat{D}(k) + 2\, \lambda\, |\mathcal{J}|\, e^{i\,\omega\,\varepsilon}\, \partial_j \widehat{D}(k)\, \frac{\partial_j \widehat{\Gamma}(K)}{\widehat{\Gamma}(K)} + \frac{\widehat{\varpi}(K)\, \partial_j^2 \widehat{\Gamma}(K)}{\varepsilon\, \widehat{\Gamma}(K)} \right\} \\ &\quad + 2\, \widehat{\varphi}(K)^3 \left\{ \lambda\, |\mathcal{J}|\, e^{i\,\omega\,\varepsilon}\, \partial_j \widehat{D}(k) + \frac{\widehat{\varpi}(K)\, \partial_j \widehat{\Gamma}(K)}{\varepsilon\, \widehat{\Gamma}(K)} \right\}^2. \end{aligned}$$

Thanks to detailed estimates of Section 5.3, $\widehat{\Gamma}(K)$ is differentiable term by term. Together with the spatial symmetry of $g_n(O, X)$, $\big|\, \partial_j^s \widehat{\Gamma}(K) \,\big| \le d^{-1} \sum_n \mathcal{G}_n^\sigma$ for $s = 1, 2$. Following the way of bounding the Gaussian quantity $\mathcal{W}_{\mathrm{G}}(X)$ and using the mean value theorem as in Section 4.3.3 (c) of [14] and the inequalities (3.10), (3.14) and (3.15), we obtain

$$\mathcal{W}(O) \le 2\, C_{\mathcal{W}}\, \rho + \mathcal{O}(\mu\, \rho) + \frac{1}{d}\, \mathcal{O}(\varrho^2), \tag{3.16}$$

and the stronger inequality for $\mathcal{W}(X)$ with $X \neq O$ by choosing $C'_{\mathcal{W}}$ sufficiently large depending only on $C_{\mathcal{T}}$, $C_{\mathcal{S}}$ and $C_{\mathcal{W}}$.

Now we have obtained the stronger statement $\mathfrak{P}_3$ under the weaker statement $\mathfrak{P}_4$. The infrared bound has been obtained in (3.14). The proof of the third statement in Section 3.2 is now completed, assuming detailed estimates in Sections 4 and 5.

3.4 Bounds on the Gaussian quantities

We prove that the Gaussian quantities $\mathcal{T}_{\mathrm{G}}(X)$, $\mathcal{S}_{\mathrm{G}}(X)$ and $\mathcal{W}_{\mathrm{G}}(X)$ are bounded as in (3.3).

We begin with estimating $\mathcal{T}_{\mathrm{G}}(X)$. We obtain a bound on $\mathcal{S}_{\mathrm{G}}$ on the way of estimating $\mathcal{T}_{\mathrm{G}}(X)$. Using the identity

$$\begin{aligned} &|\, \widehat{G}(K) \,|^2\, \widehat{G}(K) - |\, \widehat{\Delta}(K) \,|^2\, \widehat{\Delta}(K) \\ &\quad = \widehat{G}(K)^*\, \{\, \widehat{G}(K) - \widehat{\Delta}(K) \,\}^2 + 2\, |\, \widehat{G}(K) - \widehat{\Delta}(K) \,|^2\, \widehat{\Delta}(K) \\ &\qquad + 2\, \{\, \widehat{G}(K) - \widehat{\Delta}(K) \,\}\, |\, \widehat{\Delta}(K) \,|^2 + \{\, \widehat{G}(K) - \widehat{\Delta}(K) \,\}^*\, \{\, \widehat{\Delta}(K) \,\}^2, \end{aligned}$$

we have

$$\mathcal{T}_{\mathrm{G}}(X) \le \int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\pi)^{d+1}} \Big\{ \big|\widehat{G}(K)\big| \big|\widehat{G}(K) - \widehat{\Delta}(K)\big|^2 + 2\big|\widehat{G}(K) - \widehat{\Delta}(K)\big|^2 \big|\widehat{\Delta}(K)\big| \Big\}$$
$$+ 2\varepsilon^2 \sum_{Y, Z} \Delta_{O, Z} \{ G(X, Y) - \Delta_{X, Y} \} \Delta_{Y, Z} + \varepsilon^2 \sum_{Y, Z} \{ G(O, Z) - \Delta_{O, Z} \} \Delta_{X, Y} \Delta_{Y, Z}. \quad (3.17)$$

First we estimate the integral of (3.17). Taking the Fourier transform of the convolution equation (3.2), we have

$$\widehat{G}(K) = \frac{\varepsilon}{1 - \widehat{\varpi}_{\mathrm{G}}(K)} = \frac{1}{1 - \widehat{D}(k) + \widehat{\varpi}_{\mathrm{G}}(k, 0)\, \frac{1 - e^{i\omega\varepsilon}}{\varepsilon}}, \quad (3.18)$$

where $\widehat{\varpi}_{\mathrm{G}}(K) = \sum_X p_{\mathrm{G}}(O, X)\, e^{i K \cdot X}$. By using the trivial inequalities

$$0 \le 1 - \widehat{D}(k) \le 2, \qquad \Re\big[1 - e^{i\omega\varepsilon}\big] \ge 0, \qquad \big|1 - e^{i\omega\varepsilon}\big| \ge \frac{2\varepsilon}{\pi} |\omega|, \qquad \widehat{\varpi}_{\mathrm{G}}(k, 0) \ge 1 - 2\varepsilon \ge \frac{1}{3},$$

the integral of $\big|\widehat{G}(K)\big| \big|\widehat{G}(K) - \widehat{\Delta}(K)\big|^2$ in (3.17) is bounded by

$$\int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\pi)^{d+1}} \frac{\widehat{D}(k)^2}{\Big\{ \frac{1}{\sqrt{2}} \{ 1 - \widehat{D}(k) + \frac{2}{3\pi} |\omega| \} \Big\}^3} \le \int_{\Pi_d} \frac{d^d k}{(2\pi)^d} \frac{9\, \widehat{D}(k)^2}{\sqrt{2} \{ 1 - \widehat{D}(k) \}^2}, \quad (3.19)$$

and the integral of $\big|\widehat{G}(K) - \widehat{\Delta}(K)\big|^2 \big|\widehat{\Delta}(K)\big|$ in (3.17) is bounded by

$$\int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\pi)^{d+1}} \frac{\widehat{D}(k)^2}{\Big\{ \frac{1}{\sqrt{2}} \{ 1 - \widehat{D}(k) + \frac{2}{3\pi} |\omega| \} \Big\}^2} \le \int_{\Pi_d} \frac{d^d k}{(2\pi)^d} \frac{3\, \widehat{D}(k)^2}{1 - \widehat{D}(k)}. \quad (3.20)$$

It has been proved in Appendix A of [19] that the right hand sides of (3.19) and (3.20) are bounded by a d-independent multiple of $(d-4)^{-1}$ for the nearest-neighbor model, or by a L-independent multiple of $L^{-d} (\ln L)^{d/2}$ for the spread-out model. The quantity $\mathcal{S}_{\mathrm{G}}$ can be bounded by the right hand side of (3.20).

Next we estimate the second sum of (3.17)

$$\varepsilon^2 \sum_{Y, Z} \{ G(O, Z) - \Delta_{O, Z} \} \Delta_{X, Y} \Delta_{Y, Z}.$$

The first sum of (3.17) can be estimated in the same way. Using the Schwarz inequality, we have

$$\varepsilon^2 \sum_{Y, Z} \{ G(O, Z) - \Delta_{O, Z} \} \Delta_{X, Y} \Delta_{Y, Z}$$
$$\le \sqrt{\varepsilon \sum_{m=0}^{\infty} \Big(\varepsilon \sum_{n=0}^{m} \Delta_{X, X+n\Theta} \Delta_{X+n\Theta, X+m\Theta} \Big)^2} \sqrt{\varepsilon \sum_{m=0}^{\infty} \{ G(O, X+m\Theta) - \Delta_{O, X+m\Theta} \}^2}, \quad (3.21)$$

where $\Theta = (o,\, \varepsilon)$. The quantity in the former square root sign is equal to

$$\varepsilon^3 \sum_{m=0}^{\infty} (\, m+1\,)^2 \,(\, 1-\varepsilon\,)^{2\,m} = \frac{1+(\,1-\varepsilon\,)^2}{(\,2-\varepsilon\,)^3},$$

which is bounded by 1 for $\varepsilon < 1$. The sum in the latter square root sign of (3.21) must be estimated separately depending on whether $\sigma(X)$ is equal to o or not. If $\sigma(X) \neq o$, then we use the spatial symmetry of the two-point function to obtain

$$2\,d\,\varepsilon \sum_{m=0}^{\infty} \big\{\, G(O,\, X+m\,\Theta) - \Delta_{O,\,X+m\,\Theta} \,\big\}^2 \le \varepsilon \sum_{Z} \big\{\, G(O,\, Z) - \Delta_{O,\,Z} \,\big\}^2 = \mathcal{S}_{\mathrm{G}}. \tag{3.22}$$

If on the other hand $\sigma(X) = o$, we can not directly use the spatial symmetry of the two-point function as above. However if a random walker reaches $X + m\,\Theta = (o,\, \tau(X) + m\,\varepsilon)$ from O by taking at least one spatial step, then there exist $n < m + \tau(X)/\varepsilon$ and $y \in \mathbb{Z}^d$ with $J_{o,\,y} \neq 0$ such that a random walker taking only temporal steps to $(o,\, n\,\varepsilon)$ takes the first spatial step to $(y,\, (n+1)\,\varepsilon)$ and reaches $(o,\, \tau(X) + m\,\varepsilon)$. Therefore

$$\begin{aligned} &G(O,\, X+m\,\Theta) - \Delta_{O,\,X+m\,\Theta} \\ &\quad\le \sum_{n=0}^{m+\tau(X)/\varepsilon} (\,1-\varepsilon\,)^n \sum_{Y} \frac{\varepsilon\,\mathcal{J}_{O,\,Y}}{|\mathcal{J}|} \Big\{\, G\big(Y,\, X+(m-n)\,\Theta\big) - \Delta_{Y,\,X+(m-n)\,\Theta} \,\Big\}. \end{aligned}$$

Using this inequality and the Schwarz inequality, we have

$$\begin{aligned} &\varepsilon \sum_{m=0}^{\infty} \big\{\, G(O,\, X+m\,\Theta) - \Delta_{O,\,X+m\,\Theta} \,\big\}^2 \\ &\quad\le \sum_{n,\,n'} (\,1-\varepsilon\,)^{n+n'} \sum_{Y,\,Y'} \frac{\varepsilon\,\mathcal{J}_{O,\,Y}}{|\mathcal{J}|}\, \frac{\varepsilon\,\mathcal{J}_{O,\,Y'}}{|\mathcal{J}|} \Bigg\{\, \varepsilon \sum_{m} \Big\{\, G\big(Y,\, X+(m-n)\,\Theta\big) - \Delta_{Y,\,X+(m-n)\,\Theta} \,\Big\} \\ &\qquad\qquad \times \Big\{\, G\big(Y',\, X+(m-n')\,\Theta\big) - \Delta_{Y',\,X+(m-n')\,\Theta} \,\Big\} \Bigg\} \\ &\quad\le \Bigg\{\, \varepsilon \sum_{n} (\,1-\varepsilon\,)^n \sum_{Y} \frac{\mathcal{J}_{O,\,Y}}{|\mathcal{J}|} \sqrt{\varepsilon \sum_{m} \Big\{\, G\big(Y,\, X+(m-n)\,\Theta\big) - \Delta_{Y,\,X+(m-n)\,\Theta} \,\Big\}^2} \,\Bigg\}^2. \end{aligned} \tag{3.23}$$

Now we can use the spatial symmetry of the two-point function as in (3.22). The proof of bounding the Gaussian triangle diagram $\mathcal{T}_{\mathrm{G}}(X)$ is completed.

Finally we bound $\mathcal{W}_{\mathrm{G}}(X)$. Integrating by parts, we have

$$\mathcal{W}_{\mathrm{G}}(X) = \begin{cases} \displaystyle\sum_{j=1}^{d} \int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(\,2\,\pi\,)^{d+1}} \,\big|\, \partial_j \widehat{G}(K) \,\big|^2, & \text{if } X = O, \\ \displaystyle -\sum_{j=1}^{d} \int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(\,2\,\pi\,)^{d+1}} \,\big\{\, \partial_j^2 \widehat{G}(K) \,\big\}\, \widehat{G}(-K)\, e^{-i\,K\cdot X}, & \text{if } X \neq O. \end{cases}$$

We differentiate the right hand side of (3.18) to obtain

$$\partial_j \widehat{G}(K) = \widehat{G}(K)^2\, e^{i\,\omega\,\varepsilon}\, \partial_j \widehat{D}(k),$$
$$\partial_j^2 \widehat{G}(K) = 2\, \widehat{G}(K)^3 \left\{ e^{i\,\omega\,\varepsilon}\, \partial_j \widehat{D}(k) \right\}^2 + \widehat{G}(K)^2\, e^{i\,\omega\,\varepsilon}\, \partial_j^2 \widehat{D}(k). \tag{3.24}$$

Therefore $\mathcal{W}_{\mathrm{G}}(O)$ and the contribution from the term $\widehat{G}(K)^3 \left\{ e^{i\,\omega\,\varepsilon}\, \partial_j \widehat{D}(k) \right\}^2$ to $\mathcal{W}_{\mathrm{G}}(X)$ with $X \neq O$ are bounded by

$$\sum_{j=1}^{d} \int_{\Pi_d \times \frac{\Pi_1}{\varepsilon}} \frac{dK}{(2\,\pi)^{d+1}} \frac{\left\{ \partial_j \widehat{D}(k) \right\}^2}{\left\{ \frac{1}{\sqrt{2}} \left\{ 1 - \widehat{D}(k) + \frac{2}{3\,\pi}\, |\omega| \right\} \right\}^4} \leq \sum_{j=1}^{d} \int_{\Pi_d} \frac{d^d k}{(2\,\pi)^d} \frac{18 \left\{ \partial_j \widehat{D}(k) \right\}^2}{\left\{ 1 - \widehat{D}(k) \right\}^3}.$$

In Appendix B of [13] and Appendix A of [19], it has been proved that the right hand side is bounded by a d-independent multiple of $(d-4)^{-1}$ for the nearest-neighbor model, or by a L-independent multiple of $L^{2-d}\, (\ln L)^d$ for the spread-out model. The contribution from the second term of (3.24) is equal to the Fourier transform of

$$\varepsilon \sum_{Z} G(X,\, Z) \sum_{Y,\, O'} G(O,\, Y)\, \frac{\varepsilon\, \mathcal{J}_{O,\, O'}}{|\mathcal{J}|}\, \|\, \sigma(O')\, \|^2\, G(Y + O',\, Z).$$

This is bounded by

$$\begin{aligned}
&\varepsilon^2 \sum_{Y,\, Z,\, O'} \frac{\mathcal{J}_{O,\, O'}}{|\mathcal{J}|}\, \|\, \sigma(O')\, \|^2 \Big\{ \left\{ G(X,\, Z)\, G(O',\, Y + O')\, G(Y + O',\, Z) - \Delta_{X,\, Z}\, \Delta_{O',\, Y+O'}\, \Delta_{Y+O',\, Z} \right\} \\
&\qquad + \Delta_{X,\, Z}\, \Delta_{O',\, Y+O'}\, \Delta_{Y+O',\, Z} \Big\} \\
&\leq \sum_{O'} \frac{\mathcal{J}_{O,\, O'}}{|\mathcal{J}|}\, \|\, \sigma(O')\, \|^2 \left\{ \mathcal{T}_{\mathrm{G}}(X - O') + \delta_{\sigma(X),\, \sigma(O')} \right\} \leq |\mathcal{V}|\, \overline{\mathcal{T}}_G + \overline{\mathcal{V}},
\end{aligned} \tag{3.25}$$

where $\overline{\mathcal{T}}_G = \sup_X \mathcal{T}_{\mathrm{G}}(X)$ and

$$|\mathcal{V}| = \sum_{X} \frac{\mathcal{J}_{O,\, X}}{|\mathcal{J}|}\, \|\, \sigma(X)\, \|^2, \qquad \overline{\mathcal{V}} = \sup_X \frac{\mathcal{J}_{O,\, X}}{|\mathcal{J}|}\, \|\, \sigma(X)\, \|^2. \tag{3.26}$$

For the nearest-neighbor model, $|\mathcal{V}| = 1$ and $\overline{\mathcal{V}} = (2\, d)^{-1}$, and thus the right hand side of (3.25) is of order d^{-1}. For the spread-out model, $|\mathcal{V}|$ and $\overline{\mathcal{V}}$ are of order L^2 and L^{2-d} respectively, and thus the right hand side of (3.25) is of order $L^{(4-d)/2}\, (\ln L)^{d/4}$. The proof of bounding the Gaussian quantity $\mathcal{W}_{\mathrm{G}}(X)$ is now completed.

4 The lace expansion

We use the lace expansion to investigate an infrared behavior for the connectivity function. The lace expansion produces a convolution equation (4.1) for the connectivity function. The expansion can be derived through the inclusion-exclusion approach or the algebraic method analogous to that used in [20]. Both derivations lead to the same result.

First we make several definitions concerning connection events. X is said to be *doubly connected* to Y if $X = Y$ or there exist at least two non-zero distinct open paths from X to Y which have no bonds in common (but they may have common sites). Such two paths are said to be *disjoint*. We write '$X \rightrightarrows Y$' for the event that X is doubly connected to Y, and $g_0(X, Y)$ for the probability of this event. For a bond set $\mathbf{B}$, '$X \to Y$ *off* $\mathbf{B}$' means that X is connected to Y without using bonds in $\mathbf{B}$. If $\mathbf{B}$ consists of only one bond, such as (U, V), we simply write as '$X \to Y$ off (U, V)'. We define $\mathbf{C}_{(U, V)}(X)$ for the set of sites which are connected from X off (U, V). A bond (U, V) is said to be *pivotal* for the connection from X to Y if X is connected to U and V is connected to $Y \notin \mathbf{C}_{(U, V)}(X)$: if removing (U, V) would prevent the inhabitant at Y from being infected with the disease which has broken out at X. We define $\mathbf{B}_{\text{piv}}(X, Y)$ for the set of bonds which are pivotal for the connection from X to Y. We note that the event '(U, V) is pivotal for the connection from X to Y' depends only on the states of bonds other than (U, V). For a site set $\mathbf{C}$, we define $\mathfrak{E}_{(V, X\,;\,\mathbf{C})}$ for the event that there is at least one open path from V to X passing through $\mathbf{C}$, whereas there are no open paths from V to the bottom of the last bond of $\mathbf{B}_{\text{piv}}(V, X)$ passing through $\mathbf{C}$.

Proposition 4.1 (The lace expansion). *For $N \geq 0$,*

$$\varphi(O, X) = \Gamma_N(O, X) + \sum_{(U, V)} \Gamma_N(O, U)\, p(U, V)\, \varphi(V, X) + (-1)^{N+1} R_{N+1}(O, X), \tag{4.1}$$

where $\Gamma_N(O, X) = \sum_{n=0}^{N} (-1)^n g_n(O, X)$. The irreducible two-point function $g_n(O, X)$ is defined as

$$g_n(O, X) = \begin{cases} \mathbb{P}\left(O \rightrightarrows X \right), & \text{for } n = 0, \\ \displaystyle\sum_{(U_1, V_1)} p(U_1, V_1) \cdots \sum_{(U_n, V_n)} p(U_n, V_n)\, \mathbb{P}\left(O \rightrightarrows U_1, \bigcap_{i=1}^{n} \mathfrak{E}_{(V_i, U_{i+1}\,;\,\mathbf{C}_{(U_i, V_i)}(V_{i-1}))} \right), & \text{for } n \geq 1, \end{cases}$$

and the remainder $R_N(O, X)$ is defined as

$$R_N(O, X) = \begin{cases} \displaystyle\sum_{(U, V)} p(U, V)\, \mathbb{P}\left(O \rightrightarrows U,\, V \to X \in \mathbf{C}_{(U, V)}(O) \right), & \text{for } N = 1, \\ \displaystyle\sum_{(U_1, V_1)} p(U_1, V_1) \cdots \sum_{(U_N, V_N)} p(U_N, V_N) & \\ \quad \times \mathbb{P}\left(O \rightrightarrows U_1, \bigcap_{i=1}^{N-1} \mathfrak{E}_{(V_i, U_{i+1}\,;\,\mathbf{C}_{(U_i, V_i)}(V_{i-1}))}, V_N \to X \in \mathbf{C}_{(U_N, V_N)}(V_{N-1}) \right), & \text{for } N \geq 2, \end{cases}$$

provided that $V_0 = O$ and $U_{n+1} = X$.

Remark 4.1. We prove below the lace expansion through repeated applications of the inclusion-exclusion relation. The proof is similar to that of Proposition 2.3 of [14]. However by the construction of the models, we can utilize the Markov property, and thus no nested expectations as in [14] appear.

Proof. First we divide the event '$O \to X$' into two disjoint cases: either $\mathbf{B}_{\text{piv}}(O, X)$ is empty or not. If O is connected to X and $\mathbf{B}_{\text{piv}}(O, X)$ is empty, then O is doubly connected to X. Therefore

$$\varphi(O, X) = \mathbb{P}\left(O \rightrightarrows X \right) + \mathbb{P}\left(O \to X, \mathbf{B}_{\text{piv}}(O, X) \neq \varnothing \right)$$

$$= g_0(O, X) + \sum_{(U,V)} \mathbb{P}\big(O \to X,\, (U, V) \text{ is the first bond of } \mathbf{B}_{\rm piv}(O, X) \big)$$
$$= g_0(O, X) + \sum_{(U,V)} p(U, V)\, \mathbb{P}\big(O \rightrightarrows U,\, (U, V) \in \mathbf{B}_{\rm piv}(O, X) \big),$$

because the state of (U, V) is independent of the event that (U, V) is the first bond of $\mathbf{B}_{\rm piv}(O, X)$. By the definition of the pivotal bond and the inclusion-exclusion relation, we have

$$\begin{aligned}\mathbb{P}\big(O \rightrightarrows U,\, (U, V) \in \mathbf{B}_{\rm piv}(O, X) \big) &= \mathbb{P}\left(O \rightrightarrows U,\, V \to X \notin \mathbf{C}_{(U,V)}(O) \right)\\ &= \mathbb{P}\left(O \rightrightarrows U,\, V \to X \right) - \mathbb{P}\left(O \rightrightarrows U,\, V \to X \in \mathbf{C}_{(U,V)}(O) \right).\end{aligned}$$

By the construction of the models, we can use the Markov property. Therefore the first term of the right hand side equals $g_0(O, U)\, \varphi(V, X)$. We have thus obtained the result (4.1) for $N = 0$.

Next we expand $R_1(O, X)$. We divide the event that V is connected to $X \in \mathbf{C}_{(U,V)}(O)$ into two cases: either a bond of $\mathbf{B}_{\rm piv}(V, X)$ whose bottom is contained in $\mathbf{C}_{(U,V)}(O)$ does exist or not. If there are no such pivotal bonds, then the event $\mathfrak{E}_{(V, X\,;\, \mathbf{C}_{(U,V)}(O))}$ occurs. If on the other hand there is such a pivotal bond, then we distinguish the event by the first bond among them. Therefore

$$\begin{aligned}&\mathbb{P}\left(O \rightrightarrows U,\, V \to X \in \mathbf{C}_{(U,V)}(O) \right)\\
&\quad = \mathbb{P}\left(O \rightrightarrows U,\, \mathfrak{E}_{(V, X\,;\, \mathbf{C}_{(U,V)}(O))} \right)\\
&\qquad + \sum_{(U',V')} \mathbb{P}\Big(O \rightrightarrows X,\, V \to X,\, (U', V') \text{ is the first bond of } \mathbf{B}_{\rm piv}(V, X) \text{ with } U' \in \mathbf{C}_{(U,V)}(O) \Big)\\
&\quad = \mathbb{P}\left(O \rightrightarrows U,\, \mathfrak{E}_{(V, X\,;\, \mathbf{C}_{(U,V)}(O))} \right)\\
&\qquad + \sum_{(U',V')} p(U', V')\, \mathbb{P}\left(O \rightrightarrows U,\, \mathfrak{E}_{(V, U'\,;\, \mathbf{C}_{(U,V)}(O))},\, (U', V') \in \mathbf{B}_{\rm piv}(V, X) \right),\end{aligned} \tag{4.2}$$

because the state of (U', V') is independent of the event that (U', V') is the first bond of $\mathbf{B}_{\rm piv}(V, X)$ whose bottom is contained in $\mathbf{C}_{(U,V)}(O)$. As in the case of $N = 0$, by the definition of the pivotal bond, the inclusion-exclusion relation and the Markov property, we have

$$\begin{aligned}&\mathbb{P}\left(O \rightrightarrows U,\, \mathfrak{E}_{(V, U'\,;\, \mathbf{C}_{(U,V)}(O))},\, (U', V') \in \mathbf{B}_{\rm piv}(V, X) \right)\\
&\quad = \mathbb{P}\left(O \rightrightarrows U,\, \mathfrak{E}_{(V, U'\,;\, \mathbf{C}_{(U,V)}(O))},\, V' \to X \notin \mathbf{C}_{(U',V')}(V) \right)\\
&\quad = \mathbb{P}\left(O \rightrightarrows U,\, \mathfrak{E}_{(V, U'\,;\, \mathbf{C}_{(U,V)}(O))} \right) \varphi(V', X)\\
&\qquad - \mathbb{P}\left(O \rightrightarrows U,\, \mathfrak{E}_{(V, U'\,;\, \mathbf{C}_{(U,V)}(O))},\, V' \to X \in \mathbf{C}_{(U',V')}(V) \right).\end{aligned}$$

Finally we substitute the above identity to (4.2) to obtain

$$R_1(O, X) = g_1(O, X) + \sum_{(U',V')} g_1(O, U')\, p(U', V')\, \varphi(V', X) - R_2(O, X).$$

We have thus proved the result (4.1) for $N = 1$.

The remaining expansion for $N \geq 2$ can be proved inductively. □

5 Estimates

In this section, we estimate $\widehat{\Gamma}_N(K)$, $\widehat{R}_{N+1}(K)$ and $\widehat{\Gamma}_N(O) - \widehat{\Gamma}_N(K)$, which are used in Section 3.3 to prove the infrared bound on the connectivity function.

In order to estimate $\widehat{\Gamma}_N(K)$, $\widehat{R}_{N+1}(K)$ and $\widehat{\Gamma}_N(O) - \widehat{\Gamma}_N(K)$, we use a correlation inequality, namely *the Berg-Kesten inequality.* We begin with introducing this inequality. A configuration $\mathcal{C}$ is a collection of states of bonds, and we define

$$\mathcal{C}_{(X,\,Y)} = \mathbb{I}_{\{(X,\,Y)\text{ is open on }\mathcal{C}\}}.$$

We can introduce a natural partial order among configurations, denoted by $\mathcal{C} \le \mathcal{C}'$, which is defined to hold $\mathcal{C}_{(X,\,Y)} \le \mathcal{C}'_{(X,\,Y)}$ for any $(X,\,Y)$. An event E is said to be *increasing* if $\mathbb{I}_E$ is an increasing function with respect to partially ordered configurations. An example of an increasing event is $\{\, O \to X \text{ off } (U,\,V) \,\}$ for some bond $(U,\,V)$. For two events $\mathfrak{E}_1$ and $\mathfrak{E}_2$, the event $\mathfrak{E}_1 \circ \mathfrak{E}_2$ means that the events $\mathfrak{E}_1$ and $\mathfrak{E}_2$ *occur disjointly.* For example,

$$\{\, O \to X \,\} \circ \{\, O' \to X' \,\} = \bigcup_{\mathbf{B}} \{\, \{\, O \to X \text{ off } \mathbf{B} \,\} \cap \{\, O' \to X' \text{ off } \mathbf{B}^{\mathrm{c}} \,\} \,\},$$

where $\mathbf{B}^{\mathrm{c}}$ is a complement of a bond set $\mathbf{B}$.

Proposition 5.1 (The Berg-Kesten inequality).

$$\mathbb{P}\,(\,\mathfrak{E}_1 \circ \mathfrak{E}_2\,) \le \mathbb{P}\,(\,\mathfrak{E}_1\,)\,\mathbb{P}\,(\,\mathfrak{E}_2\,)\,,$$

holds for any two increasing events $\mathfrak{E}_1$ and $\mathfrak{E}_2$.

The proof of this inequality can be found in [12].

Next we explain a potential difficulty involved in taking $\varepsilon \downarrow 0$, and how to overcome it. We have obtained the lace expansion for the connectivity function, and obtained (3.5). $\widehat{\Gamma}_N(K)$ is an alternating sum of $\{\,\widehat{g}_n(K)\,\}_{n=0}^N$. Suppose that we directly use the Berg-Kesten inequality to estimate the irreducible two-point functions as in the way of [14] and [20]. Then we have, for example,

$$\left|\,\widehat{g}_0(K)\,\right| \le \varepsilon \sum_X \varphi(O,\,X)^2 = \varepsilon + \varepsilon \sum_X \psi(O,\,X)^2, \tag{5.1}$$

where $\psi(O,\,X) = \mathbb{P}\,(\,O \rightsquigarrow X\,)$ and '$O \rightsquigarrow X$' means that there exists a non-zero open path from O to X. The latter term of the right hand side is $\mathcal{O}(1)$; what is worse, $\left|\,\widehat{g}_n(K)\,\right| \le \mathcal{O}(\varepsilon^{-2\,n})$ for $n \ge 1$. Therefore we can not obtain a meaningful upper bound on $\left|\,\widehat{\Gamma}_N(K)\,\right|$, and hence the infrared behavior for the connectivity function can not be seen. This is due to the fact that the number of summations is larger than that of factors of ε. This is the difficulty involved in naive estimates in $\varepsilon \downarrow 0$.

The key observation to overcome the above difficulty is that we can derive a factor of ε from a point where at least two disjoint open paths leave or enter. This idea is an extension of the idea used in [4] to extract correct factors of ε. Let us consider the case of $g_0(O,\,X) = \mathbb{P}\,(\,O \rightrightarrows X\,)$ with $X \ne O$ for example. This is the probability that there exist at least two non-zero disjoint open paths from O to X. Since at most one temporal bond grows out of every site, at least one of two non-zero disjoint open paths goes out of O with a *spatial* bond, and at least one of those comes into X with another *spatial* bond. Thanks to this close observation, we use the Berg-Kesten inequality to obtain

$$\left|\,\widehat{g}_0(K)\,\right| \le \varepsilon + \lambda^2\,\varepsilon^3 \sum_{X,\,O',\,X'} \mathcal{J}_{O,\,O'}\,\mathcal{J}_{X',\,X}\,\{\,\psi(O,\,X)\,\varphi(O',\,X') + \varphi(O',\,X)\,\mathbb{P}\,(\,O \rightarrowtail X'\,)\,\}. \tag{5.2}$$

The latter term of the right hand side is $\mathcal{O}(\varepsilon^2)$ in contrast with (5.1). We have thus achieved deriving extra factors of ε, which enable us to control a limit $\varepsilon \downarrow 0$.

At the end of the introduction of this section, we define several connection notations.

$$
\begin{aligned}
&\{\, O \rightarrowtail X \,\} = \{\, (O,\, \Theta) \text{ is open} \,\} \circ \{\, \Theta \to X \,\}, \\
&\{\, O \Rightarrow\!\!\!\rightarrow X \,\} = \bigcup_{O' \,:\, \sigma(O') \neq o} \{\, \{\, (O,\, O') \text{ is open} \,\} \circ \{\, O' \to X \,\} \,\}, \\
&\{\, O \twoheadrightarrow X \,\} = \{\, O \to X - \Theta \,\} \circ \{\, (X - \Theta,\, X) \text{ is open} \,\}, \\
&\{\, O \rightarrow\!\!\!\Rightarrow X \,\} = \bigcup_{X' \,:\, \sigma(X') \neq \sigma(X)} \{\, \{\, O \to X' \,\} \circ \{\, (X',\, X) \text{ is open} \,\} \,\}, \\
&\{\, O \rightarrowtail\!\!\!\!\twoheadrightarrow X \,\} = \{\, (O,\, \Theta) \text{ is open} \,\} \circ \{\, \Theta \to X - \Theta \,\} \circ \{\, (X - \Theta,\, X) \text{ is open} \,\}, \\
&\{\, O \Rightarrow\!\!\!\rightarrow\!\!\!\Rightarrow X \,\} = \bigcup_{\substack{O' \,:\, \sigma(O') \neq o \\ X' \,:\, \sigma(X') \neq \sigma(X)}} \{\, \{\, (O,\, O') \text{ is open} \,\} \circ \{\, O' \to X' \,\} \circ \{\, (X',\, X) \text{ is open} \,\} \,\}.
\end{aligned}
$$

We organize the rest of this section as follows: $\widehat{\Gamma}_N(K)$ is estimated in Section 5.1. Since $\widehat{\Gamma}_N(K)$ is an alternating sum of $\{\, \widehat{g}_n(K) \,\}_{n=0}^{N}$, Section 5.1 is mainly devoted to estimating $\widehat{g}_n(K)$. These estimates are directly used to bound $\big|\, \widehat{R}_N(K) \,\big|$ in Section 5.2. Finally in Section 5.3, we estimate $\widehat{\Gamma}_N(O) - \widehat{\Gamma}_N(K)$.

5.1 Estimate of $\widehat{\Gamma}_N(K)$ for $N \geq 0$

5.1.1 Estimate of $\widehat{g}_0(K)$

In order to estimate $\widehat{g}_0(K)$, we introduce *the bubble diagram* defined as

$$
\mathcal{B}(X) = \varepsilon \sum_{Y} \{\, \varphi(O,\, Y)\, \varphi(X,\, Y) - \Delta_{O,\, Y}\, \Delta_{X,\, Y} \,\},
$$

and $\overline{\mathcal{B}} \equiv \sup_X \mathcal{B}(X)$. The bubble diagram can be estimated with the triangle diagram $\mathcal{T}(X)$.

Lemma 5.2. *The inequality $\mathcal{B}(X) \leq 2\, \mathcal{T}(X)$ holds for any X.*

Proof. We can prove this inequality as in the analogous way of proving (2.20) in [4]. Since the inequality $\varphi(O,\, Z) \geq \varphi(O,\, Y)\, \Delta_{Y,\, Z}$ holds for any Y, Z with $0 \leq \tau(Y) \leq \tau(Z)$, we obtain

$$
\begin{aligned}
\mathcal{T}(X) &= \varepsilon^2 \sum_{Y,\, Z} \Big\{\, \varphi(O,\, Z)\, \varphi(X,\, Y)\, \{\, \varphi(Y,\, Z) - \Delta_{Y,\, Z} \,\} + \varphi(O,\, Z)\, \{\, \varphi(X,\, Y) - \Delta_{X,\, Y} \,\}\, \Delta_{Y,\, Z} \\
&\qquad + \{\, \varphi(O,\, Z) - \Delta_{O,\, Z} \,\}\, \Delta_{X,\, Y}\, \Delta_{Y,\, Z} \Big\} \\
&\geq \varepsilon^2 \sum_{Y,\, Z} \Big\{\, \varphi(O,\, Y)\, \{\, \varphi(X,\, Y) - \Delta_{X,\, Y} \,\}\, \big(\, \Delta_{Y,\, Z} \,\big)^2 + \{\, \varphi(O,\, Y) - \Delta_{O,\, Y} \,\}\, \Delta_{X,\, Y}\, \big(\, \Delta_{Y,\, Z} \,\big)^2 \Big\} \\
&= \varepsilon \sum_{Y} \{\, \varphi(O,\, Y)\, \varphi(X,\, Y) - \Delta_{O,\, Y}\, \Delta_{X,\, Y} \,\}\, \varepsilon \sum_{n \geq 0} (\, 1 - \varepsilon \,)^{2\, n} \;=\; \frac{1}{2 - \varepsilon}\, \mathcal{B}(X).
\end{aligned}
$$

The proof is completed. $\square$

Using this lemma and following the idea explained in the introduction of this section, we can estimate $\widehat{g}_0(K)$.

Lemma 5.3.

$$\left|\,\widehat{g}_0(K)\,\right| \le \varepsilon \sum_X g_0(O,\,X) \le \varepsilon + 2\,\lambda\,|\mathcal{J}|\,\xi\,\varepsilon^2, \tag{5.3}$$

where $\xi = \lambda\,|\mathcal{J}|\,\big(\,2\,\overline{\mathcal{T}} + \overline{\mathcal{J}}/|\mathcal{J}|\,\big)$ *and* $\overline{\mathcal{J}} = \sup_X \mathcal{J}_{O,\,X}$.

Proof. We begin with proving the inequality

$$g_0(O,\,X) \le \delta_{O,\,X} + \sum_{O',\,X'} (\,\lambda\,\varepsilon\,)^2\,\mathcal{J}_{O,\,O'}\,\mathcal{J}_{X',\,X}\,\big\{\,\psi(O,\,X)\,\varphi(O',\,X') + \varphi(O',\,X)\,\mathbb{P}\,(\,O \rightarrowtail X'\,)\,\big\}. \tag{5.4}$$

Since a site is regarded as being doubly connected to itself, we divide the event '$O \rightrightarrows X$' into two cases: either X is equal to O or not. When X ($\neq O$) is the arrival point of two non-zero disjoint open paths, at least one of those enters X with a spatial bond. The contribution from this case is bounded by

$$\mathbb{P}\big(\,\{\,O \rightsquigarrow X\,\}\circ\{\,O \twoheadrightarrow\!\!\Rightarrow X\,\}\,\big) \le \sum_{X'\,:\,\sigma(X')\neq\sigma(X)} \mathbb{P}\big(\,\{\,O \rightsquigarrow X\,\}\circ\{\,O \to X'\,\}\circ\{\,(X',\,X)\text{ is open}\,\}\,\big). \tag{5.5}$$

By the construction of the models, there is no contribution from the case of $O = X'$ to the right hand side. If two non-zero disjoint open paths emanate from the origin, at least one of those leaves the origin with a spatial bond. Therefore we use the Berg-Kesten inequality to bound the right hand side of (5.5) by

$$\begin{aligned}
&\sum_{X'} \lambda\,\varepsilon\,\mathcal{J}_{X',\,X}\,\mathbb{P}\big(\,\{\,O \rightsquigarrow X\,\}\circ\{\,O \rightsquigarrow X'\,\}\,\big)\\
&\quad\le \sum_{X'} \lambda\,\varepsilon\,\mathcal{J}_{X',\,X}\,\Big\{\,\mathbb{P}\big(\,\{\,O \Rightarrow\!\!\rightarrowtail X\,\}\circ\{\,O \Rightarrow\!\!\rightarrowtail X'\,\}\,\big) + \mathbb{P}\big(\,\{\,O \Rightarrow\!\!\rightarrowtail X\,\}\circ\{\,O \rightarrowtail X'\,\}\,\big)\,\Big\}\\
&\quad\le \sum_{O',\,X'} (\,\lambda\,\varepsilon\,)^2\,\mathcal{J}_{O,\,O'}\,\mathcal{J}_{X',\,X}\,\big\{\,\psi(O,\,X)\,\varphi(O',\,X') + \varphi(O',\,X)\,\mathbb{P}\,(\,O \rightarrowtail X'\,)\,\big\},
\end{aligned}$$

and hence the inequality (5.4) is obtained.

We then use (5.4) to estimate $\widehat{g}_0(K)$. The contribution from the second term of (5.4) is bounded by

$$\begin{aligned}
&\lambda^2\,\varepsilon^3 \sum_{X,\,O',\,X'} \mathcal{J}_{O,\,O'}\,\mathcal{J}_{X',\,X}\,\Big\{\,\big\{\,\psi(O,\,X)\,\varphi(O',\,X') - \Delta'_{O,\,X}\,\Delta_{O',\,X'}\,\big\} + \Delta'_{O,\,X}\,\Delta_{O',\,X'}\\
&\qquad + \big\{\,\varphi(O',\,X)\,\mathbb{P}\,(\,O \rightarrowtail X'\,) - \Delta_{O',\,X}\,\Delta'_{O,\,X'}\,\big\} + \Delta_{O',\,X}\,\Delta'_{O,\,X'}\,\Big\}\\
&\quad\le (\,\lambda\,\varepsilon\,)^2 \sum_{O',\,O''} \mathcal{J}_{O,\,O'}\,\mathcal{J}_{O,\,O''}\,\big\{\,\mathcal{B}(O'+O'') + \delta_{o,\,\sigma(O'+O'')} + \mathcal{B}(O'-O'') + \delta_{o,\,\sigma(O'-O'')}\,\big\}\\
&\quad\le 2\,(\,\lambda\,|\mathcal{J}|\,)^2\,\big(\,\overline{\mathcal{B}} + \overline{\mathcal{J}}/|\mathcal{J}|\,\big)\,\varepsilon^2,
\end{aligned} \tag{5.6}$$

where $\Delta'_{O,\,X} = \Delta_{O,\,X} - \delta_{O,\,X}$. We used the identity

$$\begin{aligned}
&\varphi(O,\,X)\,\varphi(O',\,X') - \Delta_{O,\,X}\,\Delta_{O',\,X'}\\
&\quad = \psi(O,\,X)\,\varphi(O',\,X') - \Delta'_{O,\,X}\,\Delta_{O',\,X'} + \delta_{O,\,X}\,\big\{\,\varphi(O',\,X') - \Delta_{O',\,X'}\,\big\}.
\end{aligned}$$

Finally we use Lemma 5.2 to obtain the result. □

Under the inequalities in (3.4) with $a = 4$, $\xi \le \mathcal{O}(\mu)$ and the right hand side of (5.3) is bounded by $\varepsilon + \mathcal{O}(\mu)\,\varepsilon^2$. This proves (3.7) for $n = 0$.

5.1.2 Estimate of $\widehat{g}_n(K)$ for $n \ge 1$

In this section, we estimate $\widehat{g}_n(K)$ for $n \ge 1$. We begin with constructing an increasing event in which the event in the definition of $g_n(O, X)$ is contained. Before applying the Berg-Kesten inequality to this increasing event, we must pay attention to points in diagrams each of which at least two disjoint open paths go out or come in, as in the estimate of $\widehat{g}_0(K)$. Thanks to this close observation and the Berg-Kesten inequality, we can obtain a bound (5.8) on $\big|\,\widehat{g}_n(K)\,\big|$, which enables us to take a meaningful limit in $\varepsilon \downarrow 0$.

First we construct the increasing event stated above.

Lemma 5.4. *For $n \ge 1$,*

$$\begin{aligned}
&\Big\{ O \rightrightarrows U_1,\ \bigcap_{i=1}^{n} \mathfrak{E}_{\big(V_i,\, U_{i+1}\,;\, \mathbf{C}_{(U_i,\, V_i)}(V_{i-1})\big)} \Big\} \\
&\quad \subset \bigcup_{W_1,\,\cdots,\,W_n} \Big\{ \mathfrak{L}_{(O,\, W_1,\, U_1)} \circ \Big\{ \mathfrak{M}^{(U_1,\, V_1)}_{(W_1,\, V_1\,;\, W_2,\, U_2)} \cup \mathfrak{N}^{(U_1,\, V_1)}_{(W_1,\, V_1\,;\, W_2,\, U_2)} \Big\} \circ \cdots \\
&\qquad\qquad \cdots \circ \Big\{ \mathfrak{M}^{(U_{n-1},\, V_{n-1})}_{(W_{n-1},\, V_{n-1}\,;\, W_n,\, U_n)} \cup \mathfrak{N}^{(U_{n-1},\, V_{n-1})}_{(W_{n-1},\, V_{n-1}\,;\, W_n,\, U_n)} \Big\} \circ \mathfrak{R}^{(U_n,\, V_n)}_{(W_n,\, V_n,\, X)} \Big\},
\end{aligned} \tag{5.7}$$

provided that $V_0 = O$ and $U_{n+1} = X$, where

$$\begin{aligned}
\mathfrak{L}_{(O,\, W,\, U)} &= \{\, O \to W \,\} \circ \{\, O \to U \,\} \circ \{\, W \to U \,\}, \\
\mathfrak{R}^{(U,\, V)}_{(W,\, V,\, X)} &= \{\, W \rightsquigarrow X \ \textit{off}\ (U,\, V) \,\} \circ \{\, V \to X \,\}, \\
\mathfrak{M}^{(U,\, V)}_{(W,\, V\,;\, W',\, U')} &= \{\, W \rightsquigarrow U' \ \textit{off}\ (U,\, V) \,\} \circ \{\, V \to W' \,\} \circ \{\, W' \to U' \,\}, \\
\mathfrak{N}^{(U,\, V)}_{(W,\, V\,;\, W',\, U')} &= \{\, W \rightsquigarrow W' \ \textit{off}\ (U,\, V) \,\} \circ \{\, V \to U' \,\} \circ \{\, W' \to U' \,\}.
\end{aligned}$$

The events $\mathfrak{R}^{(U,\, V)}_{(W,\, V,\, X)}$, $\mathfrak{M}^{(U,\, V)}_{(W,\, V\,;\, W',\, U')}$ and $\mathfrak{N}^{(U,\, V)}_{(W,\, V\,;\, W',\, U')}$ are defined only in the case of $\tau(W) \le \tau(U)$ and $\tau(V) \le \tau(X)$ or $\tau(V) \le \tau(W') \le \tau(U')$.

Proof. First we consider the case of $n = 1$. If the event in the definition of $g_1(O, X)$ occurs, then there exist sites W, Y and Z such that the following events occur disjointly:

(a) there are at least two disjoint open paths from O to U one of which passes through W,

(b) V is connected to Y,

(c) there are at least two disjoint open paths from Y to X one of which passes through Z,

(d) W is connected to Z off (U, V).

By the construction of the models, a union of two connections from V to Y and from Y to X is equal to a connection from V to X passing through Y. Moreover two connections from Y to X passing through

Z and from W to Z off (U, V) lead to a connection from W to X passing through Z off (U, V), which is realized disjointly with the other connections. Therefore

$$\Big\{ O \rightrightarrows U,\ \mathfrak{E}_{(V, X\,;\,\mathbf{C}_{(U,V)}(O))} \Big\} \subset \bigcup_{W} \Big\{ \mathfrak{L}_{(O, W, U)} \circ \mathfrak{R}^{(U,V)}_{(W, V, X)} \Big\}.$$

This is the result (5.7) for $n = 1$.

The case of $n \geq 2$ are more involved. Let us consider the case of $n = 2$. An argument analogous to the above leads to the relation

$$\Big\{ O \rightrightarrows U,\ \mathfrak{E}_{(V, U'\,;\,\mathbf{C}_{(U,V)}(O))},\ \mathfrak{E}_{(V', X\,;\,\mathbf{C}_{(U',V')}(V))} \Big\} \subset \bigcup_{W'} \Big\{ \mathfrak{D}^{(U,V)}_{(O, W', U')} \circ \mathfrak{R}^{(U',V')}_{(W', V', X)} \Big\},$$

where

$$\mathfrak{D}^{(U,V)}_{(O, W', U')} = \Big\{ O \rightrightarrows U,\ \mathfrak{E}_{(V, U'\,;\,\mathbf{C}_{(U,V)}(O))},\ V \to W',\ W' \to U' \Big\}.$$

If the event $\mathfrak{D}^{(U,V)}_{(O, W', U')}$ occurs and the *bottom* of the last bond of $\mathbf{B}_{\mathrm{piv}}(V, U') \neq \varnothing$ is connected *from* W', then there exist sites W, Y and Z such that the following events occur disjointly:

(a) there are at least two disjoint open paths from O to U one of which passes through W,

($\dot{\mathrm{b}}$) there is at least one open path from V to Y passing through W',

($\dot{\mathrm{c}}$) there are at least two disjoint open paths from Y to U' one of which passes through Z,

(d) W is connected to Z off (U, V).

Through an argument analogous to the case of $n = 1$, we can obtain

$$\mathfrak{D}^{(U,V)}_{(O, W', U')} \cap \left\{ \begin{array}{l} W' \text{ is connected to the bottom} \\ \text{of the last bond of } \mathbf{B}_{\mathrm{piv}}(V, U') \neq \varnothing \end{array} \right\} \subset \bigcup_{W} \Big\{ \mathfrak{L}_{(O, W, U)} \circ \mathfrak{M}^{(U,V)}_{(W, V\,;\,W', U')} \Big\}.$$

If on the other hand the event $\mathfrak{D}^{(U,V)}_{(O, W', U')}$ occurs and the *top* of the last bond of $\mathbf{B}_{\mathrm{piv}}(V, U')$ is connected *to* W' (or $\mathbf{B}_{\mathrm{piv}}(V, U') = \varnothing$), then there exist sites W, Y and Z such that the following events occur disjointly:

(a) there are at least two disjoint open paths from O to U one of which passes through W,

(b) V is connected to Y,

($\ddot{\mathrm{c}}$) there are at least two disjoint open paths from Y to U' such that

- ($\ddot{\mathrm{c}}$-1) one of which passes through Z and the other passes through W', or
- ($\ddot{\mathrm{c}}$-2) one of which passes through Z and W' in order, or
- ($\ddot{\mathrm{c}}$-3) one of which passes through W' and Z in order,

(d) W is connected to Z off (U, V).

By an argument analogous to the case of $n=1$, it is not difficult to analyze the cases of (c̈-1) and (c̈-2). Let us consider the case of (c̈-3). By the construction of the models, we can unite two connections from Y to W' and from W' to X off $(U',\,V')$ (which is contained in $\mathfrak{R}^{(U',\,V')}_{(W',\,V',\,X)}$) into a connection from Y to X passing through W' off $(U',\,V')$, which is realized disjointly with the other connections. We then interchange the indices Y and W'. It thus follows that

$$\mathfrak{D}^{(U,\,V)}_{(O,\,W',\,U')} \cap \left\{ \{\, \mathbf{B}_{\mathrm{piv}}(V,\,U') = \varnothing \,\} \cup \left\{ \begin{array}{l} W' \text{ is connected from the top} \\ \text{of the last bond of } \mathbf{B}_{\mathrm{piv}}(V,\,U') \neq \varnothing \end{array} \right\} \right\}$$
$$\subset \bigcup_{W} \left\{ \mathfrak{L}_{(O,\,W,\,U)} \circ \left\{ \mathfrak{M}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} \cup \mathfrak{N}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} \right\} \right\}.$$

This proves the case of $n=2$.

The case of $n \geq 3$ can be proved similarly by induction. $\square$

Next we use the above lemma to estimate $\widehat{g}_n(K)$. However as we mentioned before, we must observe more closely points in diagrams each of which at least two disjoint open paths go out or come in.

Lemma 5.5. *For $n \geq 1$,*

$$\big|\, \widehat{g}_n(K) \,\big| \leq \varepsilon^2\, \overline{\mathcal{L}} \cdot \overline{\mathcal{H}}^{\,n-1} \cdot \overline{\mathcal{R}}, \tag{5.8}$$

where $\overline{\mathcal{L}} = \big(\, \overline{\mathcal{L}}_i \,\big)_{i=1}^{3}$ is a row vector, $\overline{\mathcal{R}} = \big(\, \overline{\mathcal{R}}_i \,\big)_{i=1}^{3}$ is a column vector and $\overline{\mathcal{H}} = \big(\, \overline{\mathcal{H}}_{ij} \,\big)_{ij}$ is a 3×3 matrix, defined as

$$\overline{\mathcal{L}} = \big(1 + 2\,\lambda\,|\mathcal{J}|\,\xi\,\varepsilon \quad 4\,\lambda\,|\mathcal{J}|\,\xi\,(\,1 + \lambda\,|\mathcal{J}|\,) \quad 2\,\lambda\,|\mathcal{J}|\,\xi\,(\,2 + \varepsilon\,) \big)\,,$$
$$\overline{\mathcal{R}} = 2\,\xi \begin{pmatrix} 2\,\lambda\,|\mathcal{J}| \\ (\,1 + \lambda\,|\mathcal{J}|\,\varepsilon\,) \\ \lambda\,|\mathcal{J}|\,(\,1 + \lambda\,|\mathcal{J}|\,\varepsilon\,) \end{pmatrix},$$
$$\overline{\mathcal{H}} = 2\,\xi \begin{pmatrix} 4\,\lambda\,|\mathcal{J}|\,\varepsilon & 2\,\lambda\,|\mathcal{J}|\,(\,1 + 2\,\lambda\,|\mathcal{J}|\,) & 4\,\lambda\,|\mathcal{J}| \\ 2\,(\,1 + \lambda\,|\mathcal{J}|\,\varepsilon\,)\,\varepsilon & (\,1 + 2\,\lambda\,|\mathcal{J}|\,)\,\widehat{\varpi}(O) & 2\,\widehat{\varpi}(O) \\ 2\,\lambda\,|\mathcal{J}|\,(\,1 + \lambda\,|\mathcal{J}|\,\varepsilon\,)\,\varepsilon & \lambda\,|\mathcal{J}|\,(\,1 + 2\,\lambda\,|\mathcal{J}|\,)\,\widehat{\varpi}(O) & 2\,\lambda\,|\mathcal{J}|\,\widehat{\varpi}(O) \end{pmatrix}.$$

Proof. As in the proof of Lemma 5.3, we first prove the inequality

$$\begin{aligned} g_n(O,\,X) \leq \sum_{\substack{W_1,\,\cdots,\,W_n,\\ U_1,\,\cdots,\,U_n}} &\mathcal{L}_{(O,\,W_1,\,U_1)} \cdot \big\{\, \mathcal{M}_{(W_1,\,U_1\,;\,W_2,\,U_2)} + \mathcal{N}_{(W_1,\,U_1\,;\,W_2,\,U_2)} \,\big\} \cdots \\ &\cdots \big\{\, \mathcal{M}_{(W_{n-1},\,U_{n-1}\,;\,W_n,\,U_n)} + \mathcal{N}_{(W_{n-1},\,U_{n-1}\,;\,W_n,\,U_n)} \,\big\} \cdot \mathcal{R}_{(W_n,\,U_n,\,X)}, \end{aligned} \tag{5.9}$$

where $\mathcal{L}_{(O,\,W,\,U)} = \big(\, \mathcal{L}^{(i)}_{(O,\,W,\,U)} \,\big)_{i=1}^{3}$ is a row vector, $\mathcal{R}_{(W,\,U,\,X)} = \big(\, \mathcal{R}^{(i)}_{(W,\,U,\,X)} \,\big)_{i=1}^{3}$ is a column vector, and $\mathcal{M}_{(W,\,U\,;\,W',\,U')} = \big(\, \mathcal{M}^{(ij)}_{(W,\,U\,;\,W',\,U')} \,\big)_{ij}$ and $\mathcal{N}_{(W,\,U\,;\,W',\,U')} = \big(\, \mathcal{N}^{(ij)}_{(W,\,U\,;\,W',\,U')} \,\big)_{ij}$ are 3×3 matrices. Every components of these quantities are given below.

We begin with the case of $n=1$. By Lemma 5.4, the Markov property and the Berg-Kesten inequality, we can bound $g_1(O,\,X)$ by

$$\sum_{W,\,(U,\,V)} p(U,\,V)\, \mathbb{P}\left(\mathfrak{L}_{(O,\,W,\,U)} \circ \mathfrak{R}^{(U,\,V)}_{(W,\,V,\,X)} \right) \,\{\, \delta_{W,\,U} + (\,1 - \delta_{W,\,U}\,) \,\}$$

$$
\begin{aligned}
&\le \sum_{W,\,(U,\,V)} p(U,\,V) \Big\{ g_0(O,\,U)\, \delta_{W,\,U}\, \mathbb{P}\left(\mathfrak{R}^{(U,\,V)}_{(U,\,V,\,X)} \right) + \mathbb{P}\left({}^{\sigma}\mathfrak{L}_{(O,\,W,\,U)} \circ \mathfrak{R}^{(U,\,V)}_{(W,\,V,\,X)} \right) \\
&\qquad + \sum_{W'} \mathbb{P}\left({}^{\tau}\mathfrak{L}_{(O,\,W,\,U)} \circ \{\, (W,\,W') \text{ is open} \,\} \circ \mathfrak{R}^{(U,\,V)}_{(W',\,V,\,X)} \right) \Big\}, \\
&\le \sum_{W,\,U} \Big\{ g_0(O,\,U)\, \delta_{W,\,U} \sum_{V} p(U,\,V)\, \mathbb{P}\left(\mathfrak{R}^{(U,\,V)}_{(U,\,V,\,X)} \right) + \mathbb{P}\left({}^{\sigma}\mathfrak{L}_{(O,\,W,\,U)} \right) \sum_{V} p(U,\,V)\, \mathbb{P}\left(\mathfrak{R}^{(U,\,V)}_{(W,\,V,\,X)} \right) \\
&\qquad + \mathbb{P}\left({}^{\tau}\mathfrak{L}_{(O,\,W,\,U)} \right) \sum_{W',\,V} \lambda\,\varepsilon\, \mathcal{J}_{W,\,W'}\, p(U,\,V)\, \mathbb{P}\left(\mathfrak{R}^{(U,\,V)}_{(W',\,V,\,X)} \right) \Big\},
\end{aligned}
\tag{5.10}
$$

where

$$
\begin{aligned}
{}^{\sigma}\mathfrak{L}_{(O,\,W,\,U)} &= \{\, O \rightarrow W \,\} \circ \{\, O \rightsquigarrow U \,\} \circ \{\, W \mathrel{\Rightarrow\!\!\rightarrow} U \,\}\,, \\
{}^{\tau}\mathfrak{L}_{(O,\,W,\,U)} &= \{\, O \rightarrow W \,\} \circ \{\, O \rightsquigarrow U \,\} \circ \{\, W \rightarrowtail U \,\}\,.
\end{aligned}
$$

In the rest of this paper, we use the Berg-Kesten inequality without further mention.

First we estimate the quantity $\sum_V p(U,\,V)\, \mathbb{P}\left(\mathfrak{R}^{(U,\,V)}_{(W,\,V,\,X)} \right)$. If $W = U$, then there is no contribution from the case of $V = X$, because the bond $(U,\,V)$ is unique. Therefore we obtain a bound

$$
\begin{aligned}
&\sum_{V} p(U,\,V)\, \mathbb{P}\big(\{\, U \rightsquigarrow X \text{ off } (U,\,V) \,\} \circ \{\, V \rightsquigarrow X \,\} \big) \,\{\, \mathbb{I}_{\{\sigma(V)=\sigma(U)\}} + \mathbb{I}_{\{\sigma(V)\neq\sigma(U)\}} \,\} \\
&\quad \le (\, 1-\varepsilon \,)\, \mathbb{P}\big(\{\, U \mathrel{\Rightarrow\!\!\rightarrow} X \,\} \circ \{\, U+\Theta \rightsquigarrow X \,\} \big) + \sum_{V} \lambda\,\varepsilon\, \mathcal{J}_{U,\,V}\, \mathbb{P}\big(\{\, U \rightsquigarrow X \,\} \circ \{\, V \rightsquigarrow X \,\} \big) \\
&\quad \le (\, 1-\varepsilon \,) \sum_{U'} \lambda\,\varepsilon\, \mathcal{J}_{U,\,U'}\, \mathbb{P}\big(\{\, U' \rightarrow X \,\} \circ \{\, U+\Theta \rightsquigarrow X \,\} \big) + \sum_{V} \lambda\,\varepsilon\, \mathcal{J}_{U,\,V}\, \mathbb{P}\big(\{\, U \rightsquigarrow X \,\} \circ \{\, V \rightsquigarrow X \,\} \big).
\end{aligned}
\tag{5.11}
$$

It is impossible that U' equals X in the first term, because of the inequality $\tau(U') = \tau(U) + \varepsilon < \tau(X)$. Therefore the right hand side of (5.11) is bounded by

$$
\begin{aligned}
&(\, 1-\varepsilon \,) \sum_{U'} \lambda\,\varepsilon\, \mathcal{J}_{U,\,U'} \Big\{ \mathbb{P}\big(\{\, U' \rightsquigarrow X \,\} \circ \{\, U+\Theta \mathrel{\rightarrow\!\!\Rightarrow} X \,\} \big) + \mathbb{P}\big(\{\, U' \mathrel{\rightarrow\!\!\Rightarrow} X \,\} \circ \{\, U+\Theta \twoheadrightarrow X \,\} \big) \Big\} \\
&\qquad + \sum_{V} \lambda\,\varepsilon\, \mathcal{J}_{U,\,V} \Big\{ \mathbb{P}\big(\{\, U \rightsquigarrow X \,\} \circ \{\, V \mathrel{\rightarrow\!\!\Rightarrow} X \,\} \big) + \mathbb{P}\big(\{\, U \mathrel{\rightarrow\!\!\Rightarrow} X \,\} \circ \{\, V \twoheadrightarrow X \,\} \big) \Big\} \\
&\quad \le \sum_{V,\,X'} (\, \lambda\,\varepsilon \,)^2\, \mathcal{J}_{U,\,V}\, \mathcal{J}_{X',\,X} \,\{\, \psi(V,\,X)\, \mathbb{P}\,(\, U \rightarrowtail X' \,) + \varphi(V,\,X')\, \mathbb{P}\,(\, U \mathrel{\succ\!\!\!\twoheadrightarrow} X \,) \\
&\qquad + \psi(U,\,X)\, \varphi(V,\,X') + \varphi(U,\,X')\, \mathbb{P}\,(\, V \twoheadrightarrow X \,) \,\},
\end{aligned}
\tag{5.12}
$$

where we used the identities

$$
(\, 1-\varepsilon \,)\, \varphi(U+\Theta,\,X') = \mathbb{P}\,(\, U \rightarrowtail X' \,)\,, \qquad (\, 1-\varepsilon \,)\, \mathbb{P}\,(\, U+\Theta \twoheadrightarrow X \,) = \mathbb{P}\,(\, U \mathrel{\succ\!\!\!\twoheadrightarrow} X \,)\,. \tag{5.13}
$$

We define $\mathcal{R}^{(1)}_{(W,\,U,\,X)}$ for the right hand side of (5.12) multiplied by $\delta_{W,\,U}$. If on the other hand $W \neq U$, then we have

$$
\sum_{V} p(U,\,V)\, \mathbb{P}\left(\mathfrak{R}^{(U,\,V)}_{(W,\,V,\,X)} \right) \{\, \delta_{V,\,X} + (\, 1-\delta_{V,\,X} \,) \,\}
$$

$$
\begin{aligned}
&\le \{\, \lambda\,\varepsilon\,\mathcal{J}_{U,\,X} + (\,1-\varepsilon\,)\,\delta_{U,\,X-\Theta}\,\}\,\mathbb{P}\big(\{\,W \rightsquigarrow X \text{ off } (U,\,X)\,\}\big) \\
&\quad + \sum_{V} \{\,(\,1-\varepsilon\,)\,\delta_{U,\,V-\Theta} + \lambda\,\varepsilon\,\mathcal{J}_{U,\,V}\,\}\,\Big\{\,\mathbb{P}\big(\{\,W \rightsquigarrow X\,\}\circ\{\,V \twoheadrightarrow\!\!\Rightarrow X\,\}\big) \\
&\qquad + \mathbb{P}\big(\{\,W \twoheadrightarrow\!\!\Rightarrow X\,\}\circ\{\,V \twoheadrightarrow X\,\}\big)\,\Big\} \\
&\le \lambda\,\varepsilon\,\mathcal{J}_{U,\,X}\,\psi(W,\,X) + (\,1-\varepsilon\,)\,\delta_{U,\,X-\Theta}\sum_{X'}\lambda\,\varepsilon\,\mathcal{J}_{X',\,X}\,\varphi(W,\,X') \\
&\quad + \sum_{X'}(\,1-\varepsilon\,)\,\lambda\,\varepsilon\,\mathcal{J}_{X',\,X}\,\{\,\psi(W,\,X)\,\varphi(U+\Theta,\,X') + \varphi(W,\,X')\,\mathbb{P}\,(\,U+\Theta \twoheadrightarrow X\,)\,\} \\
&\quad + \sum_{V,\,X'}(\,\lambda\,\varepsilon\,)^2\,\mathcal{J}_{U,\,V}\,\mathcal{J}_{X',\,X}\,\{\,\psi(W,\,X)\,\varphi(V,\,X') + \varphi(W,\,X')\,\mathbb{P}\,(\,V \twoheadrightarrow X\,)\,\}. \qquad (5.14)
\end{aligned}
$$

By the identities (5.13), the third term of the right hand side is equal to

$$
\sum_{X'}\lambda\,\varepsilon\,\mathcal{J}_{X',\,X}\,\{\,\psi(W,\,X)\,\mathbb{P}\,(\,U \rightarrowtail X'\,) + \varphi(W,\,X')\,\mathbb{P}\,(\,U \rightarrowtail\!\!\!\twoheadrightarrow X\,)\,\}.
$$

Summing the above expression and the first and the second terms of the right hand side of (5.14) by using the inequalities

$$
\delta_{U,\,X'} + \mathbb{P}\,(\,U \rightarrowtail X'\,) \le \varphi(U,\,X'), \qquad (\,1-\varepsilon\,)\,\delta_{U,\,X-\Theta} + \mathbb{P}\,(\,U \rightarrowtail\!\!\!\twoheadrightarrow X\,) \le \mathbb{P}\,(\,U \twoheadrightarrow X\,), \qquad (5.15)
$$

we obtain a bound on (5.14) as

$$
\begin{aligned}
\varepsilon^{-1}\,\mathcal{R}^{(2)}_{(W,\,U,\,X)} \equiv &\sum_{X'}\lambda\,\varepsilon\,\mathcal{J}_{X',\,X}\,\{\,\psi(W,\,X)\,\varphi(U,\,X') + \varphi(W,\,X')\,\mathbb{P}\,(\,U \twoheadrightarrow X\,)\,\} \\
&+ \sum_{V,\,X'}(\,\lambda\,\varepsilon\,)^2\,\mathcal{J}_{U,\,V}\,\mathcal{J}_{X',\,X}\,\{\,\psi(W,\,X)\,\varphi(V,\,X') + \varphi(W,\,X')\,\mathbb{P}\,(\,V \twoheadrightarrow X\,)\,\}. \qquad (5.16)
\end{aligned}
$$

We define $\mathcal{R}^{(3)}_{(W,\,U,\,X)}$ for $\sum_{W'}\lambda\,\mathcal{J}_{W,\,W'}\,\mathcal{R}^{(2)}_{(W',\,U,\,X)}$.

To prove (5.9) for $n=1$, it is sufficient to estimate $\mathbb{P}\,\big({}^{\sigma}\mathfrak{L}_{(O,\,W,\,U)}\big)$ and $\mathbb{P}\,\big({}^{\tau}\mathfrak{L}_{(O,\,W,\,U)}\big)$ in (5.10). First $\mathbb{P}\,\big({}^{\sigma}\mathfrak{L}_{(O,\,W,\,U)}\big)$ is bounded by

$$
\begin{aligned}
&\sum_{W'}\mathbb{P}\big(\{\,O \rightsquigarrow U\,\}\circ\{\,O \to W\,\}\circ\{\,(W,\,W') \text{ is open}\,\}\circ\{\,W' \to U\,\}\big) \\
&\quad \le \sum_{W'}\lambda\,\varepsilon\,\mathcal{J}_{W,\,W'}\,\mathbb{P}\big(\{\,O \rightsquigarrow U\,\}\circ\{\,O \to W\,\}\circ\{\,W' \to U\,\}\big)\,\big(\,1-\delta_{O,\,W}\,\delta_{W',\,U}\,\big).
\end{aligned}
$$

The estimates of the case of $O=W$ and $W' \ne U$ and the case of $O \ne W$ and $W' = U$ are obtained in the same way as in proving (5.4). In the case of $O \ne W$ and $W' \ne U$, we have

$$
\begin{aligned}
&\mathbb{P}\big(\{\,O \rightsquigarrow W\,\}\circ\{\,O \rightsquigarrow U\,\}\circ\{\,W' \rightsquigarrow U\,\}\big) \\
&\quad \le \mathbb{P}\big(\{\,O \rightsquigarrow W\,\}\circ\{\,O \rightsquigarrow U\,\}\circ\{\,W' \twoheadrightarrow\!\!\Rightarrow U\,\}\big) + \mathbb{P}\big(\{\,O \rightsquigarrow W\,\}\circ\{\,O \twoheadrightarrow\!\!\Rightarrow U\,\}\circ\{\,W' \twoheadrightarrow U\,\}\big) \\
&\quad \le \sum_{U'}\lambda\,\varepsilon\,\mathcal{J}_{U',\,U}\,\Big\{\,\mathbb{P}\big(\{\,O \rightsquigarrow W\,\}\circ\{\,O \rightsquigarrow U\,\}\big)\,\varphi(W',\,U') \\
&\qquad + \mathbb{P}\big(\{\,O \rightsquigarrow W\,\}\circ\{\,O \to U'\,\}\big)\,\mathbb{P}\,(\,W' \twoheadrightarrow U\,)\,\Big\}.
\end{aligned}
$$

It is impossible that U' equals O, because of the inequality $\tau(U') \geq \tau(W') \geq \tau(W) + \varepsilon > 0$. Hence the right hand side is bounded by

$$\sum_{U'} \lambda \varepsilon \mathcal{J}_{U',U} \Big\{ \Big\{ \mathbb{P}\big(\{ O \Rightarrow\!\!\!\rightarrow W \} \circ \{ O \rightsquigarrow U \}\big) + \mathbb{P}\big(\{ O \rightarrowtail W \} \circ \{ O \Rightarrow\!\!\!\rightarrow U \}\big) \Big\} \varphi(W', U')$$
$$+ \Big\{ \mathbb{P}\big(\{ O \Rightarrow\!\!\!\rightarrow W \} \circ \{ O \rightsquigarrow U' \}\big) + \mathbb{P}\big(\{ O \rightarrowtail W \} \circ \{ O \Rightarrow\!\!\!\rightarrow U' \}\big) \Big\} \mathbb{P}(W' \twoheadrightarrow U) \Big\}$$
$$\leq \sum_{O', U'} (\lambda \varepsilon)^2 \mathcal{J}_{O,O'} \mathcal{J}_{U',U} \Big\{ \{ \psi(O, U) \varphi(O', W) + \varphi(O', U) \mathbb{P}(O \rightarrowtail W) \} \varphi(W', U')$$
$$+ \{ \psi(O, U') \varphi(O', W) + \varphi(O', U') \mathbb{P}(O \rightarrowtail W) \} \mathbb{P}(W' \twoheadrightarrow U) \Big\}.$$

Therefore $\varepsilon^{-1} \mathbb{P}\left({}^{\sigma}\mathfrak{L}_{(O, W, U)} \right)$ is bounded by

$$\mathcal{L}^{(2)}_{(O,W,U)} \equiv \sum_{O', U'} \lambda^2 \varepsilon \mathcal{J}_{O,O'} \mathcal{J}_{U',U} \Big\{ \{ \psi(O, U) \varphi(O', U') + \varphi(O, U') \mathbb{P}(O' \twoheadrightarrow U) \} \delta_{O,W}$$
$$+ \{ \psi(O, U) \varphi(O', U') + \varphi(O', U) \mathbb{P}(O \rightarrowtail U') \} \delta_{W,U'}$$
$$+ \sum_{W'} \lambda \varepsilon \mathcal{J}_{W,W'} \Big\{ \{ \psi(O, U) \varphi(O', W) + \varphi(O', U) \mathbb{P}(O \rightarrowtail W) \} \varphi(W', U')$$
$$+ \{ \psi(O, U') \varphi(O', W) + \varphi(O', U') \mathbb{P}(O \rightarrowtail W) \} \mathbb{P}(W' \twoheadrightarrow U) \Big\} \Big\}. \tag{5.17}$$

Second $\mathbb{P}\left({}^{\tau}\mathfrak{L}_{(O, W, U)} \right)$ can be estimated similarly and is bounded by

$$\mathcal{L}^{(3)}_{(O,W,U)} \equiv \sum_{O', U'} (\lambda \varepsilon)^2 \mathcal{J}_{O,O'} \mathcal{J}_{U',U} \Big\{ \{ \psi(O', U) \mathbb{P}(O \rightarrowtail U') + \varphi(O', U') \mathbb{P}(O \rightarrowtail\!\!\!\!\twoheadrightarrow U) \} \delta_{O,W}$$
$$+ \{ \psi(O, U) \varphi(O', W) + \varphi(O', U) \mathbb{P}(O \rightarrowtail W) \} \mathbb{P}(W \rightarrowtail U')$$
$$+ \{ \psi(O, U') \varphi(O', W) + \varphi(O', U') \mathbb{P}(O \rightarrowtail W) \} \mathbb{P}(W \twoheadrightarrow U) \Big\}. \tag{5.18}$$

We define $\mathcal{L}^{(1)}_{(O,W,U)}$ for the right hand side of (5.4) with $X = U$ multiplied by $\delta_{W,U}$. The proof of (5.9) for $n = 1$ is now completed.

We can prove (5.9) for $n \geq 2$ along the same line. Since the case of $n \geq 3$ can be proved by induction, we consider the case of $n = 2$ only. By Lemma 5.4, the Markov property and (5.9) for $n = 1$, $g_2(O, X)$ can be bounded by

$$\sum_{\substack{W, (U,V), \\ W', (U',V')}} p(U, V) p(U', V') \Big\{ \mathbb{P}\left(\mathfrak{L}_{(O,W,U)} \circ \mathfrak{M}^{(U,V)}_{(W,V;W',U')} \circ \mathfrak{R}^{(U',V')}_{(W',V',X)} \right)$$
$$+ \mathbb{P}\left(\mathfrak{L}_{(O,W,U)} \circ \mathfrak{N}^{(U,V)}_{(W,V;W',U')} \circ \mathfrak{R}^{(U',V')}_{(W',V',X)} \right) \Big\} \{ \delta_{W',U'} + (1 - \delta_{W',U'}) \}$$
$$\leq \sum_{W', U'} \Big\{ 2 g_1(O, U') \delta_{W',U'} \mathcal{R}^{(1)}_{(W',U',X)} + \Big\{ \mathcal{P}^{\sigma\mathfrak{M}}_{(O,W',U')} + \mathcal{P}^{\sigma\mathfrak{N}}_{(O,W',U')} \Big\} \mathcal{R}^{(2)}_{(W',U',X)}$$

$$+ \left\{ \mathcal{P}^{\tau\mathfrak{M}}_{(O,\,W',\,U')} + \mathcal{P}^{\tau\mathfrak{N}}_{(O,\,W',\,U')} \right\} \mathcal{R}^{(3)}_{(W',\,U',\,X)} \Bigg\}, \tag{5.19}$$

where

$$\mathcal{P}^{\sigma\mathfrak{M}}_{(O,\,W',\,U')} = \varepsilon^{-1} \sum_{W,\,(U,\,V)} p(U,\,V)\,\mathbb{P}\left(\mathfrak{L}_{(O,\,W,\,U)} \circ {}^{\sigma}\mathfrak{M}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} \right),$$

$$\mathcal{P}^{\sigma\mathfrak{N}}_{(O,\,W',\,U')} = \varepsilon^{-1} \sum_{W,\,(U,\,V)} p(U,\,V)\,\mathbb{P}\left(\mathfrak{L}_{(O,\,W,\,U)} \circ {}^{\sigma}\mathfrak{N}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} \right),$$

$$\mathcal{P}^{\tau\mathfrak{M}}_{(O,\,W',\,U')} = \sum_{W,\,(U,\,V)} p(U,\,V)\,\mathbb{P}\left(\mathfrak{L}_{(O,\,W,\,U)} \circ {}^{\tau}\mathfrak{M}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} \right),$$

$$\mathcal{P}^{\tau\mathfrak{N}}_{(O,\,W',\,U')} = \sum_{W,\,(U,\,V)} p(U,\,V)\,\mathbb{P}\left(\mathfrak{L}_{(O,\,W,\,U)} \circ {}^{\tau}\mathfrak{N}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} \right),$$

and where

$${}^{\sigma}\mathfrak{M}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} = \{\, W \rightsquigarrow U' \text{ off } (U,\,V) \,\} \circ \{\, V \to W' \,\} \circ \{\, W' \Rrightarrow\!\!\to U' \,\},$$

$${}^{\sigma}\mathfrak{N}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} = \{\, W \rightsquigarrow W' \text{ off } (U,\,V) \,\} \circ \{\, V \rightsquigarrow U' \,\} \circ \{\, W' \Rrightarrow\!\!\to U' \,\},$$

$${}^{\tau}\mathfrak{M}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} = \{\, W \rightsquigarrow U' \text{ off } (U,\,V) \,\} \circ \{\, V \to W' \,\} \circ \{\, W' \rightarrowtail U' \,\},$$

$${}^{\tau}\mathfrak{N}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} = \{\, W \rightsquigarrow W' \text{ off } (U,\,V) \,\} \circ \{\, V \rightsquigarrow U' \,\} \circ \{\, W' \rightarrowtail U' \,\}.$$

These events are defined only in the case of $\tau(W) \le \tau(U)$ and $\tau(V) \le \tau(W') \le \tau(U')$, because of the definitions of $\mathfrak{M}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')}$ and $\mathfrak{N}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')}$.

By (5.9) for $n = 1$ and (5.19), it is appropriate to define for $i = 1,\,2,\,3$,

$$\mathcal{M}^{(i1)}_{(W,\,U\,;\,W',\,U')} = \mathcal{N}^{(i1)}_{(W,\,U\,;\,W',\,U')} = \mathcal{R}^{(i)}_{(W,\,U,\,U')}\,\delta_{W',\,U'}.$$

As in the same way of deriving (5.10), $\mathcal{P}^{\sigma\mathfrak{M}}_{(O,\,W',\,U')}$ is bounded by

$$\sum_{W,\,U} \Bigg\{ \mathcal{L}^{(1)}_{(O,\,W,\,U)}\,\varepsilon^{-1}\,\delta_{W,\,U} \sum_{V} p(U,\,V)\,\mathbb{P}\left({}^{\sigma}\mathfrak{M}^{(U,\,V)}_{(U,\,V\,;\,W',\,U')} \right) + \mathcal{L}^{(2)}_{(O,\,W,\,U)} \sum_{V} p(U,\,V)\,\mathbb{P}\left({}^{\sigma}\mathfrak{M}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} \right)$$
$$+ \mathcal{L}^{(3)}_{(O,\,W,\,U)} \sum_{W'',\,V} \lambda\,\mathcal{J}_{W,\,W''}\,p(U,\,V)\,\mathbb{P}\left({}^{\sigma}\mathfrak{M}^{(U,\,V)}_{(W'',\,V\,;\,W',\,U')} \right) \Bigg\},$$

and we estimate the term $\sum_V p(U,\,V)\,\mathbb{P}\left({}^{\sigma}\mathfrak{M}^{(U,\,V)}_{(W,\,V\,;\,W',\,U')} \right)$. We follow the way of getting bounds (5.12) and (5.16). If $W = U$,

$$\sum_{V,\,W''} (\,\lambda\,\varepsilon\,)^2\,\mathcal{J}_{U,\,V}\,\mathcal{J}_{W',\,W''} \Bigg\{ \left\{\, \psi(U,\,U')\,\varphi(V,\,W') + \varphi(V,\,U')\,\mathbb{P}\,(\,U \rightarrowtail W'\,) \,\right\} \delta_{W'',\,U'}$$
$$+ \sum_{U''} \lambda\,\varepsilon\,\mathcal{J}_{U'',\,U'} \Big\{ \left\{\, \psi(U,\,U')\,\varphi(V,\,W') + \mathbb{P}\,(\,U \rightarrowtail W'\,)\,\psi(V,\,U') \,\right\} \varphi(W'',\,U'')$$
$$+ \left\{\, \varphi(U,\,U'')\,\varphi(V,\,W') + \mathbb{P}\,(\,U \rightarrowtail W'\,)\,\varphi(V,\,U'') \,\right\} \mathbb{P}\,(\,W'' \twoheadrightarrow U'\,) \Big\} \Bigg\}. \tag{5.20}$$

If on the other hand $W \neq U$,

$$\sum_{V,\,W''} p(U,\,V)\,\lambda\,\varepsilon\,\mathcal{J}_{W',\,W''} \Big\{ \psi(W,\,U')\,\varphi(V,\,W')\,\delta_{W'',\,U'} \\ + \sum_{U''} \lambda\,\varepsilon\,\mathcal{J}_{U'',\,U'} \left\{ \psi(W,\,U')\,\varphi(V,\,W')\,\varphi(W'',\,U'') + \varphi(W,\,U'')\,\varphi(V,\,W')\,\mathbb{P}\,(\,W'' \twoheadrightarrow U'\,)\,\right\} \Big\}. \tag{5.21}$$

We define $\mathcal{M}^{(12)}_{(W,\,U\,;\,W',\,U')}$ for the expression (5.20) multiplied by $\varepsilon^{-1}\,\delta_{W,\,U}$, and $\mathcal{M}^{(22)}_{(W,\,U\,;\,W',\,U')}$ for the expression (5.21). Moreover we define

$$\mathcal{M}^{(32)}_{(W,\,U\,;\,W',\,U')} = \sum_{W''} \lambda\,\mathcal{J}_{W,\,W''}\,\mathcal{M}^{(22)}_{(W'',\,U\,;\,W',\,U')}. \tag{5.22}$$

By (5.13), (5.15) and an argument analogous to the above, we can estimate $\mathcal{P}^{\tau\mathfrak{M}}_{(O,\,W',\,U')}$, $\mathcal{P}^{\sigma\mathfrak{N}}_{(O,\,W',\,U')}$ and $\mathcal{P}^{\tau\mathfrak{N}}_{(O,\,W',\,U')}$. We define $\mathcal{M}^{(i3)}_{(W,\,U\,;\,W',\,U')}$ for $i = 1,\,2$ as

$$\mathcal{M}^{(13)}_{(W,\,U\,;\,W',\,U')} \\ = \delta_{W,\,U} \sum_{V,\,U''} (\,\lambda\,\varepsilon\,)^2\,\mathcal{J}_{U,\,V}\,\mathcal{J}_{U'',\,U'} \Big\{ \left\{ \psi(U,\,U')\,\varphi(V,\,W') + \mathbb{P}\,(\,U \rightarrowtail W'\,)\;\psi(V,\,U')\,\right\} \mathbb{P}\,(\,W' \rightarrowtail U''\,) \\ + \left\{ \varphi(U,\,U'')\,\varphi(V,\,W') + \mathbb{P}\,(\,U \rightarrowtail W'\,)\;\varphi(V,\,U'')\,\right\} \mathbb{P}\,(\,W' \twoheadrightarrow U'\,) \Big\},$$

$$\mathcal{M}^{(23)}_{(W,\,U\,;\,W',\,U')} \\ = \sum_{V,\,U''} p(U,\,V)\,\lambda\,\varepsilon^2\,\mathcal{J}_{U'',\,U'}\,\varphi(V,\,W') \left\{ \psi(W,\,U')\,\mathbb{P}\,(\,W' \rightarrowtail U''\,) + \varphi(W,\,U'')\,\mathbb{P}\,(\,W' \twoheadrightarrow U'\,)\,\right\},$$

and $\mathcal{N}^{(ij)}_{(W,\,U\,;\,W',\,U')}$ for $i = 1,\,2$ and $j = 2,\,3$ as

$$\mathcal{N}^{(12)}_{(W,\,U\,;\,W',\,U')} \\ = \delta_{W,\,U} \sum_{V,\,W''} \lambda^2\,\varepsilon\,\mathcal{J}_{U,\,V}\,\mathcal{J}_{W',\,W''} \Big\{ \left\{ \psi(U,\,W')\,\psi(V,\,U') + \varphi(V,\,W')\,\mathbb{P}\,(\,U \rightarrowtail U'\,)\,\right\} \delta_{W'',\,U'} \\ + \sum_{U''} \lambda\,\varepsilon\,\mathcal{J}_{U'',\,U'} \Big\{ \left\{ \psi(U,\,W')\,\psi(V,\,U') + \varphi(V,\,W')\,\mathbb{P}\,(\,U \rightarrowtail U'\,)\,\right\} \varphi(W'',\,U'') \\ + \left\{ \psi(U,\,W')\,\varphi(V,\,U'') + \varphi(V,\,W')\,\mathbb{P}\,(\,U \rightarrowtail U''\,)\,\right\} \mathbb{P}\,(\,W'' \twoheadrightarrow U'\,) \Big\} \Big\},$$

$$\mathcal{N}^{(22)}_{(W,\,U\,;\,W',\,U')} \\ = \sum_{V,\,W''} p(U,\,V)\,\lambda\,\varepsilon\,\mathcal{J}_{W',\,W''}\,\psi(W,\,W') \Big\{ \psi(V,\,U')\,\delta_{W'',\,U'} \\ + \sum_{U''} \lambda\,\varepsilon\,\mathcal{J}_{U'',\,U'} \left\{ \psi(V,\,U')\,\varphi(W'',\,U'') + \varphi(V,\,U'')\,\mathbb{P}\,(\,W'' \twoheadrightarrow U'\,)\,\right\} \Big\},$$

$$\mathcal{N}^{(13)}_{(W,U\,;\,W',U')}$$
$$= \delta_{W,U} \sum_{V,U''} (\lambda\varepsilon)^2 \mathcal{J}_{U,V}\, \mathcal{J}_{U'',U'} \Big\{ \big\{ \psi(U,W')\,\psi(V,U') + \varphi(V,W')\,\mathbb{P}(U \rightarrowtail U') \big\}\,\mathbb{P}(W' \rightarrowtail U'')$$
$$+ \big\{ \psi(U,W')\,\varphi(V,U'') + \varphi(V,W')\,\mathbb{P}(U \rightarrowtail U'') \big\}\,\mathbb{P}(W' \twoheadrightarrow U') \Big\},$$
$$\mathcal{N}^{(23)}_{(W,U\,;\,W',U')}$$
$$= \sum_{V,U''} p(U,V)\,\lambda\,\varepsilon^2\, \mathcal{J}_{U'',U'}\, \psi(W,W') \big\{ \psi(V,U')\,\mathbb{P}(W' \rightarrowtail U'') + \varphi(V,U'')\,\mathbb{P}(W' \twoheadrightarrow U') \big\}.$$

As in (5.22), we define $\mathcal{M}^{(33)}_{(W,U\,;\,W',U')}$ and $\mathcal{N}^{(3j)}_{(W,U\,;\,W',U')}$ for $j = 2, 3$ as

$$\mathcal{M}^{(33)}_{(W,U\,;\,W',U')} = \sum_{W''} \lambda\, \mathcal{J}_{W,W''}\, \mathcal{M}^{(23)}_{(W'',U\,;\,W',U')}, \qquad \mathcal{N}^{(3j)}_{(W,U\,;\,W',U')} = \sum_{W''} \lambda\, \mathcal{J}_{W,W''}\, \mathcal{N}^{(2j)}_{(W'',U\,;\,W',U')}.$$

We thus obtain (5.9) for $n = 2$.

Now we can estimate $\widehat{g}_n(K)$. Thanks to Lemma 5.2, $\overline{\mathcal{L}} = \big(\overline{\mathcal{L}}_i\big)_{i=1}^3$, $\overline{\mathcal{H}} = \big(\overline{\mathcal{H}}_{ij}\big)_{ij}$ and $\overline{\mathcal{R}} = \big(\overline{\mathcal{R}}_i\big)_{i=1}^3$ satisfy the inequalities

$$\overline{\mathcal{L}}_i \ge \sum_{Y,Z} \mathcal{L}^{(i)}_{(O,Y,Z)}, \qquad \varepsilon\,\overline{\mathcal{R}}_i \ge \sup_X \sum_Z \mathcal{R}^{(i)}_{(O,X,Z)}, \qquad \overline{\mathcal{H}}_{ij} \ge \sup_X \sum_{Y,Z} \Big\{ \mathcal{M}^{(ij)}_{(O,X\,;\,Y,Z)} + \mathcal{N}^{(ij)}_{(O,X\,;\,Y,Z)} \Big\}.$$

We use (5.9) and translation invariance to obtain a bound on $\big|\,\widehat{g}_n(K)\,\big|$ as

$$\begin{aligned}
\varepsilon \sum_X g_n(O,X) &\le \varepsilon^2 \sum_{\substack{W_1,\cdots,W_n,\\ U_1,\cdots,U_n}} \mathcal{L}_{(O,W_1,U_1)} \cdot \big\{ \mathcal{M}_{(W_1,U_1\,;\,W_2,U_2)} + \mathcal{N}_{(W_1,U_1\,;\,W_2,U_2)} \big\} \cdots \\
&\qquad \cdots \big\{ \mathcal{M}_{(W_{n-1},U_{n-1}\,;\,W_n,U_n)} + \mathcal{N}_{(W_{n-1},U_{n-1}\,;\,W_n,U_n)} \big\} \cdot \overline{\mathcal{R}} \\
&\le \varepsilon^2 \sum_{\substack{W_1,\cdots,W_{n-1},\\ U_1,\cdots,U_{n-1}}} \mathcal{L}_{(O,W_1,U_1)} \cdot \big\{ \mathcal{M}_{(W_1,U_1\,;\,W_2,U_2)} + \mathcal{N}_{(W_1,U_1\,;\,W_2,U_2)} \big\} \cdots \\
&\qquad \cdots \big\{ \mathcal{M}_{(W_{n-2},U_{n-2}\,;\,W_{n-1},U_{n-1})} + \mathcal{N}_{(W_{n-2},U_{n-2}\,;\,W_{n-1},U_{n-1})} \big\} \cdot \overline{\mathcal{H}} \cdot \overline{\mathcal{R}} \\
&\;\;\vdots \\
&\le \varepsilon^2 \sum_{W_1,U_1} \mathcal{L}_{(O,W_1,U_1)} \cdot \overline{\mathcal{H}}^{\,n-1} \cdot \overline{\mathcal{R}} \;\le \varepsilon^2\, \overline{\mathcal{L}} \cdot \overline{\mathcal{H}}^{\,n-1} \cdot \overline{\mathcal{R}}.
\end{aligned}$$

This completes the proof of the lemma. □

Since $\xi \le \mathcal{O}(\mu)$ under the inequalities in (3.4) with $a = 4$, the right hand side of (5.8) is bounded by $\mathcal{O}(\mu)^n\,\varepsilon^2$. This proves (3.7) for $n \ge 1$. Together with the inequality (5.3), $\sum_n \big|\,\widehat{g}_n(K)\,\big|$ is bounded by $\varepsilon + \mathcal{O}(\mu)\,\varepsilon^2$ for sufficiently large d or L.

5.2 Estimate of $\widehat{R}_N(K)$ for $N \geq 1$

By the Markov property, we obtain

$$\big|\,\widehat{R}_N(K)\,\big| \leq \sum_{(U,\,V)} g_{N-1}(O,\,U)\,p(U,\,V)\,\varepsilon \sum_X \varphi(V,\,X) \leq \varepsilon^{-1}\,\widehat{g}_{N-1}(O)\,\widehat{\varpi}(O)\,\chi(\lambda).$$

This is the inequality (3.8) used in Section 3.3. We have already seen that $\widehat{g}_{N-1}(O)$ is bounded as in (5.3) and (5.8). Therefore when $\lambda < \lambda_c$, i.e., $\chi(\lambda) < \infty$, $\widehat{R}_N(K)$ converges to 0 as $N \uparrow \infty$ for sufficiently high dimensional nearest-neighbor model or for sufficiently spread-out model in $d > 4$.

5.3 Estimate of $\widehat{\Gamma}_N(O) - \widehat{\Gamma}_N(K)$ for $N \geq 0$

By the spatial symmetry of $g_n(O,\,X)$ and the inequalities

$$\big|\,1 - e^{i\,\omega\,\tau(X)}\,\big| \leq \big|\,\omega\,\tau(X)\,\big|, \qquad\qquad \big|\,1 - \cos k\cdot\sigma(X)\,\big| \leq \frac{1}{2}\,\big\{\,k\cdot\sigma(X)\,\big\}^2,$$

we obtain a bound on $\big|\,\widehat{\Gamma}_N(O) - \widehat{\Gamma}_N(K)\,\big|$ as

$$\sum_{n=0}^{N}\bigg\{\,\varepsilon\sum_X g_n(O,\,X)\,\big|\,1 - e^{i\,\omega\,\tau(X)}\,\big| + \varepsilon\sum_X g_n(O,\,X)\,\big|\,1-\cos k\cdot\sigma(X)\,\big|\bigg\} \leq \sum_{n=0}^{N}\bigg(\,|\omega|\,\mathcal{G}_n^{\tau} + \frac{\|k\|^2}{2\,d}\,\mathcal{G}_n^{\sigma}\,\bigg), \tag{5.23}$$

where

$$\mathcal{G}_n^{\tau} = \varepsilon\sum_X g_n(O,\,X)\,\tau(X), \qquad\qquad \mathcal{G}_n^{\sigma} = \varepsilon\sum_X g_n(O,\,X)\,\|\,\sigma(X)\,\|^2.$$

This proves the inequality (3.12).

Now we proceed to estimate $\mathcal{G}_n^{\sigma}$ and $\mathcal{G}_n^{\tau}$, that is to prove (3.13). In order to estimate $\mathcal{G}_n^{\tau}$, we use the inequality

$$\begin{aligned}
\tau(X)\,\varphi(O,\,X) &= \varepsilon\sum_{n=1}^{\tau(X)/\varepsilon}\psi(O,\,X)\\
&\leq \varepsilon\sum_{n=1}^{\tau(X)/\varepsilon}\sum_{z\in\mathbb{Z}^d}\psi\big(O,\,(z,\,n\,\varepsilon)\big)\,\varphi\big((z,\,n\,\varepsilon),\,X\big) = \varepsilon\sum_Z\psi(O,\,Z)\,\varphi(Z,\,X),
\end{aligned} \tag{5.24}$$

which follows from the Markov property. We note that this inequality implies equivalence between the estimate of $\mathcal{G}_n^{\tau}$ and that of a quantity concerning the irreducible two-point function $g_n(O,\,X)$ with an extra vertex.

We begin with the case of $n = 0$. We use the bound (5.4) on $g_0(O,\,X)$ and (5.24) to obtain

$$\begin{aligned}
\mathcal{G}_0^{\tau} \leq \lambda^2\,\varepsilon^3 \sum_{O',\,X',\,X} &\mathcal{J}_{O,\,O'}\,\mathcal{J}_{X',\,X}\,\big\{\,\varepsilon\,\varphi(O',\,X)\,\mathbb{P}\,(\,O \rightarrowtail X'\,)\\
&+ \tau(X)\,\psi(O,\,X)\,\varphi(O',\,X') + \big\{\,\tau(X) - \varepsilon\,\big\}\,\varphi(O',\,X)\,\mathbb{P}\,(\,O \rightarrowtail X'\,)\,\big\}
\end{aligned}$$

$$\leq \lambda^2 \varepsilon^4 \sum_{O',X',X} \mathcal{J}_{O,O'} \mathcal{J}_{X',X} \Big\{ \varphi(O',X)\, \mathbb{P}\,(O \rightarrowtail X') + \sum_Z \varphi(Z,X) \{ \psi(O,Z)\, \varphi(O',X') + \psi(O',Z)\, \mathbb{P}\,(O \rightarrowtail X') \} \Big\}$$

$$\leq (\lambda \varepsilon)^2 \sum_{O',O''} \mathcal{J}_{O,O'} \mathcal{J}_{O,O''} \Big\{ \varepsilon \{ \mathcal{B}(O'-O'') + \delta_{\sigma(O'),\sigma(O'')} \} + \{ \mathcal{T}(O'+O'') + \delta_{o,\sigma(O'+O'')} \} + \{ \mathcal{T}(O'-O'') + \delta_{\sigma(O'),\sigma(O'')} \} \Big\} \leq \lambda |\mathcal{J}|\, \xi\, (2+\varepsilon)\, \varepsilon^2. \quad (5.25)$$

We also use (5.4) to obtain

$$\mathcal{G}_0^\sigma \leq \lambda^2 \varepsilon^3 \sum_{O',X',X} \mathcal{J}_{O,O'} \mathcal{J}_{X',X} \Big\{ \|\sigma(X)\|^2 \psi(O,X)\, \varphi(O',X') + 2 \{ \|\sigma(X-O')\|^2 + \|\sigma(O')\|^2 \} \varphi(O',X)\, \mathbb{P}\,(O \rightarrowtail X') \Big\}$$

$$\leq (\lambda \varepsilon)^2 \sum_{O',O''} \mathcal{J}_{O,O'} \mathcal{J}_{O,O''} \Big\{ \mathcal{W}(O'+O'') + 2\mathcal{W}(O'-O'') + 2 \|\sigma(O')\|^2 \{ \mathcal{B}(O'-O'') + \delta_{\sigma(O'),\sigma(O'')} \} \Big\}.$$

Here we must estimate $\mathcal{W}(O'+O'')$ and $\mathcal{W}(O'-O'')$ more minutely.

$$\begin{aligned} \mathcal{W}(O'+O'') \leq \varepsilon \sum_X \|\sigma(X)\|\, \|\sigma(X-O'-O'')\|\, \psi(O,X)\, \varphi(O'+O'',X) \\ + \|\sigma(O'+O'')\|\, \varepsilon \sum_X \|\sigma(X)\|\, \psi(O,X) \{ \varphi(O'+O'',X) - \Delta_{O'+O'',X} \} \\ + \|\sigma(O'+O'')\|\, \varepsilon \sum_X \|\sigma(X)\|\, \psi(O,X)\, \Delta_{O'+O'',X}. \end{aligned} \quad (5.26)$$

By using the Schwarz inequality, the contribution from the first and the second terms is bounded by

$$\mathcal{W}(O) + \|\sigma(O'+O'')\| \sqrt{\mathcal{W}(O)\,\mathcal{S}}.$$

In order to estimate the third term of (5.26), we use the Schwarz inequality and the spatial symmetry of the two-point function as in (3.22) to obtain

$$\|\sigma(O'+O'')\|\, \varepsilon \sum_{X,\, n \geq 0} \|\sigma(X)\| \{ \psi(O,X) - \Delta'_{O,X} \} (1-\varepsilon)^n\, \mathbb{I}_{\{X = O'+O''+n\Theta\}}$$

$$\leq \|\sigma(O'+O'')\|^2 \varepsilon \sum_{n \geq 0} \{ \psi(O, O'+O''+n\Theta) - \Delta'_{O,O'+O''+n\Theta} \} (1-\varepsilon)^n$$

$$\leq \|\sigma(O'+O'')\|^2 \sqrt{\frac{\varepsilon}{2d} \sum_X \{ \psi(O,X) - \Delta'_{O,X} \}^2} = \|\sigma(O'+O'')\|^2 \sqrt{\frac{\mathcal{S}}{2d}}.$$

We can bound $\mathcal{W}(O'-O'')$ in the same way. Therefore $\mathcal{G}_0^\sigma$ is bounded by

$$2\lambda |\mathcal{J}|\, |\mathcal{V}|\, \xi\, \varepsilon^2 + 3\, (\lambda |\mathcal{J}|)^2\, \mathcal{W}(O)\, \varepsilon^2$$

$$+\sup_{O_1,\,O_2}\left\{\,\|\,\sigma(O_1+O_2)\,\|\,\sqrt{\mathcal{W}(O)\,\mathcal{S}}+\|\,\sigma(O_1+O_2)\,\|^2\sqrt{\frac{\mathcal{S}}{2\,d}}\,\right\}\varepsilon^2, \tag{5.27}$$

where the supremum is taken over the sites O_i, $i=1,\,2$ with $\mathcal{J}_{O,\,O_i}>0$, and $|\mathcal{V}|$ is given in (3.26). Under the inequalities in (3.4) with $a=4$, $\mathcal{G}^\tau_0\le\mathcal{O}(\mu)\,\varepsilon^2$ and $\mathcal{G}^\sigma_0\le\mathcal{O}(\varrho)\,\varepsilon^2$. This proves (3.13) for $n=0$.

For $n\ge1$, we use (5.9) to obtain

$$\mathcal{G}^\tau_n\le\sum_{l=0}^{n}\varepsilon\sum_X\sum_{\substack{W_1,\,\cdots,\,W_n,\\U_1,\,\cdots,\,U_n}}\{\,\tau(\Xi_{l+1})-\tau(\Xi_l)\,\}\,\mathcal{L}_{(O,\,W_1,\,U_1)}\cdot\{\,\mathcal{M}_{(W_1,\,U_1\,;\,W_2,\,U_2)}+\mathcal{N}_{(W_1,\,U_1\,;\,W_2,\,U_2)}\,\}\cdots$$
$$\cdots\{\,\mathcal{M}_{(W_{n-1},\,U_{n-1}\,;\,W_n,\,U_n)}+\mathcal{N}_{(W_{n-1},\,U_{n-1}\,;\,W_n,\,U_n)}\,\}\cdot\mathcal{R}_{(W_n,\,U_n,\,X)},$$

where $\{\,\Xi_l\,:\,l=0,\,1,\,\cdots,\,n+1\,\}$ satisfies the condition that $\Xi_0=O$, $\Xi_1=U_1$, $\Xi_{n+1}=X$, and for $l=2,\,\cdots,\,n$,

$$\Xi_{l+1}=\begin{cases}W_{l+1}\,\mathbb{I}_{\{\Xi_l=U_l\}}+U_{l+1}\,\mathbb{I}_{\{\Xi_l=W_l\}}, & \text{on }\mathcal{M}_{(W_l,\,U_l\,;\,W_{l+1},\,U_{l+1})},\\ U_{l+1}\,\mathbb{I}_{\{\Xi_l=U_l\}}+W_{l+1}\,\mathbb{I}_{\{\Xi_l=W_l\}}, & \text{on }\mathcal{N}_{(W_l,\,U_l\,;\,W_{l+1},\,U_{l+1})}.\end{cases}$$

We also use (5.9) and the triangle inequality to obtain

$$\mathcal{G}^\sigma_n\le(\,n+1\,)\sum_{l=0}^{n}\varepsilon\sum_X\sum_{\substack{W_1,\,\cdots,\,W_n,\\U_1,\,\cdots,\,U_n}}\|\,\sigma(\Xi_{l+1})-\sigma(\Xi_l)\,\|^2\,\mathcal{L}_{(O,\,W_1,\,U_1)}\cdot\{\,\mathcal{M}_{(W_1,\,U_1\,;\,W_2,\,U_2)}$$
$$+\mathcal{N}_{(W_1,\,U_1\,;\,W_2,\,U_2)}\,\}\cdots\{\,\mathcal{M}_{(W_{n-1},\,U_{n-1}\,;\,W_n,\,U_n)}+\mathcal{N}_{(W_{n-1},\,U_{n-1}\,;\,W_n,\,U_n)}\,\}\cdot\mathcal{R}_{(W_n,\,U_n,\,X)}.$$

We can estimate $\mathcal{G}^\tau_n$ and $\mathcal{G}^\sigma_n$ by reconstructing the components of vectors and matrices appropriately. We give an outline of the reconstruction. We define

$$\mathcal{A}^{(22)}_{(W,\,U)}=\sum_{W',\,U'}\lambda^2\,\varepsilon\,\mathcal{J}_{W,\,W'}\,\mathcal{J}_{U',\,U}\,\varphi(W',\,U'),\qquad \mathcal{A}^{(32)}_{(W,\,U)}=\sum_{W'}\lambda\,\varepsilon\,\mathcal{J}_{W,\,W'}\,\mathbb{P}\,(\,W'\twoheadrightarrow U\,)\,,$$
$$\mathcal{A}^{(23)}_{(W,\,U)}=\sum_{U'}\lambda\,\varepsilon\,\mathcal{J}_{U',\,U}\,\mathbb{P}\,(\,W\rightarrowtail U'\,)\,,\qquad \mathcal{A}^{(33)}_{(W,\,U)}=\varepsilon\,\mathbb{P}\,(\,W\twoheadrightarrow U\,)\,.$$

These enable us to rewrite $\mathcal{L}^{(i)}_{(O,\,W,\,U)}$ for $i=2,\,3$ as

$$\mathcal{L}^{(i)}_{(O,\,W,\,U)}=\mathcal{L}^{\circ\,(i)}_{(O,\,W,\,U)}+\mathcal{L}'^{\,(2)}_{(O,\,W,\,U)}\,\mathcal{A}^{(2i)}_{(W,\,U)}+\mathcal{L}'^{\,(3)}_{(O,\,W,\,U)}\,\mathcal{A}^{(3i)}_{(W,\,U)},$$

for some $\mathcal{L}'^{\,(i)}_{(O,\,W,\,U)}$ and $\mathcal{L}^{\circ\,(i)}_{(O,\,W,\,U)}$ for $i=2,\,3$ (see the expressions (5.17) and (5.18) for $\mathcal{L}^{(2)}_{(O,\,W,\,U)}$ and $\mathcal{L}^{(3)}_{(O,\,W,\,U)}$ respectively). Moreover $\mathcal{M}^{(i2)}_{(W,\,U\,;\,W',\,U')}$ and $\mathcal{M}^{(i3)}_{(W,\,U\,;\,W',\,U')}$ for $i=1,\,2,\,3$ can be rewritten as

$$\mathcal{M}^{(i2)}_{(W,\,U\,;\,W',\,U')}=\mathcal{M}^{\circ\,(i2)}_{(W,\,U\,;\,W',\,U')}+\mathcal{M}'^{\,(i2)}_{(W,\,U\,;\,W',\,U')}\,\mathcal{A}^{(22)}_{(W',\,U')}+\mathcal{M}'^{\,(i3)}_{(W,\,U\,;\,W',\,U')}\,\mathcal{A}^{(32)}_{(W',\,U')},$$
$$\mathcal{M}^{(i3)}_{(W,\,U\,;\,W',\,U')}=\mathcal{M}'^{\,(i2)}_{(W,\,U\,;\,W',\,U')}\,\mathcal{A}^{(23)}_{(W',\,U')}+\mathcal{M}'^{\,(i3)}_{(W,\,U\,;\,W',\,U')}\,\mathcal{A}^{(33)}_{(W',\,U')},$$

for some $\mathcal{M}'^{(ij)}_{(W,U\,;\,W',U')}$ with $j=2,\,3$ and $\mathcal{M}^{\circ\,(i2)}_{(W,U\,;\,W',U')}$. By replacing $\mathcal{M}$ with $\mathcal{N}$, we can also rewrite $\mathcal{N}^{(ij)}_{(W,U\,;\,W',U')}$ as above. We use these identities to obtain, for example,

$$\begin{aligned}
&\mathcal{L}_{(O,W,U)}\cdot\mathcal{M}_{(W,U\,;\,W',U')}\cdot\mathcal{R}_{(W',U',X)}=\sum_{i,j}\mathcal{L}^{(i)}_{(O,W,U)}\,\mathcal{M}^{(ij)}_{(W,U\,;\,W',U')}\,\mathcal{R}^{(j)}_{(W',U',X)}\\
&\quad=\sum_{i}\mathcal{L}^{(i)}_{(O,W,U)}\bigg\{\mathcal{M}^{(i1)}_{(W,U\,;\,W',U')}\,\mathcal{R}^{(1)}_{(W',U',X)}+\mathcal{M}^{\circ\,(i2)}_{(W,U\,;\,W',U')}\,\mathcal{R}^{(2)}_{(W',U',X)}\\
&\qquad+\mathcal{M}'^{(i2)}_{(W,U\,;\,W',U')}\Big\{\mathcal{A}^{(22)}_{(W',U')}\,\mathcal{R}^{(2)}_{(W',U',X)}+\mathcal{A}^{(23)}_{(W',U')}\,\mathcal{R}^{(3)}_{(W',U',X)}\Big\}\\
&\qquad+\mathcal{M}'^{(i3)}_{(W,U\,;\,W',U')}\Big\{\mathcal{A}^{(32)}_{(W',U')}\,\mathcal{R}^{(2)}_{(W',U',X)}+\mathcal{A}^{(33)}_{(W',U')}\,\mathcal{R}^{(3)}_{(W',U',X)}\Big\}\bigg\}\\
&\quad\le\sum_{i,j}\mathcal{L}^{(i)}_{(O,W,U)}\,\mathcal{M}'^{(ij)}_{(W,U\,;\,W',U')}\,\mathcal{R}'^{(j)}_{(W',U',X)}\;\equiv\mathcal{L}_{(O,W,U)}\cdot\mathcal{M}'_{(W,U\,;\,W',U')}\cdot\mathcal{R}'_{(W',U',X)} \qquad (5.28)\\
&\quad=\sum_{j}\bigg\{\mathcal{L}^{(1)}_{(O,W,U)}\,\mathcal{M}'^{(1j)}_{(W,U\,;\,W',U')}+\mathcal{L}^{\circ\,(2)}_{(O,W,U)}\,\mathcal{M}'^{(2j)}_{(W,U\,;\,W',U')}+\mathcal{L}^{\circ\,(3)}_{(O,W,U)}\,\mathcal{M}'^{(3j)}_{(W,U\,;\,W',U')}\\
&\qquad+\mathcal{L}'^{(2)}_{(O,W,U)}\Big\{\mathcal{A}^{(22)}_{(W',U')}\,\mathcal{M}'^{(2j)}_{(W,U\,;\,W',U')}+\mathcal{A}^{(23)}_{(W',U')}\,\mathcal{M}'^{(3j)}_{(W,U\,;\,W',U')}\Big\}\\
&\qquad+\mathcal{L}'^{(3)}_{(O,W,U)}\Big\{\mathcal{A}^{(32)}_{(W',U')}\,\mathcal{M}'^{(2j)}_{(W,U\,;\,W',U')}+\mathcal{A}^{(33)}_{(W',U')}\,\mathcal{M}'^{(3j)}_{(W,U\,;\,W',U')}\Big\}\bigg\}\,\mathcal{R}'^{(j)}_{(W',U',X)}\\
&\quad\le\sum_{i,j}\mathcal{L}'^{(i)}_{(O,W,U)}\,\mathcal{M}''^{(ij)}_{(W,U\,;\,W',U')}\,\mathcal{R}'^{(j)}_{(W',U',X)}\;\equiv\mathcal{L}'_{(O,W,U)}\cdot\mathcal{M}''_{(W,U\,;\,W',U')}\cdot\mathcal{R}'_{(W',U',X)}, \qquad (5.29)
\end{aligned}$$

where $\mathcal{L}'_{(O,W,U)}=\big(\mathcal{L}'^{(i)}_{(O,W,U)}\big)_{i=1}^{3}$ is a row vector, $\mathcal{R}'_{(W,U,X)}=\big(\mathcal{R}'^{(i)}_{(W,U,X)}\big)_{i=1}^{3}$ is a column vector, and $\mathcal{M}'_{(W,U\,;\,W',U')}=\big(\mathcal{M}'^{(ij)}_{(W,U\,;\,W',U')}\big)_{ij}$ and $\mathcal{M}''_{(W,U\,;\,W',U')}=\big(\mathcal{M}''^{(ij)}_{(W,U\,;\,W',U')}\big)_{ij}$ are 3×3 matrices such that

$$\begin{aligned}
\mathcal{L}'^{(1)}_{(O,W,U)}&=\mathcal{L}^{(1)}_{(O,W,U)}+\mathcal{L}^{\circ\,(2)}_{(O,W,U)}+\mathcal{L}^{\circ\,(3)}_{(O,W,U)},\\
\mathcal{R}'^{(j)}_{(W',U',X)}&=\begin{cases}\mathcal{R}^{(1)}_{(W',U',X)}+\mathcal{R}^{(2)}_{(W',U',X)}+\mathcal{R}^{(3)}_{(W',U',X)}, & \text{for } j=1,\\ \mathcal{A}^{(j2)}_{(W',U')}\,\mathcal{R}^{(2)}_{(W',U',X)}+\mathcal{A}^{(j3)}_{(W',U')}\,\mathcal{R}^{(3)}_{(W',U',X)}, & \text{for } j=2,\,3,\end{cases}\\
\mathcal{M}'^{(i1)}_{(W,U\,;\,W',U')}&=\mathcal{M}^{(i1)}_{(W,U\,;\,W',U')}+\mathcal{M}^{\circ\,(i2)}_{(W,U\,;\,W',U')},\\
\mathcal{M}''^{(ij)}_{(W,U\,;\,W',U')}&=\begin{cases}\mathcal{M}'^{(1j)}_{(W,U\,;\,W',U')}+\mathcal{M}'^{(2j)}_{(W,U\,;\,W',U')}+\mathcal{M}'^{(3j)}_{(W,U\,;\,W',U')}, & \text{for } i=1,\\ \mathcal{A}^{(i2)}_{(W,U)}\,\mathcal{M}'^{(2j)}_{(W,U\,;\,W',U')}+\mathcal{A}^{(i3)}_{(W,U)}\,\mathcal{M}'^{(3j)}_{(W,U\,;\,W',U')}, & \text{for } i=2,\,3.\end{cases}
\end{aligned}$$

We note that $\mathcal{L}'_{(O,W,U)}$ and $\mathcal{M}'_{(W,U\,;\,W',U')}$ consist of products of two connectivity functions, while $\mathcal{L}_{(O,W,U)}$ and $\mathcal{M}_{(W,U\,;\,W',U')}$ contain products of three connectivity functions. Also $\mathcal{M}''^{(lj)}_{(W,U\,;\,W',U')}$ and $\mathcal{R}'^{(l)}_{(W',U',X)}$ for $l=2,\,3$ consist of products of three connectivity functions. We use these facts and the inequalities (5.28) and (5.29) to estimate the contribution to $\mathcal{G}_2^\tau$ from $\mathcal{M}_{(W,U\,;\,W',U')}$ by

$$\begin{aligned}\varepsilon \sum_X \sum_{\substack{W,U,\\W',U'}} &\{\, \tau(U)\, \mathcal{L}'_{(O,W,U)} \cdot \mathcal{M}''_{(W,U\,;\,W',U')} \cdot \mathcal{R}'_{(W',U',X)} \\ &+ \{\, \tau(W') - \tau(U)\,\}\, \mathcal{L}_{(O,W,U)} \cdot \mathcal{M}'_{(W,U\,;\,W',U')} \cdot \mathcal{R}'_{(W',U',X)} \\ &+ \{\, \tau(X) - \tau(W')\,\}\, \mathcal{L}_{(O,W,U)} \cdot \mathcal{M}_{(W,U\,;\,W',U')} \cdot \mathcal{R}_{(W',U',X)} \,\}.\end{aligned}$$

It is easily seen that the above bound is $\mathcal{O}(\mu)^2\,\varepsilon^2$ under the inequalities in (3.4) with $a = 4$. Furthermore by translation invariance and the way of proving (5.25) and (5.27), we can see that

$$\mathcal{G}_n^\tau \le n\,\mathcal{O}(\mu)^n\,\varepsilon^2, \qquad\qquad \mathcal{G}_n^\sigma \le n^2\,\mathcal{O}(\varrho)\,\mathcal{O}(\mu)^{n-1}\,\varepsilon^2,$$

under the inequalities in (3.4) with $a = 4$. This proves (3.13) for $n \ge 1$. Therefore $\sum_n \mathcal{G}_n^\tau \le \mathcal{O}(\mu)\,\varepsilon^2$ and $\sum_n \mathcal{G}_n^\sigma \le \mathcal{O}(\varrho)\,\varepsilon^2$ for sufficiently large d or L. Now the proof is completed.

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