On Level Clustering in Regular Spectra

Nariyuki MINAMI*
Institute of Mathematics, University of Tsukuba
Tsukuba-shi, Ibaraki, 305-8571 Japan

1 Introduction.

Let $H(\hbar)$ be the quantum Hamiltonian describing a bounded, isolated physical system of finite degrees of freedom. ($\hbar$ is the Planck's constant.) Then the spectrum of $H(\hbar)$ is purely discrete, consisting of energy levels $\{E_n(\hbar)\}_{n\geq 1}$. Suppose further that $H(\hbar)$, or its spectrum itself, is obtained by quantizing (in a certain way) the classical Hamiltonian $H(p,q)$ defining the system. Now the classical dynamical system corresponding to $H(p,q)$ may belong to one of two extreme cases of being completely integrable or chaotic. It was Percival [12] who proposed to distinguish the spectrum of $H(\hbar)$ according to the category to which the classical counterpart of $H(\hbar)$ belongs. He called the spectrum regular [resp. irregular] if $H(p,q)$ is integrable [resp. chaotic], and discussed possible difference between these two kinds of spectra. Later, Berry and Tabor [2] argued that regular and irregular spectra are distinguished by looking at the probability distribution of the spacing between adjacent energy levels. Namely they argued that in the regular spectra, the level spacing obeys the exponential distribution $e^{-t}dt$, so that the spectrum looks like a typical realization of the Poisson point process, which is the phenomenon they called "level clustering ". On the other hand, they conjectured that in the irregular spectra, the level spacing distribution $p(t)dt$ satisfies $p(t) \sim \mathrm{const.}t^{\gamma}$, $t \searrow 0$, with $\gamma > 0$ ("level repulsion "). Hence irregular spectra should typically look like the spectra of large random matrices. This conjecture is supported by numerical studies performed later. (See e.g. [3] for a review.)

However, neither the precise meaning of the "probability distribution "of level spacing, nor a mathematical formulation of the statistics for the spectrum of $H(\hbar)$ –the level statistics–in general seems to have been explicitly given in physics litterature, although some important ideas are stated in [2]. The present paper, which is an elaboration of a part of the author's previous note [10], aims at giving a mathematical formulation of level statistics based on the idea of Berry and Tabor, and applying it to regular spectra. In §2, we give the definition of strict (Definition 1) and wide (Definition 2) sense level clustering, and prove some preliminary results for later references. These results are formulated in analogy with corresponding propositions in the theory of stationary point processes. In §3, we shall apply the level statistics thus formulated to regular spectra, and discuss the closely related theorems by Sinai [15] and Major [7]. We argue that although it is probably very difficult to apply theorems of Sinai and Major to prove the strict sense

*e-mail: minami@sakura.cc.tsukuba.ac.jp

level clustering in generic regular spectra, there is some hope in proving the wide sense level clustering for some concrete Hamiltonian such as rectangular billiard. Finally in §4, we shall propose another formulation of level statistics, and with an example, shall argue again that "level clustering "should not always mean strict Poissonian property of the spectrum.

2 Strict and wide sense level clustering.

2.1 The unfolding of Berry and Tabor.

When one speaks of the "probability distribution "of the spacing of the adjacent energy levels of $H(\hbar)$, the follwing two questions immediately arise:

1. There is no stochasticity in $H(\hbar)$, hence in $\{E_n(\hbar)\}_{n\geq 1}$ too. Therfore, it will be necessary to take the semiclassical limit $\hbar \searrow 0$ to have sufficiently many levels in a fixed energy intervals, and we will have to take statistics among these levels.
2. The mean level density is not uniform over the entire spectrum, so that the statistical property of $E_{n+1} - E_n$ may not be uniform in n . Hence we will need to "unfold "the spectrum so that it will look like a uniformly distributed sequence.

Inspired by [2], we make a normalization of the spectrum $\{E_n(\hbar)\}_{n\geq 1}$ of $H(\hbar)$ which meets the above two requirements simultaneously. For this purpose, we make the following assumptions:

(A1) All levels are non-degenerate and $E_n(\hbar) \geq 0$;

(A2) $E_n(\hbar) \searrow 0$ monotonically as $\hbar \searrow 0$;

(A3) For each fixed $E > 0$, there is a constant $\nu(E) > 0$ such that

$$N_\hbar(E) \equiv \sharp\{n \geq 1 \mid E_n(\hbar) \leq E\} \sim \nu(E)\hbar^{-d} \ (\hbar \searrow 0) . \tag{1}$$

Let us take $E > 0$, which we shall fix throughout. By (A1) and (A2), we can define $\hbar_j = \hbar_j(E)$ as the unique solution of the equation $E_j(\hbar) = E$. If we define

$$\lambda_j = \nu(E)\hbar_j(E)^{-d} , \tag{2}$$

then from (A3), it is easily seen that

$$n(L) \equiv \sharp\{j \geq 1 \mid \lambda_j \leq L\} \sim L \quad (L \to \infty) . \tag{3}$$

Thus we have obtained a sequence $\{\lambda\}_j$ which has asymptotically uniform distribution. We will call the λ_j's the unfolded levels, and call the above procedure "the unfolding of Berry and Tabor ". (It is also called "quantization of Planck's constant "in [2].)

2.2 Statistics for asymptotically uniformly distributed sequences.

In this subsection, we suppose that $\{\lambda_j\}_{j=0}^{\infty}$ is simply a strictly increasing sequence of real numbers satisfying (3) and $\lambda_0 = 0$, but still call λ_j the "j-th level ". We introduce the following notation:

$$N(t) = N(t;c) = \sharp\{j \geq 1 \mid \lambda_j \in (t, t+c]\} = \sum_{j\geq 1} 1_{(t,t+c]}(\lambda_j) \ ; \tag{4}$$

$$\pi_k(c;L) = \frac{1}{L}\int_0^L 1_{\{N(t;c)=k\}} dt \ ; \tag{5}$$

$$\mu_k(c;L) = \frac{1}{L}\int_0^L \left(\begin{array}{c} N(t) \\ k \end{array} \right) dt = \sum_{j\geq k} \left(\begin{array}{c} j \\ k \end{array} \right) \pi_j(c;L) \ , \tag{6}$$

where

$$\left(\begin{array}{c} j \\ k \end{array} \right) = \frac{1}{k!} j(j-1)\cdots(j-k+1) \ ; \tag{7}$$

and finally

$$\rho(c;L) = \frac{\sharp\{j \geq 1 \mid \lambda_j \leq L \ , \ \lambda_{j+1} - \lambda_j \leq c\}}{\sharp\{j \geq 1 \mid \lambda_j \leq L\}} \ . \tag{8}$$

Here $c > 0$ and $k = 0, 1, 2, \dots$.

These notions have the following obvious probabilistic meanings. $N(t;c)$ is nothing but the number of levels in the interval $(t, t+c]$. $\pi_k(c;L)$ is the probability that this interval catches exactly k levels when t is randomly chosen from $(0, L]$ according to the uniform distribution, and $\mu_k(c;L)$ is the k-th factorial moment of the random varible $N(\cdot;c)$. Finally, $\rho(c;L)$ is the relative frequency of those pairs of levels below L which have spacing not exceeding c .

We also write

$$\bar{\mu}_k(c) = \limsup_{L\to\infty} \mu_k(c;L) \ ; \ \underline{\mu}_k(c) = \liminf_{L\to\infty} \mu_k(c;L) \ ; \tag{9}$$

$$\bar{\rho}(c) = \limsup_{L\to\infty} \rho(c;L) \ ; \ \underline{\rho}(c) = \liminf_{L\to\infty} \rho(c;L) \ , \tag{10}$$

and also

$$\pi_k(c) = \lim_{L\to\infty} \pi_k(c;L) \ ; \ \mu_k(c) = \lim_{L\to\infty} \mu_k(c;L) \ ; \ \rho(c) = \lim_{L\to\infty} \rho(c;L) \tag{11}$$

whenever these limits exist. This $\rho(c)$ will be called the limiting level spacing distribution function.

Proposition 1 *Under (3), we have $\mu_1(c) = c$ for any $c > 0$.*

Proof. By the definition,

$$\mu_1(c;L) = \frac{1}{L}\sum_j \int_0^L 1_{[\lambda_j - c, \lambda_j)}(t)dt \ . \tag{12}$$

But if $L > c > 0$, one has

$$\int_0^L 1_{[\lambda_j - c,\lambda_j)}(t)dt = \begin{cases} c, & \text{if } c \le \lambda_j \le L \\ 0, & \text{if } \lambda_j > L + c \end{cases}$$

and

$$\int_0^L 1_{[\lambda_j - c,\lambda_j)}(t)dt \le c \ , \quad \text{if } \lambda_j < c \quad \text{or} \quad L < \lambda_j \le L + c \ .$$

Hence for each fixed $c > 0$, it is obvious that

$$\mu_1(c;L) \sim c\frac{n(L)}{L} \quad \text{as } L \to \infty \ ,$$

so that

$$\mu_1(c) = \lim_{L\to\infty} \mu_1(c;L) = c \lim_{L\to\infty} \frac{n(L)}{L} = c \ ,$$

which was to be proved.

Now we give a definition of $\{\lambda_j\}$'s looking like Poisson point process in the following way:

Definition 1 *We shall say that one has the strict sense level clustering if*

$$\pi_k(c) \equiv \lim_{L\to\infty} \pi_k(c;L) = e^{-c}\frac{c^k}{k!} \tag{13}$$

holds for each $c > 0$ and $k \ge 0$.

Note that if $\{\lambda_j\}_{j\ge1}$ actually is the realization of Poisson point process on $[0,\infty)$, then by the ergodic theorem, one has the strict sense level clustering with probability one.

Obviously, the strict sense level clustering is equivalent to the weak convergence of the family of probability distributions $P(c;L) \equiv \{\pi_k(c;L)\}_{k=0}^{\infty}$ $(L > 0)$, on $\mathbf{Z}_+$ to the Poisson distribution $\{e^{-c}c^k/k!\}_{k=0}^{\infty}$ as $L \to \infty$. As is well known in elementary probability theory, a sufficient condition for this weak convergence is that the factorial moments $\mu_k(c;L)$ of all order of $P(c;L)$ converges to those corresponding to the Poisson distribution. Namely we have

Proposition 2 *If*

$$\mu_k(c) = \lim_{L\to\infty} \mu_k(c;L) = \frac{c^k}{k!} \tag{14}$$

holds for each $c > 0$ and $k = 1, 2, \dots$, then one has the strict sense level clustering.

If one has the strict sense level clustering, then the limiting level spacing distribution exists and is the exponential distribution $e^{-c}dc$. This is a direct consequence of the following more general proposition:

Proposition 3 *If $\pi_0(c) = \lim_{L\to\infty} \pi_0(c; L)$ exists and is differentiable with respect to c , then $\rho(c) = \lim_{L\to\infty} \rho(c; L)$ also exists, and is given by*

$$\rho(c) = 1 + \frac{d}{dc}\pi_0(c) \ . \tag{15}$$

In fact, if the strict sense level clustering holds, then one has in particular $\pi_0(c) = e^{-c}$. Hence by the above proposition, $\rho(c) = 1 - e^{-c}$.

In §1 of [14], Sinai gave two definitions of $\{\lambda_j\}$'s similarity to the Poisson point process. One is the strict sense level clustering as defined above, and the other is the existence of the limiting level spacing distribution $\rho(c)$ and its equality to the exponential distribution. The above proposition shows that these two definitions are not independent.

Proof of Proposition 3. If $\lambda_j \le L$ and $\lambda_{j+1} - \lambda_j > c$, then for any $\delta \in (0, c)$, we have the implication

$$t \in [\lambda_j - \delta, \lambda_j) \implies N(t; \delta) > 0 \ , \ N(t + \delta; c - \delta) = 0 \ ,$$

where the intervals $[\lambda_j - \delta, \lambda_j)$ are disjoint. Hence if we set

$$n(c; L) = \sharp\{j \ge 1 \mid \lambda_j \le L \ , \ \lambda_{j+1} - \lambda_j > c\} \ ,$$

then we have

$$\begin{aligned} \delta n(c; L) &= \sum_{\lambda_j \le L \ ; \ \lambda_{j+1}-\lambda_j > c} \int_{-\delta}^{L} 1_{[\lambda_j-\delta,\lambda_j)}(t)dt \\ &\le \delta + \int_0^L 1_{\{N(t;c)>0, N(t+\delta;c-\delta)=0\}}(t)dt \\ &= \delta + \int_0^L 1_{\{N(t+\delta;c-\delta)=0\}}(t)dt - \int_0^L 1_{\{N(t;c)=0\}}(t)dt \ , \end{aligned}$$

namely

$$\delta\frac{n(c; L)}{L} \le \frac{\delta}{L} + \pi_0(c - \delta; L) - \pi_0(c; L) \ .$$

Letting $L \to \infty$ and noting $n(L) \sim L$, we get

$$\delta \limsup_{L\to\infty} \frac{n(c; L)}{n(L)} = \delta(1 - \underline{\rho}(c)) \le \pi_0(c - \delta) - \pi_0(c) \ ,$$

or

$$\underline{\rho}(c) \ge 1 + \frac{\pi_0(c) - \pi_0(c - \delta)}{\delta} \ .$$

Letting $\delta \searrow 0$, we arrive at

$$\underline{\rho}(c) \ge 1 + \frac{d}{dc}\pi_0(c) \ .$$

On the other hand, if $N(t;\delta) > 0$ and $N(t+\delta;c) = 0$, then there is a $\lambda_j \in (t, t+\delta]$ such that $\lambda_{j+1} - \lambda_j > c$. Hence the set

$$\{t \in [0, L] \mid N(t;\delta) > 0 \ , \ N(t+\delta;c) = 0\}$$

is the union of finitely many disjoint intervals I_ℓ each with length no greater than δ , and for each I_ℓ , there corresponds a $\lambda_j \le L + \delta$ satisfying $\lambda_{j+1} - \lambda_j > c$. Hence the number of these intervals does not exceed $1 + n(c;L)$, and so

$$\begin{aligned} \delta(1 + n(c;L)) &\ge \int_0^L 1_{\{N(t;\delta)>0, N(t+\delta;c)=0\}}(t)dt \\ &= \int_0^L 1_{\{N(t+\delta;c)=0\}}(t)dt - \int_0^L 1_{\{N(t;c+\delta)=0\}}(t)dt \ . \end{aligned}$$

Dividing by L and letting $L \to \infty$, we have

$$\delta \liminf_{L\to\infty} \frac{n(c;L)}{n(L)} = \delta(1 - \bar{\rho}(c)) \ge \pi_0(c) - \pi_0(c+\delta) \ ,$$

or

$$\bar{\rho}(c) \le 1 + \frac{\pi_0(c+\delta) - \pi_0(c)}{\delta} \ .$$

Again letting $\delta \searrow 0$, we get

$$\bar{\rho}(c) \le 1 + \frac{d}{dc}\pi_0(c) \ ,$$

completing the proof.

As will be explained in §3, it seems to be a difficult problem to prove the strict sense level clustering for any concrete Hamiltonian. In fact, no explicit example of regular spectrum is known which shows strict sense level clustering. Moreover, as was shown in [8] and will be discussed in §4, there is an example of one-dimensional Hamiltonian for which the level statistics, in a somewhat different formulation, can be rigorously performed, but the obtained level spacing distribution is different from the exponential distribution. In that case, we have still $\rho'(0+) > 0$, so that we should say that the level clustering is taking place, and if we take the high disorder limit in the system, then the data $\rho(c)$ converges to the distribution function of $e^{-c}dc$. This situation suggests us that the strict sense level clustering can only be proved in some ideal limit, and generically level clustering should not mean the strict Poissonian character of the unfolded spectrum. Thus we are led to define the notion of level clustering in much weaker sense, nevertheless retaining some physical significance. Taking the broadest statistical sense of the words "clustering "and "repulsion ", we now make the following definition.

Definition 2 *We shall say that one has the wide sense level clustering [resp. repulsion] if*

$$\liminf_{c\searrow 0} \frac{\underline{\rho}(c)}{c} > 0 \quad \left[resp. \ \limsup_{c\searrow 0} \frac{\bar{\rho}(c)}{c} = 0 \right] \tag{16}$$

holds.

We can give a criteria for the wide sense repulsion and clustering in terms of $\mu_k(c)$, $k \leq 3$. Recall that we always have $\mu_1(c) = c$.

Proposition 4 *(i) If $\bar{\mu}_2(c) = o(c^2)$ as $c \searrow 0$, then one has the wide sense level repulsion.*

(ii) If $\mu_2(c) = \frac{1}{2}c^2$ for any $c > 0$ and $\bar{\mu}_3(c) = o(c^2)$ as $c \searrow 0$, then one has the wide sense level clustering

Proof. Successively applying the equality

$$\mu_k(c; L) = \sum_{j \geq k} \binom{j}{k} \pi_j(c; L) \ , \ k \geq 1 \ ,$$

we obtain

$$1 - \pi_0(c; L) = \sum_{j \geq 1} \pi_j(c; L) = \sum_{j=1}^{k} (-1)^{j-1} \mu_j(c; L) + \sum_{j=k+1}^{\infty} \sum_{i=0}^{k} (-1)^i \binom{j}{i} \pi_j(c; L)$$

for all $k \geq 1$. But sinse

$$\sum_{i=0}^{k} (-1)^i \binom{j}{i} = (-1)^k \binom{j-1}{k} \ , \ j \geq k+1 \ , \ k \geq 0 \ ,$$

we have the inequality

$$\pi_0(c; L) \leq 1 - \mu_1(c; L) + \mu_2(c; L) - \cdots + (-1)^k \mu_k(c; L) \tag{17}$$

when k is even and

$$\pi_0(c; L) \geq 1 - \mu_1(c; L) + \mu_2(c; L) - \cdots + (-1)^k \mu_k(c; L) \tag{18}$$

when k is odd.

On the other hand, if $L > 0$, $N = n(L) + 1$, namely $\lambda_N \leq L < \lambda_{N+1}$, then

$$\begin{aligned}
L\pi_0(c; L) &= \sum_{n=0}^{N-1} (\lambda_{n+1} - \lambda_n - c)_+ + \{(\lambda_{N+1} - \lambda_N - c)_+ \wedge (L - \lambda_N)\} \\
&= \sum_{n=0}^{N-1} (\lambda_{n+1} - \lambda_n - c) + \sum_{n=0}^{N-1} \{c - (\lambda_{n+1} - \lambda_n)\}_+ \\
&\quad + \{(\lambda_{N+1} - \lambda_N - c)_+ \wedge (L - \lambda_N)\} \\
&\leq \lambda_N - cN + c\sharp\{n \leq N-1 | \lambda_{n+1} - \lambda_n \leq c\} + (L - \lambda_N) \\
&= L - cN + cL\rho(c; L) \ .
\end{aligned}$$

Dividing both sides by L and letting $L \to \infty$, we obtain from (18),

$$\begin{aligned}
1 + c\underline{\rho}(c) - c &\geq \underline{\pi}_0(c) \\
&\geq 1 - \mu_1(c) + \mu_2(c) - \bar{\mu}_3(c) \\
&= 1 - c + \frac{1}{2}c^2 + o(c^2) \ ,
\end{aligned}$$

when the conditions of (ii) hold.

Hence we get

$$\liminf_{c\searrow} \frac{1}{c}\underline{\rho}(c) \geq \frac{1}{2} > 0 \ ,$$

namely the wide sense level clustering.

Now suppose the condition of (i) hold. For each $0 < \delta < 1$, one has

$$\begin{aligned} L\pi_0(c;L) &= \sum_{n=0}^{N-1}(\lambda_{n+1} - \lambda_n - c) + \sum_{n=0}^{N-1}\{c - (\lambda_{n+1} - \lambda_n)\}_+ \\ &\quad +\{(\lambda_{N+1} - \lambda_N - c)_+ \wedge (L - \lambda_N)\} \\ &\geq \lambda_N - cN + \sum_{\lambda_{n+1}-\lambda_n \leq \delta c} \{c - (\lambda_{n+1} - \lambda_n)\} + (L - \lambda_N - c) \\ &\geq (L-c) - cN + (1-\delta)c\rho(\delta c; L) \ . \end{aligned}$$

Again dividing by L and letting $L \to \infty$ and noting (17),

$$\begin{aligned} 1 + (1-\delta)c\bar{\rho}(\delta c) - c &\leq \bar{\pi}_0(c) \\ &\leq 1 - \mu_1(c) + \bar{\mu}_2(c) \\ &= 1 - c + o(c^2) \ . \end{aligned}$$

Hence we have

$$\limsup_{c\searrow} \frac{1}{c}\bar{\rho}(c) = \limsup_{c\searrow} \frac{1}{\delta c}\bar{\rho}(\delta c) = 0 \ ,$$

namely the wide sense level repulsion.

At this point, let us discuss the relation of the factorial moment $\mu_k(c)$ to the so called k-th correlation $R_k(c)$ defined by

$$R_k(c) = \lim_{L\to\infty} R_k(c;L) \ ; \tag{19}$$

$$R_k(c;L) = \sharp\{(\lambda_{j_1}, \dots, \lambda_{j_k}) \mid \lambda_{j_1} < \cdots < \lambda_{j_k} \leq L \ , \ \lambda_{j_k} - \lambda_{j_1} \leq c\} \ , \tag{20}$$

whenever the limit exists. In particular, $R_2(c)$ is called the pair correlation (see e.g. [13]).

Proposition 5 *(i) If $\mu_k(c)$ exists for each $c > 0$ and is differentiable with respect to c , then $R_k(c)$ also exists and*

$$R_k(c) = \frac{d}{dc}\mu_k(c) \ . \tag{21}$$

(ii) If $R_k(c)$ exists for each $c > 0$, then $\mu_k(c)$ also exists and

$$\mu_k(c) = \int_0^L R_k(c')dc' \ . \tag{22}$$

Proof. Since

$$\begin{aligned}
\binom{N(t)}{k} &= \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}} 1_{(t,t+c]}(\lambda_{j_1})\cdots 1_{(t,t+c]}(\lambda_{j_k}) \\
&= \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}} 1_{[\lambda_{j_1}-c,\lambda_{j_1})}(t)\cdots 1_{[\lambda_{j_k}-c,\lambda_{j_k})}(t) \ ,
\end{aligned}$$

we can compute

$$\begin{aligned}
L\mu_k(c;L) &= \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}} |[0,L]\cap\cap_{p=1}^{k}[\lambda_{j_p}-c,\lambda_{j_p})| \\
&= \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}} \{(L\wedge\lambda_{j_1})-(\lambda_{j_k}-c)_+\}_+ \\
&= \sum_{\substack{\lambda_{j_1}<\cdots<\lambda_{j_k};\\ \lambda_{j_1}\le L,\lambda_{j_k}\ge c}} (c-\lambda_{j_k}+\lambda_{j_1})_+ \\
&\quad + \sum_{L<\lambda_{j_1}<\cdots<\lambda_{j_k}} (c-\lambda_{j_k}+L)_+ + \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}<c} \lambda_{j_1} \\
&= \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}\le L} (c-\lambda_{j_k}+\lambda_{j_1})_+ \\
&\quad + \left[\sum_{\substack{\lambda_{j_1}<\cdots<\lambda_{j_k};\\ \lambda_{j_1}\le L<\lambda_{j_k}}} (c-\lambda_{j_k}+\lambda_{j_1})_+ + \sum_{L<\lambda_{j_1}<\cdots<\lambda_{j_k}} (c-\lambda_{j_k}+L)_+\right] \\
&\quad + \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}<c} [\lambda_{j_1}-(c-\lambda_{j_k}+\lambda_{j_1})_+] \ .
\end{aligned}$$

The third term is independent of L , hence is $\mathcal{O}(1)$ as $L\to\infty$. On the other hand, the second term, which we denote by $M(L)$, can be estimated in two ways. To begin with, we have

$$\begin{aligned}
M(L) &\le \sum_{\substack{\lambda_{j_1}<\cdots<\lambda_{j_k};\\ \lambda_{j_1}\le L<\lambda_{j_k} \ \ or\ \ L<\lambda_{j_1}<\cdots<\lambda_{j_k}\le L+c}} (c-\lambda_{j_k}+\lambda_{j_1})_+ \\
&= \sum_{\substack{\lambda_{j_1}<\cdots<\lambda_{j_k};\\ \lambda_{j_1}\le L<\lambda_{j_k} \ \ or\ \ L<\lambda_{j_1}<\cdots<\lambda_{j_k}\le L+c}} \int_{-\infty}^{\infty} 1_{(t,t+c]}(\lambda_{j_1})\cdots 1_{(t,t+c]}(\lambda_{j_k})dt \ .
\end{aligned}$$

Suppose $\lambda_{j_1}\le L<\lambda_{j_k}$. Then we have $1_{(t,t+c]}(\lambda_{j_1})=0$ for $t>L+c$ and $1_{(t,t+c]}(\lambda_{j_k})=0$ for $t<L-c$. If, on the other hand, $L<\lambda_{j_1}<\cdots<\lambda_{j_k}\le L+c$, then we have $1_{(t,t+c]}(\lambda_{j_k})=0$ for $t>L+c$ and $1_{(t,t+c]}(\lambda_{j_1})=0$ for $t<L-c$. Therefore

$$\begin{aligned}
M(L) &\le \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}} \int_{L-c}^{L+c} 1_{(t,t+c]}(\lambda_{j_1})\cdots 1_{(t,t+c]}(\lambda_{j_k})dt \\
&= \int_{L-c}^{L+c} \binom{N(t)}{k} dt = o(L) \ , \ \ L\to\infty
\end{aligned}$$

when $\mu_k(c)$ exists.

On the other hand, we also have the inequality

$$\begin{aligned} M(L) &\leq c \sum_{\substack{\lambda_{j_1}<\cdots<\lambda_{j_k};\\ \lambda_{j_1}\leq L<\lambda_{j_k} \text{ or } L<\lambda_{j_1}<\cdots<\lambda_{j_k}\leq L+c}} 1_{\{\lambda_{j_k}-\lambda_{j_1}\leq c\}} \\ &\leq c \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}\leq L+c} 1_{\{\lambda_{j_k}-\lambda_{j_1}\leq c\}} - c \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}\leq L} 1_{\{\lambda_{j_k}-\lambda_{j_1}\leq c\}} \ . \end{aligned}$$

Obviously, the right hand side is of $o(L)$ as $L \to \infty$ when $R_k(c)$ exists. Therefore, under either of the conditions of (i) or (ii), one has

$$L\mu_k(c;L) = \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}\leq L} (c-\lambda_{j_k}+\lambda_{j_1})_+ + o(L) \ .$$

Let us prove (i). From the above remark and

$$(c+\delta-\lambda_{j_k}+\lambda_{j_1})_+ - (c-\lambda_{j_k}+\lambda_{j_1})_+ \geq \delta 1_{\{\lambda_{j_k}-\lambda_{j_1}\leq c\}} \ ,$$

we see

$$\mu_k(c+\delta)-\mu_k(c) \geq \delta \limsup_{L\to\infty} \frac{1}{L} \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}\leq L} 1_{\{\lambda_{j_k}-\lambda_{j_1}\leq c\}} \ .$$

Similarly, we have

$$\mu_k(c)-\mu_k(c-\delta) \leq \delta \liminf_{L\to\infty} \frac{1}{L} \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}\leq L} 1_{\{\lambda_{j_k}-\lambda_{j_1}\leq c\}} \ .$$

Under the assumption of the differentiability of $\mu_k(c)$, we can let $\delta \searrow 0$, to obtain

$$R_k(c) = \lim_{L\to\infty} \frac{1}{L} \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}\leq L} 1_{\{\lambda_{j_k}-\lambda_{j_1}\leq c\}} = \frac{d}{dc}\mu_k(c) \ .$$

We turn to the proof of (ii). Let $\ell \in \mathbf{N}$ and write $\delta = c/\ell$ and $\Delta = \lambda_{j_k}-\lambda_{j_1}$. Then

$$\delta \sum_{m=1}^{\ell} 1_{\{c\geq\Delta+m\delta\}} \leq (c-\Delta)_+ \leq \delta \sum_{m=0}^{\ell} 1_{\{c\geq\Delta+m\delta\}} \ .$$

Hence

$$\begin{aligned} \frac{c}{\ell}\sum_{m=1}^{\ell}\frac{1}{L} \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}\leq L} 1_{\{\lambda_{j_k}-\lambda_{j_1}\leq c-mc/\ell\}} &\leq \frac{1}{L} \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}\leq L} (c-\lambda_{j_k}+\lambda_{j_1})_+ \\ &\leq \frac{c}{\ell}\sum_{m=0}^{\ell}\frac{1}{L} \sum_{\lambda_{j_1}<\cdots<\lambda_{j_k}\leq L} 1_{\{\lambda_{j_k}-\lambda_{j_1}\leq c-mc/\ell\}}. \end{aligned}$$

Letting $L \to \infty$, we obtain

$$\frac{c}{\ell}\sum_{m=1}^{\ell} R_k(c(1-\frac{m}{\ell})) \leq \liminf_{L\to\infty} \mu_k(c;L) \leq \limsup_{L\to\infty} \mu_k(c;L) \leq \frac{c}{\ell}\sum_{m=0}^{\ell} R_k(c(1-\frac{m}{\ell})) \ ,$$

for all $\ell \in \mathbf{N}$. Hence letting $\ell \to \infty$, we arrive at

$$\mu_k(c) = \lim_{L\to\infty} \mu_k(c;L) = c\int_0^1 R_k(c(1-x))dx = \int_0^c R_k(c')dc' \ ,$$

completing the proof.

2.3 Comparison with the theory of stationary point processes.

Let us consider a point process $N_\omega(dx)$ on $\mathbf{R}$ defined for ω in a probability space $(\Omega, \mathcal{F}, P)$ equipped with an ergodic measure preserving flow $\theta = \{\theta_t\}_{t\in\mathbf{R}}$. We assume that N is θ-stationary in the sense that $N_{\theta_t\omega} = N_\omega \circ \tau_t^{-1}$, where $\tau_t x = x - t$ is the translation on $\mathbf{R}$, and that N_ω has no multiple points so that one can write $N_\omega = \sum_j \delta_{\lambda_j(\omega)}$ with

$$\cdots < \lambda_{-1}(\omega) < \lambda_0(\omega) \leq 0 < \lambda_1(\omega) < \lambda_2(\omega) < \cdots \tag{23}$$

Let us furhter suppose that $m \equiv E[N(0,1]] = 1$ and that $E[(N(0,1])^k] < \infty$ for all $k \geq 2$. Now it is known that there is a measure $\hat{P}(d\omega)$ on $(\Omega, \mathcal{F})$, which is called the Palm measure of the stationary point process N , such that for all jointly measurable function $f(\omega, s) \geq 0$ on $\Omega \times \mathbf{R}$, the following formula holds:

$$\int_\Omega P(d\omega) \int_{\mathbf{R}} N_\omega(ds) f(\theta_s\omega, s) = \int_\Omega \hat{P}(d\omega) \int_{\mathbf{R}} ds f(\omega, s) \ . \tag{24}$$

$\hat{P}$ is concentrated on the set $\hat{\Omega} = \{\omega | N_\omega(\{0\}) > 0\}$ and turns out to be a probability measure when $m = 1$. In this case, $\hat{P}(d\omega)$ has an intuitive meaning of the conditional probability $P(d\omega | N(\{0\}) > 0)$. (See [11] for detail.)

Now we can apply the individual ergodic theorem and (24), to prove that the following relations hold with probability one:

(0)

$$\lim_{L\to\infty} \frac{1}{L} \sharp\{j \geq 1 | \lambda_j(\omega) \leq L\} = \lim_{L\to\infty} \frac{1}{L} N_\omega(0, L] = E[N_\omega(0,1]] = 1 \ ;$$

(i)

$$\pi_k(c) \equiv \lim_{L\to\infty} \frac{1}{L} \int_0^L 1_{\{N_\omega(t,t+c]=k\}} dt = \lim_{L\to\infty} \frac{1}{L} \int_0^L 1_{\{N_{\theta_t\omega}(0,c]=k\}} dt = P(N(0,c] = k) \ ;$$

(ii)

$$\rho(c) \equiv \lim_{L\to\infty} \frac{1}{L} \sharp\{j \geq 1 | \lambda_j(\omega) \leq L \ , \ \lambda_{j+1}(\omega) - \lambda_j(\omega) \leq c\} = \hat{P}(N(0,c] > 0) \ ;$$

(iii)

$$\mu_k(c) \equiv \lim_{L\to\infty} \frac{1}{L} \int_0^L \left(\begin{array}{c} N_\omega(t; t+c] \\ k \end{array} \right) dt = E\left[\left(\begin{array}{c} N(0; c] \\ k \end{array} \right) \right] \ ;$$

(iv)

$$R_k(c) \equiv \lim_{L\to\infty} \frac{1}{L} \int_0^L \sum_{\lambda_{j_1}(\omega) < \cdots < \lambda_{j_k}(\omega) \leq L} 1_{\{\lambda_{j_k}(\omega) - \lambda_{j_1}(\omega) \leq c\}} = \hat{E}\left[\left(\begin{array}{c} N(0; c] \\ k-1 \end{array} \right) \right] \ ,$$

where $E[\cdot]$ and $\hat{E}[\cdot]$ denote the integration with respect to the measures P and $\hat{P}$ respectively.

If we let

$$f(\omega, s) = 1_{(-\infty,0]}(s) 1_{\{N_\omega(0,x-s]=j\}}$$

in (24), we obtain the so called Palm-Khinchin formula:

$$P(N(0,x] \leq j) = \int_x^\infty \hat{P}(N(0,s] = j) ds \ . \tag{25}$$

(See [6] for another approach.)

This can be used to relate the right hand sides of (i) and (ii), (iii) and (iv). Indeed, letting $j = 0$ in Palm-Khinchin formula above, one obtains

$$P(N(0,c] = 0) = \int_c^\infty \hat{P}(N(0,s] = 0) ds \ ,$$

namely

$$\pi_0(c) = \int_c^\infty (1 - \rho(s)) ds \ ,$$

or

$$\rho(c) = 1 + \frac{d}{dc}\pi_0(c) \ ,$$

whenever $\rho(\cdot)$ is continuous at c . On the other hand,

$$\begin{aligned} E\left[\begin{pmatrix} N(0;c] \\ k \end{pmatrix}\right] &= \sum_{n \geq k} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} P(N(0,c] \geq n) \\ &= \sum_{n \geq k} \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \int_0^c \hat{P}(N(0,s] = n-1) ds \\ &= \int_0^c \hat{E}\left[\begin{pmatrix} N(0;c] \\ k-1 \end{pmatrix}\right] ds \ , \end{aligned}$$

namely

$$\mu_k(c) = \int_0^c R_k(s) ds \ .$$

(Compare [5].)

Hence when the sequence $\{\lambda_j\}$ is the typical realization of a stationary point process, then $\pi_0(c)$, $\rho(c)$, $\mu_k(c)$ and $R_k(c)$ all exist with probability one, and are expressed as appropriate expectation values which are related with each other through Palm- Khinchin formula. By this observation, we are inclined to call the considerations developed in §2 "deterministic point process theory ". After all, the energy level statistics is based upon the hypothesis, often tacitly supposed, that the spectrum of a quantum Hamiltonian looks, after a suitable normalization, like a typical realization of a stationary point process. Hence the phenomenological side of the theory of energy level statistics should be developed in analogy of the theory of point processes.

3 Level statistics for regular spectra.

Suppose that the classical Hamiltonian system associated to $H(p,q)$ is completely integrable. Then one can transform the variable $(p,q) \in \mathbf{R}^d \times \mathbf{R}^d$ into the action–angle variable $(I,\varphi) = (I_1, \ldots, I_d; \varphi_1, \ldots, \varphi_d)$, and $H(p,q) = H(I)$ depends only on the action variable (see [1]).

Now following Percival ([12]) , Berry and Tabor ([2]), we shall say that $H(\hbar)$, the quantizationof $H(p,q)$,has regular spectrum if its energy levels are (approximately) given by quantizing the action variables $I_1,\ldots,I_d$ appearing in $H(I)$. Namely, we suppose that the energy levels of $H(\hbar)$ are

$$E_n(\hbar) = H(\hbar(n+\frac{1}{4}\alpha)) \quad ; \quad n=(n_1,\ldots,n_d)\in \mathbf{Z}_+^d \ , \tag{26}$$

where $\alpha \in \mathbf{Z}_+^d$ is the Maslov index. This procedure is called the EBK quantization after the names of Einstein, Brillouin and Keller, and it gives the exact spectrum in such concrete examples as the rectangular billiards and the harmonic oscillators.

Under mild conditions on $H(I)$ (e.g. $H(I)$ being convex, positive with $H(0)=0$) , regular spectra satisfy the conditions (A1)–(A3) stated in §1 with

$$\nu(E) = \int\cdots\int_{\mathbf{R}_+^d} 1_{\{H(I)\le E\}} dI_1\cdots dI_d \ . \tag{27}$$

Let us apply the unfolding of Berry and Tabor as formulated in §2-1 to our regular spectrum. Note that the suffix n distinguishing the levels is now d-dimensional.

For $x\in \mathbf{R}_+^d\setminus\{0\}$ define $\hbar(x)=\hbar_E(x)\ge 0$ by the equation $H(\hbar(x)\cdot x)=E$, where $E>0$ will be fixed throughout, and let $\lambda(x)=\nu(E)\hbar_E(x)^{-d}$. Our unfolded levels are then given by $\lambda(n+\frac{1}{4}\alpha)$, $n\in\mathbf{Z}_+^d$. Since $\lambda(\beta x)=\beta^d\lambda(x)$ for $\beta>0$, we have the equivalence

$$\lambda(n+\frac{1}{4}\alpha)\in(t,t+c] \ \Leftrightarrow n+\frac{1}{4}\alpha\in\Pi_c(t) \ , \tag{28}$$

where we have defined

$$\Pi_c(t)=\{x\in\mathbf{R}_+^d \mid t^{1/d}\lambda(\varphi(x))^{-1/d}<|x|\le (t+c)^{1/d}\lambda(\varphi(x))^{-1/d}\} \ , \tag{29}$$

with

$$\varphi(x)=\frac{x}{|x|} \ .$$

It is easy to see $|\Pi_c(t)|=c$, where $|\Pi_c(t)|$ is the volume of $\Pi_c(t)$.

Thus the number $N(t)=N(t;c)$ of unfolded levels in $(t,t+c]$ is equal to the number of lattice points (which are shifted by $\frac{1}{4}\alpha$) in the domain $\Pi_c(t)$, namely we have

$$N(t)=\sharp\{\Pi_c(t)\cap(\mathbf{Z}_+^d+\frac{1}{4}\alpha)\} \ . \tag{30}$$

As t gets large, then the domain $\Pi_c(t)$ expands in the space $\mathbf{R}_+^d$, at the same time getting thinner and thinner to keep its volume constant. Hence provided the boundary of $\Pi_c(t)$ is not too degenerate, e.g. flat, then each lattice point in $\mathbf{Z}_+^d$ would randomly belong to $\Pi_c(t)$ if t were chosen at random from a long interval $[0,L]$, so that the total number of lattice points in $\Pi_c(t)$, namely $N(t)$, would obey the Poisson distribution. This is conceivable if one recalls how Poisson's law of small numbers was proved in elementary probability theory. But it must be a very difficult peoblem to justify this intuitive idea for concretely specified $\Pi_c(t)$. In fact no example of Hamiltonian is known for which this program is rigorously performed, to prove the strict sense level clustering.

At this point, it is appropriate to discuss the results of Sinai [15] and Major [7] (see also [9]). They considered the case in which $d = 2$, $\alpha = 0$ and the curve (written in polar coordinate) $r = f(\varphi) = \lambda(\varphi)^{-1/d}$ which defines the boundary of $\Pi_c(t)$ is very random. Especially, Major proved that $\mu_k(c) = c^k/k!$, $k \geq 1$ holds for almost all realization of the random curve $r = f(\varphi)$. Although the randomness they assume is so strong that the curve $r = f(\varphi)$ consisting the boundary of $\Pi_c(t)$ cannot be smooth, violating the natural connection between level statistics and the lattice points counting, their proof suggests us that it would be possible to prove $\mu_k(c) = c^k/k!$ for $k = 1, \ldots, K$ with a finite K if the boundary of $\Pi_c(t)$ has finite dimensional randomness as we are going to argue now.

The conditions for the random curve $r = f(\varphi)$, $0 \leq \varphi \leq \pi/2$ assumed by Sinai and Major (see [7] or [9]) are the following:

(a) $b_1 \leq f(\varphi) \leq b_2$, $|f(\varphi_2) - f(\varphi_1)| \leq b_3|\varphi_2 - \varphi_1|$ for some positive constants b_j , $j = 1, 2, 3$

(b) For any $k \geq 1$, $0 \leq \varphi_1 < \cdots < \varphi_k \leq 2\pi$, the joint probability distribution of $f(\varphi_j)$, $j = 1, \ldots, k$ has a C^1 density

$$p_k(y_1, \ldots, y_k|\varphi_1, \ldots, \varphi_k) .$$

(c) For some $\tau \in (1, 2)$, one has

$$p_k(y_1, \ldots, y_k|\varphi_1, \ldots, \varphi_k) \leq \text{const.} \prod_{j=2}^{k} (\varphi_j - \varphi_{j-1})^{-\tau}$$

(d) Similar conditions for derivatives of p_k .

Define

$$\Pi_c(t; f) = \{x \in \mathbf{R}^2 | 0 \leq \varphi(x) \leq \Theta , \ \sqrt{t} f(\varphi(x)) < |x| < \sqrt{t + c} f(\varphi(x))\} . \tag{31}$$

Then the area of $\Pi_c(t; f)$ equals $\lambda(f) = \frac{c}{2} \int_0^\Theta f(\varphi)^2 d\varphi$. Let $\xi(t; f) = \sharp(\Pi_c(t; f) \cap \mathbf{Z}^2)$ be the number of lattice points caught in $\Pi_c(t; f)$. Then the following proposition holds ([15], [7] and [9]):

Proposition 6 *(i) Under the conditions* **(a)** *to* **(c)** *above, we have*

$$\lim_{t \to \infty} E_P \left[\binom{\xi(t; f)}{k} \right] = \frac{1}{k!} E_P[\lambda(f)^k] , \ k \geq 1 . \tag{32}$$

(ii) Under the conditions **(a)** *to* **(d)** *above, we have with probability one,*

$$\mu_k(c) = \lim_{L \to \infty} \frac{1}{(a_2 - a_1)L} \int_{a_1 L}^{a_2 L} \binom{\xi(t; f)}{k} dt = \frac{1}{k!} \lambda(f)^k , \ k \geq 1 , \tag{33}$$

where $0 < a_1 < a_2$.

We make the following observations:

(1) Under (b) and (c), the curve $r = f(\varphi)$ is not of C^2 , as noticed by Major [7].

(2) Condition (b) means that one has "infinite dimensional randomness "of the curve $r = f(\varphi)$. But the proof of the convergence of up to $k-$th moments requires the existence of the density p_k only, in the assertion (i) of the above proposition, and p_{2k} , in the assertion (ii).

(3) The hardest part of the proof consists in obtaining good bounds of the number of lattice points in $\Pi_c(t; f)$ for which their angles φ are very close together. For this purpose, it is necessary to impose a condition like (c) .

Now let us consider, for example, the d-dimensional billiard of a particle of mass 1 in the rectangle $\Lambda = \Pi_{j=1}^d [0, a_j]$. Then its classical Hamiltonian is $H(I) = \frac{\pi^2}{2} \sum_{j=1}^d (I_j/a_j)^2$, if expressed in terms of action variables, and its quantization is $H(\hbar) = -\frac{\hbar^2}{2}\Delta$ with the Dirichlet boundary condition on $\partial\Lambda$. Its energy levels (eigenvalues) are exactly given by

$$E_n(\hbar) = H(\hbar n) = \frac{\pi^2 \hbar^2}{2} \sum_{j=1}^d (\frac{n_j}{a_j})^2 \quad (n = (n_1, \dots, n_d) \ , \ n_j \geq 1) \ , \tag{34}$$

and $\Pi_c(t)$ is given by

$$\Pi_c(t) = \{x \in \mathbf{R}_+^d \mid t^{2/d} < (\frac{c_d}{2^d}) \left(\prod_{j=1}^d a_j\right)^{2/d} \sum_{j=1}^d (\frac{x_j}{a_j})^2 < (t+c)^{2/d}\} \ , \tag{35}$$

where c_d is the volume of the unit ball. We can then conjecture that if $(a_1 \dots, a_d)$ is a d-dimensional random variable with smooth density, then the surface of $\Pi_c(t)$ has d-dimensional randomness and for sufficiently large d , one would have $\mu_k(c) = c^k/k!$ for $k = 2$ and 3 for almost all $(a_1 \dots, a_d)$, yielding the wide sense level clustering according to Proposition 4.

Before closing this section, we remark that Sarnak [13] proved $R_2(c) = c$ by a number theoretic method for the quantized uniform motion on a two dimensional flat torus. According to Proposition 5 in §2, this is equivalent to $\mu_2(c) = c^2/2$. Unfortunately, we cannot apply Proposition 4 to conclude the wide sense level clustering in this case, because we have no information on $\bar{\mu}_3(c)$.

We also remark that if $\{\lambda_j\}$ is a typical realization of renewal process, namely a stationary point process N in which $\lambda_{n+1} - \lambda_n$, $n \geq 1$ are i.i.d under the Palm measure $\hat{P}$, then $R_2(c) = c$ implies that N is Poissonian. Indeed, if we set

$$F(c) = \hat{P}(N(0, c] > 0) = \hat{P}(\lambda_{n+1} - \lambda_n \leq c) \ ,$$

which does not depend on n , then

$$R_2(c) = \hat{E}[N(0, c]] = \sum_{n=1}^\infty \hat{P}(N(0, c] \geq n) = \sum_{n=1}^\infty \hat{P}(\sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j \leq c) = \sum_{n=1}^\infty (F^{n*})(c) \ ,$$

where F^{n*} is the n-fold convolution of F . Hence if $\mathcal{L}(\xi) = \int_0^\infty e^{-\xi c} dF(c)$ is the Laplace transform of $dF(c)$, then from $R_2(c) = c$, we get

$$\sum_{n=1}^\infty (\mathcal{L}(\xi))^n = \frac{\mathcal{L}(\xi)}{1 - \mathcal{L}(\xi)} = \int_0^\infty e^{-\xi c} dc = \frac{1}{\xi} \ ,$$

namely $\mathcal{L}(\xi) = 1/(1+\xi)$. Hence $dF(c) = e^{-c}dc$. But the Poisson point process is the only renewal process for which $\lambda_{n+1} - \lambda_n$ obeys the exponential distribution.

4 An example of level statistics, in which non-Poissonian level clustering is observed.

Consider the one-dimensional Schrödinger operator

$$H_v(\hbar) = -\hbar^2 \frac{d^2}{dx^2} + v \sum_{j=1}^{n} \delta(x - x_j) \quad 0 \le x \le 1 \tag{36}$$

with Dirichlet boundary condition at $x = 0, 1$. Here $v > 0$ and

$$0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1 \ .$$

Let $E_n(\hbar)$ be the energy levels of $H_v(\hbar)$ and let $\kappa_n(\hbar) = \frac{1}{\pi}\sqrt{E_n(\hbar)}$. Then

$$\sharp\{n | \kappa_n(\hbar) \le E\} \sim \frac{E}{\hbar} \ , \ E/\hbar \to \infty \ . \tag{37}$$

It is not obvious if Assumption **(A2)** holds, but the above asymptotics is stronger than Assumption **(A3)**, so we shall directly consider the statistics for $\{\kappa_n(\hbar)\}$.

We can prove the following ([8]):

Let $0 \le a_1 < a_2$ and let

$$N_\hbar(t; c) = \sharp\{n | \kappa_n(\hbar) \in (t, t + c\hbar]\} \ . \tag{38}$$

Then the limit

$$\pi_k(c) = \lim_{\hbar \searrow 0} \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} 1_{\{N_\hbar(t;c) = k\}} dt \ , \ k \ge 1 \tag{39}$$

always exists, and when $y_j = x_{j+1} - x_j$, $j = 0, 1, \ldots, n$ are rationally independent, they are explicitly computable. In particular

$$\pi_0(c) = \prod_{j=0}^{n} (1 - cy_j) \tag{40}$$

for $c > 0$ such that $cy_j < 1$, $j = 0, 1, \ldots, n$. Hence by Proposition 3 in §2,

$$\begin{aligned} \rho(c) &= 1 - \Big(\sum_{j=0}^{n} \frac{y_j}{1 - cy_j}\Big) \prod_{j=0}^{n} (1 - cy_j) \\ &\sim c\Big(1 - \sum_{j=0}^{n} y_j^2\Big) \ , \ c \searrow 0 \end{aligned}$$

so that $\rho'(0+) = 1 - \sum_{j=0}^{n} y_j^2 > 0$. Hence we have level clustering, but the level spacing distribution $d\rho(c)$ is different from the exponential distribution.

Now let $X_1, X_2, \dots$ be i.i.d random variables uniformly distributed in $(0,1)$, and let $X_1^{(n)} < \cdots < X_n^{(n)}$ be the rearrangement of $X_n, \dots, X_n$ according to its magnitude. Let $x_j = X_j^{(n)}$, $j = 1, \dots, n$, in (36) . Then it can be shown that with probability one,

$$\lim_{n\to\infty} \pi_k(c; X_1^{(n)}, \dots, X_n^{(n)}) = e^{-c}\frac{c^k}{k!} \ , \ k \geq 0 \tag{41}$$

Moreover,

$$\lim_{n\to\infty} \rho(c; X_1^{(n)}, \dots, X_n^{(n)}) = 1 - e^{-c} \ . \tag{42}$$

Thus we obtain Poisson distribution in the high disorder limit.

We note that considering $\kappa_n(\hbar)$ instead of $E_n(\hbar)$ amounts to unfolding the spectrum $\{E_n(\hbar)\}$ so that it will have asymptotically uniform distribution with mean density $1/\hbar$, as can be seen from (37) . We then take the limit $\hbar \searrow 0$ to accomplish the level statistics. This procedure, compared to the unfolding of Berry and Tabor, looks more natural in spirit, but if we apply it to regular spectra, the connection of the level statistics to lattice points counting is not as clear-cut as in §3.

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