ON THE POISSON LIMIT THEOREMS OF SINAI AND MAJOR

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Abstract. Let $f(\varphi)$ be a positive continuous function on $0 \le \varphi \le \Theta$, where $\Theta \le 2\pi$, and let ξ be the number of two-dimensional lattice points in the domain $\Pi_R(f)$ between the curves $r = (R + c_1/R)f(\varphi)$ and $r = (R + c_2/R)f(\varphi)$, where $c_1 < c_2$ are fixed. Randomizing the function f according to a probability law $\mathbf{P}$, and the parameter R according to the uniform distribution μ_L on the interval $[a_1 L, a_2 L]$, Sinai showed that the distribution of ξ under $\mathbf{P} \times \mu_L$ converges to a mixture of the Poisson distributions as $L \to \infty$. Later Major showed that for $\mathbf{P}$-almost all f , the distribution of ξ under μ_L converges to a Poisson distribution as $L \to \infty$. In this note, we shall give shorter and more transparent proofs to these interesting theorems, at the same time extending the class of $\mathbf{P}$ and strengthening the statement of Sinai.

§1 Introduction

In connection to a question in the theory of quantum chaos, Sinai proposed ([S1]) and solved ([S2]) the following problem.

Given a planar curve which is described in the polar coordinate as

$$r = f(\varphi) \ , \ 0 \le \varphi \le \Theta \ , \tag{1.1}$$

with $\Theta \in (0, 2\pi]$ and $f(\cdot) > 0$ continous, consider the domain $\Pi_R(f)$ defined by

$$\begin{aligned} \Pi_R(f) = \{x \in \mathbf{R}^2 |\ 0 \le \varphi(x) \le \Theta \ , \\ (R + c_1/R)f(\varphi(x)) < |x| < (R + c_2/R)f(\varphi(x))\} \ . \end{aligned} \tag{1.2}$$

Here $c_1 < c_2$ are real constants and $R > 0$ is a parameter which we shall let tend to infinity. Moreover $|x| = r = \sqrt{x_1^2 + x_2^2}$ is the distance of x from the origin and $\varphi(x)$ is the angle between the vector $x = (x_1, x_2)$ and the positive real axis. If we set

$$\lambda(f) = (c_2 - c_1) \int_0^\Theta f(\varphi)^2 d\varphi \ , \tag{1.3}$$

then as $R \to \infty$, the area of $\Pi_R(f)$ is asymptotically given by $\lambda(f)(1 + O(R^{-2}))$. Thus the domain $\Pi_R(f)$ becomes thinner and thinner but its area is asymptotically constant. Now consider the quantity

$$\xi = \xi(R; f) = \sharp\{\Pi_R(f) \cap \mathbf{Z}^2\} \ , \tag{1.4}$$

which is the number of the lattice points in $\Pi_R(f)$. Then the problem, which is of physical significance, is the following:

Suppose the parameter $R > 0$ is randomly distributed on the interval $I_L = (a_1 L, a_2 L)$, $0 < a_1 < a_2$, according to the uniform distribution μ_L . What is the limiting probability distribution of $\xi(\cdot; f)$ under μ_L as $L \to \infty$? Numerical studies suggest that for typical smooth curves f which are obtained from Hamiltonians of integrable dynamical systems, the limiting distribution is Poissonian. To give a rigorous proof to this conjecture for a concretely given f remains as a difficult open question. What Sinai did was to randomize the function f according to some probability law $\mathbf{P}$ on some function space. Then he showed that under the probability measure $\mathbf{P} \times \mu_L$, the distribution of $\xi(\cdot; \cdot)$ converges, as $L \to \infty$, to a mixture of Poisson distributions each of

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which having the parameter $\lambda(f)$. He also showed that there is a sequence $\bar{L}_n \to \infty$ such that for $\mathbf{P}$-almost all f , the law of $\xi(\cdot; f)$ under $\mu_{\bar{L}_n}$ converges as $n \to \infty$ to the Poisson distribution with parameter $\lambda(f)$. Later Major [M] extended Sinai's idea and succeeded in getting rid of subsequences. Namely by a more elaborate analysis, he proved that as $L \to \infty$, the distribution of $\xi(\cdot; f)$ under μ_L converges to the Poisson distribution with parameter $\lambda(f)$, for $\mathbf{P}$-almost all f .

Under the conditions on $\mathbf{P}$ posed by Sinai and Major (and which we shall assume also in this note), the sample functions $f(\varphi)$ is not smooth in φ (see Remark 2 of [M]), so that the direct connection of their results to the problem of quantum chaos is lost. But still the theorems of Sinai and Major are interesting from the viewpoint of elementary probability theory, since they provide us a new limit theorem which gives the Poisson distribution. The present author has the opinion that these theorems deserve to be made more popular among probabilists who are not necessarily interested in such topics as quantum chaos. The author also believes that in the future study of quantum chaos too, one will need to prove variants of these theorems under various different conditions, e.g. for higher dimensional lattices. For these purposes, however, the proofs of these theorems should be made as transparent and readable as possible. In fact, Sinai's paper [S2] contains many unnecessary arguments and is obscure at some crucial points. (For example, the interrelation among his conditions I) to VI) are not clear enough.) There is no such ambiguity in Major's paper [M], but some of his arguments are also dispensable.

In this paper, we will make a more direct attack to the theorems of Sinai and Major, to improve them in the following way:

(1) The distribution under $\mathbf{P}$ of $\xi(R; \cdot)$ converges, as $R \to \infty$, to a mixture of Poisson distribution. This result has a much simpler proof than Sinai's one and implies Sinai's original assertion as a trivial corollary.
(2) We shall give a slightly different formulation of Major's theorem, which turns out to be equivalent to the original one but technically more tractable and more closely related to the problem in quantum chaos raised by Berry and Tabor [BT] . (See also [Mi].)
(3) The theorems are extended to the case where $f(\cdot)$ are not Lipschitz continuous. Major's proof cannot cover this case because he makes explicit use of the sample Lipschitz continuity. But in fact, we need to use the samplewise Lipschitz continuity only in combination with the upper bound of the probability density, so that these two conditions ((1.7) and (1.8) in (II-2) below) can entirely be replaced by a single condition (II-1). As we shall see below, the number $\sigma \in (0, 1)$ in (II-1) plays the roll of $\tau - 1$ in (II-2), τ appearing in (1.8), where the number 1 comes from the Lipschitz continuity of $f(\cdot)$.
(4) We can slightly loosen the technical condition on the partial derivative of the probability density as in (III). (Compare this with (2b) of Property A of [M].) Moreover even if we do not assume this condition, we still have a partial result, Thorem 2.
(5) We prove the limit theorems by directly proving the convergence of factorial moments of all order as in (2.2) and (3.5). These assertions are actually stronger than those of Sinai and Major because they proved the convergence of moments not for the random variables $\xi(R; f)$ themselves but for their modifications. As discussed by the present author in [Mi], the convergence of k-th moments may have applications in a problem of quantum chaos even if it is proven only for finite numbers of k's, so that the Poisson limit theorem is not available.

Now let us formulate our conditions and theorems. We suppose that our probability measure $\mathbf{P}$ is defined on the function space

$$\mathcal{X} = \mathcal{X}_{b_1, b_2} = \{f | f \text{ is continuous on } [0, \Theta] \text{ and } b_1 \leq f(\varphi) \leq b_2\} \tag{1.5}$$

with $0 < b_1 < b_2$, and we assume that $\mathbf{P}$ satisfies the following two conditions (I) and (II).

Condition (I). For each $k \geq 1$, and $(\varphi_1, \dots, \varphi_k)$ $(\varphi_i \neq \varphi_j)$ in $[0, \Theta]^k$, the distribution of $(f(\varphi_1), \dots, f(\varphi_k))$ has a density

$$p_k(y_1, \dots, y_k | \varphi_1, \dots, \varphi_k)$$

which is C^1 on

$$[b_1, b_2]^k \times \{(\varphi_1, \dots, \varphi_k) \in [0, \Theta]^k \mid \varphi_i \neq \varphi_j \ (i \neq j)\} .$$

We note that if π is a permutation of $\{1, \dots, k\}$, then

$$p_k(y_{\pi(1)}, \dots, y_{\pi(k)} | \varphi_{\pi(1)}, \dots, \varphi_{\pi(k)}) = p_k(y_1, \dots, y_k | \varphi_1, \dots, \varphi_k) .$$

Condition (II). Either of the conditions (II-1) and (II-2) below holds:

(II-1) There exists a $\sigma \in (0,1)$ such that the inequality

$$p_k(y_1, \dots, y_k|\varphi_1, \dots, \varphi_k) \ \leq \ B_k \prod_{j=2}^{k} (\varphi_j - \varphi_{j-1})^{-\sigma} \tag{1.6}$$

holds with some constant B_k for all $k \geq 2$ and $0 \leq \varphi_1 < \cdots < \varphi_k \leq \Theta$.

(II-2) There is a constant $b_3 > 0$ such that for $\mathbf{P}$-almost all $f \in \mathcal{X}$,

$$Y(f) \equiv \sup_{0 \leq \varphi_1 < \varphi_2 \leq \Theta} \frac{|f(\varphi_2) - f(\varphi_1)|}{\varphi_2 - \varphi_1} \ \leq \ b_3 \ . \tag{1.7}$$

Moreover there exists a $\tau \in (1,2)$ such that the inequality

$$p_k(y_1, \dots, y_k|\varphi_1 \dots, \varphi_k) \ \leq \ C_k \prod_{j=2}^{k} (\varphi_j - \varphi_{j-1})^{-\tau} \tag{1.8}$$

holds with some constant C_k for all $k \geq 2$ and $0 \leq \varphi_1 < \cdots < \varphi_k \leq \Theta$.

Under these conditions, we can prove Theorems 1 and 2 below:

Theorem 1. *As $R \to \infty$, the distribution of $\xi(R;\cdot)$ under the probability measure $\mathbf{P}$ converges to the mixture of Poisson distributions each of which having the parameter $\lambda(f)$, where $\lambda(f)$ was defined in (1.3). Namely we have*

$$\lim_{R\to\infty} \mathbf{P}(\xi(R;\cdot) = k) = \mathbf{E_P}\left[e^{-\lambda(f)}\frac{\lambda(f)^k}{k!}\right] \ , \ k = 0, 1, \dots . \tag{1.9}$$

As an immediate corollary, we obtain Sinai's theorem ([S2]).

Corllary. *As $L \to \infty$, the distribution of $\xi(\cdot;\cdot)$ under the probability measure $\mu_L \times \mathbf{P}$ converges to the mixture of Poisson distribution, each of which having the parameter $\lambda(f)$, namely*

$$\lim_{L\to\infty} (\mu_L \times \mathbf{P})(\xi = k) = \mathbf{E_P}\left[e^{-\lambda(f)}\frac{\lambda(f)^k}{k!}\right] \ , \ k = 0, 1, \dots . \tag{1.10}$$

Sinai proved this assertion under conditions which are close to our conditions (I) and (II-2) with $\tau = 1$. Under the same conditions, he also showed the following theorem, which we shall prove, simultaneously with Theorem 3, under wider conditions (I) and (II).

Theorem 2. *There is a sequence $\bar{L}_n \to \infty$ such that for $\mathbf{P}$-almost all $f \in \mathcal{X}$, the distribution of $\xi(\cdot;f)$ under $\mu_{\bar{L}_n}$ converges to the Poisson distribution with parameter $\lambda(f)$, namely we have*

$$\lim_{n\to\infty} \mu_{\bar{L}_n}(\xi(\cdot;f) = k) = e^{-\lambda(f)}\frac{\lambda(f)^k}{k!} \ , \ k = 0, 1, \dots , \tag{1.11}$$

for $\mathbf{P}$-almost all $f \in \mathcal{X}$.

If we assume, in addition to Conditions (I) and (II), the following technical condition, we can get rid of subsequences.

Condition (III). For some constants $A_k > 0$, $\nu_k \in (0, d/2)$, where $k \geq 2$ and $d = 1-\sigma$ or $= 2-\tau$ according to which of (II-1) and (II-2) $\mathbf{P}$ satisfies, we have

$$|\partial_i p_k(y_1, \dots, y_k|\varphi_1, \dots, \varphi_k)| \ \leq \ A_k \exp(\beta^{-\nu_k}) \tag{1.12}$$

whenever $k \geq 2$, $b_1 \leq y_j \leq b_2$, $|\varphi_i - \varphi_j| \geq \beta > 0$, $i, j = 1, \dots, k$. Here ∂_i denotes either $\partial/\partial y_i$ or $\partial/\partial \varphi_i$.

Theorem 3. *If we assume Condition (III) in addition to Conditions (I) and (II), then for* $\mathbf{P}$*-almost all* $f \in \mathcal{X}$ *, the distribution of* $\xi(\cdot; f)$ *under* μ_L *converges to the Poisson distribution with parameter* $\lambda(f)$ *as* $L \to \infty$ *, namely we have*

$$\lim_{L\to\infty} \mu_L(\xi(\cdot; f) = k) = e^{-\lambda(f)} \frac{\lambda(f)^k}{k!} \ , \ k = 0, 1 \dots , \tag{1.13}$$

for $\mathbf{P}$*-almost all* $f \in \mathcal{X}$ *.*

This was proved by Major under Conditions (I), (II-2) and a slightly more restrictive condition than (III).

Before closing this introduction, we shall give an example of $\mathbf{P}$ satisfying our conditions (I), (II-1) and (III). (An example which satisfies (II-2) instead of (II-1) was discussed by Sinai [S2] and Major [M].)

Let $\{X_t; t \geq 0\}$ be the reflecting Brownian motion on the interval $S = [b_1, b_2]$ with fixed $X_0 = b$, and let $\mathbf{P}$ be its probability law. Then if we set $f(\varphi) = X_{1+\varphi}$, the law of $\{f(\varphi); 0 \leq \varphi \leq \Theta\}$ under $\mathbf{P}$ satisfies our requirements.

In fact, the infinitesimal generator of $\{X_t\}$ is the Neumann Laplacian $\frac{1}{2}\Delta$ on the interval S . Let $\cdots < -\lambda_n < \cdots < -\lambda_1 < -\lambda_0 = 0$ be its eigenvalues and $\psi_n(x)$, $n \geq 0$ be the corresponding normalized eigenfunctions. The transition density $P_t(x, y)$ of $\{X_t\}$ has the eigenfunction expansion:

$$P_t(x, y) = \sum_{n\geq 0} e^{-\lambda_n t} \psi_n(x)\psi_n(y) \ , \tag{1.16}$$

and if $0 \leq \varphi_1 < \cdots < \varphi_k \leq \Theta$, the joint distribution of $(f(\varphi_1), \dots, f(\varphi_k))$ under $\mathbf{P}$ has the density

$$\begin{aligned} &p_k(y_1, \dots, y_k|\varphi_1, \dots, \varphi_k) \\ &= P_{1+\varphi_1}(b, y_1) P_{\varphi_2-\varphi_1}(y_1, y_2) \cdots P_{\varphi_k - \varphi_{k-1}}(y_{k-1}, y_k) \ . \end{aligned} \tag{1.17}$$

Now it is elementary to calculate λ_n and $\psi_n(x)$ explicitly:

$$\lambda_n = \frac{\pi^2 n^2}{2(b_2 - b_1)^2} \quad ; \ \psi_n(x) = \sqrt{\frac{2}{b_2 - b_1}} \cos \frac{n\pi}{b_2 - b_1}(x - b_1) \ , \ n \geq 0 \ . \tag{1.18}$$

¿From this, it is clear that $P_t(x, y)$ is C^1 on $[0, \infty) \times [b_1, b_2]^2$ and hence from (1.17), our $p_k(\cdot|\cdot)$ is C^1 in all of its variables $y_1, \dots, y_k$ and $\varphi_1, \dots, \varphi_k$ as required in Condition (I).

It is clear from (1.16), (1.17) and (1.18) that there is a constant B'_k such that

$$p_k(y_1, \dots, y_k|\varphi_1, \dots, \varphi_k) \leq B'_k \prod_{j=2}^{k} \left(\sum_{n\geq 0} e^{-\lambda_n(\varphi_j - \varphi_{j-1})} \right) \ . \tag{1.19}$$

On the other hand, if we let

$$N(\lambda) = \sum_{\lambda_n \leq \lambda} 1 = \left[\frac{\sqrt{2\lambda}}{\pi}(b_2 - b_1) \right] \ , \tag{1.20}$$

then

$$\sum_{n\geq 0} e^{-\lambda_n t} = \int_{0-}^{\infty} e^{-\lambda t} dN(\lambda) \tag{1.21}$$

and since

$$N(\lambda) \sim \frac{\sqrt{2}(b_2 - b_1)}{\pi}\sqrt{\lambda} \tag{1.22}$$

as $\lambda \to +\infty$, we can use the Abelian theorem to obtain

$$\sum_{n\geq 0} e^{-\lambda_n t} \sim \text{const.} t^{-1/2} \tag{1.23}$$

as $t \downarrow 0$, where the constant can be explicitly given. Hence by (1.19), there is another constant $B_k > 0$ such that

$$p_k(y_1, \ldots, y_k | \varphi_1, \ldots, \varphi_k) \leq B_k \prod_{j=2}^{k} (\varphi_j - \varphi_{j-1})^{-1/2} \ , \tag{1.24}$$

so that Condition (II-1) is valid with $\sigma = 1/2$.

As is obvious from (1.17), in order to verify Condition (III), it suffices to obtain an upper bound of $|(\partial/\partial y)P_t(x,y)|$ and $|(\partial/\partial t)P_t(x,y)|$. Differentiating (1.16) term by term, we obtain

$$\left|\frac{\partial}{\partial y}P_t(x,y)\right| \leq \text{const.} \int_{0-}^{\infty} e^{-\lambda t} dN_1(\lambda) \tag{1.25}$$

and

$$\left|\frac{\partial}{\partial t}P_t(x,y)\right| \leq \text{const.} \int_{0-}^{\infty} e^{-\lambda t} dN_2(\lambda) \ , \tag{1.26}$$

where we have set

$$N_1(\lambda) = \sum_{n;\lambda_n \leq \lambda} n \quad ; \ N_2(\lambda) = \sum_{n;\lambda_n \leq \lambda} n^2 \ . \tag{1.27}$$

Since as $\lambda \to \infty$ we have

$$N_1(\lambda) \sim \text{const.}\lambda \quad ; \ N_2(\lambda) \sim \text{const.}\lambda^{3/2} \ , \tag{1.28}$$

we can again use the Abelian theorem to obtain

$$\left|\frac{\partial}{\partial y}P_t(x,y)\right| = O(t^{-1}) \tag{1.29}$$

and

$$\left|\frac{\partial}{\partial t}P_t(x,y)\right| = O(t^{-3/2}) \tag{1.30}$$

as $t \downarrow 0$. Hence we can conclude that for some constant $A_k > 0$,

$$|\partial_i p_k(y_1, \ldots, y_k | \varphi_1, \ldots, \varphi_k)| \leq A_k \beta^{-3/2} \tag{1.31}$$

whenever $b_1 \leq y_j \leq b_2$, $|\varphi_i - \varphi_j| \geq \beta > 0$, $i, j = 1, \ldots, k$, which is much stronger than (1.12).

§2 Proof of Theorem 1

For $k \geq 1$, define

$$\xi_k = \xi_k(R; f) = \binom{\xi}{k} 1_{\{\xi \geq k\}} = \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} 1_{\{\xi = n\}} . \tag{2.1}$$

As in [S2], for the proof of Theorem 1, it suffices to prove

$$\lim_{R\to\infty} \mathbf{E}_{\mathbf{P}}[\xi_k(R;\cdot)] = \frac{1}{k!}\mathbf{E}_{\mathbf{P}}[\lambda(f)^k] \tag{2.2}$$

for each $k \geq 1$. On the other hand, by a simple combinatorial argument, we can write

$$\xi_k = \sum_{\sharp A = k} 1_{\{A \subset \Pi_R(f)\}} , \tag{2.3}$$

where the summation ranges over all k-point subsets A of $\mathbf{Z}^2$.

If m_1 , $m_2 \in \mathbf{Z}^2$ are distinct but $\varphi(m_1) = \varphi(m_2)$, then we must have $|m_1 - m_2| \geq 1$. Hence by the definition of $\Pi_R(f)$, if $R > 0$ is large, then m_1 , $m_2 \in \Pi_R(f)$ with $m_1 \neq m_2$ is possible only if $\varphi(m_1) \neq \varphi(m_2)$. Noting $b_1 \leq f \leq b_2$, we can therefore rewrite (2.3) as

$$\xi_k = \sum_{(m_1,\dots,m_k)\in \mathcal{Z}_k(R)} 1_{\{m_1,\dots,m_k \in \Pi_R(f)\}} \tag{2.4}$$

when $R > 0$ is large, where we have set

$$\begin{aligned}\mathcal{Z}_k(R) = \{(m_1, \dots, m_k) \in (\mathbf{Z}^2)^k | \ & \frac{b_1}{2} R \leq |m_j| \leq 2b_2 R \ , j = 1, \dots, k \\ & \text{and } 0 \leq \varphi(m_1) < \cdots < \varphi(m_k) \leq \Theta\} .\end{aligned} \tag{2.5}$$

Now it is easy to see that $m \in \Pi_R(f)$ is equivalent to

$$f(\varphi(m)) \in I_m^R \equiv \left(\frac{|m|}{R + c_2/R}, \frac{|m|}{R + c_1/R} \right) , \tag{2.6}$$

and hence

$$\mathbf{E}_{\mathbf{P}}[\xi_k(R;\cdot)] = \sum_{\mathcal{Z}_k(R)} \mathbf{P}(f(\varphi(m_j)) \in I_{m_j}^R \ , \ j = 1, \dots, k) . \tag{2.7}$$

Take a $\beta > 0$ and define

$$\mathcal{Z}_k(\beta; R) = \{(m_1, \dots, m_k) \in \mathcal{Z}_k(R) | \ \varphi(m_j) - \varphi(m_{j-1}) \geq \beta \ , \ j = 2, \dots, k\} ; \tag{2.8}$$

and

$$\mathcal{Z}_k'(\beta; R) = \mathcal{Z}_k(R) \setminus \mathcal{Z}_k(\beta; R) \tag{2.9}$$

so that

$$\sum_{\mathcal{Z}_k(R)} = \sum_{\mathcal{Z}_k(\beta;R)} + \sum_{\mathcal{Z}_k'(\beta;R)} . \tag{2.10}$$

Let us first estimate the sum over $\mathcal{Z}_k(\beta; R)$. By the definition (2.6) of I_m^R , we have as $R \to \infty$,

$$\text{the center of } I_m^R = \frac{|m|}{R}(1 + O(R^{-2})) \tag{2.11}$$

and

$$\text{the length of } I_m^R = \frac{(c_2-c_1)|m|}{R^3}(1+O(R^{-2})) \ . \tag{2.12}$$

In particular, if $(m_1,\dots,m_k)\in\mathcal{Z}_k(R)$, the length of I_m^R is of $O(R^{-2})$ for $j=1,\dots,k$.

Define for $\beta>0$

$$\Psi_k(\beta) = \sup_{\substack{(y_1,\dots,y_k)\\ \in[b_1,b_2]^k}} \sup_{\substack{(\varphi_1,\dots,\varphi_k)\in[0,\Theta]^k\\ \varphi_j-\varphi_{j-1}\ge\beta\\ j=2,\dots,k}} \max_{1\le i\le k} \max\left\{ |\frac{\partial}{\partial y_i}p_k(y_1,\dots,y_k|\varphi_1,\dots,\varphi_k)| \ , \ |\frac{\partial}{\partial\varphi_i}p_k(y_1,\dots,y_k|\varphi_1,\dots,\varphi_k)| \right\} \tag{2.13}$$

and

$$\psi_k(\beta) = \sup_{\substack{(y_1,\dots,y_k)\\ \in[b_1,b_2]^k}} \sup_{\substack{(\varphi_1,\dots,\varphi_k)\in[0,\Theta]^k\\ \varphi_j-\varphi_{j-1}\ge\beta\\ j=2,\dots,k}} p_k(y_1,\dots,y_k|\varphi_1,\dots,\varphi_k) \ . \tag{2.14}$$

Then by Condition (I), and (2.11), (2.12) and the remark following it, we will have

$$\begin{aligned} &\mathbf{P}(f(\varphi(m_j))\in I_{m_j}^R \ , \ j=1,\dots,k) \\ &= \int_{I_{m_1}^R} dy_1\cdots\int_{I_{m_k}^R} dy_k \ p_k(y_1,\dots,y_k|\varphi(m_1),\dots,\varphi(m_k)) \\ &= p_k(\frac{|m_1|}{R},\dots,\frac{|m_k|}{R}|\varphi(m_1),\dots,\varphi(m_k))\prod_{j=1}^k\frac{(c_2-c_1)|m_j|}{R^3} \\ &\quad + O((\psi_k(\beta)+\Psi_k(\beta))R^{-2k-2}) \ . \end{aligned} \tag{2.15}$$

Hence noting $\sharp\mathcal{Z}_k(R)=O(R^{2k})$, we get

$$\begin{aligned} &\sum_{\mathcal{Z}_k(\beta;R)}\mathbf{P}(f(\varphi(m_j))\in I_{m_j}^R \ , \ j=1,\dots,k) \\ &= (c_2-c_1)^k\sum_{\mathcal{Z}_k(\beta;R)} p_k(\frac{|m_1|}{R},\dots,\frac{|m_k|}{R}|\varphi(m_1),\dots,\varphi(m_k))\left(\prod_{j=1}^k\frac{|m_j|}{R}\right)\left(\frac{1}{R^2}\right)^k \\ &\quad + O((\psi_k(\beta)+\Psi_k(\beta))R^{-2}) \ . \end{aligned} \tag{2.16}$$

Now let us consider a function on $(\mathbf{R}^2)^k$ defined by

$$R(x_1,\dots,x_k) = p_k(|x_1|,\dots,|x_k| \mid \varphi(x_1),\dots,\varphi(x_k))\left(\prod_{j=1}^k|x_j|\right) \ . \tag{2.17}$$

Then the sum on the right hand side of (2.16) is the approximation by Riemann sum of the integral

$$(c_2-c_1)^k\int\cdots\int_{D_k(\beta)} dx_1\cdots dx_k \ R(x_1,\dots,x_k) \ , \tag{2.18}$$

where the domain $D_k(\beta)$ of integration is given by

$$\begin{aligned} D_k(\beta) = \{(x_1,\dots,x_k)\in(\mathbf{R}^2)^k|&\frac{b_1}{2}\le|x_j|\le 2b_2 \ , \ j=1,\dots,k \ , \\ &0\le\varphi(x_j)\le\Theta \ , \\ &\varphi(x_j)-\varphi(x_{j-1})\ge\beta \ , \ j=2,\dots,k \ \} \ . \end{aligned} \tag{2.19}$$

It is easy to see that

$$\max_{1\le j\le k}\left\|\frac{\partial R}{\partial x_j}\right\| \le \text{const.}(\psi_k(\beta)+\Psi_k(\beta)) \tag{2.20}$$

holds on $D_k(\beta)$. This gives the accuracy of the above Riemann sum approximation, and we obtain, after replacing the domain of integration $D_k(\beta)$ by

$$\begin{aligned} D_k = \{(x_1,\dots,x_k)\in(\mathbf{R}^2)^k | \frac{b_1}{2} \le |x_j| \le 2b_2 \ , \ j=1,\dots,k \ , \\ 0\le\varphi(x_{j-1})<\varphi(x_j)\le\Theta \ , \ j=2,\dots,k\} \ , \end{aligned} \tag{2.21}$$

and tranforming the variable into the polar coordinate,

$$\begin{aligned} &\sum_{\mathcal{Z}_k(\beta;R)} \mathbf{P}(f(\varphi(m_j))\in I_m^R \ , \ j=1,\dots,k) \\ &= (c_2-c_1)^k \int\cdots\int_{0\le\varphi_1<\cdots<\varphi_k\le\Theta} d\varphi_1\cdots d\varphi_k \int\cdots\int_{b_1\le y_j\le b_2 \ , \ j=1,\dots,k} dy_1\cdots dy_k \times \\ &\quad\times\left(\prod_{j=1}^k y_j^2\right) p_k(y_1,\dots,y_k|\varphi_1,\dots,\varphi_k) \\ &\quad + O((\psi_k(\beta)+\Psi_k(\beta))R^{-1}+\beta) \\ &= \frac{1}{k!}\mathbf{E}_{\mathbf{P}}[\lambda(f)^k] + O((\psi_k(\beta)+\Psi_k(\beta))R^{-1}+\beta) \ . \end{aligned} \tag{2.22}$$

In order to complete the proof of (2.2), it remains to prove

$$\lim_{\beta\downarrow 0}\overline{\lim_{R\uparrow\infty}} \sum_{\mathcal{Z}'_k(\beta;R)} \mathbf{P}(f(\varphi(m_j))\in I_{m_j}^R \ , \ j=1,\dots,k) = 0 \ . \tag{2.23}$$

Suppose first that $\mathbf{P}$ satisfies (II-1). By the remark which follows (2-12), we see that

$$\mathbf{P}(f(\varphi(m_j))\in I_{m_j}^R \ , \ j=1,\dots,k) \le \text{const.}R^{-2k}\prod_{j=2}^k(\varphi(m_j)-\varphi(m_{j-1}))^{-\sigma} \ . \tag{2.24}$$

Hence Lemma 1 below gives

$$\overline{\lim_{R\uparrow\infty}} \sum_{\mathcal{Z}'_k(\beta;R)} \mathbf{P}(f(\varphi(m_j))\in I_m^R \ , \ j=1,\dots,k) = O(\beta^{1-\sigma}) \tag{2.25}$$

for small $\beta>0$, and (2.23) follows immediately from this.

Lemma 1. *For large $R>0$ and for small $\beta>0$, we have the estimate*

$$\max_{m\in\mathcal{Z}_1(R)} \sum_{\substack{m'\in\mathcal{Z}_1(R) \ ; \\ 0<\varphi(m')-\varphi(m)<\beta}} (\varphi(m')-\varphi(m))^{-\sigma} = O(R^2(R^{\sigma-1}+\beta^{1-\sigma})) \ .$$

Proof. We may assume $0\le\varphi(m)\le\pi/4$, because $\mathbf{Z}^2$ is invariant under the rotation by $\pi j/2$, $j=1,2,3$, and because the lattice obtained by rotating $\mathbf{Z}^2$ by $\pi/4+\pi j/2$, $j=1,2,3$ is contained in the lattice $(1/\sqrt{2})\mathbf{Z}^2$. But in this case, we have $\rho\equiv\tan\varphi(m)\in[0,1]$.

Writing $m' = (\mu_1, \mu_2)$, we get

$$\sum_{\substack{m' \in \mathcal{Z}_1(R)\ ;\\ 0<\varphi(m')-\varphi(m)<\beta}} (\varphi(m') - \varphi(m))^{-\sigma} \leq \sum_{1 \leq \mu_1 \leq 2b_2 R} \ \sum_{\mu_2 ; \rho < \frac{\mu_2}{\mu_1} < \tan(\varphi(m)+\beta)} \left\{ \tan^{-1} \frac{\mu_2}{\mu_1} - \varphi(m) \right\}^{-\sigma} . \tag{2.26}$$

Since

$$\min\{\mu_2 \mid \frac{\mu_2}{\mu_1} > \rho\} = [\rho\mu_1] + 1 , \tag{2.27}$$

the inner summation on the right hand side of (2.26) is further estimated as below:

$$\begin{aligned} &\sum_{\mu_2 ; \rho < \frac{\mu_2}{\mu_1} < \tan(\varphi(m)+\beta)} \left\{ \tan^{-1} \frac{\mu_2}{\mu_1} - \varphi(m) \right\}^{-\sigma} \\ &\leq \mu_1 \int_{\rho}^{\tan(\varphi(m)+\beta)} (\tan^{-1} y - \varphi(m))^{-\sigma} dy \\ &+ \quad \frac{\{\rho\mu_1\}}{\mu_1} \left(\tan^{-1} \frac{[\rho\mu_1]+1}{\mu_1} - \tan^{-1} \rho \right)^{-\sigma} \\ &\equiv F(\mu_1) + G(\mu_1) \ . \end{aligned} \tag{2.28}$$

Here $[a]$ and $\{a\}$ are respectively the integer and the fractional parts of a real number a . Now we have

$$\begin{aligned} \sum_{1 \leq \mu_1 \leq 2b_2 R} F(\mu_1) &= \left(\sum_{1 \leq \mu_1 \leq 2b_2 R} \mu_1 \right) \int_{\varphi(m)}^{\varphi(m)+\beta} (\varphi - \varphi(m))^{-\sigma} \frac{d\varphi}{\cos^2 \varphi} \\ &= O(R^2 \beta^{1-\sigma}) \ , \end{aligned} \tag{2.29}$$

where we have used the assumption that $0 \leq \varphi(m) \leq \pi/4$ and that $\beta > 0$ is small so that $\cos^2 \varphi$ is bounded away from 0 on the range of integration.

In order to estimate the sum of $G(\mu_1)$ we set $\rho = p/q$ with p and q mutually prime integers. Since $m \in \mathcal{Z}_1(R)$, and $0 \leq \rho \leq 1$, we must have $0 \leq p \leq q \leq 2b_2 R$. Moreover for any integer ℓ ,

$$\begin{aligned} \{\{\rho n\} \mid n \in \mathbf{N}\} &= \{\{\rho(q\ell + j)\} \mid j = 1, \dots , q\} \\ &= \{0, \frac{1}{q}, \dots , \frac{q-1}{q}\} \ . \end{aligned} \tag{2.30}$$

Then noting that

$$0 \leq \{\rho\mu_1\} \leq 1 \ ; \ \frac{[\rho\mu_1]+1}{\mu_1} = \rho + \frac{1 - \{\rho\mu_1\}}{\mu_1} \tag{2.31}$$

hold, we can compute

$$
\begin{aligned}
(2.32)\qquad & \sum_{1\le \mu_1\le 2b_2R} G(\mu_1)\\
&\le \sum_{1\le j\le q-1} G(j) + \sum_{1\le \ell\le 2b_2R/q}\ \sum_{0\le j\le q-1} G(q\ell+j)\\
&\le \sum_{1\le j\le q-1}\left\{\tan^{-1}(\rho+\frac{1-j/q}{q})-\varphi(m)\right\}^{-\sigma}\\
&+\sum_{1\le \ell\le 2b_2R/q}\ \sum_{0\le j\le q-1}\frac{1}{q\ell}\left\{\tan^{-1}(\rho+\frac{1-j/q}{q(\ell+1)})-\varphi(m)\right\}^{-\sigma}\\
&\le q\int_0^1\left\{\tan^{-1}(\rho+\frac{1-x}{q})-\varphi(m)\right\}^{-\sigma}dx\\
&\quad+\sum_{1\le \ell\le 2b_2R/q}\frac{1}{\ell}\int_0^1\left\{\tan^{-1}(\rho+\frac{1-x}{q(\ell+1)})-\varphi(m)\right\}^{-\sigma}dx\\
&=q^2\int_{\varphi(m)}^{\tan^{-1}(\rho+1/q)}(\varphi-\varphi(m))^{-\sigma}\frac{d\varphi}{\cos^2\varphi}\\
&\quad+\sum_{1\le \ell\le 2b_2R/q}\frac{q(\ell+1)}{\ell}\int_{\varphi(m)}^{\tan^{-1}(\rho+1/q(\ell+1))}(\varphi-\varphi(m))^{-\sigma}\frac{d\varphi}{\cos^2\varphi}\\
&=O(q^{\sigma+1}+R^{\sigma})\\
&=O(R^{\sigma+1})\ ,
\end{aligned}
$$

completing the proof of Lemma 1.

Next we consider the case where $\mathbf{P}$ satisfies (II-2). By (2.11) and (2.12), there is a constant $C>0$ such that the conditions

$$\frac{1}{2}b_1R\le |m|\le 2b_2R$$

and

$$f(\varphi(m))\in I_m^R$$

imply the condition

$$\left|f(\varphi(m))-\frac{|m|}{R}\right|\le\frac{1}{2}CR^{-2}\ . \tag{2.33}$$

By the Lipschitz continuity of $f(\cdot)$, we have the following implication for large $R>0$:

$$
\begin{aligned}
(2.34)\qquad & f(\varphi(m_j))\in I_{m_j}^R\ ,\ \frac{1}{2}b_1R\le |m_j|\le 2b_1R\ ,\ j=1,\ldots,k\\
&\Longrightarrow \left|\frac{|m_j|}{R}-\frac{|m_{j-1}|}{R}\right|\le\frac{C}{R^2}+b_3|\varphi(m_j)-\varphi(m_{j-1})|\ ,\ j=2,\ldots,k\ .
\end{aligned}
$$

On the other hand, it is easy to see that for some absolute constant $M>0$,

$$\min_{(m,m')\in\mathcal{Z}_2(R)}(\varphi(m')-\varphi(m))\ge MR^{-2} \tag{2.35}$$

for large $R>0$. Hence if we set $K=b_3+C/M$, the right hand side of (2.34) can be rewritten as

$$||m_j|-|m_{j-1}||\le KR|\varphi(m_j)-\varphi(m_{j-1})|\ ,\ j=2,\ldots,k\ . \tag{2.36}$$

¿From this consideration and (1-8), we obtain

$$\sum_{\mathcal{Z}'_k(\beta;R)} \mathbf{P}(f(\varphi(m_j)) \in I^R_{m_j}\ ,\ j=1,\dots,k) \le \text{const.} \sum_{\mathcal{Z}''_k(\beta;R)} R^{-2k} \prod_{j=2}^{k} (\varphi(m_j)-\varphi(m_{j-1}))^{-\tau}\ , \tag{2.37}$$

where $\mathcal{Z}''_k(\beta;R)$ is the totality of $(m_1,\dots,m_k) \in \mathcal{Z}'_k(\beta;R)$ for which (2.36) holds. Then by Lemma 3 below, we will have

$$\overline{\lim_{R\to\infty}} \sum_{\mathcal{Z}''_k(\beta;R)} R^{-2k} \prod_{j=2}^{k} (\varphi(m_j)-\varphi(m_{j-1}))^{-\tau} = O(\beta^{2-\tau})\ ,\ \beta\downarrow 0\ , \tag{2.38}$$

which completes the proof of Theorem 1.

Lemma 2. *Let $A>0$, $S>0$, $T>0$ and $R\ge 1$, where $T\ge \delta R$ and $S\ge\delta R$ for some $\delta>0$. Let further $\beta>(AR)^{-1}$. Then there is a constant $C>0$ such that*

$$\sum_{m\in\mathbf{Z}^2;(AR)^{-1}<\varphi(m)-\varphi_0<\beta} 1_{\{||m|-S|\le T(\varphi(m)-\varphi_0)\}}(\varphi(m)-\varphi_0)^{-\tau} \le CST\beta^{2-\tau}$$

for any $\varphi_0\in[0,\Theta)$.

Proof. Divide the interval $(\varphi_0+(AR)^{-1},\varphi_0+\beta]$ into subintervals Δ_ℓ of length $(AR)^{-1}$, where $\ell=1,\dots,[AR\beta]-1$, and an interval $\Delta_{[AR\beta]}$ of length $\le (AR)^{-1}$. Let N_ℓ be the number of lattice points $m\in\mathbf{Z}^2$ such that $\varphi(m)\in\Delta_\ell$ and that

$$||m|-S|\le T(\varphi(m)-\varphi_0)\ .$$

Then $N_\ell\le \text{const.}R^{-2}ST\ell$ and we have

$$\begin{aligned}
&\sum_{m\in\mathbf{Z}^2;(AR)^{-1}<\varphi(m)-\varphi_0<\beta} 1_{\{||m|-S|\le T(\varphi(m)-\varphi_0)\}}(\varphi(m)-\varphi_0)^{-\tau} \\
&= \sum_{\ell=1}^{[R\beta]} \sum_{m;\varphi(m)\in\Delta_\ell} 1_{\{||m|-S|\le T(\varphi(m)-\varphi_0)\}}(\varphi(m)-\varphi_0)^{-\tau} \\
&\le \sum_{\ell=1}^{[R\beta]} \sum_{m;\varphi(m)\in\Delta_i} 1_{\{||m|-S|\le T(\varphi(m)-\varphi_0)\}}(iR^{-1})^{-\tau} \\
&\le \text{const.} \sum_{\ell=1}^{[R\beta]} \frac{ST}{R^2}\ell^{1-\tau}R^{\tau} \\
&\le \text{const.} \frac{ST}{R^{2-\tau}}(R\beta)^{2-\tau} \\
&= CST\beta^{2-\tau}\ ,
\end{aligned} \tag{2.39}$$

and this estimate is uniform in φ_0 .

As a corollary of Lemma 2, we get the following

Lemma 3. *For large $R>0$ and small $\beta>0$, we have*

$$\max_{m'\in\mathcal{Z}_1(R)} \sum_{m;(m',m)\in\mathcal{Z}''_2(\beta;R)} (\varphi(m)-\varphi(m'))^{-\tau} = O(KR^2\beta^{2-\tau})\ .$$

Proof. By an elementary geometric consideration, we can show the existence of an absolute constant $M'>0$ such that

$$\min_{m\in\mathcal{Z}_1(R)} \min_{\substack{m';\varphi(m')>\varphi(m),\\ ||m'|-|m||\le R(\varphi(m')-\varphi(m))}} (\varphi(m')-\varphi(m)) \ge M'R^{-1}\ . \tag{2.40}$$

Hence if we take $A^{-1}=M'$, $S=|m'|$, $T=KR$ and $\varphi_0=\varphi(m')$ in Lemma 2, then the condition $(AR)^{-1}<\varphi(m)-\varphi_0$ under the summation symbol can be dropped and we obtain the desired assertion.

§3 Proof of Theorems 2 and 3

In this section, we shall prove simultaneously Theorems 2' and 3' , which are equivalent variants of Theorems 2 and 3, admitting a technical lemma which will be proved in §4. After that, we shall sketch how to transform the modified version of our theorems back to the original ones.

In order to state the modified version of our theorems–which the author believs are in fact closer to the original problem raised by Berry and Tabor [BT]– let us introduce the planar domain $\tilde{\Pi}_R(f)$ by

$$\begin{aligned}\tilde{\Pi}_R(f) = \{x \in \mathbf{R}^2 |\ 0 \le \varphi(x) \le \Theta\ ,\\ \sqrt{R+2c_1}f(\varphi(x)) < |x| < \sqrt{R+2c_2}f(\varphi(x))\}\ .\end{aligned} \tag{3.1}$$

Then the area of $\tilde{\Pi}_R(f)$ is not only asymptotic to, but is equal to $\lambda(f)$ for all $R > 0$. Given positive numbers $0 < a_1 < a_2$ and $L > 0$, let $\mu_L^{a_1,a_2}(dR)$ be the uniform probability distribution on the interval $I_L = (a_1 L, a_2 L)$. Finally we define

$$\tilde{\xi} = \tilde{\xi}(R; f) = \sharp\{\tilde{\Pi}_R(f) \cap \mathbf{Z}^2\}\ , \tag{3.2}$$

to be the number of lattice points in the domain $\tilde{\Pi}_R(f)$.

Now we shall prove the following two theorems:

Theorem 2'. *Suppose Conditions (I) and (II) hold. Then there is a sequence $\bar{L}_n \to \infty$ such that for $\mathbf{P}$-almost all $f \in \mathcal{X}$, the distribution of $\tilde{\xi}(\cdot; f)$ under $\mu_{\bar{L}_n}^{a_1,a_2}$ converges to the Poisson distribution with parameter $\lambda(f)$.*

Theorem 3'. *If we assume Condition (III) in addition to Conditions (I) and (II), then for $\mathbf{P}$-almost all $f \in \mathcal{X}$, the distribution of $\tilde{\xi}(\cdot; f)$ under $\mu_L^{a_1,a_2}$ converges to the Poisson distribution with parameter $\lambda(f)$ as $L \to \infty$.*

Let us turn to the proof of Theorem 3'.

For $k = 1, 2, \dots$, set

$$\mathbf{E}_L = \mathbf{E}_{L,k}(f) = \int_{I_L} \tilde{\xi}(R; f)\mu_L(dR)\ , \tag{3.3}$$

where $\mu_L = \mu_L^{a_1,a_2}$, and

$$\tilde{\xi}_k = \tilde{\xi}_k(R; f) = \binom{\tilde{\xi}}{k} 1_{\{\tilde{\xi} \ge k\}} = \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} 1_{\{\tilde{\xi}=n\}}\ . \tag{3.4}$$

As was done in [M], Theorem 3' will be proved as soon as we have shown

$$\lim_{L\to\infty} \mathbf{E}_{L,k}(f) = \frac{1}{k!}\lambda(f)^k\ ,\ k = 1, 2, \dots \tag{3.5}$$

for $\mathbf{P}$ -almost all $f \in \mathcal{X}$. Similarly, Theorem 2' will be proved as soon as we have shown the existence of a sequence $\bar{L}_j \to \infty$ such that (3.5) holds along this sequence for $\mathbf{P}$ -almost all $f \in \mathcal{X}$.

For a lattice point $m \in \mathbf{Z}^2$, let

$$D_m = D_m(f) = \{R > 0 \mid \tilde{\Pi}_R(f) \ni m\}\ . \tag{3.6}$$

By the definition (3.1) of $\tilde{\Pi}_R(f)$, we have

$$D_m = \left(\frac{|m|^2}{f(\varphi(m))^2} - 2c_2, \frac{|m|^2}{f(\varphi(m))^2} - 2c_1\right)\ . \tag{3.7}$$

In particular, $|D_m| = 2(c_2 - c_1)$. For convenience, we let

$$\gamma_m = \gamma_m(f) = \frac{|m|^2}{f(\varphi(m))^2} \ . \tag{3.8}$$

Now set $L_n = 2^n$, $n = 1, 2, \ldots$. Then if $\tilde{\Pi}_R(f) \ni m$, $R \in I_L$ and if $L_n \leq L < L_{n+1}$, then we will have

$$b_1\sqrt{a_1 L_n} \leq |m| \leq 2b_2\sqrt{a_2 L_{n+1}} \ . \tag{3.9}$$

Under the same condition, it is also easy to see that if $m \neq m'$ but $\varphi(m) = \varphi(m')$, then $D_m \cap D_{m'} = \phi$. Hence if we define, as in §2,

$$\begin{aligned} \mathscr{Z}_{k,n} = \{(m_1, \ldots, m_k) \in (\mathbf{Z}^2)^k \ |m_j \text{ satisfies (3.9) for } j = 1, \ldots, k \text{ and} \\ 0 \leq \varphi(m_1) < \cdots < \varphi(m_k) \leq \Theta\} \ , \end{aligned} \tag{3.10}$$

then for large n and for $L_n \leq L < L_{n+1}$, we have

$$\begin{aligned} \mathbf{E}_{L,k} &= \sum_{(m_1,\ldots,m_k)\in\mathscr{Z}_{k,n}} \mu_L(m_1, \ldots, m_k \in \tilde{\Pi}_R(f)) \\ &= \frac{1}{|I_L|} \sum_{(m_1,\ldots,m_k)\in\mathscr{Z}_{k,n}} |I_L \cap D_{m_1} \cap \cdots \cap D_{m_k}| \ . \end{aligned} \tag{3.11}$$

For $k = 1$, (3.5) can be proved without any reference to the randomness of $f(\cdot)$. Indeed in this case,

$$\mathbf{E}_{L,1}(f) = \frac{1}{|I_L|} \sum_{m\in\mathscr{Z}_{1,n}} |I_L \cap D_m| \ . \tag{3.12}$$

Since

$$D_m \cap I_L \neq \phi \Longleftrightarrow a_1 L + 2c_1 < \gamma_m < a_2 L + 2c_2 \ , \tag{3.13}$$

and

$$D_m \subset I_L \Longleftrightarrow a_1 L + 2c_2 \leq \gamma_m \leq a_2 L + 2c_1 \ , \tag{3.14}$$

we see

$$\begin{aligned} \mathbf{E}_{L,1}(f) &\leq \frac{1}{|I_L|} \sum_{m\in\mathbf{Z}^2} 1_{(a_1L+2c_1, a_2L+2c_2)}(\gamma_m)|D_m| \\ &= \frac{2(c_2 - c_1)}{|I_L|} \sum_{m\in\mathbf{Z}^2} 1_{(a_1L+2c_1, a_2L+2c_2)}\left(\frac{|m|^2}{f(\varphi(m))^2}\right) \ , \end{aligned} \tag{3.14}$$

and similarly

$$\mathbf{E}_{L,1}(f) \geq \frac{2(c_2 - c_1)}{|I_L|} \sum_{m\in\mathbf{Z}^2} 1_{[a_1L+2c_2, a_2L+2c_1]}\left(\frac{|m|^2}{f(\varphi(m))^2}\right) \ . \tag{3.15}$$

Define $\bar{\mathbf{E}}_{L,1}(f)$ and $\underline{\mathbf{E}}_{L,1}(f)$ to be the right hand sides of (3.14) and (3.15) respectively.

For each $\epsilon > 0$, consider the planar domain

$$\begin{aligned} \Lambda_\epsilon &= \Lambda_\epsilon(f) \\ &= \{x \in \mathbf{R}^2 \mid 0 \leq \varphi(x) \leq \Theta \text{ and } a_1 f(\varphi(x))^2 < |x|^2 < (a_2 + \epsilon) f(\varphi(x))^2\} \ . \end{aligned} \tag{3.17}$$

Since $f(\cdot)$ is a continuous function, the boundary of Λ_ϵ has zero Lesbesgue measure, and hence the indicator of Λ_ϵ is Riemann integrable. We have therefore

$$\begin{aligned} \varlimsup_{L\to\infty} \bar{\mathbf{E}}_{L,1}(f) &\le \varlimsup_{L\to\infty} \frac{2(c_2-c_1)}{a_2-a_1}\left(\frac{1}{\sqrt{L}}\right)^2 \sum_{m\in\mathbf{Z}^2} 1_{\Lambda_\epsilon}\left(\frac{m}{\sqrt{L}}\right) \\ &= \frac{2(c_2-c_1)}{a_2-a_1} \times \text{the area of } \Lambda_\epsilon \\ &= \left(1+\frac{\epsilon}{a_2-a_1}\right)\lambda(f) \ . \end{aligned} \tag{3.18}$$

Letting $\epsilon \searrow 0$, we obtain

$$\varlimsup_{L\to\infty} \bar{\mathbf{E}}_{L,1}(f) \le \lambda(f) \ . \tag{3.19}$$

By a similar argument, we can also prove

$$\varliminf_{L\to\infty} \underline{\mathbf{E}}_{L,1}(f) \ge \lambda(f) \ . \tag{3.20}$$

So that

$$\lim_{L\to\infty} \mathbf{E}_{L,1}(f) = \lambda(f) \tag{3.21}$$

for every $f \in \mathcal{X}$.

Next we consider the case $k \ge 2$.

If $L_n \le L < L_{n+1}$, then $I_L \subset [a_1L_n, a_2L_{n+1})$. We divide the interval $[a_1L_n, a_2L_{n+1})$ into $q_n = n^\eta$ subintervals $[z_{i+1}, z_i)$ having equal length, where $\eta > 0$ is arbitrary. Obviously

$$z_i = a_1L_n + \frac{i}{q_n}(a_2L_{n+1} - a_1L_n) \ , \ i = 1, \dots, q_n \ . \tag{3.22}$$

If we let

$$\bar{A}_n(L) = \frac{a_1(L-L_n)+2c_1}{a_2L_{n+1}-a_1L_n}q_n \ ; \ \bar{B}_n(L) = 1 + \frac{a_2L - a_1L_n + 2c_2}{a_2L_{n+1}-a_1L_n}q_n \ , \tag{3.22}$$

and

$$\underline{A}_n(L) = 1 + \frac{a_1(L-L_n)+2c_2}{a_2L_{n+1}-a_1L_n}q_n \ ; \ \underline{B}_n(L) = \frac{a_2L - a_1L_n + 2c_1}{a_2L_{n+1}-a_1L_n}q_n \ , \tag{3.23}$$

then we have

$$\gamma_m \in [z_{i-1}, z_i) \ , \ D_m \cap I_L \ne \phi \Longrightarrow \bar{A}_n(L) < i < \bar{B}_n(L) \tag{3.24}$$

and

$$\gamma_m \in [z_{i-1}, z_i) \ , \ \underline{A}_n(L) \le i \le \underline{B}_n(L) \Longrightarrow D_m \subset I_L \ . \tag{3.25}$$

Hence for n large, we can write

$$\begin{aligned} \mathbf{E}_{k,L}(f) &= \frac{1}{|I_L|} \sum_{(m_1,\dots,m_k)\in\mathcal{Z}_{k,n}} |I_L \cap D_{m_1} \cap \cdots \cap D_{m_k}| \\ &\le \frac{1}{|I_L|} \sum_{\bar{A}_n(L)<i<\bar{B}_n(L)} \sum_{m_1} 1_{\{\gamma_{m_1}\in[z_{i-1},z_i)\}} \sum_{\mathcal{Z}_{k,n}(m_1)} |D_{m_1} \cap \cdots \cap D_{m_k}| \ , \end{aligned} \tag{3.26}$$

and

$$\mathbf{E}_{k,L}(f) \geq \frac{1}{|I_L|} \sum_{\underline{A}_n(L) \leq i \leq \underline{B}_n(L)} \sum_{m_1} 1_{\{\gamma_{m_1} \in [z_{i-1}, z_i)\}} \sum_{\mathcal{Z}_{k,n}(m_1)} |D_{m_1} \cap \cdots \cap D_{m_k}| \ , \tag{3.27}$$

where

$$\mathcal{Z}_{k,n}(m_1) \equiv \{(m_2, \ldots, m_k) \in (\mathbf{Z}^2)^{k-1} |\ (m_1, m_2, \ldots, m_k) \in \mathcal{Z}_{k,n}\} \ . \tag{3.28}$$

As before, we define $\bar{E}^{(1)}_{k,L}(f)$ and $\underline{E}^{(1)}_{k,L}(f)$ to be the right hand sides of (3.26) and (3.27) respectively. We also define $\bar{E}^{(2)}_{k,L}(f)$ by

$$\begin{aligned} \bar{E}^{(2)}_{k,L}(f) = \frac{1}{|I_L|} \sum_{\bar{A}_n(L) < i < \bar{B}_n(L)} \sum_{m_1} 1_{\{\gamma_{m_1} \in [z_{i-1}, z_i)\}} \times \\ \times 2(c_2 - c_1)^k \int_{\varphi(m_1)}^{\Theta} d\varphi_2 \int_{\varphi_2}^{\Theta} d\varphi_3 \cdots \int_{\varphi_{k-1}}^{\Theta} d\varphi_k \ f(\varphi_2)^2 \cdots f(\varphi_k)^2 \ , \end{aligned} \tag{3.29}$$

and $\underline{E}^{(2)}_{k,L}(f)$ by the same formula as (3.29) except that the condition under the summation symbol "$\bar{A}_n(L) < i < \bar{B}_n(L)$"is replaced by "$\underline{A}_n(L) \leq i \leq \underline{B}_n(L)$".

Now it is easy to prove

$$\lim_{L \to \infty} \bar{E}^{(2)}_{k,L}(f) = \lim_{L \to \infty} \underline{E}^{(2)}_{k,L}(f) = \frac{\lambda(f)^k}{k!} \tag{3.30}$$

for every $f \in \mathcal{X}$, so that we are left with estimating the differences

$$\bar{G}_{k,L}(f) \equiv \bar{E}^{(1)}_{k,L}(f) - \bar{E}^{(2)}_{k,L}(f) \ ; \ \underline{G}_{k,L}(f) \equiv \underline{E}^{(1)}_{k,L}(f) - \underline{E}^{(2)}_{k,L}(f) \ . \tag{3.31}$$

More precisely, writing

$$\begin{aligned} \Gamma_n(f) \equiv & \sup_{L_n \leq L < L_{n+1}} \max\{|\underline{G}_{k,L}(f)|, |\bar{G}_{k,L}(f)|\} \\ & \leq \frac{1}{|I_{L_n}|} \sum_{i=1}^{q_n+1} \Bigg| \sum_{m_1} 1_{\{\gamma_{m_1} \in [z_{i-1}, z_i)\}} \ \times \\ & \times \Bigg\{ \sum_{(m_2, \ldots, m_k) \in \mathcal{Z}_{k,n}(m_1)} |D_{m_1} D_{m_2} \cap \cdots \cap D_{m_k}| \ - \\ & -2(c_2 - c_1)^k \int_{\varphi(m_1)}^{\Theta} d\varphi_2 \int_{\varphi_2}^{\Theta} d\varphi_3 \cdots \int_{\varphi_{k-1}}^{\Theta} d\varphi_k \ f(\varphi_2)^2 \cdots f(\varphi_k)^2 \Bigg\} \Bigg| \ , \end{aligned} \tag{3.32}$$

we shall prove

$$\sum_{n=1}^{\infty} \mathbf{E}_{\mathbf{P}}[\Gamma_n(\cdot)] = \mathbf{E}_{\mathbf{P}}[\sum_{n=1}^{\infty} \Gamma_n(\cdot)] < \infty \ , \tag{3.33}$$

which immediately gives

$$\lim_{L \to \infty} \max\{|\underline{G}_{k,L}(f)|, |\bar{G}_{k,L}(f)|\} \leq \overline{\lim_{n \to \infty}} \ \Gamma_n(f) = 0 \tag{3.34}$$

for $\mathbf{P}$-almost all $f \in \mathcal{X}$.

Let us finish the proof of (3.33), and hence of Theorem 3', admitting the following technical lemma:

Lemma 4. *For $n \geq 1$ and $1 \leq i \leq q_n + 1$, let*

$$U_{i,n} = \mathbf{E_P}\left[\left|\sum_{m_1} 1_{\{\gamma_{m_1}\in[z_{i-1},z_i)\}}\left\{\sum_{(m_2,\dots,m_k)\in\mathcal{Z}_{k,n}(m_1)} |D_{m_1}\cap D_{m_2}\cap\cdots\cap D_{m_k}| - \right.\right.\right.$$

$$\left.\left.\left. -2(c_2-c_1)^k\int_{\varphi(m_1)}^{\Theta} d\varphi_2\int_{\varphi_2}^{\Theta} d\varphi_3\cdots\int_{\varphi_{k-1}}^{\Theta} d\varphi_k\; f(\varphi_2)^2\cdots f(\varphi_k)^2\right\}\right|^2\right] . \tag{3.35}$$

Let further $T^{-1} \leq a_1 < a_2 \leq T$ with $T \geq 1$. Then there is a contant $C_T > 0$ which depends only on $T > 0$ such that for each $n \geq 1$ and $\beta > L_n^{-1/2}$,

$$\max_{1\leq i\leq q_n+1} U_{i,n} \leq C_T(\Psi_{2k}(\beta) + \sqrt{L_n}\beta^d)L_n^{3/2}q_n^{-2} , \tag{3.36}$$

where $d = 1-\sigma$ or $d = 2-\tau$ according to which of the conditions (II-1) or (II-2) is satisfied by $\mathbf{P}$.

Take $\beta = \beta_n = An^{-1/\nu_k}$ in the above lemma. Then by (3.32) and (3.35),

$$\begin{aligned} \mathbf{E}[\Gamma_n(\cdot)] &\leq \frac{1}{|I_{L_n}|}\sum_{i=1}^{q_n+1}\sqrt{U_{i,n}} \\ &\leq \text{const.}L_n^{-1}q_n(\sqrt{\Psi_{2k}(\beta_n)} + L_n^{1/4}\beta_n^{d/2})L_n^{3/4}q_n^{-1} \\ &= \text{const.}(L_n^{-1/4}\sqrt{\Psi_{2k}(\beta_n)} + \beta_n^{d/2}) \\ &\leq \text{const.}(L_n^{-1/4}\exp(\frac{1}{2}A^{-\nu_k}n) + n^{-d/2\nu_k}) . \end{aligned} \tag{3.37}$$

If we take $A > 0$ sufficiently large, then the first term on the right hand side decays exponentially first, while $d/2\nu_k > 1$. Hence we have

$$\sum_{n=1}^{\infty}\mathbf{E}[\Gamma_n(\cdot)] < \infty , \tag{3.38}$$

which completes the proof of Theorem 3.

When we do not assume Condition (III), $\Psi_{2k}(\beta)$ can be any positive function monotonically tending to ∞ as $\beta \searrow 0$. It is possible to choose a decreasing sequence $\{\beta_n\}$ tending to 0 so slowly that

$$L_n^{-1/2}\Psi_{2k}(\beta_n) \longrightarrow 0 \tag{3.39}$$

holds. Then choose a subsequence $\{n'\}$ such that

$$\sum_{n'}(L_{n'}^{-1/4}\sqrt{\Psi_{2k}(\beta_{n'})} + \beta_{n'}^{d/2}) < \infty . \tag{3.40}$$

Choose a number $\bar{L}_{n'}$ arbitrarily from the interval $[L_{n'}, L_{n'+1})$ for each n' . Then obvioiusly the sequence $\{\bar{L}_{n'}\}$ meets the requirement of Theorem 2' .

Before closing this section, we sketch how to obtain Theorems 2 and 3 from what we have shown up to now. For this purpose, we first note that for any $f \in \mathcal{X}$ and any $0 < a_1 < a_2$,

$$\lim_{L\to\infty}\mu_L^{a_1,a_2}(\{R > 0|\tilde{\xi}(R;f) \neq \xi(\sqrt{R};f)\}) = 0 . \tag{3.41}$$

Indeed, $\tilde{\xi}(R;f) \neq \xi(\sqrt{R};f)$ implies

$$\{\tilde{\Pi}_R(f)\Delta\Pi_R(f)\}\cap\mathbf{Z}^2 \neq \phi , \tag{3.42}$$

and since $a_1L \le R \le a_2L$, $b_1 \le f(\cdot) \le b_2$, it holds that

$$\begin{aligned}\mu_L^{a_1,a_2}(\{R>0|\tilde{\xi}(R;f)\neq\xi(\sqrt{R};f)\})\\ \le \mu_L^{a_1,a_2}(\{R>0|(\tilde{\Pi}_R(f)\Delta\Pi_R(f))\cap\mathbf{Z}^2\neq\phi\})\\ \le \sum_{m\in\mathcal{Z}_{1,n}}\mu_L^{a_1,a_2}(\{R>0|\ m\in\tilde{\Pi}_R(f)\Delta\Pi_R(f)\})\ .\end{aligned}\tag{3.43}$$

If we set for each $m \in \mathbf{Z}^2$ with $0 \le \varphi(m) \le \Theta$,

$$J_m=\{R>0|\ m\in\tilde{\Pi}_R(f)\Delta\Pi_R(f)\}\ ,\tag{3.44}$$

and

$$J_m^{(i)}=\{R>0|\ \sqrt{R+2c_i}f(\varphi(m))\le|m|\le(\sqrt{R}+\frac{c_i}{\sqrt{R}})f(\varphi(m))\}\ ,\ i=1,2\ ,\tag{3.45}$$

then $J_m \subset J_m^{(1)} \cup J_m^{(2)}$, and it suffices for our purpose to show

$$\lim_{L\to\infty}\sum_{m\in\mathcal{Z}_{1,n}}\mu_L^{a_1,a_2}(J_m^{(i)})=0\ ,\ i=1,2\ .\tag{3.46}$$

Now it is easy to see that $R \in J_m^{(i)}$ implies

$$\frac{|m|^2}{f(\varphi(m))^2}-2c_i-\frac{c_1^2}{a_1L}\le R\le\frac{|m|^2}{f(\varphi(m))^2}-2c_i\ ,\tag{3.47}$$

and hence

$$\mu_L^{a_1,a_2}(J_m^{(i)})\le\frac{c_i^2}{(a_2-a_1)a_1}\frac{1}{L^2}\ .\tag{3.48}$$

Since the number of lattice points in $\mathcal{Z}_{1,n}$ is bounded by a constant times L , we finally arrive at

$$\sum_{m\in\mathcal{Z}_{1,n}}\mu_L^{a_1,a_2}(J_m^{(i)})=\mathcal{O}(L^{-1})\tag{3.49}$$

as desired.

Now fix $0 < a_1 < a_2$ and apply Theorem 3' with $a_1^2 < a_2^2$ in use instead of $0 < a_1 < a_2$. Then from the above argument, for $\mathbf{P}$-almost all $f \in \mathcal{X}$, one has

$$\lim_{L\to\infty}\frac{1}{L^2}\int_{a_1^2L^2}^{a_2^2L^2}1_{\{\xi(\sqrt{R};f)=k\}}dR=(a_2^2-a_1^2)p_k(f)\ ,\tag{3.50}$$

where we have set for brevity $p_k(f) = e^{-\lambda(f)}\lambda(f)/k!$. On the other hand, the estimate in Lemma 4 is uniform in a_1 and a_2 such that $T^{-1} \le a_1 < a_2 \le T$, for any $T \ge 1$. Hence with the same choice $\beta_n = An^{-1/\nu_k}$ as before, we see

$$\sum_{n=1}^{\infty}\int_{a_1<\alpha<T}d\alpha\ \mathbf{E}[\Gamma_n(\cdot)]<\infty\tag{3.51}$$

for any $T > a_1$. This means that for $\mathbf{P}-$almost all $f \in \mathcal{X}$ and for Lesbesgue almost all $\alpha > a_1$, one has

$$\lim_{L\to\infty}\frac{1}{L^2}\int_{a_1^2L^2}^{\alpha L^2}1_{\{\tilde{\xi}(R;f)=k\}}dR=\lim_{L\to\infty}\frac{1}{L^2}\int_{a_1^2L^2}^{\alpha L^2}1_{\{\xi(\sqrt{R};f)=k\}}dR=(\alpha-a_1^2)p_k(f)\ ,\ k\ge1\ .\tag{3.52}$$

We now fix $f \in \mathcal{X}$ for which (3.50) and (3.52) are valid, and letting $g_k(R) = 1_{\{\xi(R;f)=k\}}$ for brevity, we further compute as follows:

$$\begin{aligned}(3.53)\qquad & \mu_L^{a_1,a_2}(\{R>0|\ \xi(R;f)=k\}) \\ &= \frac{1}{(a_2-a_1)L}\int_{a_1L}^{a_2L} g_k(R)dR \\ &= \frac{1}{2(a_2-a_1)L}\int_{a_1^2L^2}^{a_2^2L^2} g_k(\sqrt{s})\frac{ds}{\sqrt{s}} \\ &= \frac{1}{2(a_2-a_1)a_2}\frac{1}{L^2}\int_{a_1^2L^2}^{a_2^2L^2} g_k(\sqrt{s})ds + \frac{1}{2(a_2-a_1)}\int_{a_1}^{a_2}\frac{d\tau}{\tau^2}\left\{\frac{1}{L^2}\int_{a_1^2L^2}^{\tau^2L^2} g(\sqrt{s})ds\right\} .\end{aligned}$$

The first term on the right hand side converges to $\frac{a_2^2-a_1^2}{2(a_2-a_1)a_2}p_k(f)$ as $L\to\infty$ because of (3.50). On the other hand, we have

$$(3.54)\qquad \left|\frac{1}{L^2}\int_{a_1^2L^2}^{\tau^2L^2} g_k(\sqrt{s})ds\right| \le a_2^2-a_1^2$$

for all $\tau\in[a_1,a_2]$ and

$$(3.55)\qquad \lim_{L\to\infty}\frac{1}{L^2}\int_{a_1^2L^2}^{\tau^2L^2} g(\sqrt{s})ds = (\tau^2-a_1^2)p_k(f)$$

for Lesbesgue almost all $\tau\in[a_1,a_2]$ because of (3.52). Now apply Lesbesgue's dominated convergence theorem to the second term on the right hand side, to obtain

$$\begin{aligned}(3.56)\qquad & \lim_{L\to\infty}\mu_L^{a_1,a_2}(\{R>0|\ \xi(R;f)=k\}) \\ &= \frac{a_2^2-a_1^2}{2(a_2-a_1)a_2}p_k(f) + \frac{1}{2(a_2-a_1)}\int_{a_1}^{a_2}\frac{d\tau}{\tau^2}(\tau^2-a_1^2)p_k(f) \\ &= p_k(f)\ ,\end{aligned}$$

completing the proof of Theorem 3.

Finally let $\{\bar{L}_n\}$ be the sequence taken according to Theorem 2'. We can repeat the above argument to see that the sequence $\{\sqrt{\bar{L}_n}\}$ meets the requirement of Theorem 2.

§4 Proof of Lemma 4

Let us fix an $i\in\{1,\dots,1+q_n\}$ and a $T>1$ and assume $T^{-1}\le a_1<a_2\le T$. We make the following definitions:

$$\begin{aligned}(4.1)\qquad & Q^i(m_1,\dots,m_k;m_1',\dots,m_k') = Q^i(\mathbf{m};\mathbf{m}') \\ &= \mathbf{E_P}[1_{\{\gamma_{m_1},\gamma_{m_1}'\in[z_{i-1},z_i)\}}|D_{m_1}\cap\cdots\cap D_{m_k}|\ |D_{m_1'}\cap\cdots\cap D_{m_k'}|]\ ,\end{aligned}$$

where we write $(\mathbf{m},\mathbf{m}')=(m_1,\dots,m_k;m_1',\dots,m_k')$ for brevity;

$$(4.2)\qquad X_n^i = \sum_{(\mathbf{m};\mathbf{m}')\in\mathcal{Z}_{k,n}\times\mathcal{Z}_{k,n}} Q^i(\mathbf{m};\mathbf{m}')\ ;$$

$$\begin{aligned}(4.3)\qquad & R^i(m_1,\dots,m_1;m_1') = R^i(\mathbf{m};m_1') \\ &= \mathbf{E_P}[1_{\{\gamma_{m_1},\gamma_{m_1}'\in[z_{i-1},z_i)\}}|D_{m_1}\cap\cdots\cap D_{m_k}|\times \\ &\quad\times\left(\int_{\varphi(m_1')}^{\Theta}d\varphi_2'\int_{\varphi_2'}^{\Theta}d\varphi_3'\cdots\int_{\varphi_{k-1}'}^{\Theta}d\varphi_k'\ f(\varphi_2')^2\cdots f(\varphi_k')^2\right)\Bigg]\ ;\end{aligned}$$

$$Y^i_n = \sum_{(\mathbf{m};m'_1)\in\mathcal{Z}_{k,n}\times\mathcal{Z}_{1,n}} R^i(\mathbf{m};m'_1) \ ; \tag{4.4}$$

$$\begin{aligned} S^i(m_1,m'_1) = \mathbf{E_P}[1_{\{\gamma_{m_1},\gamma'_{m_1}\in[z_{i-1},z_i)\}}\times \\ \times\left(\int_{\varphi(m_1)}^{\Theta} d\varphi_2\int_{\varphi_2}^{\Theta} d\varphi_3\cdots\int_{\varphi_{k-1}}^{\Theta} d\varphi_k\ f(\varphi_2)^2\cdots f(\varphi_k)^2\right)\times \\ \times\left(\int_{\varphi(m'_1)}^{\Theta} d\varphi'_2\int_{\varphi'_2}^{\Theta} d\varphi'_3\cdots\int_{\varphi'_{k-1}}^{\Theta} d\varphi'_k\ f(\varphi'_2)^2\cdots f(\varphi'_k)^2\right)\Bigg] \ ; \end{aligned} \tag{4.5}$$

$$Z^i_n = \sum_{(m_1,m'_1)\in\mathcal{Z}_{1,n}\times\mathcal{Z}_{1,n}} S^i(m_1,m'_1) \ ; \tag{4.6}$$

Then

$$U_{i,n} = X^i_n - 4(c_2-c_1)^k Y^i_n + 4(c_2-c_1)^{2k} Z^i_n \ . \tag{4.7}$$

We shall prove that for some positive constant C_T which depends only on T , the following two inequlities hold as far as we have $T^{-1}\le a_1 < a_2 \le T$:

$$\mid X^i_n - 4(c_2-c_1)^{2k}Z^i_n| \le C_T(L_n^{3/2}q_n^{-2}(\Psi_{2k}(\beta)+\sqrt{L_n}\beta^d)) \ ; \tag{4.8}$$

and

$$\mid Y^i_n - 2(c_2-c_1)^{k}Z^i_n| \le C_T(L_n^{3/2}q_n^{-2}(\Psi_{2k}(\beta)+\sqrt{L_n}\beta^d)) \ , \tag{4.9}$$

from which the assertion of Lemma 4 follows immediately.

At this point, we make the following convention. We denote by C or C_T positive constants, of which we are not interested in exact values, and which may differ from inequality to inequality. We distinguish C_T from C when it is depndent on T , while C does not depend on a_1 and a_2 . Similarly, when something is bounded by C [resp. C_T] times something else, we shall say it is of $\mathcal{O}(\cdot)$ [resp. $\mathcal{O}_T(\cdot)$].

In order to estimate X^i_n , let us make the representation (4.1) of $Q^i(\mathbf{m};\mathbf{m}')$ more explicit. To begin with, from the definition (3.8) of γ_m , we see that

$$\gamma_{m_1}\in[z_{i-1},z_i) \iff \frac{|m_1|}{\sqrt{z_i}} < f(\varphi(m_1)) \le \frac{|m_1|}{\sqrt{z_{i-1}}} \ , \tag{4.10}$$

where

$$\frac{|m_1|}{\sqrt{z_{i-1}}} - \frac{|m_1|}{\sqrt{z_i}} = \frac{|m_1|}{\sqrt{z_i z_{i-1}}}\left(\frac{z_i - z_{i-1}}{\sqrt{z_i}+\sqrt{z_{i-1}}}\right) = \mathcal{O}((a_2/a_1)^{3/2}q_n^{-1}) = \mathcal{O}_T(q_n^{-1}) \ , \tag{4.11}$$

if $m_1\in\mathcal{Z}_{1,n}$. Recall (3.9) and $a_1L_n\le z_i\le a_2L_{n+1}$. Here we have simply written $\mathcal{O}_T$ because the dependence of the estimate on a_1 and a_2 need not be made explicit. Next, writing $x_+ = \max\{x,0\} = x\vee 0$ for $x\in\mathbf{R}$, we obtain from (3.7) and (3.8)

$$|D_{m_1}\cap\cdots\cap D_{m_k}| = \left\{A - 0\vee\max_{2\le j\le k}(\gamma_{m_j}-\gamma_{m_1}) + 0\wedge\min_{2\le j\le k}(\gamma_{m_j}-\gamma_{m_1})\right\}_+ \ , \tag{4.12}$$

where we have let $A = 2(c_2-c_1)$ for brevity. Letting $y_s = f(\varphi(m_s))$ and $y'_t = f(\varphi(m'_t))$, we can write, when $\varphi(m_s)$, $\varphi(m'_t)$, $s,t=1,\dots,k$ are all different,

$$\begin{aligned} &Q^i(\mathbf{m};\mathbf{m}') \\ &= \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1\int_{|m'_1|/\sqrt{z_i}}^{|m'_1|/\sqrt{z_{i-1}}} dy'_1\int_0^\infty dy_2\cdots\int_0^\infty dy_k\int_0^\infty dy'_2\cdots\int_0^\infty dy'_k\times \\ &\times\left\{A - 0\vee\max_{2\le s\le k}\left(\frac{|m_s|^2}{y_s^2}-\frac{|m_1|^2}{y_1^2}\right) + 0\wedge\min_{2\le s\le k}\left(\frac{|m_s|^2}{y_s^2}-\frac{|m_1|^2}{y_1^2}\right)\right\}_+\times \\ &\times\left\{A - 0\vee\max_{2\le t\le}\left(\frac{|m'_t|^2}{y_t'^2}-\frac{|m'_1|^2}{y_1'^2}\right) + 0\wedge\min_{2\le t\le}\left(\frac{|m'_t|^2}{y_t'^2}-\frac{|m'_1|^2}{y_1'^2}\right)\right\}_+\times \\ &\times p_{2k}(y_1,\dots,y_k,y'_1,\dots,y'_k \mid \varphi(m_1),\dots,\varphi(m_k),\varphi(m'_1),\dots,\varphi(m'_k)) \end{aligned} \tag{4.13}$$

Note that p_{2k} , as a function of $y's$, is supported on the set of $(y_1,\dots,y_k,y'_1,\dots,y'_k)$ such that $b_1\le y_s,y_t\le b_2$, $s,t=1,\dots,k$.

Let us make the following change of variables:

$$\zeta_s=\frac{|m_s|^2}{y_s^2}-\frac{|m_1|^2}{y_1^2}\ ;\ \zeta'_s=\frac{|m'_s|^2}{y_s'^2}-\frac{|m'_1|^2}{y_1'^2}\ ,\ s=2,\dots,k\ . \tag{4.14}$$

Then we have

$$y_s=\frac{|m_s|}{\ell_s}\ ,\ y'_s=\frac{|m'_s|}{\ell'_s}\ ,\quad s=2,\dots,k \tag{4.15}$$

and

$$dy_s=-\frac{1}{2}\frac{|m_s|}{\ell_s^3}d\zeta_s\ ,\ dy'_s=-\frac{1}{2}\frac{|m_s|}{\ell_s'^3}d\zeta'_s\ ,\quad s=2,\dots,k \tag{4.16}$$

where we have defined

$$\ell_s=\ell_s(y_1,m_1,\zeta_s)=\frac{|m_1|}{y_1}\sqrt{1+\frac{y_1^2}{|m_1|^2}\zeta_s}\ , \tag{4.17}$$

and similarly for $\ell'_s=\ell'_s(y'_1,m'_1,\zeta'_s)$. (4.13) can now be written as follows:

(4.18)
$$\begin{aligned}
&Q^i(\mathbf{m};\mathbf{m}')\\
&=\int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}}dy_1\int_{|m'_1|/\sqrt{z_i}}^{|m'_1|/\sqrt{z_{i-1}}}dy'_1\int_{-A}^{A}d\zeta_2\cdots\int_{-A}^{A}d\zeta_k\int_{-A}^{A}d\zeta'_2\cdots\int_{-A}^{A}d\zeta'_k\times\\
&\times\left\{A-(0\vee\max_{2\le s\le k}\zeta_s)+(0\wedge\min_{2\le s\le k}\zeta_s)\right\}_+\left\{A-(0\vee\max_{2\le t\le k}\zeta_s)+(0\wedge\min_{2\le t\le k}\zeta_s)\right\}_+\times\\
&\times\prod_{s=2}^{k}\left\{\left(\frac{1}{2}\left|\frac{m_s}{\ell_s}\right|\right)\left(\frac{1}{2}\left|\frac{m'_s}{\ell'_s}\right|\right)\frac{1}{\ell_s^2}\frac{1}{\ell_s'^2}\right\}\times\\
&\times p_{2k}(y_1,\left|\frac{m_2}{\ell_2}\right|,\dots,\left|\frac{m_k}{\ell_k}\right|,y'_1,\left|\frac{m'_2}{\ell'_2}\right|,\dots,\left|\frac{m'_k}{\ell'_k}\right|\ \mid\varphi(m_1),\dots,\varphi(m_k),\varphi(m'_1),\dots,\varphi(m'_k))
\end{aligned}$$

Here the variables ζ's range in $(-A,A)$ because the integrand vanishes if at least one of them is outside of this interval.

Given $\beta>L_n^{-1/2}$, define a subset $\mathcal{W}_n(\beta)$ of $\mathcal{Z}_{k,n}\times\mathcal{Z}_{k,n}$ to be the totality of those $(\mathbf{m};\mathbf{m}')=(m_1,\dots,m_k;m'_1,\dots,m'_k)$ such that

$$\varphi(m_s)-\varphi(m_{s-1})\ge\beta\ ,\ \varphi(m'_s)-\varphi(m'_{s-1})\ge\beta\ ,\ s=2,\dots,k \tag{4.19}$$

and that

$$|\varphi(m_s)-\varphi(m'_t)|\ge\beta\ ,\ s,t=1,\dots,k\ . \tag{4.20}$$

Then

$$X_n^i=\left(\sum_{\mathcal{W}_n(\beta)}+\sum_{(\mathcal{Z}_{k,n}\times\mathcal{Z}_{k,n})\setminus\mathcal{W}_n(\beta)}\right)Q^i(\mathbf{m};\mathbf{m}')\ . \tag{4.21}$$

Let $X_n^i(\beta)$ be the first sum on the right hand side of (4.21). If for given $m_1,m'_1\in\mathcal{Z}_{1,n}$ with $|\varphi(m_1)-\varphi(m'_1)|\ge\beta$ one defines

(4.22)
$$\begin{aligned}
&\mathcal{W}_n(\beta;m_1,m'_1)\\
&=\{(m_2,\dots,m_k;m'_2,\dots,m'_k)\ ;\ (m_1,m_2,\dots,m_k;m'_1,m'_2,\dots,m'_k)\in\mathcal{W}_n(\beta)\}\ ,
\end{aligned}$$

then

$$X_n^i(\beta) = \sum_{\substack{m_1, m_1' \in \mathcal{Z}_{1,n} \\ |\varphi(m_1) - \varphi(m_1')| \geq \beta}} \sum_{\mathcal{W}_n(\beta; m_1, m_1')} Q^i(m_1, \dots, m_k; m_1', \dots, m_k') \ . \tag{4.23}$$

Fix $m_1, m_1' \in \mathcal{Z}_{1,n}$ such that

$$|\varphi(m_1) - \varphi(m_1')| \geq \beta \tag{4.24}$$

and y_1 , y_1' such that

$$\frac{|m_1|}{\sqrt{z_i}} \leq y_1 \leq \frac{|m_1|}{\sqrt{z_{i-1}}} \ ; \ \frac{|m_1'|}{\sqrt{z_i}} \leq y_1' \leq \frac{|m_1'|}{\sqrt{z_{i-1}}} \ . \tag{4.25}$$

Take also ζ_s , ζ_s' , $s = 2, \dots, k$ from the interval $(-A, A)$. Then for some positive constant $C_T > 0$, it holds that

$$C_T^{-1} L_n^{1/2} \leq \ell_s \ , \ \ell_s' \leq C_T L_n^{1/2} \ , \ s = 2, \dots, k \ . \tag{4.26}$$

Now we shall evaluate approximately the sum

(4.27)

$$\sum_{\substack{(m_2, \dots, m_k; m_2', \dots, m_k') \\ \in \mathcal{W}_n(\beta; m_1, m_1')}} \prod_{s=2}^{k} \left\{ \left(\frac{1}{2} \left| \frac{m_s}{\ell_s} \right| \right) \left(\frac{1}{2} \left| \frac{m_s'}{\ell_s'} \right| \right) \frac{1}{\ell_s^2} \frac{1}{\ell_s'^2} \right\} \times$$
$$\times \, p_{2k}(y_1, \left|\frac{m_2}{\ell_2}\right|, \dots, \left|\frac{m_k}{\ell_k}\right|, y_1', \left|\frac{m_2'}{\ell_2'}\right|, \dots, \left|\frac{m_k'}{\ell_k'}\right| \mid \varphi(m_1), \dots, \varphi(m_k), \varphi(m_1'), \dots, \varphi(m_k')) \ ,$$

the approximation being uniform in $\zeta_2, \dots, \zeta_k$, $\zeta_2', \dots, \zeta_k'$. Note that the summand appears as a part of the integrand in (4.18).

For this purpose, we introduce a function

$$\begin{aligned} F &= F_{m_1, m_1', y_1, y_1'}(x_2, \dots, x_k; x_2', \dots, x_k') \\ &\equiv \prod_{s=2}^{k} (\frac{|x_s|}{2} \frac{|x_s'|}{2}) p_{2k}(y_1, |x_2|, \dots, |x_k|, y_1', |x_2'|, \dots, |x_k'| \mid \\ &\quad \varphi(m_1), \varphi(x_2), \dots, \varphi(x_k), \varphi(m_1'), \varphi(x_2'), \dots, \varphi(x_k')) \end{aligned} \tag{4.28}$$

which is defined on the set

$$\begin{aligned} \mathcal{D}_{m_1, m_1'} \equiv \{ &(x_2, \dots, x_k; x_2', \dots, x_k') \in (\mathbf{R}^2)^{2(k-1)} \mid b_1 \leq |x_s|, |x_s'| \leq b_2 \ , \\ &\varphi(m_1) < \varphi(x_2) < \cdots < \varphi(x_k) \ ; \ \varphi(m_1') < \varphi(x_2') < \cdots < \varphi(x_k')) \} \ . \end{aligned} \tag{4.29}$$

It is easy to see that for some $C > 0$,

$$\left\| \frac{\partial F}{\partial x_j} \right\| \ , \ \left\| \frac{\partial F}{\partial x_j'} \right\| \ \leq \ C(\psi_{2k}(\theta) + \Psi_{2k}(\theta)) \tag{4.30}$$

holds on $\mathcal{D}_{m_1, m_1'}$, where θ is the minimum of $\varphi(x_2) - \varphi(m_1)$, $\varphi(x_3) - \varphi(x_2) \dots$ etc..

In view of Conditions (II) and (III) , we may suppose $\psi_{2k}(\theta) < \Psi_{2k}(\theta)$, so that (4.30) is actually

$$\left\| \frac{\partial F}{\partial x_j} \right\| \ , \ \left\| \frac{\partial F}{\partial x_j'} \right\| \ \leq \ C \Psi_{2k}(\theta) \ . \tag{4.31}$$

The sum (4.27) can then be viewed as a Riemann sum approximation for the integral of the function F , and (4.26), (4.31) can be used to estimate its accuracy. Namely (4.27) is equal to

$$
\begin{aligned}
(4.32) \qquad & \sum_{\substack{(m_2,\dots,m_k;m_2',\dots,m_k')\\ \in\mathcal{W}_n(\beta;m_1,m_1')}} \left(\prod_{s=2}^{k}\frac{1}{\ell_s^2}\frac{1}{\ell_s'^2}\right) F_{m_1,m_1',y_1,y_1'}\left(\frac{m_2}{\ell_2},\dots,\frac{m_k}{\ell_k},\frac{m_2'}{\ell_2'},\dots,\frac{m_k'}{\ell_k'}\right)\\
&= \int dx_2\cdots\int dx_k\int dx_2'\cdots\int dx_k' F_{m_1,m_1',y_1,y_1'}(x_2,\dots,x_k,x_2',\dots,x_k')\\
&\quad + \mathcal{O}_T(L_n^{-1/2}\Psi_{2k}(\beta)) \ ,
\end{aligned}
$$

where the integration ranges over those $(x_2,\dots,x_k,x_2',\dots,x_k')\in\mathcal{D}_{m_1,m_1'}$ for which

$$
\begin{aligned}
(4.33) \qquad & \varphi(x_2)-\varphi(m_1)\ge\beta\ ,\ \varphi(x_s)-\varphi(x_{s-1})\ge\beta\ ,\ s=3,\dots,k\ ;\\
& \varphi(x_2')-\varphi(m_1')\ge\beta\ ,\ \varphi(x_t')-\varphi(x_{t-1}')\ge\beta\ ,\ t=3,\dots,k\ ;\\
& |\varphi(x_s)-\varphi(x_t')|\ge\beta\ ,\ s,t=2,\dots,k\ .
\end{aligned}
$$

Transforming each of x_s and x_s' into the polar coordinates, the above integral becomes

$$
\begin{aligned}
(4.34) \qquad & \int d\varphi_2\cdots\int d\varphi_k\int d\varphi_2'\cdots\int d\varphi_k'\int_0^\infty dr_2\cdots\int_0^\infty dr_k\int_0^\infty dr_2'\cdots\int_0^\infty dr_k'\times\\
&\times\prod_{s=2}^{k}\left(\frac{r_s^2}{2}\frac{r_s'^2}{2}\right)\times\\
&\times p_{2k}(y_1,r_2,\dots,r_k,y_1',r_2',\dots,r_k'\mid\varphi(m_1),\varphi_2,\dots,\varphi_k,\varphi(m_1'),\varphi_2',\dots,\varphi_k')\ ,
\end{aligned}
$$

where the integration ranges over those φ_s , φ_s' such that

$$
\begin{aligned}
(4.35) \qquad & \varphi(x_2)-\varphi(m_1)\ge\beta\ ,\ \varphi(x_s)-\varphi(x_{s-1})\ge\beta\ ,\ s=3,\dots,k\ ;\\
& \varphi(x_2')-\varphi(m_1')\ge\beta\ ,\ \varphi(x_t')-\varphi(x_{t-1}')\ge\beta\ ,\ t=3,\dots,k\ ;
\end{aligned}
$$

and that

$$
(4.36) \qquad |\varphi_s-\varphi_t'|\ge\beta\ ,\ s,t=1,\dots,k\ .
$$

If we integrate out with respect to the variables r_s , r_s' , $s=2,\dots,k$, we see without difficulty that the difference made by letting $\beta=0$ in the range of integration of (4.34) is less than

$$
(4.37) \qquad C\beta p_2(y_1,y_1'\mid\varphi(m_1),\varphi(m_1'))\ .
$$

We have thus approximated the sum (4.27) uniformly in the parameters $\zeta_2,\dots,\zeta_k$, $\zeta_2',\dots,\zeta_k'$. In order to finish the estimation of $X_n^i(\beta)$ (recall (4.23) for the definition), we need to get an approximation of the sum of $Q^i(m_1,\dots,m_k;m_1',\dots,m_k')$ over $\mathcal{W}(\beta;m_1,m_1')$. This is done by integrating the sum (4.27) over parameters y_1 , y_1' , $\zeta_2,\dots,\zeta_k$, $\zeta_2',\dots,\zeta_k'$. By noting the above mentioned uniformity of the integral-approximation in ζ_s and ζ_s' , and by using the formula

$$
(4.38) \qquad \int_{-A}^{A}d\zeta_2\cdots\int_{-A}^{A}d\zeta_k\{A-(0\vee\zeta_2\vee\cdots\vee\zeta_k)+(0\wedge\zeta_2\wedge\cdots\wedge\zeta_k)\}_+=A^k\ ,\ k\ge2\ ,
$$

which we shall prove as Lemma 5 at the end of this section, we obtain

$$
\begin{aligned}
(4.39)\qquad X_n^i(\beta) = & \sum_{\substack{m_1, m_1' \in \mathcal{Z}_{1,n} \\ |\varphi(m_1)-\varphi(m_1')|\ge\beta}} A^{2k} \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{|m_1'|/\sqrt{z_i}}^{|m_1'|/\sqrt{z_{i-1}}} dy_1' \times \\
& \times \Bigg\{ \int\cdots\int_{\varphi(m_1)<\varphi_2<\cdots<\varphi_k} d\varphi_2\cdots d\varphi_k \int\cdots\int_{\varphi(m_1')<\varphi_2'<\cdots<\varphi_k'} d\varphi_2'\cdots d\varphi_k' \times \\
& \times \int_0^\infty dr_2 \cdots \int_0^\infty dr_k \int_0^\infty dr_2' \cdots \int_0^\infty dr_k' \prod_{s=2}^k \left(\frac{r_s^2}{2}\frac{r_s'^2}{2}\right) \times \\
& \times p_{2k}(y_1, r_2, \ldots, r_k, y_1', r_2', \ldots, r_k' \mid \varphi(m_1), \varphi_2, \ldots, \varphi_k, \varphi(m_1'), \varphi_2', \ldots, \varphi_k') + \\
& \quad + \mathcal{O}_T(L_n^{-1/2}\Psi_{2k}(\beta) + \beta p_2(y_1, y_1' | \varphi(m_1), \varphi(m_1'))\Bigg\} \\
= & \; 4(c_2-c_1)^{2k} \sum_{\substack{m_1, m_1' \in \mathcal{Z}_{1,n} \\ |\varphi(m_1)-\varphi(m_1')|\ge\beta}} S^i(m_1, m_1') + \mathcal{O}_T(L_n^{3/2} q_n^{-2} \Psi_{2k}(\beta)) \\
& + \mathcal{O}\left(\sum_{\substack{m_1, m_1' \in \mathcal{Z}_{1,n} \\ |\varphi(m_1)-\varphi(m_1')|\ge\beta}} \beta \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{|m_1'|/\sqrt{z_i}}^{|m_1'|/\sqrt{z_{i-1}}} dy_1' p_2(y_1, y_1' \mid \varphi(m_1), \varphi(m_1')) \right) ,
\end{aligned}
$$

where we have used (4.11) and $\sharp(\mathcal{Z}_{1,n} \times \mathcal{Z}_{1,n}) = \mathcal{O}_T(L_n)$ to get the bound of the first error term. See also (4.5) for the definition of $S^i(m_1, m_1')$. In order to estimate the second error term, let us first consider the case where our probability measure $\mathbf{P}$ satisfies Condition (II-1). Then the sum

$$
(4.40)\qquad \sum_{\substack{m_1, m_1' \in \mathcal{Z}_{1,n} \\ |\varphi(m_1)-\varphi(m_1')|\ge\beta}} \beta \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{|m_1'|/\sqrt{z_i}}^{|m_1'|/\sqrt{z_{i-1}}} dy_1' p_2(y_1, y_1' \mid \varphi(m_1), \varphi(m_1'))
$$

is bounded by

$$
\begin{aligned}
(4.41)\qquad & C\beta q_n^{-2} \sum_{\substack{m_1, m_1' \in \mathcal{Z}_{1,n} \\ |\varphi(m_1)-\varphi(m_1')|\ge\beta}} |\varphi(m_1)-\varphi(m_1')|^{-\sigma} \\
& \le C\beta\beta^{-\sigma} q_n^{-2} \sum_{m_1, m_1' \in \mathcal{Z}_{1,n}} 1 \\
& \le C_T L_n^2 q_n^{-2} \beta^{1-\sigma} \ .
\end{aligned}
$$

Next suppose that $\mathbf{P}$ satisfies Condition (II-2). Then we have

$$
(4.42)\qquad p_2(y_1, y_1' \mid \varphi(m_1), \varphi(m_1')) \le C 1_{\{|y_1-y_1'|\le b_3|\varphi(m_1)-\varphi(m_1')|\}} |\varphi(m_1)-\varphi(m_1')|^{-\tau} \ .
$$

Then the sum (4.40) is bounded by

$$
\begin{aligned}
(4.43)\qquad & C\beta \int_{b_1}^{b_2} dy_1 \int_{b_1}^{b_2} dy_1' \sum_{m_1, m_1' \in \mathcal{Z}_{1,n}} 1_{\{y_1\sqrt{z_{i-1}}\le|m_1|y_1\le\sqrt{z_i}\}} \times \\
& \times 1_{\{y_1'\sqrt{z_{i-1}}\le|m_1'|\le y_1'\sqrt{z_i}\}} 1_{\{|y_1-y_1'|\vee(b_3\beta)\le|\varphi(m_1)-\varphi(m_1')|\}} \times \\
& \times |\varphi(m_1)-\varphi(m_1')|^{-\tau} \ .
\end{aligned}
$$

For fixed y_1, y_1' and m_1 , we shall estimate

$$
(4.44)\qquad \sum_{m_1' \in \mathcal{Z}_{1,n}} 1_{\{y_1'\sqrt{z_{i-1}}\le|m_1'|\le y_1'\sqrt{z_i}\}} 1_{\{|y_1-y_1'|\vee(b_3\beta)\le|\varphi(m_1)-\varphi(m_1')|\}} |\varphi(m_1)-\varphi(m_1')|^{-\tau} \ .
$$

It is sufficient to do this assuming $\varphi(m_1) > \varphi(m_1')$. To this end, let us divide the interval $[\beta \vee (|y_1 - y_1'|/b_3), \Theta]$ into subintervals Δ_j of length $L_n^{-1/2}$. Since the number of those $m_1' \in \mathcal{Z}_{1,n}$ for which $y_1'\sqrt{z_{i-1}} \le |m_1'| \le y_1'\sqrt{z_i}$ and $\varphi(m_1') \in \Delta_j$ hold is bounded by $C_T L_n^{1/2} q_n^{-1}$, the sum (4.44) is less than

$$\begin{aligned} &C_T L_n^{1/2} q_n^{-1} \sum_{j=1}^{\infty} \{ j L_n^{-1/2} + (\beta \vee \frac{|y_1 - y_1'|}{b_3}) \}^{-\tau} \\ &\le C_T L_n q_n^{-1} \int_{\beta \vee (\frac{|y_1 - y_1'|}{b_3})}^{\infty} t^{-\tau} dt \\ &= C_T L_n q_n^{-1} (\beta^{1-\tau}) \wedge (|y_1 - y_1'|^{1-\tau}) \ . \end{aligned} \tag{4.45}$$

Inserting this estimate into (4.43), we obtain the following new bound of (4.40):

$$\begin{aligned} &C_T \beta \int_{b_1}^{b_2} dy_1 \int_{b_1}^{b_2} dy_1' \left(\sum_{m_1 \in \mathcal{Z}_{1,n}} 1_{\{y_1\sqrt{z_{i-1}} \le |m_1| \le y_1 \sqrt{z_i}\}} \right) L_n q_n^{-1} \{ \beta^{1-\tau} \wedge |y_1 - y_1'|^{1-\tau} \} \\ &\le C_T \beta (L_n q_n^{-1})^2 \int_{b_1}^{b_2} dy_1 \int_{b_1}^{b_2} dy_1' \{ \beta^{1-\tau} \wedge |y_1 - y_1'|^{1-\tau} \} \\ &= C_T L_n^2 q_n^{-2} \beta^{2-\tau} \ . \end{aligned} \tag{4.46}$$

Returning to (4.39), we have thus proved

$$X_n^i(\beta) = 4(c_2 - c_1)^{2k} \sum_{\substack{m_1, m_1' \in \mathcal{Z}_{1,n} \\ |\varphi(m_1) - \varphi(m_1')| \ge \beta}} S^i(m_1, m_1') + \mathcal{O}_T(L_n^{3/2} q_n^{-2} \Psi_{2k}(\beta) + L_n^2 q_n^{-2} \beta^d) \ , \tag{4.47}$$

where $d = 1 - \sigma$ or $d = 2 - \tau$. (Recall $A = 2(c_2 - c_1)$.)

To finish the discussion on $X_n^i(\beta)$, we estimate the error

$$\sum_{\substack{m_1, m_1' \in \mathcal{Z}_{1,n} \\ |\varphi(m_1) - \varphi(m_1')| < \beta}} S^i(m_1, m_1') \tag{4.48}$$

which is made by dropping the condition $|\varphi(m_1) - \varphi(m_1')| \ge \beta$ under the summation symbol in (4.47). For this purpose, we divide the sum (4.48) into two parts, namely we rewrite (4.48) as

$$\sum_{0 < |\varphi(m_1) - \varphi(m_1')| < \beta} S^i(m_1, m_1') + \sum_{\varphi(m_1) = \varphi(m_1')} S^i(m_1, m_1') \equiv J_1 + J_2 \tag{4.49}$$

when Condition (II-1) holds, or

$$\sum_{L_n^{-1/2} \le |\varphi(m_1) - \varphi(m_1')| < \beta} S^i(m_1, m_1') + \sum_{|\varphi(m_1) - \varphi(m_1')| < L_n^{-1/2}} S^i(m_1, m_1') \equiv K_1 + K_2 \tag{4.50}$$

when Condition (II-2) holds.

Note that if $\varphi(m_1) \ne \varphi(m_1')$

$$\begin{aligned} S^i(m_1, m_1') &\le C \mathbf{P}(\gamma_{m_1}, \gamma_{m_1'} \in [z_{i-1}, z_i)) \\ &= C \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{|m_1'|/\sqrt{z_i}}^{|m_1'|/\sqrt{z_{i-1}}} dy_1' p_2(y_1, y_1' \mid \varphi(m_1), \varphi(m_1')) \ . \end{aligned} \tag{4.51}$$

Suppose that Condition (II-1) is satisfied. Then by (4-11),

$$S(m_1, m_1') \le C_T q_n^{-2} |\varphi(m_1) - \varphi(m_1')|^{-\sigma} \tag{4.51}$$

with $\sigma \in (0,1)$. Hence applying Lemma 1 with $R = L_n^{1/2}$ and $\beta > L_n^{-1/2}$, we see

$$J_1 \le C_T L_n^2 q_n^{-2} \beta^{1-\sigma} \ . \tag{4.52}$$

On the other hand, if $\varphi(m_1) = \varphi(m_1')$, one will have

$$S(m_1, m_1') \le C \int_{b_1}^{b_2} dy_1 1_{\{y_1\sqrt{z_{i-1}} \le |m_1|,|m_1'| \le y_1\sqrt{z_i}\}} \ . \tag{4.53}$$

Since $y_1(\sqrt{z_i} - \sqrt{z_{i-1}}) = \mathcal{O}_T(L_n^{1/2} q_n^{-1})$, we have

$$J_2 \le C_T L_n^{3/2} q_n^{-2} \ . \tag{4.54}$$

Next suppose that Condition (II-2) holds. Then if $\varphi(m_1) \ne \varphi(m_1')$,

$$\begin{aligned} S(m_1, m_1') \le C \int_{b_1}^{b_2} dy_1 \int_{b_1}^{b_2} dy_1' 1_{\{y_1\sqrt{z_{i-1}} \le |m_1| \le y_1\sqrt{z_i}\}} 1_{\{y_1'\sqrt{z_{i-1}} \le |m_1'| \le y_1'\sqrt{z_i}\}} \times \\ \times 1_{\{|y_1 - y_1'| \le b_3|\varphi(m_1) - \varphi(m_1')|\}} |\varphi(m_1) - \varphi(m_1')|^{-\tau} \ . \end{aligned} \tag{4.55}$$

Hence we have

$$\begin{aligned} K_1 \le C \int_{b_1}^{b_2} dy_1 \int_{b_1}^{b_2} dy_1' 1_{\{|y_1 - y_1'| \le \beta\}} \sum_{m_1 \in \mathcal{Z}_{1,n}} 1_{\{y_1\sqrt{z_{i-1}} \le |m_1| \le y_1\sqrt{z_i}\}} \times \\ \times \sum_{m_1' \in \mathcal{Z}_{1,n}} 1_{\{y_1'\sqrt{z_{i-1}} \le |m_1'| \le y_1'\sqrt{z_i}\}} 1_{\{|y_1 - y_1'| \vee (b_3 L_n^{-1/2}) \le |\varphi(m_1) - \varphi(m_1')|\}} \\ \times |\varphi(m_1) - \varphi(m_1')|^{-\tau} \ . \end{aligned} \tag{4.56}$$

Let us estimate the summation over m_1' . We can assume $\varphi(m_1') > \varphi(m_1)$. As we did in the estimation of the sum (4.44) , we divide the interval $[L_n^{-1/2} \vee (\frac{1}{b_3}|y_1 - y_1'|), \Theta]$ into subintervals of length $L_n^{-1/2}$, and obtain

$$\begin{aligned} &\sum_{m_1' \in \mathcal{Z}_{1,n}} 1_{\{y_1'\sqrt{z_{i-1}} \le |m_1'| \le y_1'\sqrt{z_i}\}} 1_{\{|y_1 - y_1'| \vee (b_3 L_n^{-1/2}) \le |\varphi(m_1) - \varphi(m_1')|\}} |\varphi(m_1) - \varphi(m_1')|^{-\tau} \\ &\le C_T L_n^{1/2} q_n^{-1} \sum_{j=0}^{\infty} \{(L_n^{-1/2} \vee \frac{1}{b_3}|y_1 - y_1'|) + j L_n^{-1/2}\}^{-\tau} \\ &\le C_T L_n q_n^{-1} (L_n^{-1/2} \vee \frac{1}{b_3}|y_1 - y_1'|)^{1-\tau} \ . \end{aligned} \tag{4.57}$$

Inserting this inequality into (4.56), one easily get the bound

$$K_1 \le C_T L_n^2 q_n^{-2} \beta^{2-\tau} \ . \tag{4.58}$$

We now turn to estimating K_2 . By $\sqrt{z_i} \le C_T L_n^{1/2}$ and $|\varphi(m_1) - \varphi(m_1')| \le L_n^{-1/2}$, one easily sees that there is a constant $C_T > 0$ such that the condition,

$$f(\varphi(m_1'))\sqrt{z_{i-1}} \le |m_1'| \le f(\varphi(m_1'))\sqrt{z_i} \tag{4.59}$$

implies

$$f(\varphi(m_1))\sqrt{z_{i-1}} - C_T \le |m_1'| \le f(\varphi(m_1))\sqrt{z_i} + C_T \ . \tag{4.60}$$

For each fixed $m_1 \in \mathcal{Z}_{1,n}$, the number of m_1' which satisfies the condition (4.60) and $|\varphi(m_1)-\varphi(m_1')|<L_n^{-1/2}$ is again bounded by $C_T L_n^{1/2} q_n^{-1}$. Hence

$$\sum_{\substack{m_1,m_1'\in\mathcal{Z}_{1,n}\\ |\varphi(m_1)-\varphi(m_1')|<L_n^{-1/2}}} S(m_1,m_1') \le C_T L_n^{1/2} q_n^{-1} \sum_{m_1\in\mathcal{Z}_{1,n}} \mathbf{P}(\gamma_{m_1}\in[z_{i-1},z_i)) \le C_T L_n^{3/2} q_n^{-2} \; . \tag{4.61}$$

We have thus shown that the sum (4.48) is bounded by

$$C_T(L_n^{3/2}q_n^{-2} + L_n^2 q_n^{-2}\beta^d) \tag{4.62}$$

with $d=1-\sigma$ or $d=2-\tau$. Since $1=\mathcal{O}(\Psi_{2k}(\beta))$, we finally arrive at

$$X_n^i(\beta) = 4(c_2-c_1)^{2k} \sum_{m_1,m_1'\in\mathcal{Z}_{1,n}} S(m_1,m_1') + \mathcal{O}_T(L_n^{3/2}q_n^{-2}(\Psi_{2k}(\beta)+L_n^{1/2}\beta^d)) \; . \tag{4.63}$$

We are still left with the task of estimating

$$X_n^i - X_n^i(\beta) = \sum_{(\mathcal{Z}_{1,n}\times\mathcal{Z}_{1,n})\setminus\mathcal{W}_n(\beta)} Q^i(\mathbf{m};\mathbf{m}') \; . \tag{4.64}$$

Again we divide this sum into two parts, in a slightly different ways according to which of the conditions (II-1) or (II-2) is satisfied. Namely we let

$$X_n^i - X_n^i(\beta) \equiv X_n'(\beta) + X_n'' = \left(\sum_{\mathcal{W}_n'(\beta)} + \sum_{\mathcal{W}_n''}\right) Q^i(\mathbf{m};\mathbf{m}') \; , \tag{4.65}$$

where we define, in case Condition (II-1) is satisfied,

$$\mathcal{W}_n'' \equiv \{(\mathbf{m};\mathbf{m}')\in\mathcal{Z}_{k,n}\times\mathcal{Z}_{k,n} \mid \varphi(m_s)=\varphi(m_t') \text{ for some } s,t=1,\dots,k\} \; , \tag{4.66}$$

and define, in case Condition (II-2) is satisfied,

$$\mathcal{W}_n'' \equiv \{(\mathbf{m};\mathbf{m})\in\mathcal{Z}_{k,n}\times\mathcal{Z}_{k,n} \mid |\varphi(m_s)-\varphi(m_t')|<M'L_n^{-1/2} \text{ for some } s,t=1,\dots,k\} \; , \tag{4.67}$$

the constant $M'>0$ being chosen later.

In either case, we let

$$\mathcal{W}_n'(\beta) = (\mathcal{Z}_{1,n}\times\mathcal{Z}_{1,n}) \setminus (\mathcal{W}_n(\beta)\cup\mathcal{W}_n'') \; . \tag{4.68}$$

Recall $\mathcal{W}_n(\beta)$ was defined just before (4.21).

Suppose (II-1) is satisfied. We shall estimate $X_n'(\beta)$. By (4.66) and (4.68), $X_n'(\beta)$ is the sum of $Q^i(\mathbf{m};\mathbf{m}')$ over those $(\mathbf{m};\mathbf{m}')\in\mathcal{Z}_{k,n}\times\mathcal{Z}_{k,n}$ such that $\varphi(m_s)$ and $\varphi(m_t')$ are all different but that at least one of the following is less than β :

$$\begin{aligned} &\varphi(m_s)-\varphi(m_{s-1}) \; , \; \varphi(m_s')-\varphi(m_{s-1}') \; , s=2,\dots,k \; ; \\ &|\varphi(m_s)-\varphi(m_t')| \; , \; s,t=1,2,\dots,k \; . \end{aligned} \tag{4.69}$$

Now let $\bar m_1 \dots, \bar m_{2k}$ be the rearrangement of $m_1,\dots,m_k$, $m_1',\dots,m_k'$ in the increasing order of $\varphi(\cdot)$, namely

$$\varphi(\bar m_1)<\dots<\varphi(\bar m_{2k}) \; , \; \{\bar m_1 \dots, \bar m_{2k}\} = \{m_1,\dots,m_k,m_1',\dots,m_k'\} \; . \tag{4.70}$$

There are $(2k)!/(k!)^2$ ways of dividing a given $\{\bar m_1 \dots ,\bar m_{2k}\}$ into two classes $\{m_1, \dots , m_k\}$ and $\{m'_1, \dots , m'_k\}$, a typical one, let us call it σ, being as follows:

$$\begin{aligned} &\varphi(m_1) < \dots < \varphi(m_r) < \varphi(m'_1) < \varphi(\bar m_{r+2}) < \dots < \varphi(\bar m_{2k}) \ ; \\ &\{\bar m_{r+2}, \dots , \bar m_{2k}\} = \{m_{r+1}, \dots , m_k, m'_2, \dots , m'_k\} \ . \end{aligned} \tag{4.71}$$

Accordingly $\mathcal{W}'_n(\beta)$ is divided into $(2k)!/(k!)^2$ disjoint classes:

$$\mathcal{W}'_n(\beta) = \bigcup_\sigma \mathcal{W}'_{n,\sigma}(\beta) \ , \tag{4.72}$$

and obviously it is sufficient to estimate the sum $X'_{n,\sigma}(\beta)$ of $Q^i(m_1, \dots , m_k; m'_1, \dots , m'_k)$ over one of these classes. Recalling the definition (3.10) of $\mathcal{Z}_{k,n}$, of which $\mathcal{W}'_n(\beta)$ is a subset, and the definition (4.17) of ℓ_s , we see that $|m_s|/\ell_s$ and $|m'_s|/\ell'_s$ are of $\mathcal{O}_T(1)$, and hence

$$\begin{aligned} &X'_n(\beta) \\ &\le C_T L_n^{-2(k-1)} \sum_{\mathcal{W}'_n(\beta)} \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{|m'_1|/\sqrt{z_i}}^{|m'_1|/\sqrt{z_{i-1}}} dy'_1 \int_{-A}^{A} d\zeta_2 \cdots \int_{-A}^{A} d\zeta_k \int_{-A}^{A} d\zeta'_2 \cdots \int_{-A}^{A} d\zeta'_k \times \\ &\times p_{2k}(y_1, \left|\frac{m_2}{\ell_2}\right|, \dots , \left|\frac{m_k}{\ell_k}\right|, y'_1, \left|\frac{m'_2}{\ell'_2}\right|, \dots , \left|\frac{m'_k}{\ell'_k}\right| \ |\varphi(m_1), \dots , \varphi(m_k), \varphi(m'_1), \dots , \varphi(m'_k)) \ . \end{aligned} \tag{4.73}$$

At this point, we assume that $(m_1, \dots , m_k; m'_1, \dots , m'_k)$ are arranged like (4.71) and set

$$\bar y_1 = y_1 \ , \ \bar y_j = \frac{|m_s|}{\ell_s} \ ; \ \bar\zeta_j = \zeta_s \quad \text{if } \bar m_j = m_s \text{ for some } s = 2, \dots , k \tag{4.74}$$

and

$$\bar y_{r+1} = y'_1 \ , \ \bar y'_j = \frac{|m'_t|}{\ell'_t} \ ; \ \bar\zeta_j = \zeta'_t \quad \text{if } \bar m_j = m'_t \text{ for some } t = 2, \dots , k \tag{4.75}$$

Then by Condition (II-1), we get

$$\begin{aligned} &X'_n(\beta) \\ &\le C_T L_n^{-2(k-1)} \sum_{\mathcal{W}'_{n,\sigma}(\beta)} \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{-A}^{A} d\zeta_2 \cdots \int_{-A}^{A} d\zeta_r \times \\ &\times \int_{|m'_1|/\sqrt{z_i}}^{|m'_1|/\sqrt{z_{i-1}}} dy'_1 \int_{-A}^{A} d\bar\zeta_{r+2} \cdots \int_{-A}^{A} d\bar\zeta_{2k} \times \\ &\times p_{2k}(y_1, \bar y_2, \dots , \bar y_r, y'_1, \bar y_{r+2}, \dots , \bar y_{2k} \ |\varphi(m_1), \dots , \varphi(m_r), \varphi(m'_1), \dots , \varphi(\bar m_{r+2}), \dots , \varphi(\bar m_{2k})) \\ &\le C_T L_n^{-2(k-1)} \sum_{\substack{\bar m_1 \dots , \bar m_{2k}, \\ 0 < \bar m_j - \bar m_{j-1} < \beta_j}} q_n^{-2} \prod_{j=2}^{2k} (\varphi(\bar m_j) - \varphi(\bar m_{j-1}))^{-\sigma} \ , \end{aligned} \tag{4.76}$$

where $\beta_j = \beta$ or $\beta_j = \Theta$ and at least one of β_j is equal to β . We now apply Lemma 1 with $R = L_n^{1/2}$, successively to the summation with respect to $\bar m_j$, $j = 2k, 2k-1, \dots , 2$, to obtain

$$X'_n(\beta) \le C_T q_n^{-2} L_n^2 \prod_{j=2}^{2k} \bar\beta_j^{1-\sigma} \le C_T L_n^2 q_n^{-2} \beta^{1-\sigma} \ . \tag{4.77}$$

Next let us estimate X''_n , still assuming Condition (II-1). By definition, we can write

$$X''_n = \sum_{\ell=1}^{k} \sum_{1\le s_1<\cdots<s_\ell\le k} \sum_{1\le t_1<\cdots<t_\ell\le k} \sum_{\substack{(\mathbf{m};\mathbf{m}')\in\mathcal{Z}_{k,n}\times\mathcal{Z}_{k,n};\\ \varphi(m_{s_j})=\varphi(m_{t_j}),j=1,\dots,\ell}} Q^i(\mathbf{m};\mathbf{m}') \ . \tag{4.78}$$

Let us denote the last sum by $X''_n(s_1,\dots,s_\ell;t_1,\dots,t_\ell)$. We shall prove

$$X''_n(s_1,\dots,s_\ell;t_1,\dots,t_\ell) \le C_T L_n^{3/2} q_n^{-2} \ , \tag{4.79}$$

which is sufficient for our purpose. We first consider the special case of $s_j = t_j = j$, $j = 1,\dots,\ell$. Letting as before $y_s = f(\varphi(m_s))$, $s = 1,\dots,k$ and $y'_t = f(\varphi(m'_t))$, $t = \ell+1,\dots,k$, we can write, noting $f(\varphi(m_j)) = f(\varphi(m'_j))$, $j = 1,\dots,\ell$,

$$\begin{aligned}
&Q^i(\mathbf{m};\mathbf{m}') \\
&= \int_{(\frac{|m_1|}{\sqrt{z_i}},\frac{|m_1|}{\sqrt{z_{i-1}}})\cap(\frac{|m'_1|}{\sqrt{z_i}},\frac{|m'_1|}{\sqrt{z_{i-1}}})} dy_1 \int dy_2\cdots\int dy_k \int dy'_{\ell+1}\cdots\int dy'_k \times \\
&\quad\times \left[A - \{0\vee \max_{2\le s\le k}(\frac{|m_s|^2}{y_s^2}-\frac{|m_1|^2}{y_1^2})\} + \{0\wedge \min_{2\le s\le k}(\frac{|m_s|^2}{y_s^2}-\frac{|m_1|^2}{y_1^2})\}\right]_+ \times \\
&\quad\times \Bigg[A - \{0\vee \max_{2\le t\le \ell}(\frac{|m'_t|^2}{y_t^2}-\frac{|m'_1|^2}{y_1^2}) \vee \max_{\ell<t\le k}(\frac{|m'_t|^2}{y_t'^2}-\frac{|m'_1|^2}{y_1'^2})\} \\
&\quad + \{0\wedge \min_{2\le t\le \ell}(\frac{|m'_t|^2}{y_t^2}-\frac{|m'_1|^2}{y_1^2}) \wedge \min_{\ell<t\le k}(\frac{|m'_t|^2}{y_t'^2}-\frac{|m'_1|^2}{y_1'^2})\}\Bigg]_+ \times \\
&\quad\times p_{2k-\ell}(y_1,\dots,y_k,y'_{\ell+1},\dots,y'_k \mid \varphi(m_1),\dots,\varphi(m_k),\varphi(m'_{\ell+1}),\dots,\varphi(m'_k)) \ .
\end{aligned} \tag{4.80}$$

If we make the change of variables

$$\zeta_s = \frac{|m_s|^2}{y_s^2} - \frac{|m_1|^2}{y_1^2} \ , \ s = 2,\dots,k \ ; \ \zeta'_t = \frac{|m'_t|^2}{y_t'^2} - \frac{|m'_1|^2}{y_1'^2} \ , t = \ell+1,\dots,k \ , \tag{4.81}$$

and if we note that for $t = 2,\dots,\ell$,

$$\frac{|m'_t|^2}{y_t^2} - \frac{|m'_1|^2}{y_1^2} = \left(\frac{\zeta_t}{|m_t|^2} + \frac{|m_1|^2}{y_1^2|m_t|^2}\right)|m'_t|^2 - \frac{|m'_1|^2}{y_1^2} \ , \tag{4.82}$$

we can further rewrite (4.80) as

$$\begin{aligned}
&Q^i(\mathbf{m};\mathbf{m}') \\
&= \int_{(\frac{|m_1|}{\sqrt{z_i}},\frac{|m_1|}{\sqrt{z_{i-1}}})\cap(\frac{|m'_1|}{\sqrt{z_i}},\frac{|m'_1|}{\sqrt{z_{i-1}}})} dy_1 \int_{-A}^{A} d\zeta_2 \dots \int_{-A}^{A} d\zeta_k \int_{-A}^{A} d\zeta'_{\ell+1} \dots \int_{-A}^{A} d\zeta'_k \times \\
&\times \prod_{s=2}^{k}\left(\frac{1}{2}\left|\frac{m_s}{\ell_s}\right|\right)\frac{1}{\ell_s^2} \prod_{t=\ell+1}^{k}\left(\frac{1}{2}\left|\frac{m'_t}{\ell'_t}\right|\right)\frac{1}{\ell_t'^2} \times \\
&\times [A - (0\vee \max_{2\le s\le k}\zeta_s) + (0\wedge \min_{2\le s\le k}\zeta_s)]_+ \times \\
&\times \Bigg[A - \left\{0\vee \max_{2\le t\le\ell}\left(\left(\frac{\zeta_t}{|m_t|^2}+\frac{|m_1|^2}{y_1^2|m_t|^2}\right)|m'_t|^2 - \frac{|m'_1|^2}{y_1^2}\right)\vee \max_{\ell<t\le k}\zeta'_t\right\} \\
&+ \left\{0\wedge \min_{2\le t\le\ell}\left(\left(\frac{\zeta_t}{|m_t|^2}+\frac{|m_1|^2}{y_1^2|m_t|^2}\right)|m'_t|^2 - \frac{|m'_1|^2}{y_1^2}\right)\wedge \min_{\ell<t\le k}\zeta'_t\right\}\Bigg]_+ \times \\
&\times p_{2k-\ell}(y_1,\left|\frac{m_2}{\ell_2}\right|,\dots,\left|\frac{m_k}{\ell_k}\right|,\left|\frac{m'_{\ell+1}}{\ell'_{\ell+1}}\right|,\dots,\left|\frac{m'_k}{\ell'_k}\right| |\varphi(m_1),\dots,\varphi(m_k),\varphi(m'_{\ell+1}),\dots,\varphi(m'_k)) \ .
\end{aligned} \tag{4.83}$$

In order to estimate the summation over those $(\mathbf{m};\mathbf{m}') \in \mathcal{Z}_{k,n} \times \mathcal{Z}_{k,n}$ such that $\varphi(m_j) = \varphi(m'_j)$, $j = 1, \dots, \ell$, we make the following observation.

First, we have

$$\ell_s \ge C_T \sqrt{L_n} \quad , \quad \left|\frac{m_s}{\ell_s}\right| \le C_T \ , \ s = 2, \dots, k \tag{4.84}$$

and

$$\ell'_t \ge C_T \sqrt{L_n} \quad , \quad \left|\frac{m'_t}{\ell'_t}\right| \le C_T \ , \ t = \ell+1, \dots, k \tag{4.85}$$

Moreover $Q^i(\mathbf{m};\mathbf{m}')$ must vanish unless for $t = 2, \dots, \ell$ and for some value of y_1 and ζ_t , one has

$$\left|\left(\frac{\zeta_t}{|m_t|^2} + \frac{|m_1|^2}{y_1^2 |m_t|^2}\right)|m'_t|^2 - \frac{|m'_1|^2}{y_1^2}\right| < A \ . \tag{4.86}$$

It is easy to see that for given values of m_1 , m_t , m'_1 and y_1 , ζ_t , the number of m'_t which satisfies this condition and $\varphi(m'_t) = \varphi(m_t)$ is at most one. Hence if we make summation over $m'_2, \dots, m'_\ell$ first, then

$$\begin{aligned} &X''_n(1,\dots,\ell;1,\dots,\ell) \\ &\le C_T (L_n^{-1})^{2k-\ell-1} \sum_{\substack{m_1,\dots,m_k;m'_1,\\ m'_{\ell+1},\dots,m'_k}} \int_{(\frac{|m_1|}{\sqrt{z_i}},\frac{|m_1|}{\sqrt{z_{i-1}}})\cap(\frac{|m'_1|}{\sqrt{z_i}},\frac{|m'_1|}{\sqrt{z_{i-1}}})} dy_1 \times \\ &\quad \times \int_{-A}^{A} d\zeta_2 \cdots \int_{-A}^{A} d\zeta_k \int_{-A}^{A} d\zeta'_{\ell+1} \cdots \int_{-A}^{A} d\zeta'_k \times \\ &\quad \times p_{2k-\ell}(y_1,\dots,y_k,y'_{\ell+1},\dots,y'_k \mid \varphi(m_1),\dots,\varphi(m_k),\varphi(m'_{\ell+1}),\dots,\varphi(m'_k)) \ . \end{aligned} \tag{4.87}$$

Now we arrange $m_2, \dots, m_k$ and $m'_{\ell+1}, \dots, m'_k$ in the increasing order of φ , and call them $\bar{m}_2, \dots, \bar{m}_{2k-\ell}$, namely

$$\varphi(m_1) = \varphi(m'_1) < \varphi(\bar{m}_2) < \cdots < \varphi(\bar{m}_{2k-\ell}) \ . \tag{4.88}$$

By Condition (II-1), we then have

$$\begin{aligned} &X''_n(1,\dots,\ell;1,\dots,\ell) \\ &\le C_T (L_n^{-1})^{2k-\ell-1} \sum_{\substack{(m_1,m'_1,\bar{m}_2,\dots,\bar{m}_{2k-\ell})\in\mathcal{Z}_{2k-\ell,n}\\ \varphi(m'_1)=\varphi(m_1)}} \int_{(\frac{|m_1|}{\sqrt{z_i}},\frac{|m_1|}{\sqrt{z_{i-1}}})\cap(\frac{|m'_1|}{\sqrt{z_i}},\frac{|m'_1|}{\sqrt{z_{i-1}}})} dy_1 \times \\ &\quad \times \prod_{s=3}^{2k-\ell} (\varphi(\bar{m}_s) - \varphi(\bar{m}_{s-1}))^{-\sigma} (\varphi(\bar{m}_2) - \varphi(m_1))^{-\sigma} \ , \end{aligned} \tag{4.89}$$

Applying Lemma 1 successively to the summation over $\bar{m}_{2k-\ell}, \bar{m}_{2k-\ell-1}, \dots, \bar{m}_2$, we arrive at

$$X''_n(1,\dots,\ell;1,\dots,\ell) \le C_T \sum_{\substack{m_1,m'_1\in\mathcal{Z}_{1,n}\\ \varphi(m'_1)=\varphi(m_1)}} \left[\frac{|m_1| \wedge |m'_1|}{\sqrt{z_{i-1}}} - \frac{|m_1| \vee |m'_1|}{\sqrt{z_i}}\right]_+ \ . \tag{4.90}$$

The summand vanishes unless

$$\frac{\sqrt{z_{i-1}}}{\sqrt{z_i}} |m_1| < |m'_1| < \frac{\sqrt{z_i}}{\sqrt{z_{i-1}}} |m_1| \ . \tag{4.91}$$

But for each $m_1 \in \mathcal{Z}_{1,n}$, the number of m_1' which satisfies this condition and $\varphi(m_1') = \varphi(m_1)$ is bounded by $C_T L_n^{1/2} q_n^{-1}$. On the other hand, each summand is bounded by

$$\frac{|m_1|}{\sqrt{z_{i-1}}} - \frac{|m_1|}{\sqrt{z_i}} = \left(\frac{\sqrt{z_i}}{\sqrt{z_{i-1}}} - 1\right) \frac{|m_1|}{\sqrt{z_i}} \le C_T q_n^{-1} \ , \tag{4.92}$$

hence we finally obtain

$$X_n''(1,\dots,\ell;1,\dots,\ell) \le C_T(L_n^{1/2}q_n^{-1})q_n^{-1} \sum_{m_1 \in \mathcal{Z}_{1,n}} 1 \le C_T L_n^{3/2} q_n^{-2} \ , \tag{4.93}$$

as desired. The general $X_n''(s_1,\dots,s_\ell;t_1,\dots,t_\ell)$ can be estimated in essentially the same way. Indeed, if $\gamma_{m_1}, \gamma_{m_1'} \in [z_{i-1}, z_i)$ and if $|D_{m_1} \cap \cdots \cap D_{m_k}| > 0$, $|D_{m_1'} \cap \cdots \cap D_{m_k'}| > 0$, then we must have

$$|\gamma_{m_{s_1}} - \gamma_{m_1}| < A \ , \ |\gamma_{m_{t_1}'} - \gamma_{m_1'}| < A \ , \tag{4.94}$$

which means

$$\gamma_{m_1}, \gamma_{m_1'} \in (z_{i-1} - A, z_i + A) \ . \tag{4.95}$$

Hence

$$Q^i(\mathbf{m};\mathbf{m}') \le \mathbf{E_P}[1_{\gamma_{m_1},\gamma_{m_1'} \in (z_{i-1}-A, z_i+A)} |D_{m_1} \cap \cdots \cap D_{m_k}||D_{m_1'} \cap \cdots \cap D_{m_k'}|] \ . \tag{4.96}$$

The right hand side can be treated similarly as above, except that the role of

$$(m_1, \dots, m_\ell \ ; \ m_1', \dots, m_\ell')$$

is now played by

$$(m_{s_1}, \dots, m_{s_\ell} \ ; \ m_{t_1}', \dots, m_{t_\ell}') \ ,$$

and z_{i-1} , z_i are replaced by $z_{i-1} - A$ and $z_i + A$ respectively. Thus we obtain (4.79) again.

Combining (4.63), (4.77) and (4.79), and noting $1 = \mathcal{O}(\Psi_{2k}(\beta))$ as $\beta \searrow 0$, we now conclude

$$X_n^i = 4(c_2 - c_1)^{2k} Z_n^i + \mathcal{O}(L_n^{3/2} q_n^{-2}(\Psi_{2k}(\beta) + L_n^{1/2}\beta^d)) \tag{4.97}$$

under Condition (II-1).

Next we assume Condition (II-2) and estimate $X_n'(\beta)$ and X_n'' again. This time, $X_n'(\beta)$ is the sum of $Q^i(\mathbf{m};\mathbf{m}')$ over those $(\mathbf{m};\mathbf{m}') \in \mathcal{Z}_{k,n} \times \mathcal{Z}_{k,n}$ such that

$$|\varphi(m_s) - \varphi(m_t')| \ge M L_n^{-1/2} \ , \ s,t = 1,\dots,k \tag{4.98}$$

hold and that at least one of $\varphi(m_s) - \varphi(m_{s-1})$, $\varphi(m_s') - \varphi(m_{s-1}')$ or $|\varphi(m_s) - \varphi(m_t')|$ is less than β .

Let us take $M' > 0$ sufficiently small. Then as can be seen from the proof of Lemma 3, as far as we have $(\mathbf{m};\mathbf{m}') \in \mathcal{Z}_{k,n} \times \mathcal{Z}_{k,n}$, the condition

$$\varphi(m_s) - \varphi(m_{s-1}) \ge M' L_n^{-1/2} \ ; \ \varphi(m_s') - \varphi(m_{s-1}') \ge M' L_n^{-1/2} \ , \ s = 2,\dots,k \tag{4.99}$$

is automatically satisfied.

As before, let $\mathcal{W}_{n,\sigma}'(\beta)$ be the subset of $\mathcal{W}_n'(\beta)$ consisting of those $(\mathbf{m};\mathbf{m}')$ which can be arranged like (4.71) .

Let us define, for $j = 1,\dots,2k$,

$$\bar{y}_j = \begin{cases} y_1 & \bar{m}_j = m_1 \\ y_1' & \bar{m}_j = m_1' \\ |m_s|/\ell_s & \bar{m}_j = m_s \text{ for some } s = 2,\dots,k \\ |m_t'|/\ell_t' & \bar{m}_j = m_t' \text{ for some } t = 2,\dots,k \end{cases} \tag{4.100}$$

Then if we note that

$$C_T^{-1} L_n^{1/2} \le \ell_s, \ell'_s \le C_T L_n^{1/2} \tag{4.101}$$

for some constant $C_T \ge 1$, we see from (4.18) and Condition (II-2), that

$$\begin{aligned} &Q^i(\mathbf{m};\mathbf{m}') \\ &\le C_T L_n^{-2(k-1)} \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{-A}^{A} d\zeta_2 \cdots \int_{-A}^{A} d\zeta_r \int_{|m'_1|/\sqrt{z_i}}^{|m'_1|/\sqrt{z_{i-1}}} dy'_1 \times \\ &\quad \times \int_{-A}^{A} d\bar\zeta_{r+1} \cdots \int_{-A}^{A} d\bar\zeta_{2k} \prod_{j=2}^{2k} 1_{\{|\bar y_j - \bar y_{j-1}| \le b_3(\varphi(\bar m_j)-\varphi(\bar m_{j-1})\}} (\varphi(\bar m_j) - \varphi(\bar m_{j-1}))^{-\tau} , \end{aligned} \tag{4.102}$$

where $\bar\zeta_j = \zeta_s$ or $\bar\zeta_j = \zeta'_t$ according to $\bar m_j = m_s$ or $\bar m_j = m'_t$.

Let $X'_{n,\sigma}(\beta)$ be the sum of $Q^i(\mathbf{m};\mathbf{m}')$ over $\mathcal{W}'_{n,\sigma}(\beta)$. Noting (4.99), we apply Lemma 2 successively to the summation over $\bar m_{2k}, \dots, \bar m_{r+2}$, and integrate out over $d\bar\zeta_j$, to obtain

$$\begin{aligned} &X'_{n,\sigma}(\beta) \le C_T L_n^{-r+1} \left(\prod_{j=r+2}^{2k} \bar\beta_j^{2-\tau} \right) \times \\ &\times \sum_{m_1 \in \mathcal{Z}_{1,n}} \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \prod_{s=2}^{r} \int_{-A}^{A} d\zeta_s \sum_{m_s} 1_{\{|\bar y_s - \bar y_{s-1}| \le b_3(\varphi(\bar m_s)-\varphi(\bar m_{s-1}))\}} (\varphi(\bar m_s) - \varphi(\bar m_{s-1}))^{-\tau} \times \\ &\times \int_{b_1}^{b_2} dy'_1 \sum_{m'_1} 1_{\{y'_1\sqrt{z_{i-1}} \le |m'_1| \le y'_1 \sqrt{z_i}\}} 1_{\{|\bar y_s - \bar y_{s-1}| \le b_3(\varphi(\bar m_s)-\varphi(\bar m_{s-1}))\}} (\varphi(\bar m_s) - \varphi(\bar m_{s-1}))^{-\tau} , \end{aligned} \tag{4.103}$$

where as before $\bar\beta_j = \beta$ or $\bar\beta_j = \Theta$.

We can apply the same method which we used in obtaining (4.57), namely we see without difficulty that

$$\begin{aligned} &\sum_{m'_1} 1_{\{y'_1\sqrt{z_{i-1}} \le |m'_1| \le y'_1\sqrt{z_i}\}} 1_{\{|\bar y_s - \bar y_{s-1}| \le b_3(\varphi(\bar m_s)-\varphi(\bar m_{s-1}))\}} \times (\varphi(\bar m_s) - \varphi(\bar m_{s-1}))^{-\tau} \\ &\le C_T q_n^{-1} L_n^{1/2} (L_n^{\tau/2} \wedge b_3^{-1} |\bar y_r - y'_1|)^{1-\tau} . \end{aligned} \tag{4.104}$$

Integrating with respect to y'_1 on the region $\{|y'_1 - y_r| < \bar\beta_{r+1}\}$, we get

$$\begin{aligned} &\int dy'_1 \sum_{m'_1} 1_{\{y'_1\sqrt{z_{i-1}} \le |m'_1| \le y'_1\sqrt{z_i}\}} 1_{\{|\bar y_s - \bar y_{s-1} \le b_3(\varphi(\bar m_s)-\varphi(\bar m_{s-1}))\}} (\varphi(\bar m_s) - \varphi(\bar m_{s-1}))^{-\tau} \\ &\le C_T q_n^{-1} L_n (L_n^{\tau/2-1} + \bar\beta_{r+1}^{2-\tau}) \\ &\le C_T q_n^{-1} L_n \bar\beta_{r+1}^{2-\tau} , \end{aligned} \tag{4.105}$$

because of $\beta \ge L_n^{-1/2}$.

We insert the estimate (4.102), which is uniform in $m_1, \dots, m_r$, y_1 and in $\zeta_2, \dots, \zeta_r$, apply Lemma 2 successively to the summation over $m_r, \dots, m_2$, and then integrate out over $\zeta_2, \dots, \zeta_r$, to obtain

$$\begin{aligned} X'_{n,\sigma}(\beta) &\le C_T L_n^{-r+1} L^{r-1} \left(\prod_{j=r+2}^{2k} \bar\beta_j^{2-\tau} \right) q_n^{-1} L_n \sum_{m_1 \in \mathcal{Z}_{1,n}} \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} 1 \, dy_1 \\ &\le C_T L_n^2 q_n^{-2} \left(\prod_{j=r+2}^{2k} \bar\beta_j^{2-\tau} \right) \\ &\le C_T L_n^2 q_n^{-2} \beta^{2-\tau} , \end{aligned} \tag{4.106}$$

because at least one of $\bar{\beta}_j$'s is β .

This estimate being similar for all arrangement (4.71), we can now conclude

$$X_n'(\beta) \le C_T L_n^2 q_n^{-2} \beta^{2-\tau} \ . \tag{4.107}$$

Finally, we turn to the estimate of X_n'' under Condition (II-2), to finish the estimate of X_n .

Recall that this time, X_n'' is the sum of $Q^i(\mathbf{m};\mathbf{m}')$ over those $(\mathbf{m};\mathbf{m}') \in \mathcal{Z}_{k,n} \times \mathcal{Z}_{k,n}$ for which (4.99) holds but $|\varphi(m_s) - \varphi(m_t')| < M' L_n^{-1/2}$ for some $s, t = 1, \dots, k$. Analogously to (4.79), we let

$$X_n''(s_1, \dots, s_\ell, s_{\ell+1}, \dots, s_r; t_1, \dots, t_\ell, t_{\ell+1}, \dots, t_r)$$

to be the sum of $Q^i(\mathbf{m};\mathbf{m}')$ over those $(\mathbf{m};\mathbf{m}') \in \mathcal{Z}_{k,n} \times \mathcal{Z}_{k,n}$ such that for some $1 \le s_1 < \cdots < s_\ell \le k$, $1 \le s_{\ell+1} < \cdots < s_r \le k$, $1 \le t_1 < \cdots < t_\ell \le k$ and $1 \le t_{\ell+1} < \cdots < t_r \le k$, it holds that

$$\varphi(m_{s_j}) = \varphi(m'_{t_j}) \ , \ j = 1, \dots, \ell \tag{4.108}$$

and that

$$0 < |\varphi(m_{s_j}) - \varphi(m'_{t_j})| < M' L_n^{-1/2} \ , \ j = \ell+1, \dots, r \ . \tag{4.109}$$

(Possibly one of the equalities $\ell = 0$ or $r = \ell$ may holds, but not both.) We shall treat in detail the case of $\ell \ge 1$, $r \ge \ell$ and $s_j = t_j = j$, $j = 1, \dots, r$, and give a brief sketch of the other cases.

Since $\varphi(m_s) = \varphi(m_s')$ for $s = 1, \dots, \ell$, $Q^i(\mathbf{m};\mathbf{m}')$ has the representation (4.83) and summing over $m_2', \dots, m_\ell'$, we obtain

$$\begin{aligned} &X_n''(1, \dots, r; 1, \dots, r) \\ &\le C_T \sum_{\substack{m_1, \dots, m_k; m_1', \\ m_\ell', \dots, m_k'}} \int_{(\frac{|m_1|}{\sqrt{z_i}}, \frac{|m_1|}{\sqrt{z_{i-1}}}) \cap (\frac{|m_1'|}{\sqrt{z_i}}, \frac{|m_1'|}{\sqrt{z_{i-1}}})} dy_1 \times \\ &\quad \times \int_{-A}^{A} d\zeta_2 \cdots \int_{-A}^{A} d\zeta_k \int_{-A}^{A} d\zeta_{\ell+1}' \cdots \int_{-A}^{A} d\zeta_k' \prod_{s=2}^{k} \left(\frac{1}{2} \left| \frac{m_s}{\ell_s} \right| \right) \frac{1}{\ell_s^2} \prod_{t=\ell+1}^{k} \left(\frac{1}{2} \left| \frac{m_t'}{\ell_t'} \right| \right) \frac{1}{\ell_t'^2} \times \\ &\quad \times p_{2k-\ell}(y_1, \left| \frac{m_2}{\ell_2} \right|, \dots, \left| \frac{m_k}{\ell_k} \right|, \left| \frac{m_{\ell+1}'}{\ell_{\ell+1}'} \right|, \dots, \left| \frac{m_k'}{\ell_k'} \right| |\varphi(m_1), \dots, \varphi(m_k), \varphi(m_{\ell+1}'), \dots, \varphi(m_k')) \ . \end{aligned} \tag{4.110}$$

By Condition (II-2) and by $|\varphi(m_j) - \varphi(m_j')| < M' L_n^{-1/2}$ which holds for $j = \ell+1, \dots, r$, the summand vanishes if the condition

$$\left| \frac{|m_j'|}{\ell_j'} - \frac{|m_j|}{\ell_j} \right| < b_3 M' L_n^{-1/2} \ , \ j = \ell+1, \dots, r \tag{4.111}$$

fails to hold for all y_1 , ζ_j and ζ_j' .

Since

$$\ell_j = \frac{|m_1|}{y_1} \sqrt{1 + \frac{y_1^2}{|m_1|^2} \zeta_j} \quad ; \quad \ell_j' = \frac{|m_1'|}{y_1} \sqrt{1 + \frac{y_1^2}{|m_1|^2} \zeta_j'} \ , \tag{4.112}$$

we see

$$\frac{\ell_j'}{\ell_j} = \frac{|m_1'|}{|m_1|} + \mathcal{O}_T(L_n^{-1}) \tag{4.113}$$

uniformly in $y_1 \in (b_1, b_2)$ and in $\zeta_j, \zeta_j' \in (-A, A)$. Hence if (4.111) ever holds for some y_1 , ζ_j and ζ_j' , then there is a constant C such that

$$\left| |m_j'| - \frac{|m_1'|}{|m_1|}|m_j| \right| < C \ . \tag{4.114}$$

Now if we perform the integration with respect to ζ_j , $j = \ell+1, \dots, r$, then by (4.16), the variables of integration transform back into $y_{\ell+1}', \dots, y_r'$, and since $p_{2k-\ell}(\cdot)$ is a probability density, the summand of (4.110) is bounded by

(4.115)

$$\begin{aligned}&\int_{(\frac{|m_1|}{\sqrt{z_i}}, \frac{|m_1|}{\sqrt{z_{i-1}}}) \cap (\frac{|m_1'|}{\sqrt{z_i}}, \frac{|m_1'|}{\sqrt{z_{i-1}}})} dy_1 \int_{-A}^{A} d\zeta_2 \cdots \int_{-A}^{A} d\zeta_k \int_{-A}^{A} d\zeta_{r+1}' \cdots \int_{-A}^{A} d\zeta_k' \times \\ &\times \prod_{s=2}^{k} \left(\frac{1}{2}\left|\frac{m_s}{\ell_s}\right|\right) \frac{1}{\ell_s^2} \prod_{t=r+1}^{k} \left(\frac{1}{2}\left|\frac{m_t'}{\ell_t'}\right|\right) \frac{1}{\ell_t'^2} \prod_{j=\ell+1}^{r} 1_{\{||m_j'| - \frac{|m_1'|}{|m_1|}|m_j|| < C\}} \times \\ &\times p_{2k-r}(y_1, \left|\frac{m_2}{\ell_2}\right|, \dots, \left|\frac{m_k}{\ell_k}\right|, \left|\frac{m_{r+1}'}{\ell_{r+1}'}\right|, \dots, \left|\frac{m_k'}{\ell_k'}\right| |\varphi(m_1), \dots, \varphi(m_k), \varphi(m_{r+1}'), \dots, \varphi(m_k')) \ .\end{aligned}$$

For given values of $m_1, m_1', m_j' \in \mathcal{Z}_{1,n}$, the number of m_j' which satisfy (4.114) and $\varphi(m_j') = \varphi(m_j)$ is bounded by a constant which is independent of m_1, m_1', m_j' and also of L_n . Hence if we make summation over $m_{\ell+1}', \dots, m_r'$ first, then by (4.110) and (4.115),

(4.116)

$$\begin{aligned}&X_n''(1, \dots, r; 1, \dots, r) \\ &\le C_T \sum_{\substack{m_1, \dots, m_k; m_1', \\ m_{r+1}', \dots, m_k'}} L_n^{-(2k-r-1)} \int_{(\frac{|m_1|}{\sqrt{z_i}}, \frac{|m_1|}{\sqrt{z_{i-1}}}) \cap (\frac{|m_1'|}{\sqrt{z_i}}, \frac{|m_1'|}{\sqrt{z_{i-1}}})} dy_1 \times \\ &\times \int_{-A}^{A} d\zeta_2 \cdots \int_{-A}^{A} d\zeta_k \int_{-A}^{A} d\zeta_{r+1}' \cdots \int_{-A}^{A} d\zeta_k' \times \\ &\times p_{2k-r}(y_1, \left|\frac{m_2}{\ell_2}\right|, \dots, \left|\frac{m_k}{\ell_k}\right|, \left|\frac{m_{r+1}'}{\ell_{r+1}'}\right|, \dots, \left|\frac{m_k'}{\ell_k'}\right| |\varphi(m_1), \dots, \varphi(m_k), \varphi(m_{r+1}'), \dots, \varphi(m_k')) \ .\end{aligned}$$

Again we arrange $m_2, \dots, m_k; m_{r+1}', \dots, m_k'$ in the increasing order of φ , and relabel them as $\bar{m}_2, \dots, \bar{m}_{2k-r}$.

Introducing $\bar{y}_j$ by (4.100), we then have a bound of the summand of (4.116) similar to (4.102). Applying Lemma 2 successively to the summation over $\bar{m}_{2k-r}, \dots, \bar{m}_2$, we arrive at

$$\begin{aligned}X_n''(1, \dots, r; 1, \dots, r) &\le C_T \sum_{\substack{m_1, m_1' \in \mathcal{Z}_{1,n}, \\ \varphi(m_1) = \varphi(m_1')}} \left[\frac{|m_1| \wedge |m_1'|}{\sqrt{z_{i-1}}} - \frac{|m_1| \vee |m_1'|}{\sqrt{z_i}}\right]_+ \\ &\le C_T L_n^{3/2} q_n^{-2} \ ,\end{aligned} \tag{4.117}$$

as before.

Next suppose $\ell \ge 1$ and that for some $1 \le s_1 < \cdots < s_\ell \le k$ and $1 \le t_1 < \cdots < t_\ell \le k$ one has $\varphi(m_{s_j}) = \varphi(m_{t_j}')$, while for some other $1 \le s_{\ell+1} < \cdots < s_r \le k$ and $1 \le t_{\ell+1} < \cdots < t_r \le k$, one has $0 < |\varphi(m_{s_j}) - \varphi(m_{t_j}')| < ML_n^{-1/2}$. We can use (4.96) to reduce the analysis of this case to the case of $s_j = t_j = j$, $j = 1, \dots, r$.

In case of $\ell = 0$, $r > 0$, we assume, without loss of generality, that $s_j = t_j = j$, $j = 1, \dots, r$ holds.

This time, $\varphi(m_s)$, $\varphi(m_t')$, $s, t = 1, \dots, k$ are all different, and we have the representation (4.18) for $Q^i(\mathbf{m}; \mathbf{m}')$. Now in (4.18), y_1' can vary only within the interval $\{|y_1' - y_1| < b_3 M L_n^{-1/2}\}$, from which we deduce, instead of (4.113),

$$\frac{\ell_j'}{\ell_j} = \frac{|m_1'|}{|m_1|} + \mathcal{O}_T(L_n^{-1/2}) \ . \tag{4.118}$$

But this still gives the following condition on m_j :

$$\left| |m'_j| - \frac{|m'_1|}{|m_1|}|m_j| \right| < C \ . \tag{4.119}$$

for the non-vanishing of $Q^i(m_1, \dots, m_k; m'_1, \dots, m'_k)$.

For given values of m_1 , m'_1 and m_j , the number of m'_j satisfying (4.119) and $|\varphi(m'_j) - \varphi(m'_j)| < M' L_n^{-1/2}$ is bounded by some constant which is independent of m_1 , m'_1 , m_j and L_n . Hence the same argument which lead to (4.116) yields

$$\begin{aligned} & X''_n(1, \dots, r; 1, \dots, r) \\ & \le C_T \sum_{\substack{m_1, m'_1 \in \mathcal{Z}_{1,n}, \\ |\varphi(m_1) - \varphi(m'_1)| < M' L_n^{-1/2}}} \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{|m'_1|/\sqrt{z_i}}^{|m'_1|/\sqrt{z_{i-1}}} dy'_1 p_2(y_1, y'_1 \mid \varphi(m_1), \varphi(m'_1)) \\ & \le C_T \sum_{\substack{m_1, m'_1 \in \mathcal{Z}_{1,n}, \\ |\varphi(m_1) - \varphi(m'_1)| < M' L_n^{-1/2}}} S(m_1, m'_1) \ . \end{aligned} \tag{4.120}$$

But this sum was already estimated in (4.61). Namely we have

$$X''_n(1, \dots, r; 1, \dots, r) \le C_T L_n^{3/2} q_n^{-2} \tag{4.121}$$

once again. thus we have proved (4.97) under Condition (II-2).

Let us finish the proof of Lemma 4 by estimating Y^i_n . Since no new technical difficulty arises, we shall only sketch the outline, omitting the repetition of detailed argument.

To begin with, we divide the summation (4.4) into two parts: $Y^i_n = Y''_n + (Y^i_n - Y''_n)$, where we define

$$Y''_n = \sum_{\substack{\varphi(m'_1) = \varphi(m_s) \\ \text{for some } s = 1, \dots, k}} R^i(\mathbf{m}; m'_1) \tag{4.122}$$

in case Condition (II-1) holds, and

$$Y''_n = \sum_{\substack{|\varphi(m'_1) - \varphi(m_1)| < M L_n^{-1/2} \\ \text{for some } s = 1, \dots, k}} R^i(\mathbf{m}; m'_1) \tag{4.123}$$

in case Condition (II-2) holds.

Suppose Condition (II-1) holds, and let $\varphi(m'_1) = \varphi(m_s)$. By the same consideration which lead us to (4.96), we see

$$R^i(\mathbf{m}; m'_1) \le C \mathbf{E}_{\mathbf{P}}[1_{\{\gamma_{m_s}, \gamma_{m'_1} \in (z_{i-1} - A, z_i + A)\}} |D_{m_1} \cap \cdots \cap D_{m_k}|] \ , \tag{4.124}$$

and it is easy to obtain

$$\begin{aligned} Y''_n &= \sum_{s=1}^{k} \sum_{\varphi(m'_1) = \varphi(m_s)} R^i(\mathbf{m}; m'_1) \\ &\le C \sum_{s=1}^{k} \sum_{\varphi(m'_1) = \varphi(m_s)} \mathbf{E}_{\mathbf{P}}[1_{\{\gamma_{m_s}, \gamma_{m'_1} \in (z_{i-1} - A, z_i + A)\}} |D_{m_1} \cap \cdots \cap D_{m_k}|] \\ &\le C L_n^{3/2} q_n^{-2} \ , \end{aligned} \tag{4.125}$$

as before.

In case Condition (II-2) holds, we further divide Y_n'' into summtion over those $(\mathbf{m}; m_1')$ for which $\varphi(m_1') = \varphi(m_s)$ for some $s = 1, \ldots, k$ and over those for which $0 < |\varphi(m_1') - \varphi(m_1)| < ML_n^{-1/2}$ for some s . Recalling what we did in estimating X_n'' under Condition (II-1), there is no difficulty in obtaining (4.125) again.

Now define a subset $\mathcal{V}_n(\beta)$ of $\mathcal{Z}_{k,n} \times \mathcal{Z}_{1,n}$ to be the totality of those $(m_1, \ldots, m_k, m_1')$ such that

$$\varphi(m_s) - \varphi(m_{s-1}) \geq \beta \ , \ s = 2, \ldots, k \tag{4.126}$$

and that

$$|\varphi(m_1') - \varphi(m_s)| \geq \beta \ , \ s = 2, \ldots, k \ . \tag{4.127}$$

Corresponding to this, we let

$$Y_n(\beta) \equiv \sum_{\mathcal{V}_n(\beta)} R^i(\mathbf{m}; m_1') \tag{4.128}$$

and

$$Y_n'(\beta) \equiv Y_n - Y_n'' - Y_n(\beta) \ . \tag{4.129}$$

To estimate $Y_n'(\beta)$, we use the bound

$$\begin{aligned} R^i(\mathbf{m}; m_1') &\leq C\mathbf{E}_{\mathbf{P}}[1_{\{\gamma_{m_s}, \gamma_{m_1'} \in [z_{i-1}, z_i))\}} | D_{m_1} \cap \cdots \cap D_{m_k}|] \\ &\leq CL_n^{-2(k-1)} \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{|m_1'|/\sqrt{z_i}}^{|m_1'|/\sqrt{z_{i-1}}} dy_1' \int_{-A}^{A} d\zeta_2 \ldots \int_{-A}^{A} d\zeta_k \times \\ &\quad \times p_{k+1}(y_1, \left|\frac{m_2}{\ell_2}\right|, \ldots, \left|\frac{m_k}{\ell_k}\right|, y_1' \mid \varphi(m_1), \ldots, \varphi(m_k), \varphi(m_1')) \ . \end{aligned} \tag{4.130}$$

Now we make summation over those $(m_1, \ldots, m_k; m_1') \in \mathcal{Z}_{k,n} \times \mathcal{Z}_{1,n}$ satisfying

$$\varphi(m_s) - \varphi(m_{s-1}) < \beta \ , \ \text{for some } s = 2, \ldots, k \tag{4.131}$$

or

$$0 < |\varphi(m_1') - \varphi(m_s)| < \beta \ , \ \text{for some } s = 1, \ldots, k \tag{4.132}$$

in case Condition (II-1) holds, or over those $(m_1, \ldots, m_k; m_1') \in \mathcal{Z}_{k,n} \times \mathcal{Z}_{1,n}$ satisfying

$$ML_n^{-1/2} \leq |\varphi(m_1') - \varphi(m_s)| < \beta \text{ for some } s = 1, \ldots, k \tag{4.133}$$

in case Condition (II-2) holds. By the same argument as in the estiamtion of $X_n'(\beta)$, we then obtain

$$Y_n'(\beta) \leq CL_n^2 q_n^{-2} \beta^d \ . \tag{4.134}$$

Let us proceed to the estimation of $Y_n(\beta)$. We use the following representation of $R^i(\mathbf{m}; m_1')$:

(4.135)

$$\begin{aligned} &R^i(\mathbf{m}; m_1') \\ &= \int_{\varphi(m_1')}^{\Theta} d\varphi_2' \int_{\varphi_2'}^{\Theta} d\varphi_3' \cdots \int_{\varphi_{k-1}'}^{\Theta} d\varphi_k' \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{|m_1'|/\sqrt{z_i}}^{|m_1'|/\sqrt{z_{i-1}}} dy_1' \times \\ &\times \int_{-2(c_2-c_1)}^{2(c_2-c_1)} d\zeta_2 \cdots \int_{-2(c_2-c_1)}^{2(c_2-c_1)} d\zeta_k \int dy_2' \cdots \int dy_k' (y_2'^2 \cdots y_k'^2) \times \\ &\times [2(c_2 - c_1) - (0 \vee \max_{2 \leq s \leq k} \zeta_s) + (0 \wedge \min_{2 \leq s \leq k} \zeta_s)]_+ \prod_{s=2}^{k} \left(\frac{1}{2} \left| \frac{m_s}{\ell_s} \right| \right) \frac{1}{\ell_s^2} \times \\ &\times p_{2k}(y_1, \left|\frac{m_2}{\ell_2}\right|, \ldots, \left|\frac{m_k}{\ell_k}\right|, y_1', y_2', \ldots, y_k' \mid \varphi(m_1), \ldots, \varphi(m_k), \varphi(m_1'), \varphi_2', \ldots, \varphi_k') \ . \end{aligned}$$

Now we decompose the integral with respect to $\varphi'_2, \dots, \varphi'_k$ into to parts. Namely we set

$$D(m'_1) = \{(\varphi'_2, \dots, \varphi'_k) \mid \varphi(m'_1) < \varphi'_2 < \cdots < \varphi'_k < \Theta\} \tag{4.136}$$

and

$$\begin{aligned} D(\mathbf{m}; m'_1; \beta) = \{(\varphi'_2, \dots, \varphi'_k) \in D(m_1) \mid \varphi'_2 - \varphi(m'_1) \ge \beta \ , \\ \varphi'_s - \varphi'_{s-1} \ge \beta \ , \ s = 3, \dots, \ |\varphi'_t - \varphi(m_s)| \ge \beta \ , \forall s, \forall t\} \ ; \end{aligned} \tag{4.137}$$

$$D'(\mathbf{m}; m'_1; \beta) = D(m'_1) \setminus D(m_1, \dots, m_k; m'_1; \beta) \ . \tag{4.138}$$

Let $R^i_\beta(\mathbf{m}; m'_1)$ and $R^{ii}_\beta(\mathbf{m}; m'_1)$ be defined by the right hand side of (4.135) with the integration over $\varphi'_2, \dots, \varphi'_k$ ranging in $D(\mathbf{m}; m'_1; \beta)$ and $D'(\mathbf{m}; m'_1; \beta)$ respectively. If we integrate with respect to $y'_2, \dots, y'_k$ first, then the resulting integral is bounded with respect to $\varphi'_2, \dots, \varphi'_k$. Hence we obviously have

$$\begin{aligned} R^{ii}_\beta(\mathbf{m}; m'_1) \le C_T \beta L_n^{-(k-1)} \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{|m'_1|/\sqrt{z_i}}^{|m'_1|/\sqrt{z_{i-1}}} dy'_1 \times \\ \times p_{k+1}\left(y_1, \left|\frac{m_2}{\ell_2}\right|, \dots, \left|\frac{m_k}{\ell_k}\right|, y'_1 \mid \varphi(m_1), \dots, \varphi(m_k), \varphi(m'_1)\right) \ , \end{aligned} \tag{4.139}$$

and hence it is easy to see that

$$\sum_{\mathcal{V}_n(\beta)} R^{ii}_\beta(\mathbf{m}; m'_1) \le C \beta L_n^2 q_n^{-2} \ , \tag{4.140}$$

namely

$$Y_n(\beta) = \sum_{\mathcal{V}_n(\beta)} R^i_\beta(\mathbf{m}; m'_1) + \mathcal{O}(\beta L_n q_n^{-2}) \ . \tag{4.141}$$

At this point, we introduce a function

$$\begin{aligned} &V(x_2, \dots, x_k) \\ &= V(x_2, \dots, x_k; m_1, m'_1; y_1, y'_1, \dots, y'_k; \varphi'_2, \dots, \varphi'_k) \\ &\equiv p_{2k}(y_1, |x_2|, \dots, |x_2|, y'_1, y'_2, \dots, y'_k \mid \varphi(m_1), \varphi(x_2), \dots, \varphi(x_k), \varphi(m'_1), \varphi'_2, \dots, \varphi'_k) \times \\ &\quad \times \prod_{s=2}^k (\frac{1}{2}|x_s|) \end{aligned} \tag{4.142}$$

which is defined on

$$\begin{aligned} \mathcal{D}_{m_1} \equiv \{(x_2, \dots, x_k) \in (\mathbf{R}^2)^{k-1} \mid b_1 \le |x_s| \le b_2, s = 2, \dots, k \ ; \\ \varphi(m_1) < \varphi(x_2) < \cdots < \varphi(x_k)\} \ . \end{aligned} \tag{4.143}$$

Then we can write the first term on the right hand side of (4.141) as follows:

$$\begin{aligned} &\sum_{\mathcal{V}_n(\beta)} R^i_\beta(\mathbf{m}; m'_1) \\ &= \sum_{m_1, m'_1} \int \cdots \int_{D(m'_1)} d\varphi'_2 \cdots \varphi'_k \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{|m'_1|/\sqrt{z_i}}^{|m'_1|/\sqrt{z_{i-1}}} dy'_1 \int_{-A}^{A} d\zeta_2 \dots \int_{-A}^{A} d\zeta_k \times \\ &\quad \times \int dy'_2 \cdots \int dy'_k (y'^2_2 \cdots y'^2_k) \times \\ &\quad \times [A - (0 \vee \max_{2 \le s \le k} \zeta_s) + (0 \wedge \min_{2 \le s \le k} \zeta_s)]_+ \times \\ &\quad \times \sum_{m_2, \dots, m_k} 1_{D(m_1, \dots, m_k; m'_1; \beta)} V\left(\left|\frac{m_2}{\ell_2}\right|, \dots, \left|\frac{m_k}{\ell_k}\right|\right) \prod_{s=2}^k \frac{1}{\ell_s^2} \ . \end{aligned} \tag{4.144}$$

As before, the summation over $(m_2, \ldots, m_k)$ is nothing but the Riemann sum approximation of the integral of V , namely we have

$$\begin{aligned} &\sum_{m_2,\ldots,m_k} 1_{D(m_1,\ldots,m_k;m_1';\beta)} V\left(\left|\frac{m_2}{\ell_2}\right|, \ldots, \left|\frac{m_k}{\ell_k}\right|\right) \prod_{s=2}^{k} \frac{1}{\ell_s^2} \\ &= \int \cdots \int_{\mathcal{D}_\beta(\varphi_2',\ldots,\varphi_k')} dx_2 \cdots dx_k V(x_2, \ldots, x_k) + \mathcal{O}(L_n^{-1/2}\Psi_{2k}(\beta)) \ , \end{aligned} \tag{4.145}$$

where we have defined the range of integration by

$$\mathcal{D}_\beta(\varphi_2', \ldots, \varphi_k') \equiv \{(x_2, \ldots, x_k) \in \mathcal{D}_{m_1} \mid |\varphi(x_s) - \varphi_t'| \geq \beta \ , \forall s, \forall t\} \ . \tag{4.146}$$

Inserting this approximation into (4.144) and using (4.38) , we obtain

$$\begin{aligned} &\sum_{\mathcal{V}_n(\beta)} R_\beta^i(\mathbf{m}; m_1') \\ &= \sum_{m_1,m_1'} \int \cdots \int_{D(m_1')} d\varphi_2' \cdots \varphi_K' \int_{|m_1|/\sqrt{z_i}}^{|m_1|/\sqrt{z_{i-1}}} dy_1 \int_{|m_1'|/\sqrt{z_i}}^{|m_1'|/\sqrt{z_{i-1}}} dy_1' \times \\ &\times A^k \int dy_2' \cdots \int dy_k' (y_2'^2 \cdots y_k'^2) \times \\ &\times 2^{-(k-1)} \int dy_2 \cdots \int dy_k (y_2^2 \cdots y_n^2) \int \cdots \int_{\tilde{\mathcal{D}}_\beta(\varphi_2',\ldots,\varphi_k')} d\varphi_2 \cdots \varphi_k \times \\ &\times p_{2k}(y_1, \ldots, y_k, y_1' \ldots, y_k' | \varphi(m_1), \varphi_2, \ldots, \varphi_k, \varphi(m_1'), \varphi_2', \ldots, \varphi_k') \\ &+ \mathcal{O}(L_n^{3/2} q_n^{-2} \Psi_{2k}(\beta)) \ , \end{aligned} \tag{4.147}$$

where we have set

$$\begin{aligned} &\tilde{\mathcal{D}}_\beta(\varphi_2', \ldots, \varphi_k') \\ &= \{(\varphi_2, \ldots, \varphi_k) \mid \varphi(m_1) < \varphi(x_2) < \cdots < \varphi(x_k) \ ; \ |\varphi_s - \varphi_t'| \geq \beta\} \ . \end{aligned} \tag{4.148}$$

Now (4.147) is further rewritten as

$$\begin{aligned} &\sum_{\mathcal{V}_n(\beta)} R_\beta^i(\mathbf{m}; m_1') \\ &= \sum_{\substack{m_1,m_1'; \\ |\varphi(m_1)-\varphi(m_1')|\geq\beta}} 2(c_2 - c_1)^k \mathbf{E}_{\mathbf{P}} \Big[1_{\{\gamma_{m_s}, \gamma_{m_1'} \in [z_{i-1}, z_i))\}} \times \\ &\times \int \cdots \int d\varphi_2' \cdots \varphi_k' f(\varphi_2')^2 \cdots f(\varphi_k')^2 \times \\ &\times \int \cdots \int_{\tilde{\mathcal{D}}_\beta(\varphi_2',\ldots,\varphi_k')} d\varphi_2 \cdots \varphi_k f(\varphi_2)^2 \cdots f(\varphi_k)^2 \Big] + \mathcal{O}(L_n^{3/2} q_n^{-2} \Psi_{2k}(\beta)) \\ &= 2(c_2 - c_1)^k \sum_{\substack{m_1,m_1'; \\ |\varphi(m_1)-\varphi(m_1')|\geq\beta}} S(m_1, m_1') \\ &+ \mathcal{O}\left(L_n^{3/2} q_n^{-2} \Psi_{2k}(\beta) + \sum_{\substack{m_1,m_1'; \\ |\varphi(m_1)-\varphi(m_1')|\geq\beta}} \beta \mathbf{P}(\gamma_{m_1}, \gamma_{m_1'} \in [z_{i-1}, z_i)) \right) \ . \end{aligned} \tag{4.149}$$

Since we already have the estimate

$$\sum_{\substack{m_1,m_1';\\ |\varphi(m_1)-\varphi(m_1')|\geq\beta}} \beta\mathbf{P}(\gamma_{m_1},\gamma_{m_1'}\in[z_{i-1},z_i)) = \mathcal{O}(L_n^2 q_n^{-2}\beta^d)$$

(see (4.41) and (4.46)), we finally arrive at

$$\begin{aligned} Y_n &= Y_n'' + (Y_n - Y_n'') \\ &= Y_n'' + Y_n(\beta) + Y_n'(\beta) \\ &= \sum_{\mathcal{V}_n(\beta)} R_\beta^i(m_1,\dots,m_k;m_1') + \mathcal{O}(\beta L_n^2 q_n^{-2}) + Y_n'(\beta) + Y_n'' \\ &= 2(c_2-c_1)^k \sum_{\substack{m_1,m_1';\\ |\varphi(m_1)-\varphi(m_1')|\geq\beta}} S(m_1,m_1') + \mathcal{O}(L_n^{3/2}q_n^{-2}\Psi_{2k}(\beta) + L_n^2 q_n^{-2}\beta^d) \\ &= 2(c_2-c_1)^k \sum_{m_1,m_1'} S(m_1,m_1') + \mathcal{O}(L_n^{3/2}q_n^{-2}(\Psi_{2k}(\beta) + L_n^{1/2}\beta^d)) \ . \end{aligned} \tag{4.150}$$

We have thus proved

$$\begin{aligned} U_{i,n} &= X_n^i - 4(c_2-c_1)^k Y_n^i + 4(c_2-c_1)^{2k} Z_n^i \\ &= 4(c_2-c_1)^{2k} Z_n^i - 8(c_2-c_1)^{2k} Z_n^i 4(c_2-c_1)^{2k} Z_n^i + \mathcal{O}(L_n^{3/2}q_n^{-2}(\Psi_{2k}(\beta) + L_n^{1/2}\beta^d)) \\ &= \mathcal{O}(L_n^{3/2}q_n^{-2}(\Psi_{2k}(\beta) + L_n^{1/2}\beta^d)) \ , \end{aligned} \tag{4.151}$$

completing the proof of Lemma 4.

Before closing this section, we give a proof of formula (4.38) as a lemma.

Lemma 5. *For $A>0$ and $k\geq 2$,*

$$\int_{-A}^{A} d\zeta_2 \cdots \int_{-A}^{A} d\zeta_k [A - \{0\vee \max_{2\leq s\leq k}\zeta_s\} + \{0\wedge \min_{2\leq s\leq k}\zeta_s\}]_+ = A^k \ .$$

Proof. We denote the left hand side of the above formula by I_k . Moreover, if we set $\bar{\tau}_k = 0\vee\max_{2\leq s\leq k}\zeta_s$ and $\underline{\tau}_k = 0\wedge\min_{2\leq s\leq k}\zeta_s$ for brevity, then we can compute

$$\begin{aligned} I_k &= \int\cdots\int_{\{\bar{\tau}_{k-1}-\underline{\tau}_{k-1}<A\}} d\zeta_2\cdots d\zeta_{k-1}\left[\int_{\bar{\tau}_{k-1}-A}^{\underline{\tau}_{k-1}} \{A-(\bar{\tau}_{k-1}-\zeta_k)\}d\zeta_k\right. \\ &\quad \left. +\{A-(\bar{\tau}_{k-1}-\underline{\tau}_{k-1})\}(\bar{\tau}_{k-1}-\underline{\tau}_{k-1}) + \int_{\bar{\tau}_{k-1}}^{\underline{\tau}_{k-1}+A}\{A-(\zeta_k-\underline{\tau}_{k-1})\}d\zeta_k\right] \\ &= \int\cdots\int_{\{\bar{\tau}_{k-1}-\underline{\tau}_{k-1}<A\}} d\zeta_2\cdots d\zeta_{k-1} \\ &\quad \left[\int_0^{A-\bar{\tau}_{k-1}+\underline{\tau}_{k-1}} sds + \int_0^{A-\bar{\tau}_{k-1}+\underline{\tau}_{k-1}} sds + \{A-(\bar{\tau}_{k-1}-\underline{\tau}_{k-1}\}(\bar{\tau}_{k-1}-\underline{\tau}_{k-1})\right] \\ &= A\int\cdots\int d\zeta_2\cdots d\zeta_{k-1}\{A-(\bar{\tau}_{k-1}-\underline{\tau}_{k-1})\}_+ \\ &= AI_{k-1} \ . \end{aligned} \tag{4.152}$$

Thus we obtain

$$I_k = A^{k-2}I_2 = A^{k-2}\int d\zeta_2(A-|\zeta_2|)_+ = A^k \ . \tag{4.153}$$

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