On the Fundamental Solution of Semiclassical Schrödinger Equations at Resonant Times

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Abstract

We consider perturbations of the semiclassical harmonic oscillator of the form $P = -\frac{h^2}{2}\Delta + \frac{x^2}{2} + h^\delta W(x)$, $x \in \mathbf{R}^m$, with $W(x) \sim \langle x\rangle^{2-\mu}$ as $|x| \to +\infty$ and $\delta, \mu \in (0,1)$, and we investigate the fundamental solution $E(t,x,y)$ of the corresponding time-dependent Schrödinger equation. We prove that at resonant times $t = n\pi$ ($n \in \mathbf{Z}$) it admits a semiclassical asymptotics of the form: $E(n\pi, x, y) \sim h^{-m(1+\nu)/2} a_0 e^{iS(x,y)/h}$ with $a_0 \neq 0$ and $\nu = \delta/(1-\mu)$, under the conditions $x \neq (-1)^n y$ and $\nu < 1$.

1 Introduction and Main Result

We consider time dependent Schrödinger equation in $L^2(\mathbf{R}^m)$:

$$ih\frac{\partial u}{\partial t} = -\frac{h^2}{2}\Delta u + \frac{x^2}{2}u + h^\delta W(x)u = P^h u \tag{1.1}$$

where $\delta \in (0,1)$ and $W(x) \in C^\infty(\mathbf{R}^m)$ is real valued. We assume for some constants $C > 0$ and $\mu \in (0,1)$:

$$\begin{cases} \frac{1}{C}\langle x\rangle^{-\mu} \le D^2 W(x) \le C\langle x\rangle^{-\mu}, \\ |\partial^\alpha W(x)| \le C_\alpha \langle x\rangle^{2-\mu-|\alpha|} \text{ for } |\alpha| \ge 3 \end{cases} \tag{1.2}$$

for $x \in \mathbf{R}^m$. In particular W is subquadratic at infinity. Under this assumption, P^h on $C_0^\infty(\mathbf{R}^m)$ admits a unique selfadjoint extension, which we denote by P^h again, and the solution of (1.1) with initial data $u(0,x) = \phi(x)$ is given by $u(t) = \exp(-itP^h/h)\phi$. The distribution kernel $E(t,x,y)$ of $\exp(-itP^h/h)$ is called the fundamental solution (FDS for short) of (1.1) and we investigate the behaviour as $h \to 0+$ of $E(t,x,y)$ at the resonant times $t = n\pi$ ($n \in \mathbf{Z}^*$). We set $\nu = \frac{\delta}{1-\mu}$ and assume $0 < \nu < 1$. Our main result is:

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Theorem 1.1 *Let $n \in \mathbf{Z}^*$. Then, $E(n\pi, x, y)$ is a C^∞ function of (x, y) and for h small enough it can be written in the form:*

$$E(n\pi, x, y) = h^{-m(1+\nu)/2} a(x, y, h) e^{iS(x,y)/h}, \tag{1.3}$$

where $S(x, y)$ is the action integral of classical trajectory corresponding to (1.1) connecting $x(0) = y$ and $x(n\pi) = x$, and for any compact subset K of $\mathbf{R}^{2m} \setminus \Delta$, $\Delta = \{(x, (-1)^n x) \; ; \; x \in \mathbf{R}^m\}$, $a(x, y, h)$ satisfies $0 < C^{-1} \le |a(x, y, h)| \le C < \infty$ for $(x, y) \in K$ uniformly with respect to small h.

The estimate (1.3) should be compared with the result at non-resonant time: If $t \notin \pi\mathbf{Z}$, then the FDS solution behaves as $h \to 0+$

$$E(t, x, y) = h^{-m/2} a(x, y, h) e^{iS(x,y)/h} = \mathcal{O}(h^{-m/2})$$

and (1.3) represents the anomalous increase of the amplitude as $h \to 0$. We should also remark that, if W is sublinear, viz. $W = \mathcal{O}(\langle x \rangle^{1-\varepsilon})$, then for $(x, y) \in K$, K being as above, $E(n\pi, x, y) = \mathcal{O}(h^N)$ for any N as $h \to 0+$. Indeed in this case, $E(n\pi, \cdot, y)$ has singularities at $(-1)^n y$. These remarks can be easily obtained by applying the standard stationary phase method to (1.5) below.

Motivation to this work comes from the study of the behavior at infinity $x^2 + y^2 \to \infty$ of the FDS of the equation (1.1) with fixed $h = 1$ under the condition (1.2):

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + \frac{x^2}{2}u + W(x)u. \tag{1.4}$$

When $t \notin \pi\mathbf{Z}$, $E(t, x, y)$ for (1.4) converges to the FSD solution of the harmonic oscillator as $x^2 + y^2 \to \infty$ ([Ya-1]). At resonant times, however, we believe that $E(n\pi, x, y)$, $n \neq 0$, blows up as $|x - y| \to \infty$. It turns out that proving the latter is a little too intricated and still out of reach, although intimately related to the semiclassical problem we investigate here. Indeed, in the case $W(x) = |x|^{2-\mu}$ the change of scale $u(t, x) \to u(t, x/\sqrt{h})$ converts (1.4) to (1.1) with $\delta = \mu/2$ and the study of the solution of (1.4) as $|x| \to \infty$ is equivalent to that of (1.1) at fixed x as $h \to 0$. Thus, we expect in general $|E(n\pi, x, y)| \sim |x - y|^{m\nu}$ as $|x - y| \to \infty$, with $\nu = \mu/(2 - 2\mu)$.

The strategy for proving the theorem is as follows. First of all, one can see as in [Ya-1] (see also [Ro], [KK] for a semiclassical version for short time) that $E(n\pi, x, y)$ can be written under the form of an oscillatory integral:

$$E(n\pi, x, y) = \frac{1}{(2\pi h)^m} \int e^{i(x\xi - \psi(y,\xi))/h} b(y, \xi, h) d\xi. \tag{1.5}$$

Here b is a semiclassical symbol which is uniformly bounded together with all its derivatives, and ψ is the function:

$$\psi(y, \xi) = \widetilde{x}(n\pi, y, \xi) \cdot \xi - \int_0^{n\pi} \left(\frac{\dot{\widetilde{x}}(t, y, \xi)^2}{2} - V(\widetilde{x}(t, y, \xi)) \right) dt \tag{1.6}$$

where $\widetilde{x}(t,y,\xi)$ denotes the x-projection at time t of the unique classical trajectory $t \mapsto (x(t),p(t))$ satisfying $x(0)=y$ and $p(n\pi)=\xi$. (Notice that since V depends on h, the same is true for $\widetilde{x}(t,y,\xi)$.) We then apply the stationary phase method to (1.5). It is standard to show that:

$$\nabla_\xi \psi(y,\xi) = \widetilde{x}(n\pi,y,\xi) \tag{1.7}$$

and the point of stationary phase is given as the solution of $x=\widetilde{x}(n\pi,y,\xi)$.

We study the properties of $\widetilde{x}(n\pi,y,\xi)$ as $h \to 0$ as well as $|\xi| \to \infty$ in Section 2. Section 3 is devoted to studying the phase function $\psi(y,\xi)$. We show there exits a unique point of stationary phase for $x \neq (-1)^n y$ and we estimate $|x-\nabla_\xi\psi(y,\xi)|$ from below. Estimates on the symbol b is given in Section 4 and the proof of the theorem is completed in Section 5. In the Appendix an implicit function theorem for mappings in $\mathbf{R}^m$ with positive definite differentials outside a compact set is given.

2 Estimates on the Classical Flow

The purpose of this section is to show the following proposition.

Proposition 2.1 *Let a compact set $K \subset \mathbf{R}^m$ be fixed and $\alpha,\beta \in \mathbf{N}^m$. Then:*
(1) *For all $t \in [0,n\pi]$ one has:*

$$|\partial_y^\alpha \partial_\xi^\beta (\widetilde{x}(t,y,\xi) - y\mathrm{cos}t - (-1)^n \xi \mathrm{sin}t)| = \mathcal{O}(h^\delta \langle\xi\rangle^{(1-|\beta|)_+ - \mu})$$

and

$$\widetilde{x}(t,y,\xi) = y\mathrm{cos}t + (-1)^n\xi\mathrm{sin}t + h^\delta \mathrm{cos}t \int_0^t \mathrm{sin}s \nabla W(y\mathrm{cos}s + (-1)^n \xi \mathrm{sin}s)ds$$
$$+h^{2\delta} r(t,y,\xi)$$

with

$$\partial_y^\alpha \partial_\xi^\beta r(t,y,\xi) = \mathcal{O}(\langle\xi\rangle^{(1-|\beta|)_+ - 2\mu} + |\mathrm{sin}t|\langle\xi\rangle^{1-2\mu})$$

uniformly with respect to $\xi \in \mathbf{R}^m$, $y \in K$ and $h>0$ small enough.
(2) *For any $\varepsilon > 0$, there exists $h_0 = h_0(\varepsilon, K)$ such that*

$$|\partial_y^\alpha \partial_\xi^\beta (\widetilde{x}(t,y,\xi) - y\mathrm{cos}t - (-1)^n\xi\mathrm{sin}t)| = \mathcal{O}((h^\delta\langle\xi\rangle^{-\mu})^{|\beta|}(h^\delta\langle\xi\rangle^{1-\mu})^{(1-|\beta|}$$

$)_+)uniformlywithrespectto—\xi| \geq \varepsilon h^{-\nu}$, $0<h<h_0$, $t \in [0,n\pi]$ *and* $y \in K$.

Proof - For $(y,k) \in \mathbf{R}^{2m}$, we denote $(x(t,y,k),p(t,y,k))$ the unique classical trajectory $t \mapsto (x(t),p(t))$ satisfying $(x(0),p(0)) = (y,k)$. We also denote $k(y,\xi)$ the value of k for which $p(n\pi,y,k)=\xi$ (so that we have $\widetilde{x}(t,y,\xi) = x(t,y,k(y,\xi)))$.

We use the following lemma:

Lemma 2.2 (1) *For $h>0$ small enough and for all $\alpha,\beta\in\mathbf{N}^m$, one has:*

$$|\partial_y^\alpha\partial_\xi^\beta(k(y,\xi)-(-1)^n\xi)| = \mathcal{O}(h^\delta\langle\xi\rangle^{(1-|\beta|)_+-\mu})$$

and

$$k(y,\xi)=(-1)^n\xi+h^\delta\int_0^{n\pi}\cos s\nabla W(y\cos s+(-1)^n\xi\sin s)ds+h^{2\delta}r_1(y,\xi)$$

with

$$\partial_y^\alpha\partial_\xi^\beta r_1(y,\xi)=\mathcal{O}(\langle\xi\rangle^{1-2\mu})$$

uniformly with respect to ξ and h.
(2) *For any $\varepsilon>0$, there exists $h_0=h_0(\varepsilon,K)$ such that for all $\alpha,\beta\in\mathbf{N}^m$,*

$$|\partial_y^\alpha\partial_\xi^\beta(k(y,\xi)-(-1)^n\xi)|=\mathcal{O}((h^\delta\langle\xi\rangle^{-\mu})^{|\beta|}(h^\delta\langle\xi\rangle^{1-\mu})^{(1-|\beta|)_+})$$

uniformly with respect to $|\xi|\geq\varepsilon h^{-\nu}$, $0<h<h_0$, $t\in[0,n\pi]$ and $y\in K$.

Proof of the lemma - By Duhamel principle, we have for any $(t,y,k)\in\mathbf{R}\times\mathbf{R}^{2m}$:

$$\begin{cases} x(t,y,k)=y\cos t+k\sin t-h^\delta\int_0^t\sin(t-s)\nabla W(x(s,y,k))ds, \\ p(t,y,k)=-y\sin t+k\cos t-h^\delta\int_0^t\cos(t-s)\nabla W(x(s,y,k))ds \end{cases}\tag{2.1}$$

and therefore, $k=k(y,\xi)$ is the unique solution of the equation:

$$(-1)^n\xi=k-h^\delta\int_0^{n\pi}\cos s\nabla W(x(s,y,k))ds.\tag{2.2}$$

Denoting by $F(y,k)$ the right-hand-side of (2.2), we see that:

$$\frac{\partial F}{\partial k}=I+\mathcal{O}\left(h^\delta\int_0^{n\pi}\left\|\frac{\partial x}{\partial k}(s,y,k)\right\|\langle x(s,y,k)\rangle^{-\mu}ds\right)\tag{2.3}$$

while, using Gronwall's inequality iteratively, we deduce from (2.1) that for all $\alpha,\beta\in\mathbf{N}^m$ one has:

$$\left\|\partial_y^\alpha\partial_k^\beta\left(\frac{\partial x}{\partial k}(s,y,k)-\sin s\right)\right\|=\mathcal{O}\left(h^\delta\int_0^{n\pi}\langle x(u,y,k)\rangle^{-\mu}du\right)\tag{2.4}$$

uniformly with respect to k and h. Moreover, using the same arguments as in [Ya] Lemmas 4.2-4.4, we see that:

$$\int_0^{n\pi}\langle x(s,y,k)\rangle^{-\mu}ds=\mathcal{O}(\langle k\rangle^{-\mu}).\tag{2.5}$$

In particular $\|\partial x/\partial k\|$ is uniformly bounded and we deduce from (2.3)-(2.5) that for any $\alpha,\beta\in\mathbf{N}^m$:

$$\left\|\partial_y^\alpha\partial_k^\beta\left(\frac{\partial F}{\partial k}-I\right)\right\|=\mathcal{O}(h^\delta\langle k\rangle^{-\mu})\tag{2.6}$$

uniformly. It follows from (2.6) that $k \mapsto F(y,k)$ is a global diffeomorphism in $\mathbf{R}^m$ for all y and for h small enough, and that moreover the solution $k(y,\xi)$ of (2.2) satisfies:

$$|\partial_y^\alpha \partial_\xi^\beta (k(y,\xi) - (-1)^n \xi)| = \mathcal{O}(h^\delta \langle \xi \rangle^{(1-|\beta|)_+ - \mu}) \tag{2.7}$$

for any $\alpha, \beta \in \mathbf{N}^m$. Inserting this estimate in (2.2) and using again (2.4) as well as (1.2), we get in particular:

$$k(y,\xi) = (-1)^n \xi + h^\delta \int_0^{n\pi} \cos s \nabla W(x(s,y,(-1)^n\xi)) ds + \mathcal{O}\left(h^{2\delta} \langle \xi \rangle^{1-2\mu}\right)$$

an analogous estimates for the derivatives. Then the result follows by using (see (2.1)) that $x(s,y,(-1)^n\xi) = y\cos s + (-1)^n \xi \sin s + \mathcal{O}(h^\delta \langle \xi \rangle^{1-\mu})$.

For proving the second statement, we need two lemmas.

Lemma 2.3 *Let $\rho > 0$ and $\ell \geq 0$. Let $T > 0$ and a compact set $K \subset \mathbf{R}^m$ be fixed. Then, for any $\varepsilon > 0$, there exist $h_0 = h_0(\varepsilon, K)$ and $C = C(\varepsilon, K) > 0$ such that*

$$\int_0^T \langle x(t,y,k) \rangle^{-\rho} |\sin t|^\ell dt \leq C(|k|^{-1}(h^\delta |k|^{-\mu})^\ell + |k|^{-\min(1+\ell,\rho)}) \tag{2.8}$$

for $y \in K$ and $|k| \geq \varepsilon h^{-\nu}$, $0 < h < h_0$. Here we have assumed $\rho \notin \mathbf{N}$ for simplicity.

Proof - We have $\left| h^\delta \int_0^T \sin(t-s) \nabla_x W(x(s,y,k)) ds \right| \leq C_T h^\delta |k|^{1-\mu}$ for $|k| \geq C_0$. Set $C_{\varepsilon,K} = \varepsilon^{-1/(1-\mu)} \sup_{y \in K} |y|$ so that $C_{\varepsilon,K} h^\delta |k|^{1-\mu} \geq \sup\limits_{y\in K} |y|$ for $|k| \geq \varepsilon h^{-\nu}$, $0 < h < 1$. Define

$$D_1 = \{t \in [0,T] : |\sin t| \leq 2(C_T + C_{\varepsilon,K}) h^\delta |k|^{-\mu}\}, \qquad D_2 = [0,T] \setminus D_1.$$

For $t \in D_1$, we have $|x(t,y,k)| \leq 3(C_T + C_{\varepsilon,K}) h^\delta |k|^{1-\mu} \leq \sqrt{|k|^2 + y^2}/10$ if $|k| \geq \varepsilon h^{-\nu}$ and, as in [Ya] Lemma 4.2, we see that

$$\left| \int_{D_1} \langle x(t) \rangle^{-\rho} (\sin t)^\ell dt \right| \leq C \sum_j \int_{|t - j\pi| \leq \theta} |t - j\pi|^\ell \langle (t - a_j) k \rangle^{-\rho} dt$$

where we have set $\theta = 3(C_T + C_{\varepsilon,K}) h^\delta |k|^{-\mu}$ and where the sum over j integer is finite, and $a_j \in [-2T, 2T]$ is the unique time in

$[j\pi - \pi/10, j\pi + \pi/10]$ for which $|x(t)|^2$ is minimal. In particular, using that $x(t,y,k) = y\cos t + k\sin t + \mathcal{O}(h^\delta \langle k \rangle^{1-\mu})$ we get:

$$|a_j - j\pi| = \mathcal{O}(\theta) \tag{2.9}$$

uniformly. Therefore, denoting $b_j = a_j - j\pi$ we get by a change of variable:

$$\begin{aligned}
\left|\int_{D_1} \langle x(t)\rangle^{-\rho}(\mathrm{sin}t)^{\ell}dt\right| &\leq C\sum_j \int_{|t+b_j|\leq\theta} |t+b_j|^{\ell}\langle tk\rangle^{-\rho}dt \\
&\leq C\sum_j \sum_{q=0}^{\ell} |b_j|^{\ell-q}\int_{|t|\leq C\theta} |t|^q\langle tk\rangle^{-\rho}dt \\
&\leq C\sum_{q=0}^{\ell} |k|^{-\min(q+1,\rho)}\theta^{\ell-q+(q+1-\ell)_+} \\
&\leq C(|k|^{-\rho}\theta^{\ell+1} + |k|^{-1}\theta^{\ell}).
\end{aligned}$$

Now, if $t \in D_2$, then $|x(t,y,k)| \geq (1/2)|(\mathrm{sin}t)k|$ and we have

$$\left|\int_{D_2} \langle x(t,y,k)\rangle^{-\rho}(\mathrm{sin}t)^{\ell}dt\right| \leq C\int_0^T |(\mathrm{sin}t)^{\ell}|\langle(\mathrm{sin}t)k\rangle^{-\rho}dt \leq C|k|^{-\min(\ell+1,\rho)}.$$

Adding all the contributions completes the proof of the lemma. ⋄

Lemma 2.4 *Let a compact set $K \subset \mathbf{R}^m$ and $T > 0$ be fixed. Then, for any $\varepsilon > 0$ we have for any $|\beta| \geq 1$,*

$$|\partial_y^{\alpha}\partial_k^{\beta}(x(t,y,k) - y\mathrm{cos}t - k\mathrm{sin}t)| \leq \mathcal{O}((h^{\delta}k^{-\mu})^{|\beta|}),$$

$$|\partial_y^{\alpha}\partial_k^{\beta}(p(t,y,k) + y\mathrm{sin}t - k\mathrm{cos}t)| \leq \mathcal{O}((h^{\delta}k^{-\mu})^{|\beta|})$$

for $|k| \geq \varepsilon h^{-\nu}$, $0 < h < h_0 = h_0(\varepsilon, K)$ and $y \in K$.

Proof. We prove the case $\alpha = 0$ only. The proof for other cases are similar. We write

$$\partial_k x(t) = \mathrm{sin}t - h^{\delta}\int_0^t \mathrm{sin}(t-s)\partial_x^2 W(x(s))\partial_k x(s)ds = \mathrm{sin}t + X(t). \tag{2.10}$$

Then $|X(t)| \leq Ch^{\delta}|k|^{-\mu}$ and this prove the case $|\beta| = 1$. We prove the general case by induction on $|\beta|$. We assume that the lemma holds for $|\beta| \leq \ell - 1$ and let $|\beta| = \ell$, $\ell \geq 2$. We have by Leibniz' formula

$$\begin{aligned}
\partial_k^{\beta}x(t) = h^{\delta}&\int_0^t \mathrm{sin}(t-s)\partial_x^2 W(x(s))\partial_k^{\beta}x(s)ds \\
+\sum h^{\delta}&\int_0^t \mathrm{sin}(t-s)\partial_x^{\kappa}\partial_x W(x(s))(\prod \partial_k^{\beta_j}x(s))ds.
\end{aligned}\tag{2.11}$$

where the sum is taken over β_j such that $\sum_j \beta_j = \beta$, $|\kappa| \geq 2$ and $|\kappa|$ is the number of the factors in the product. We estimate each integral under the sign of summation. Replacing all $\partial_k x(s)$ by $\sin s + X(s)$ and using the induction hypothesis for $|\partial_k^{\beta_j} x(s)|$ with $|\beta_j| \geq 2$ and $|X(s)| \leq Ch^\delta |k|^{-\mu}$, we estimate it by

$$h^\delta (h^\delta |k|^{-\mu})^{|\beta|-q} \cdot \int_0^T \langle x(s)\rangle^{1-\mu-|\kappa|} |\sin s|^q ds$$

where $q \leq |\kappa|$ is the number of $\sin s$'s which appear from the factors $\partial_k x(s) = \sin s + X(s)$. We have $\mu + |\kappa| - 1 > q + 1$ unless $q \geq |\kappa| - 1$. Hence, we have from Lemma 2.3 that, if $q \leq |\kappa| - 2$,

$$h^\delta \int_0^T \langle x(s)\rangle^{1-\mu-|\kappa|} |\sin s|^q ds \leq Ch^\delta (|k|^{-1}(h^\delta |k|^{-\mu})^q + |k|^{-(q+1)})$$

(2.12)

and if $|\kappa| - 1 \leq q \leq |\kappa|$,

$$h^\delta \int_0^T \langle x(s)\rangle^{1-\mu-|\kappa|} |\sin s|^q ds \leq Ch^\delta (|k|^{-1}(h^\delta |k|^{-\mu})^q + |k|^{1-\mu-q}). \tag{2.13}$$

Using that $|k| \geq \varepsilon h^{-\nu}$, we see that the right hand sides of (2.12) and (2.13) are bounded by $C(h^\delta |k|^{-\mu})^q$

and the lemma follows by applying Gronwall's inequality to (2.11).

◇

Completion of the proof of Lemma 2.2 - When $|\xi| \geq \varepsilon h^{-\nu}$, we improve (2.6) to

$$\left\| \partial_y^\alpha \partial_k^\beta \left(\frac{\partial F}{\partial k} - I \right) \right\| = \mathcal{O}((h^\delta \langle k \rangle^{-\mu})^{|\beta|+1})$$

by using the argument of the proof of the previous Lemma 2.4 which leads to the following improvement of (2.7):

$$|\partial_y^\alpha \partial_\xi^\beta (k(y,\xi) - (-1)^n \xi)| = \mathcal{O}((h^\delta \langle \xi \rangle^{-\mu})^{|\beta|} (h^\delta \langle \xi \rangle^{1-\mu})^{(1-|\beta|)_+}).$$

This completes the proof of the lemma. ◇

Completion of the proof of Proposition 2.1 - Going back to (2.1) and using the first estimate of Lemma 2.2, we first get:

$$\widetilde{x}(t,y,\xi) = y\cos t + k(y,\xi)\sin t - h^\delta \int_0^t \sin(t-s)\nabla W(x(s,y,(-1)^n\xi) + h^{2\delta} r_2(t,y,\xi) \tag{2.14}$$

with

$$\partial_y^\alpha \partial_\xi^\beta r_2(t,y,\xi) = \mathcal{O}(\langle \xi \rangle^{(1-|\beta|)_+ - 2\mu})$$

and the first estimate of statement (1) follows by using also (2.5). The second estimate of statement (1) is also obtained immediately from (2.14) by using the second estimate of lemma 2.2. Statement (2) may be proved by differentiating $\tilde{x}(t,y,\xi) = x(t,y,k(y,\xi))$ and applying Lemma 2.4 and Lemma 2.2. Note that $|\xi| \sim |k|$ by virtue of (2.2).

$\diamond$

3 Estimates on the Phase

Let ψ be the phase defined in (1.6). In this section we show:

Proposition 3.1 *For all $(x,y) \in \mathbf{R}^{2m}$ with $x \neq (-1)^n y$, there exists a unique $\xi_c = \xi_c(x,y,h) \in \mathbf{R}^m$ for $0 < h$ small enough such that:*

$$\nabla_\xi \psi(y,\xi_c) = x.$$

Moreover, if (x,y) remains in a compact set K of $\mathbf{R}^{2m} \cap \{x \neq (-1)^n y\}$, then there exists a constant $C_K > 0$ such that:

$$\frac{1}{C_K} h^{-\nu} \leq |\xi_c(x,y,h)| \leq C_K h^{-\nu}$$

with $\nu = \dfrac{\delta}{1-\mu}$, and for any $\xi \in \mathbf{R}^m$:

$$|x - \nabla_\xi \psi(y,\xi)| \geq \frac{h^\delta}{C_K(|\xi_c| + |\xi - \xi_c|)^\mu} |\xi - \xi_c|.$$

Proof - By (1.7) and (2.1), we have to solve the equation:

$$\int_0^{n\pi} \sin s \nabla W(x(s,y,k))ds = \frac{(-1)^n x - y}{h^\delta}$$

(3.1)

where $k = k(y,\xi)$. Actually, since the mapping $\xi \mapsto k(y,\xi)$ is one-to-one on $\mathbf{R}^m$ (for y fixed), it is enough to solve (3.1) by taking k as the unknown variable. Denote by $G(y,k)$ the function defined by the left hand side of (3.1). Then computing as

$$\nabla_k G(y,k) = \int_0^{n\pi} \sin s D^2 W(x(s,y,k)) \frac{\partial x}{\partial k}(s,y,k)ds$$

and using (2.4) and (1.2), we see as in [Ya], that $\nabla_k G(y,k)$ satisfies $\|\nabla_k G(y,k) - P(k)\| \leq C\langle k\rangle^{-2\mu}$ for some positive definite matrix $P(k)$ such that $C^{-1}\langle k\rangle^{-\mu} \leq P(k) \leq C\langle k\rangle^{-\mu}$. It follows from the global implicit function theorem given in the appendix that, for large enough $R > 0$, the mapping

$$k \mapsto G(y,k) = \int_0^{n\pi} \sin s \nabla W(x(s,y,k))ds$$

is a diffeomorphism from the exterior $B_{>R} \subset \mathbf{R}^m$ of the ball of radius R to its image and the image contains another exterior domain $B_{>R_1}$. In particular, for h small enough we get the existence of a unique solution $k_c = k(y, \xi_c)$ of (3.1).

Moreover, we have:

Lemma 3.2 *For y remaining in a fixed compact set of $\mathbf{R}^m$ and for $|k|$ large enough, one has:*

$$\frac{1}{C}|k|^{1-\mu} \leq \left| \int_0^{n\pi} \sin s \nabla W(x(s,y,k)) ds \right| \leq C|k|^{1-\mu}$$

for some constant $C > 0$ and uniformly with respect to h and k.

Proof of the lemma - Since the upper-bound is obvious, we concentrate on the lower-bound. We see from (2.1) that

$$x(s,y,k) = y\cos s + k\sin s + \mathcal{O}(h^\delta \langle k \rangle^{1-\mu})$$

and we also use (2.4) to obtain:

$$\begin{aligned}\frac{\partial}{\partial \theta} \int_0^{n\pi} \sin s \nabla W(x(s,y,\theta k)) ds = & \int_0^{n\pi} \sin^2 s D^2 W(y\cos s + \theta k \sin s) k \, ds \\ & + \mathcal{O}(h^\delta) \left(\langle \theta k \rangle^{1-\mu} |k| + \langle k \rangle^{1-\mu} \right).\end{aligned}$$

Integrating with respect to θ from 0 to 1, this gives:

$$\begin{aligned}\int_0^{n\pi} \sin s \nabla W(x(s,y,k)) ds = & \int_0^1 \int_0^{n\pi} \sin^2 s D^2 W(y\cos s + \theta k \sin s) k \, ds d\theta \\ & + \mathcal{O}(1 + h^\delta \langle k \rangle^{1-\mu}) \quad (3.2)\end{aligned}$$

Then we fix $\rho \in (\mu, 1)$, and we consider the set:

$$D_\rho = \{(\theta, s) \in [0,1] \times [0, n\pi] \; ; \; |\theta \sin s| \geq \langle k \rangle^{-\rho}\}$$

and its complementary D_ρ^C in $[0,1] \times [0, n\pi]$. Since on D_ρ we have $|\theta k \sin s| \to +\infty$ as $|k| \to +\infty$, we can use (1.2) to get:

$$\begin{aligned}& \left\langle \int_{D_\rho} \sin^2 s D^2 W(y\cos s + \theta k \sin s) k \, ds d\theta \; , \; k \right\rangle \\ & \qquad \geq \int_{D_\rho} \frac{\sin^2 s}{C} \langle y\cos s + \theta k \sin s \rangle^{-\mu} \, d\theta ds \; |k|^2\end{aligned}$$

and thus, by Cauchy-Schwarz inequality and since for $|k|$ large enough D_ρ contains $\{(\theta, s) \in [0,1] \times [0, n\pi] \; ; \; |\theta \sin s| \geq \delta_1\}$ (of measure ~ 1) for any fixed $\delta_1 > 0$ small enough, we get (with some other constant $C > 0$):

$$\left| \int_{D_\rho} \sin^2 s D^2 W(y\cos s + \theta k \sin s) k \, ds d\theta \right| \geq \frac{1}{C} \langle k \rangle^{1-\mu}. \qquad (3.3)$$

On another hand, since the Lebesgue measure of D_ρ^C is $\mathcal{O}(\langle k\rangle^{-\rho}\ln\langle k\rangle^\rho)$ as $|k| \to +\infty$, we have:

$$\left|\int_{D_\rho^C} \sin^2 s D^2 W(y\cos s + \theta k \sin s)k \, ds d\theta\right| = \mathcal{O}(\langle k\rangle^{1-\rho}\ln\langle k\rangle^\rho) = \mathcal{O}(\langle k\rangle^{1-\mu-\varepsilon}) \tag{3.4}$$

for some $\varepsilon > 0$. Then the result follows from (3.2)-(3.4). $\diamond$

Completion of the proof of Proposition 3.1 - We deduce from the lemma and from (3.1) that $|k_c|$ (and thus also $|\xi_c|$) behaves like $h^{-\nu}$ as $h \to 0$.

Now, denoting

$$G(y,\xi) := \int_0^{n\pi} \sin s \nabla W\left(x(s,y,k(y,\xi))\right) ds = \int_0^{n\pi} \sin s \nabla W(\widetilde{x}(s,y,\xi))ds, \tag{3.5}$$

we can write:

$$\begin{aligned} x - \nabla_\xi \psi(y,\xi) &= x - (-1)^n y - (-1)^n h^\delta G(y,\xi) = (-1)^n h^\delta \left(G(y,\xi_c) - G(y,\xi)\right) \\ &= (-1)^n h^\delta \int_0^1 \frac{\partial G}{\partial \xi}(y, t\xi_c + (1-t)\xi) \cdot (\xi_c - \xi) dt. \end{aligned}$$

that is:

$$x - \nabla_\xi \psi(y,\xi) = (-1)^n h^\delta \int_0^1 \int_0^{n\pi} A(t,s,x,y,\xi) ds dt \tag{3.6}$$

with

$$A(t,s,x,y,\xi) = \sin s D^2 W\left(\widetilde{x}(s,y,t\xi_c + (1-t)\xi)\right) \frac{\partial \widetilde{x}}{\partial \xi}(s,y,t\xi_c + (1-t)\xi) \cdot (\xi_c - \xi).$$

Now, given some constant $\lambda > 0$ large enough, we split the integral in (3.6) in two pieces by setting:

$$B = B(x,y,\xi) = \{(s,t) \in [0,n\pi] \times [0,1] \; ; \; |(t\xi_c + (1-t)\xi)\sin s| \geq \lambda\}$$

and

$$I_1(x,y,\xi) = (-1)^n h^\delta \int_B A(t,s,x,y,\xi) ds dt$$

$$I_2(x,y,\xi) = (-1)^n h^\delta \int_{B^C} A(t,s,x,y,\xi) ds dt.$$

If λ is taken sufficiently large, for $(s,t) \in B$ we can apply (1.2) with $x = \widetilde{x}(s,y,t\xi_c + (1-t)\xi)$. Since also, by Proposition 2.1, we have

$$\frac{\partial \widetilde{x}}{\partial \xi}(s,y,\xi) = (-1)^n \sin s + \mathcal{O}(h^\delta \langle \xi \rangle^{-\mu})$$

this permits to get:

$$\langle I_1, \xi_c - \xi\rangle \geq \frac{h^\delta}{C_0}\int_B \sin^2 s \langle \widetilde{x}(s,y,t\xi_c + (1-t)\xi)\rangle^{-\mu}|\xi_c - \xi|^2 ds dt$$
$$-\mathcal{O}(h^{2\delta})\int_B \langle t\xi_c + (1-t)\xi\rangle^{-\mu}|\xi_c - \xi|^2 ds dt. \qquad (3.7)$$

Now, let us estimate the measure of B^C. Since $|\xi_c + (1-t)(\xi - \xi_c)| \geq ||\xi_c| - (1-t)|\xi - \xi_c||$, it is easy to see that if $(s,t) \in B^C$, then t belongs to an interval of length $\mathrm{Min}\left(1, \dfrac{2\lambda}{|(\xi - \xi_c)\sin s|}\right)$. When e.g. $|\xi - \xi_c| \geq |\xi_c|/2$, this gives a set in $[0, n\pi] \times [0,1]$ of measure $\mathcal{O}\left(\dfrac{\ln|\xi - \xi_c|}{|\xi - \xi_c|}\right)$. On another hand, if $|\xi - \xi_c| \leq |\xi_c|/2$ then s belongs to a set of measure $\mathcal{O}(|\xi_c|^{-1})$. Thus we get in any case:

$$\mathrm{Measure}(B^C) = \mathcal{O}\left(\frac{\ln(|\xi_c| + |\xi - \xi_c|)}{|\xi_c| + |\xi - \xi_c|}\right). \qquad (3.8)$$

It follows that

$$|I_2| = \mathcal{O}\left(\frac{h^\delta |\xi - \xi_c| \ln(|\xi_c| + |\xi - \xi_c|)}{|\xi_c| + |\xi - \xi_c|}\right) \qquad (3.9)$$

and also, in view of (3.7):

$$\langle I_1, \xi_c - \xi\rangle \geq \frac{h^\delta}{C_0}\int_0^1\int_0^{n\pi} \sin^2 s \langle \widetilde{x}(s,y,t\xi_c + (1-t)\xi)\rangle^{-\mu}|\xi_c - \xi|^2 ds dt$$
$$-\mathcal{O}\left(\frac{h^\delta|\xi - \xi_c|^2}{|\xi_c| + |\xi - \xi_c|}\right) - \mathcal{O}(h^{2\delta})\int_0^1 \langle t\xi_c + (1-t)\xi\rangle^{-\mu}|\xi_c - \xi|^2 dt$$

that is:

$$\langle I_1, \xi_c - \xi\rangle \geq \frac{h^\delta}{C_0}\int_0^1\int_0^{n\pi} \sin^2 s \langle \widetilde{x}(s,y,t\xi_c + (1-t)\xi)\rangle^{-\mu}|\xi_c - \xi|^2 ds dt$$
$$-\mathcal{O}\left(\frac{h^\delta|\xi - \xi_c|^2}{|\xi_c| + |\xi - \xi_c|} + \frac{h^{2\delta}|\xi_c - \xi|^2}{(1 + |\xi_c| + |\xi - \xi_c|)^\mu}\right). \qquad (3.10)$$

As a consequence, since $|\widetilde{x}(s,y,\xi)| = \mathcal{O}(|\xi|)$ uniformly, we get from (3.10):

$$\langle I_1, \xi_c - \xi\rangle \geq \frac{h^\delta}{C_1}\int_0^1 \langle t\xi_c + (1-t)\xi\rangle^{-\mu}|\xi_c - \xi|^2 dt$$
$$-\mathcal{O}\left(\frac{h^\delta|\xi - \xi_c|^2}{|\xi_c| + |\xi - \xi_c|} + \frac{h^{2\delta}|\xi_c - \xi|^2}{(1 + |\xi_c| + |\xi - \xi_c|)^\mu}\right).$$

where $C_1 > 0$ is a constant, and thus (with some other constant $C_2 > 0$):

$$\langle I_1, \xi_c - \xi\rangle \geq \frac{h^\delta}{C_2}\frac{|\xi_c - \xi|^2}{(1 + |\xi_c| + |\xi - \xi_c|)^\mu} - \mathcal{O}\left(\frac{h^\delta|\xi - \xi_c|^2}{|\xi_c| + |\xi - \xi_c|} + \frac{h^{2\delta}|\xi_c - \xi|^2}{(1 + |\xi_c| + |\xi - \xi_c|)^\mu}\right). \qquad (3.11)$$

Putting together (3.6), (3.9) and (3.11), we get for h small enough:

$$|x-\nabla_\xi\psi(y,\xi)| \geq h^\delta\left(\frac{1}{C_3(1+|\xi_c|+|\xi-\xi_c|)^\mu}-\frac{C_3\ln(|\xi_c|+|\xi-\xi_c|)}{|\xi_c|+|\xi-\xi_c|}\right)|\xi_c-\xi| \tag{3.12}$$

with $C_3>0$ constant. Now the result follows by observing that $|\xi_c|\sim h^{-\nu}\sim 1+|\xi_c|$ and, for any fixed $\rho\in(\mu,1)$, $(|\xi_c|+|\xi-\xi_c|)^{-1}\ln(|\xi_c|+|\xi-\xi_c|)=\mathcal{O}((|\xi_c|+|\xi-\xi_c|)^{-\rho})=\mathcal{O}(|\xi_c|^{\mu-\rho}(|\xi_c|+|\xi-\xi_c|)^{-\mu})=\mathcal{O}(h^{\nu(\rho-\mu)}(|\xi_c|+|\xi-\xi_c|)^{-\mu})$.
$\diamond$

Remark - We deduce in particular from Proposition 3.1 that for ξ such that $|\xi-\xi_c|=\mathcal{O}(|\xi_c|)$, one has:

$$|x-\nabla\psi(y,\xi)|\geq\frac{h^\nu}{C'_K}|\xi-\xi_c|$$

with $C'_K>0$ constant.

4 Estimates on the Symbol

Let $b(y,\xi,h)$ be the amplitude function in (1.5). We denote $\langle y,\xi\rangle^{-\mu}=\langle(y,\xi)\rangle^{-\mu}$. The purpose of this section is to prove:

Proposition 4.1 *As $h\to+0$, $\partial_y^\alpha\partial_\xi^\beta(b(y,\xi,h)-1)=\mathcal{O}(h^\delta\langle y,\xi\rangle^{-\mu})+\mathcal{O}(h)$.*

Proof. For small $\varepsilon>0$ and large $T>0$ fixed, we set $\mathcal{I}_{\varepsilon,T}=\{|t|<T: |t-(m+1/2)\pi|>\varepsilon,\ \forall m\in\mathbf{Z}\}$. For $|t|<T$, we have

$$\left\|\frac{\partial p}{\partial k}(t,y,k)-\mathrm{cos}t\right\|\leq C_Th^\delta$$

and for $0<h$ small, the map $k\mapsto p(t,y,k)$ is a diffeomorphism of $\mathbf{R}^m$ for every fixed $t\in\mathcal{I}_{\varepsilon,T}$ and $y\in\mathbf{R}^m$. It follows for such t that the phase function is globally defined by

$$\psi(t,y,\xi)=x(t,y,k)\cdot\xi-\int_0^t\left(\frac{p(s,y,k)^2}{2}\right.$$

-V(x(s,y,k))ds,k$being such that \xi=p(t,y,k)$, and that $E(t,x,y)$, $(n-1/2)\pi<t<(n+1/2)\pi$, can be written ([Ya-1]) in the form

$$E(t,x,y)=\frac{i^n}{(2\pi h)^m|\mathrm{cos}t|^{m/2}}\int e^{i(x\xi-\psi(t,y,\xi))/h}b(t,y,\xi)d\xi.$$

When $|t|\leq T_1\equiv(\pi/2)-\varepsilon$, it can be shown as in ([Ya-1]) that

$$b(t,y,k)=\frac{b_0(t,y,\xi)}{(\mathrm{cos}t)^{m/2}}=\left(\det\frac{\partial p}{\partial k}(t,y,k)\right)^{-1/2}+h\mathcal{O}(1),\quad \xi=p(t,y,k).$$

Since $\partial_y^\alpha \partial_k^\beta \left(\frac{\partial p}{\partial k}(t,y,k) - \mathrm{cos} t \right) = \mathcal{O}(h^\delta \langle y,k\rangle^{-\mu})$ ([Ya], Lemma 4.4) and $\langle y,\xi\rangle \sim \langle y,k\rangle$ for $|t| \le T_1$, (4.1) holds for small t. For obtaining the proposition, it suffices via induction argument to show the following lemma. We let $t,s \in \mathcal{I}_{\varepsilon,T}$ be such that, for some $n_1,n_2,n_3 \in \mathbf{Z}$, $|s-n_1\pi| < \pi/2$, $|t-n_2\pi| < \pi/2$ and $|s+t-n_3\pi| < \pi/2$ and set

$$F(x,y,\xi) = \frac{i^{-n_1}}{(2\pi h)^m |\mathrm{cos} t|^{m/2}} e^{i(x\xi - \psi(t,y,\xi))/h}(b_0(y,\xi) + hb_1(y,\xi))$$

$$G(x,y,\xi) = \frac{i^{-n_2}}{(2\pi h)^m |\mathrm{cos} s|^{m/2}} e^{i(x\xi - \psi(s,y,\xi))/h}(c_0(y,\xi) + hc_1(y,\xi))$$

and define

$$H(y,\xi) = \int e^{-ix\xi/h} F(x,z,\eta) G(z,y,\zeta) d\zeta dz d\eta dx.$$

Lemma 4.2 *Suppose that b_0 and c_0 satisfy $\partial_y^\alpha \partial_\xi^\beta (b_0(y,\xi,h)-1) = \mathcal{O}(h^\delta \langle y,\xi\rangle^{-\mu})$ and $\partial_y^\alpha \partial_\xi^\beta (c_0(y,\xi,h)-1) = \mathcal{O}(h^\delta \langle y,\xi\rangle^{-\mu})$ and $b_1, c_1 = \mathcal{O}(1)$. Then*

$$H(y,\xi) = \frac{i^{-m_3} e^{-i\psi(t+s,y,\xi)/h}}{|\cos(t+s)|^{n/2}}(d_0(y,\xi) + hd_1(y,\xi)),$$

where $\partial_y^\alpha \partial_\xi^\beta (d_0(y,\xi,h)-1) = \mathcal{O}(h^\delta \langle y,\xi\rangle^{-\mu})$ and $d_1 = \mathcal{O}(1)$.

Proof. Set $\Phi(x,z,\eta,y,\zeta) = -x\xi + x\eta - \psi(t,z,\eta) + z\zeta - \psi(s,y,\zeta)$. The derivatives of Φ of order higher than one are bounded and $\mathrm{Hess}_{x,\eta,z,\zeta}\Phi$ is given by

$$\begin{pmatrix} 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & -\psi_{\eta\eta} & -\psi_{z\eta} & 0 \\ 0 & -\psi_{\eta z} & -\psi_{zz} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & -\psi_{\zeta\zeta} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & -\tan t & -\sec t & 0 \\ 0 & -\sec t & -\tan t & \mathbf{1} \\ 0 & 0 & \mathbf{1} & -\tan s \end{pmatrix} + \mathcal{O}(h^\delta).$$

Denote by A the matrix on the right. It is easy to see that

$$|\det A| = |\tan t \tan s - 1|^m = \left| \frac{\cos(t+s)}{\mathrm{cos} t \cdot \mathrm{cos} s} \right|^m \neq 0.$$

Thus the point of stationary phase exists uniquely for every (ξ, y) and is determined by the system of equations

$$\begin{cases} \partial_x \Phi = -\xi + \eta = 0, \\ \partial_\eta \Phi = x - \partial_\eta \psi(t,z,\eta) = 0, \\ \partial_z \Phi = -\partial_z \psi(t,z,\eta) + \zeta = 0, \\ \partial_\zeta \Phi = z - \partial_\zeta \psi(s,y,\zeta) = 0 \end{cases} \tag{4.1}$$

For any k,

$$(x,\eta,z,\zeta) = (x(t+s,y,k), p(t+s,y,k), x(s,y,k), p(s,y,k)) \tag{4.2}$$

satisfies the last three equations of (4.1) and $(y,k) \mapsto (y, p(t+s,y,k))$ is a diffeomorphism on $\mathbf{R}^{2m}$. It follows that the unique stationary phase point $(x_c, \eta_c, z_c, \zeta_c)$ is given by the right hand side of (4.2) with k being replaced by the solution $k(y,\xi)$ of $\xi = p(t+s,y,k)$. The quadratic form defined by the matrix A can be written for $\mathbf{x} = (a,b,c,d) \in \mathbf{R}^{4m}$ in the form

$$\begin{aligned}\langle A\mathbf{x}, \mathbf{x}\rangle &= \frac{\tan t \tan s - 1}{\tan t \tan s} a^2 - \frac{\tan t \tan s}{\tan t \tan s - 1}\left(b - \frac{\tan t \tan s - 1}{\tan t \tan s} a\right)^2 \\ &+ \frac{\tan t \tan s - 1}{\tan s}\left(c - \frac{\tan t \sec t}{\tan t \tan s - 1} b\right)^2 + \tan s \left(d - \frac{c}{\tan s}\right)^2,\end{aligned}$$

and we see that the signature of A is given by

$$\operatorname{sgn}(A) = \begin{cases} 0 & \text{if } \tan t \tan s < 1, \\ -2m & \text{if } \tan t \tan s > 1 \text{ and } \tan s > 0, \\ 2m & \text{if } \tan t \tan s > 1 \text{ and } \tan s < 0. \end{cases} \tag{4.3}$$

It follows by the standard stationary phase method that $H(y,\xi)$ is given by

$$\frac{i^{-n_1-n_2} e^{i\pi \operatorname{sgn}(A)/4}}{|\cos(t+s)|^{m/2}} e^{i(-\psi(t,z_c,\eta_c)+z_c\zeta_c-\psi(s,y,\zeta_c))/h}(b_0(z_c,\eta_c)c_0(y,\zeta_c) + hd_1(y,\xi)).$$

Notice that $\tan t \tan s < 1$ if and only if $|t+s-(n_1+n_2)\pi| < \pi/2$ and, $\tan t \tan s > 1$ and $\pm \tan s > 0$ if and only if $|t+s-(n_1+n_2 \pm 1)\pi| < \pi/2$ and that

$$-\psi(t,z_c,\eta_c) + z_c\zeta_c - \psi(s,y,\zeta_c) = -\psi(t+s,y,\xi).$$

Moreover, because $\langle y,\xi\rangle \sim \langle y,k\rangle \sim \langle y,\zeta_c\rangle \sim \langle z_c,\zeta_c\rangle$, we have

$$b_0(z_c,\eta_c)c_0(y,\zeta_c) - 1 = (b_0(z_c,\eta_c)-1)c_0(y,\zeta_c) + c_0(y,\zeta_c) - 1 = \mathcal{O}(h^\delta \langle y,\xi\rangle^{-\mu})$$

$\diamond$

5 Completion of the Proof

In what follows we fix a compact set $K \subset \mathbf{R}^m$ and always assume that $x, y \in K$. We apply the method of stationary phase to the integral on the right of (1.5). As the magnitude of the critical point ξ_c of the phase function $\xi \mapsto x\xi - \psi(y,\xi,h)$ is of order $h^{-\nu}$ as was shown in Proposition 3.1, we change the variables $\xi \mapsto h^{-\nu}\xi$ to make $|\xi_c| \sim 1$ in the new scale. Thus, we consider

$$E(x,y,h) = \frac{i^{-n}}{(2\pi)^m h^{(1+\nu)m}} \int e^{i(x\xi - h^\nu \psi(y,h^{-\nu}\xi,h)/h^{1+\nu}} b(y,h^{-\nu}\xi,h)d\xi. \tag{5.1}$$

Set $\Psi(x,y,\xi,h) = x\xi - h^\nu\psi(y,h^{-\nu}\xi,h)$ and denote by $\xi_c = \xi_c(x,y,h)$ the critical point of the function $\xi \mapsto \Psi(x,y,\xi,h)$. By virtue of Proposition 3.1,

$$x = \partial_\xi \psi(y,h^{-\nu}\xi_c,h), \qquad C_K^{-1} \le |\xi_c| \le C_K,$$

$$|\nabla_\xi \Psi(x,y,\xi,h)| \geq \frac{|\xi-\xi_c|}{C_K(1+|\xi-\xi_c|)^\mu}. \tag{5.2}$$

In view of (5.2), we split the integral (5.1) $E(x,y,h) = E_{\leq\varepsilon}(x,y,h)+E_{\geq\varepsilon}(x,y,h)$ by using the cutoff function $\chi_\varepsilon(\xi) = \chi\left(\dfrac{\xi-\xi_c}{\varepsilon}\right)$:

$$E_{\leq\varepsilon}(x,y,h) = \frac{i^{-n}}{(2\pi h)^{(1+\nu)m}}\int e^{i\Psi(x,y,\xi)/h^{1+\nu}}\chi_\varepsilon(\xi)b(y,h^{-\nu}\xi,h)d\xi,$$
$$E_{\geq\varepsilon}(x,y,h) = \frac{i^{-n}}{(2\pi h)^{(1+\nu)m}}\int e^{i\Psi(x,y,\xi)/h^{1+\nu}}(1-\chi_\varepsilon(\xi))b(y,h^{-\nu}\xi,h)d\xi,$$

where $\chi \in C_0^\infty(\mathbf{R}^m)$ is such that $\chi(\xi)=1$ for $|\xi|<1/2$ and $\chi(\xi)=0$ for $|\xi|>1$.

Lemma 5.1 *Let $\varepsilon>0$. For any $N=0,1,\ldots$, $\partial_x^\alpha\partial_y^\beta E_{\geq\varepsilon}(x,y,h) = \mathcal{O}(h^N)$.*

Proof. We apply integration by parts by using the identitity

$$h^{(1+\nu)N}\left(\frac{\nabla_\xi\Psi}{i|\nabla_\xi\Psi|^2}\cdot\nabla_\xi\right)^N e^{i\Psi/h^{1+\nu}} = e^{i\Psi/h^{1+\nu}}$$

and write in the form

$$E_{\geq\varepsilon}(x,y,h) = \frac{i^{-n}h^{(1+\nu)N}}{(2\pi h)^{(1+\nu)m}}\int e^{i\Psi/h^{1+\nu}}\left(\frac{\nabla_\xi\Psi}{i|\nabla_\xi\Psi|^2}\cdot\nabla_\xi\right)^{\dagger N}(1-\chi_\varepsilon)b(y,h^{-\nu}\xi,h)d\xi$$

where $\dagger$ stands for the real transpose. Since

$$\partial_\xi^\alpha\nabla_\xi\Psi(x,y,\xi) = \mathcal{O}(h^{-\nu|\alpha|}),\quad \partial_\xi^\alpha b(y,h^{-\nu}\xi,h) = \mathcal{O}(h^{-\nu|\alpha|}),\quad |\alpha|\geq 1,$$

we have by virtue of (5.2)

$$\left|\left(\frac{\nabla_\xi\Psi}{i|\nabla_\xi\Psi|^2}\cdot\nabla_\xi\right)^{\dagger N}(1-\chi_\varepsilon(\xi))b(y,h^{-\nu}\xi,h)\right| \leq \frac{C_N h^{-N\nu}}{\langle\xi-\xi_c\rangle^{N(1-\mu)}}$$

and we obtain the lemma for $\alpha=\beta=0$ by letting N large enough. The proof for the derivatives of $E_{\geq\varepsilon}$ is similar. ◇

We deal with $E_{\leq\varepsilon}(x,y,h)$ next. Assume $\varepsilon>0$ is small enough and $|\xi|\geq\dfrac{1}{2C_K}$ for $\xi\in\mathrm{supp}\chi_\varepsilon$. Since $\nabla_\xi\psi(y,\xi) = \tilde{x}(n\pi,y,\xi)$, we have $\mathrm{Hess}_\xi\Psi(x,y,\xi) = -h^{-\nu}(\partial_\xi\tilde{x})(n\pi,y,h^{-\nu}\xi)$ and by Proposition 2.1 the right hand side can be written as

$$(-1)^{n+1}h^{-\nu+\delta}\int_0^{n\pi}\sin^2 s\,\partial_\xi^2 W(y\cos s+(-1)^n h^{-\nu}\xi\sin s)ds + h^{2\delta-\nu}\mathcal{O}(\langle h^{-\nu}\xi\rangle^{-2\mu}). \tag{5.3}$$

It follows by an estimate similar to the one used in the proof of Lemma 4.2 that the symmetric matrix given by the integral (5.3) is larger than $Ch^{\mu\nu}$ on the support of χ_ε. Thus we have for $x,y\in K$ and $\xi\in\mathrm{supp}\ \chi_\varepsilon$:

$$0 < C_1 \leq (-1)^{n+1}\mathrm{Hess}_\xi\Psi(x,y,\xi) \leq C_2 < \infty. \tag{5.4}$$

Moreover, by virtue of the second statement of Proposition 2.1, we have for $x, y \in K$ and $\xi \in \operatorname{supp} \chi_\varepsilon$:

$$\partial_y^\alpha \partial_\xi^\beta \Psi = \mathcal{O}(h^{-\nu(|\beta|+1)}(h^\delta |h^{-\nu}\xi|^{-\mu})) = \mathcal{O}(1). \tag{5.5}$$

By Taylors formula we have

$$\Psi(x,y,\xi) = \Psi(x,y,\xi_c) + (\xi - \xi_c, B(x,y,\xi)(\xi-\xi_c))/2,$$

$$B(x,y,\xi) = 2\int_0^1 (1-\theta)\mathrm{Hess}_\xi \Psi(x,y,\theta\xi + (1-\theta)\xi_c)d\theta.$$

It is obvious from (5.4) that for $x, y \in K$ and $\xi \in \operatorname{supp} \chi_\varepsilon$,

$$0 < C_1 \le (-1)^{n+1}B(x,y,\xi) \le C_2 < \infty. \tag{5.6}$$

Set $M(x,y,\xi) = ((-1)^n B(x,y,\xi))^{1/2}$ and define $\eta = M(x,y,\xi)(\xi - \xi_c)$. Then

$$\partial_\xi \eta = M(x,y,\xi) + (\partial_\xi M(x,y,\xi))(\xi - \xi_c)$$

and, if we replace $\varepsilon > 0$ by a smaller one if necessary, we see from (5.6) and (5.5) that the map $\xi \mapsto \eta$ is a diffeomorphism on the ball $\{\xi : |\xi - \xi_x| < 2\varepsilon\}$ to its image with uniformly bounded derivatives with its inverse map. We change the variables in the integral for $E_{\le\varepsilon}(x,y,h)$ from ξ to η:

$$E_{\le\varepsilon}(x,y,h) = \frac{i^{-n}e^{i\Psi(x,y,\xi_c)/h^{1+\nu}}}{(2\pi)^m h^{(1+\nu)m}} \times$$
$$\int e^{i(-1)^{n+1}\eta^2/h^{1+\nu}}\chi_\varepsilon(\xi)b(y,h^{-\nu}\xi,h)\left(\det\frac{\partial\eta}{\partial\xi}\right)^{-1} d\eta$$

where $\xi = \xi(x,y,\eta)$ is the inverse of $\xi \mapsto \eta(x,y,\xi)$. Since $1+\nu > 2\nu$ by our assumption, we can apply the extended form of stationary phase and, in virtue of Proposition 4.1,

$$E_{\le\varepsilon}(x,y,h) = \frac{i^{-n}e^{i\Psi(x,y,\xi_c)/h^{1+\nu}+i\pi(-1)^{n+1}m/4}}{(2\pi)^{m/2}h^{(1+\nu)m/2}} \times$$
$$\left|\det\frac{\partial\tilde{x}}{\partial\xi}(n\pi,y,h^{-\nu}\xi_c)\right|^{-1/2}(1+\mathcal{O}(h^\mu)+\mathcal{O}(h^{1-\nu})).$$

This concludes the proof of the Theorem. $\diamond$

6 Appendix

For $R > 0$, we write $B_{>R} = \{x \in \mathbf{R}^m : |x| > R\}$, $B_{<R} = \{x \in \mathbf{R}^m : |x| < R\}$ and etc.

Lemma 6.1 *Let F be a smooth map from $\mathbf{R}^m$ to $\mathbf{R}^m$. Suppose that the differential $\partial_x F$ of F satisfies*

$$\|\partial_x F(x) - P(x)\| \le C_3\langle x\rangle^{-2\delta}, \qquad |x| \ge R_0$$

for a positive definite matrix $P(x)$ such that

$$C_1\langle x\rangle^{-\delta} \le P(x) \le C_2\langle x\rangle^{-\delta} \quad |x| \ge R_0$$

for some constants $C_1, C_2, C_3 > 0$ and $0 < \delta < 1$. Then, there exists R_1 such that $F(x)$ is a diffeomorphism from $B_{>R_1}$ onto its image and such that the image $F(B_{>R_1})$ contains the exterior domain $B_{>\rho}$ for some $\rho > 0$.

Proof. Take $R_2 > 0$ large enough such that for a constant $C_4 > 0$,

$$(\partial_x F(x)u, u) \ge C_4\langle x\rangle^{-\delta}\|u\|^2, \quad x \in B_{\ge R_2},\ u \in \mathbf{R}^m.$$

Then $\partial_x F(x)$ is non-singular and $F(x)$ is a local diffeomorphism in $B_{\ge R_2}$. We suppose $R_1 > 10R_2$ and show first that F is one to one on $B_{\ge R_1}$. Let $x, y \in B_{\ge R_1}$ and $x \ne y$. If x and y may be connected by a line segment $L \subset B_{\ge R_2}$, then we have

$$(F(x) - F(y), x - y) = \int_0^1 (\partial_x F(tx + (1-t)y)(x-y), x-y)dt > 0 \tag{6.1}$$

and $F(x) \ne F(y)$. Suppose, therefore, $L \cap B_{<R_2} \ne \emptyset$. Then, we have $(x, y) < -(49/50)|x||y|$ and, therefore $|x-y| \ge (9/10)(|x|+|y|)$. Let $M = \sup\limits_{|x|<R_2} \|\partial_x F(x)\|$ and $I = \{t \in [0,1] : tx + (1-t)y \in B_{<R_2}\}$. It then follows that $|I| \le 2R_2/|x-y| \le 20R_2/(9(|x|+|y|))$ and

$$\left\|\int_I \partial_x F(tx + (1-t)y)dt\right\| \le M|I| \le \frac{20MR_2}{9(|x|+|y|)}.$$

On the other hand, if $t \notin I$, then $(\partial_x F(tx+(1-t)y)u, u) \ge C_4\langle tx + (1-t)y\rangle^{-\delta}$ and, if we write $I_1 = [0,1] \setminus I$, we have for u with $\|u\| = 1$,

$$\begin{aligned}\int_{I_1} (\partial_x F(tx + (1-t)y)u, u)dt &\ge \int_{I_1} C_4\langle tx + (1-t)y\rangle^{-\delta}dt \\ &\ge \int_{I_1} C_4\langle |x| + |y|\rangle^{-\delta}dt \ge C_4\{2^{-\delta}(|x|+|y|)^{-\delta} - |I|\}\end{aligned}$$

Thus, we have, with $u = x - y$,

$$\int_0^1 (\partial_x F(tx + (1-t)y)u, u)dt \ge \left(C_4 2^{-\delta}(|x|+|y|)^{-\delta} - \frac{20R_2(C_4 + M)}{9(|x|+|y|)}\right)\|u\|^2,$$

and F is one to one on $B_{\ge R_1}$ if R_1 is replaced by a larger R_1 if necessary. We then want to show that the image $F(B_{\ge R_1})$ covers a ball $B_{\ge\rho}$ if we take ρ such that $F(B_{\le R_1}) \subset B_{\ge\rho/2}$. The proof follows that of the Hadamard global implicit function theorem given in [F] and goes with continuity argument. Letting $y = R_1\hat{x}$, $\hat{x} = x/|x|$ in (6.1), for $|x| \ge R_1$, we have

$$(F(x) - F(R_1\hat{x}), \hat{x}) = \int_0^1 (\partial_x F(tx + (1-t)R_1\hat{x}) \cdot (x - R_1\hat{x}), \hat{x})dt$$

$\ge C_1\langle x\rangle^{-\delta}(|x|-R_1) and we see that$—F(x)— $\to \infty$ as $|x| \to \infty$. Hence, we can find $y \in B_{\ge\rho}$ such that $y = F(x)$ for some $x \in B_{\ge R_1}$. Take any $y_1 \in B_{\ge\rho}$ and connect y and y_1 by a C^1 curve $\gamma(t)$, $0 \le t \le 1$ in $B_{\ge\rho}$, $\gamma(0) = y$ and $\gamma(1) = y_1$. We show that there exists a curve $\gamma_1(t)$ in $B_{\ge R_1}$ such that $F(\gamma_1(t)) = \gamma(t)$ for $0 \le t \le 1$. Such a curve $\gamma_1(t)$ certainly exists for small $0 \le t < c$ because F is a local diffeomorphism in $B_{\ge R_1}$ and $F(B_{\le R_1}) \subset B_{\rho/2}$. If it exists for $0 \le t < c$, then it also exists for $0 \le t \le c$. Indeed, $\gamma_1'(t) = (\partial_x F(\gamma_1(t)))^{-1}\gamma'(t)$ and, as $|\gamma_1(t)| \le C$ for $0 \le t < c$ (otherwise $\gamma(t)$ would not be bounded), we have $\|\partial_x F(\gamma_1(t))^{-1}\| \le M$ and $|\gamma_1'(t)| \le M|\gamma'(t)|$. Hence $\gamma_1(t)$ is uniformly continuous in $[0, c)$ and and it has a limit $\lim_{t\to c} \gamma_1(t) \equiv \gamma_1(c)$ and $F(\gamma_1(c)) = \gamma(c)$. $\gamma_1(c) \in B_{>R_1}$ is obvious as otherwise $|\gamma(c)| \le \rho/2$ and, again using the local diffeomorphic property of F, we further continue γ_1 beyond c. In this way we can continue $\gamma_1(t)$ upto $[0, 1]$.

$\diamond$

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