Level Statistics for Quantum Hamiltonians –Some Preliminary Ideas toward Mathematical Justification of the Theory of Berry and Tabor

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Abstract

According to Berry and Tabor, the spectrum of a classically integrable quantum Hamiltonian (the regular spectrum) should generically have the same statistical property of the Poisson point process. In particular, the spacings between energy levels obey the exponential distribution, which is the phenomena called "level clustering ". Strict justification of this assertion seems to be a very difficult mathematical problem. In this note, the author shall introduce a weakened notion of level clustering and give a sufficient condition for this in terms of factorial moments of the number of levels in intervals of fixed length. A preliminary result is also stated, which is closely related to the justification of this sufficient condition in the case of regular spectra. All proofs are omitted, and will be published elsewhere.

1 An outline of Berry-Tabor's theory and the definition of clustering and repulsion of energy levels.

Consider a bounded physical system of $d-$degrees of freedom which is described by a classical Hamiltonian $H(p,q)$, $(p,q) \in \mathbf{R}^d \times \mathbf{R}^d$. Let $H(\hbar)$ be its quantization, and $\{E_n(\hbar)\}_{n\geq 1}$ be the energy levels (eigenvalues) of $H(\hbar)$. Here $\hbar$ is the Planck's constant, and we suppose that the spectrum of $H(\hbar)$ is purely discrete.

Now the assertion of Berry and Tabor ([2]) is the following: Suppose the classical dynamical system associated to $H(p,q)$ is completely integrable. Then the spectrum $\{E_n(\hbar)\}_{n\geq 1}$ of $H(\hbar)$ resembles, in the limit of $\hbar \searrow 0$, the Poisson point process. In particular, the level spacing distribution is $e^{-t}dt$ under suitable normalization. This phenomena was called "level clustering "by Berry and Tabor because the large part of spacing distribution is concentrated near zero, which means the statistical tendency of the appearance of closely lying levels. On the other hand, when the classical dynamical system is

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chaotic, the level spacing distribution $p(t)dt$ should satisfy $p(t) \sim Ct^{\gamma}$, $t \searrow 0$ with $\gamma > 0$, which is the phenomena called level repulsion.

In short, Berry and Tabor conjectured that the regular or chaotic properties of a classical dynamical system manifest themselves in the level statistics of its quantization.

In order to give a precise meaning to the level spacing distribution, we need to somehow normalize (unfold) the spectrum of $H(\hbar)$, so that it looks like a typical realization of a stationary point process. In doing this, we follow the idea of Berry and Tabor, thereby accomplishing simultaneously this normalization and the limiting procedure $\hbar \searrow 0$. Let us assume that the specrum $\{E_n(\hbar)\}_{n\geq 1}$ satisfies the following conditions:

(A1) All levels are non-degenerate and $E_n(\hbar) \geq 0$.

(A2) $E_n(\hbar) \searrow 0$ monotonically as $\hbar \searrow 0$.

(A3) For each fixed $E > 0$, there is a constant $\nu(E) > 0$ such that

$$N_{\hbar}(E) \equiv \sharp\{n \geq 1 \mid E_n(\hbar) \leq E\} \sim \nu(E)\hbar^{-d} \ (\hbar \searrow 0) \ . \tag{1}$$

Let us take $E > 0$, which we shall fix throughout. By (A1) and (A2), we can define $\hbar_n = \hbar_n(E)$ as the unique solution of the equation $E_n(\hbar) = E$. Let us define the unfolded level λ_n by $\lambda_n = \nu(E)\hbar_n(E)^{-d}$. Then from (A3), it is easily seen that

$$\sharp\{n \geq 1 \mid \lambda_n \leq L\} \sim L \quad (L \to \infty) \ . \tag{2}$$

Thus the unfolded levels $\{\lambda\}_n$ are asymptotically uniformly distributed.

We introduce the following notation:

$$N(t) = N(t;c) = \sharp\{n \geq 1 \mid \lambda_n \in (t, t+c]\} \ ; \tag{3}$$

$$\pi_k(c;L) = \frac{1}{L}\int_0^L 1_{\{N(t;c)=k\}}dt \ ; \tag{4}$$

$$\mu_k(c;L) = \frac{1}{k!}\frac{1}{L}\int_0^L N(t)\{N(t)-1\}\cdots\{N(t)-k+1\}dt \ ; \tag{5}$$

$$\rho(c;L) = \frac{\sharp\{n \geq 1 \mid \lambda_n \leq L \ , \ \lambda_{n+1} - \lambda_n \leq c\}}{\sharp\{n \geq 1 \mid \lambda_n \leq L\}} \ . \tag{6}$$

Here $c > 0$ and $k = 0, 1, 2, \ldots$.

Definition 1 *We shall say that one has the strict sense level clustering if*

$$\pi_k(c) \equiv \lim_{L\to\infty} \pi_k(c;L) = e^{-c}\frac{c^k}{k!} \tag{7}$$

holds for each $c > 0$ *and* $k \geq 0$ *.*

Obviously, the strict sense level clustering is equivalent to the weak convergence as $L \to \infty$ of the probability distribution $\{\pi_k(c;L)\}_{k=0}^{\infty}$ on $\mathbf{Z}_+$ to the Poisson distribution with mean c . As is well known in probability theory, a sufficient condition for this weak convergence is that the moments of all orders of $\{\pi_k(c;L)\}_{k=0}^{\infty}$ converges to those of the Poisson distribution. Namely we have

Proposition 1 *If one has*

$$\mu_k(c) \equiv \lim_{L\to\infty} \mu_k(c;L) = \frac{c^k}{k!} \tag{8}$$

for each $c > 0$ and $k = 1, 2, \ldots$, then one has the strict sense level clustering.

If one has the strict sense level clustering, then the limiting level spacing distribution exists and is the exponential distribution $e^{-c}dc$. This is a direct consequence of the following more general proposition:

Proposition 2 *If $\pi_0(c) = \lim_{L\to\infty} \pi_0(c;L)$ exists and is differentiable with respect to c , then $\rho(c) = \lim_{L\to\infty} \rho(c;L)$ also exists, and one has*

$$\rho(c) = 1 + \frac{d}{dc}\pi_0(c) \ . \tag{9}$$

It seems to be a difficult problem to prove the strict sense level clustering for any concrete Hamiltonian. Moreover, as was shown in [5] , there is an example of one-dimensional Hamiltonian for which the level statistics, in a slightly different formulation, can be rigorously performed, but the obtained level spacing distribution is different from the exponential distribution. In that case, we have still $\rho'(0+) > 0$, so that we should say that the level clustering is taking place, and if we take the high disorder limit in the system, then the data $\rho(c)$ converges to the distribution function of $e^{-c}dc$. This circumstance has led the present author to the opinion that we should difine the notion of level clustering in much weaker sense, nevertheless retaining physical significance. Thus taking the broadest statistical sense of the clustering and repulsion, we make the following definition:

Definition 2 *We shall say that one has the wide sense level clustering [resp. repulsion] if*

$$\liminf_{c\searrow 0} \frac{\underline{\rho}(c)}{c} > 0 \quad \left[\textit{resp.} \ \limsup_{c\searrow 0} \frac{\bar{\rho}(c)}{c} = 0 \right] \tag{10}$$

holds. Here we have set

$$\underline{\rho}(c) = \liminf_{L\to\infty} \rho(c;L) \quad ; \quad \bar{\rho}(c) = \limsup_{L\to\infty} \rho(c;L) \ . \tag{11}$$

We can prove the following

Proposition 3 *(i) Suppose one has $\mu_1(c) = c$ for any $c > 0$ and*

$$\bar{\mu}_2(c) \equiv \limsup_{L\to\infty} \mu_2(c; L) = o(c^2) \quad (c \searrow 0) \ . \tag{12}$$

Then one has the wide sense level repulsion.
(ii) Suppose one has $\mu_1(c) = c$, $\mu_2(c) = \frac{1}{2}c^2$ for any $c > 0$ and

$$\bar{\mu}_3(c) \equiv \limsup_{L\to\infty} \mu_3(c; L) = o(c^2) \quad (c \searrow 0) \ . \tag{13}$$

Then one has the wide sense level clustering.

2 Level statistics for regular spectra

Suppose that the classical Hamiltonian system associated to $H(p,q)$ is completely integrable. Then one can transform the variable (p,q) into the action–angle variable $(I,\varphi) = (I_1,\dots,I_d;\varphi_1,\dots,\varphi_d)$, and $H(p,q) = H(I)$ depends only on the action variable (see [1]).

Now following Percival ([7]) , Berry and Tabor ([2]), we shall say that $H(\hbar)$ has regular spectrum if its energy levels are (approximately) given by quantizing the action variables $I_1,\dots,I_d$ appearing in $H(I)$. Namely, we suppose that the energy levels of $H(\hbar)$ are

$$E_n(\hbar) = H(\hbar(n + \frac{1}{4}\alpha)) \quad ; \quad n = (n_1,\dots,n_d) \in \mathbf{Z}_+^d \ , \tag{14}$$

where $\alpha \in \mathbf{Z}_+^d$ is the Maslov index. This procedure is called the EBK quantization after the names of Einstein, Brillouin and Keller, and it gives the exact spectrum in such concrete examples as the rectangular billiards and the harmonic oscillators.

Under mild conditions on $H(I)$ (e.g. $H(I)$ being convex, positive with $H(0) = 0$) , regular spectra satisfy the conditions (A1)–(A3) stated in §1 with

$$\nu(E) = \int\cdots\int_{\mathbf{R}_+^d} 1_{\{H(I)\le E\}} dI_1\cdots dI_d \ . \tag{15}$$

Let us apply the level statistics formulated in §1 to our regular spectrum. Note that the suffix n distinguishing the levels is now d-dimensional.

For $x \in \mathbf{R}_+^d \setminus \{0\}$ define $\hbar(x) = \hbar_E(x) \ge 0$ by the equation $H(\hbar(x)\cdot x) = E$, where $E > 0$ will be fixed throughout, and let $\lambda(x) = \nu(E)\hbar_E(x)^{-d}$. Our unfolded levels are then given by $\lambda(n + \frac{1}{4}\alpha)$, $n \in \mathbf{Z}_+^d$. Since $\lambda(\beta x) = \beta^d\lambda(x)$ for $\beta > 0$, we have the equivalence

$$\lambda(n + \frac{1}{4}\alpha) \in (t, t+c] \ \Leftrightarrow n + \frac{1}{4}\alpha \in \Pi_c(t) \ , \tag{16}$$

where we have defined

$$\Pi_c(t) = \{x \in \mathbf{R}_+^d \mid t^{1/d}\lambda(\varphi(x))^{-1/d} < |x| < (t+c)^{1/d}\lambda(\varphi(x))^{-1/d}\} \ , \tag{17}$$

with

$$\varphi(x) = \frac{x}{|x|} \ . \tag{18}$$

It is easy to see $|\Pi_c(t)| = c$, where $|\Pi_c(t)|$ is the volume of $\Pi_c(t)$.

Hence the number $N(t) = N(t;c)$ of unfolded levels is equal to the number of lattice points (which are shifted by $\frac{1}{4}\alpha$) in the domain $\Pi_c(t)$, namely we have

$$N(t) = \sharp\{\Pi_c(t) \cap (\mathbf{Z}_+^d + \frac{1}{4}\alpha)\} . \tag{19}$$

¿From this geometric consideration and $|\Pi_c(t)| = c$, it is obvious that we always have $\mu_1(c) = c$. It is, however, a non-trivial question to prove or disprove $\mu_k(c) = c^k/k!$ for $k \geq 2$. Sarnak ([8]) proved $\mu_2(c) = c^2/2$ by a number theoretic method for the quantized uniform motion on a two dimensional flat torus. (Actually he proved that the pair correlation function is proportional to c , which turns out to be equivalent to $\mu_2(c) = c^2/2$ after a suitable normalization.) On the other hand, Sinai [9] and Major [4] (see also [6]) considered the case in which $d = 2$, $\alpha = 0$ and the curve (written in polar coordinate) $r = f(\varphi) = \lambda(\varphi)^{-1/d}$ which defines the boundary of $\Pi_c(t)$ is very random. Especially, Major proved that $\mu_k(c) = c^k/k!$, $k \geq 1$ holds for almost all realization of the random curve $r = f(\varphi)$.

But the randomness they assume is so strong that the curve $r = f(\varphi)$ consisting the boundary of $\Pi_c(t)$ cannot be smooth, which violates the natural connection between level statistics and the lattice points counting. On the other hand, their proof suggests us that it would be possible to prove $\mu_k(c) = c^k/k!$ for $k = 1, \ldots, K$ and for any finite K , if the boundary of $\Pi_c(t)$ has "n-dimensional randomness "with n sufficiently large. (Presumably $n \geq 2K$ would suffice.)

For example, consider the d-dimensional billiard of a particle of mass 1 in the rectangle $\Lambda = \Pi_{j=1}^d [0, a_j]$. Then its classical Hamiltonian is $H(I) = \frac{\pi^2}{2}\sum_{j=1}^d (I_j/a_j)^2$, if expressed in terms of action variables, and its quantization is $H(\hbar) = -\frac{\hbar^2}{2}\Delta$ with the Dirichlet boundary condition on $\partial\Lambda$. Its energy levels (eigenvalues) are exactly given by

$$E_n(\hbar) = H(\hbar n) = \frac{\pi^2\hbar^2}{2}\sum_{j=1}^{d}(\frac{n_j}{a_j})^2 \quad (n = (n_1, \ldots, n_d)\ ,\ n_j \geq 1) . \tag{20}$$

We can then conjecture that if $(a_1 \ldots, a_d)$ is a d-dimensional random variable with smooth density, then for sufficiently large d , one has $\mu_k(c) = c^k/k!$ for $k = 2$ and 3 for almost all $(a_1 \ldots, a_d)$, yielding the wide sense level clustering according to Proposition 3.

3 A preliminary result

Let $d = 2$ in the rectangular billiard of §2. In this case we have

$$\Pi_c(t) = \{x \in \mathbf{R}_+^2 \mid t < \frac{\pi}{4}a_1a_2\sum_{j=1}^{2}(\frac{x_j}{a_j})^2 < t + c\} . \tag{21}$$

Since this depends only on the ratio a_1/a_2 , we consider

$$\tilde{\Pi}_c(t) = \{x \in \mathbf{R}_+^2 \mid t < \sum_{j=1}^{2}(\frac{x_j}{a_j})^2 < t + c\} \tag{22}$$

instead of $\Pi_c(t)$ in order to have two-dimensional randomness.

Let $\xi(t) = \xi(t; a_1, a_2)$ be the number of lattice points in $\tilde{\Pi}_c(t)$, namely $\xi(t) = \sharp(\tilde{\Pi}_c(t) \cap \mathbf{Z}_+^2)$. Finally let (a_1, a_2) be a two-dimensional random variable whose distribution has a density $p(a_1, a_2)$ which is C^1 and has a compact support in $(0, \infty)^2$. Then by a direct computation, we can prove the following

Theorem *We have*

$$\lim_{t\to\infty} E[\xi(t)] = E[\lambda_c] \; ; \tag{23}$$

and

$$\lim_{t\to\infty} E[\xi(t)(\xi(t)-1)] = E[\lambda_c^2] \; , \tag{24}$$

where $E[\cdot]$ means the integration with respect to $p(a_1, a_2)da_1da_2$, and

$$\lambda_c \equiv \frac{\pi c}{4} a_1 a_2 = |\tilde{\Pi}_c(t)| \; , \tag{25}$$

which is the area of $\tilde{\Pi}_c(t)$ independently of t .

This result is consistent with that of Cheng, Lebowitz and Major ([3]) , in which they computed the second moment of the number of lattice points in the domain which has a shape similar to our $\Pi_c(t)$ with nonrandom boundary but has randomly shited center.

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