On the Hydrodynamic Equilibrium of a Rod in a Lattice Fluid

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Abstract

We model the behavior of a big (Brazil) nut in a medium of smaller nuts with a stochastic asymmetric simple exclusion dynamics of a polymer-monomer lattice system. The polymer or 'rod' can move up or down in an external negative field, occupying N horizontal lattice sites where the monomers cannot enter. The monomers (at most one per site) or 'fluid particles' are moving symmetrically in the horizontal plane and asymmetrically in the vertical direction, also with a negative field. For a fixed position of the rod, this lattice fluid is in equilibrium with a vertical height profile reversible for the monomers' motion. Upon 'shaking' (speeding up the monomers) the motion of the 'rod' dynamically decouples from that of the monomers resulting in a reversible random walk for the rod around an average height proportional to $\log N$.

Keywords: Brazil nuts, driven diffusion, hydrodynamic equilibrium.

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1 Introduction.

Studying the coupled dynamics of granular matter of different shapes and sizes is of great interest for a range of phenomena. One example is the size segregation of particles as a result of vibrations. A typical realization is a can with nuts; upon shaking the larger (often taken to be Brazil) nuts rise to the top. Because of its wide interest the phenomenon has been considered and reconsidered and while some of the aspects are well-understood not everything has stopped surprising. If one asks for an analysis starting at the microscopic level the situation is not so satisfactory and even simple models have escaped serious mathematical handling (cf. [1], [2] for further references).

In this paper we consider such a microscopic — *albeit* stochastic — dynamics for the motion of a large particle or rod in a lattice fluid composed of monomers. The problem of the present paper is however not quite similar to the canonical Brazil nuts scenario as we are interested in the equilibrium dynamics. In fact, as we will see, on the time scale of the motion of the rod, the monomers are in equilibrium for a reversible density profile. The rod then finds its hydrodynamic equilibrium at a vertical height where the density of the fluid is about equal to its own density. Going beyond equilibrium conditions, e.g. starting from a homogeneous density for the lattice fluid, gives rise to additional mathematical problems that we will only be touching at the end of the paper (see Section 4, Remark 2), and which will be the subject of future work.
The result of this paper can be classified under the heading: how to obtain a Markovian reduced dynamics? This problem is of course a very common one in nonequilibrium statistical mechanics where one considers the system composed of various types of degrees of freedom. The dynamics is globally defined in which the various degrees of freedom are coupled. In some circumstances and under some limit procedures one then expects that some degrees of freedom of the system effectively decouple giving rise to an autonomous (in many cases, Markovian) dynamics for a subset of degrees of freedom. In our case, it is the shaking, the speeding up of the monomer dynamics in the horizontal direction, that does the job. In this way, between any two moves of the polymer, the monomer configuration has the time to relax to its reversible measure and the polymer always sees the fluid in equilibrium.

Our main result is a mathematically rigorous proof of this dynamical decoupling between the motion of the rod and the monomer fluid when the monomer dynamics is (infinitely) speeded up (at least) in the horizontal direction (orthogonal to the motion of the polymer). In that limit of excessive horizontal shaking the reduced dynamics of the polymer becomes that of a random walker with rates directly given in terms of the equilibrium fluid density. When N (the length of the polymer) is sufficiently big (depending on the rates for jumping up or down) the polymer finds its most probable height around its mean position of order $\log N$ with a variance of order 1. In the next section we describe the model and the result. The third section is devoted to the proofs. The final section contains an open problem and some additional remarks.

2 Model and Results.

2.1 Model.

2.1.1 Configuration.

For convenience we put the system on the square lattice $\mathbb{Z}^2$. A point $i = (x, y)$ of the lattice has a 'vertical' coordinate y and a 'horizontal' coordinate x. We also write $i = (i_1, i_2)$ if, in the notation of the coordinates, we want to remember the site i.
The system contains a rigid polymer (large particle, rod) whose position at time t is denoted by Y_t. For simplicity we allow the rod to move only vertically. The horizontal coordinate is fixed (at 0) and Y_t takes values in $\mathbb{Z}$ (thought of as the 'vertical' axis). The same results would hold if the polymer also jumps horizontally at rate 1. The polymer occupies $N \in \{2, 3, \ldots\}$ lattice sites. If the polymer has position $Y_t = y$, then it occupies the region

$$A_N(y) = \{(0, y), (1, y), \ldots, (N-1, y)\}\,.$$

This region is forbidden for the monomers (fluid particles). The monomer configuration is denoted by $\eta \in \{0, 1\}^{\mathbb{Z}^2}$ and we use η_t to denote the random field of monomers at time t. We have that $\eta_t(i) = 0$ if there is no monomer at site i at time t; $\eta_t(i) = 1$ if there is a monomer at site i at time t. The dynamics will always be subject to the restriction (exclusion) that $\eta_t(i) = 0$ for $i \in A_N(Y_t)$ (the rod acts as an obstacle for the fluid motion). The full configuration space is denoted by $\Omega = \{0, 1\}^{\mathbb{Z}^2} \times \mathbb{Z}$.

2.1.2 Dynamics.

We now define the coupled dynamics for the polymer-monomers system. All motion is via jumping to vacant sites. There are the horizontal jumps of the monomers (which we take symmetric and at rate γ_1), the vertical jumps of the monomers (asymmetric at rate γ_2) and the vertical jumps of the rod (asymmetric at rate 1). The asymmetry in the vertical direction models the presence of an external (e.g. gravitational) field acting on fluid matter and polymer but can in general be different for monomers and polymer. Increasing the rates γ_1 and γ_2 can be used to simulate the greater mobility of the smaller particles upon shaking. We are most interested in the case where $\gamma_2 \simeq 1$ and $\gamma_1 \gg 1$ (horizontal shaking).

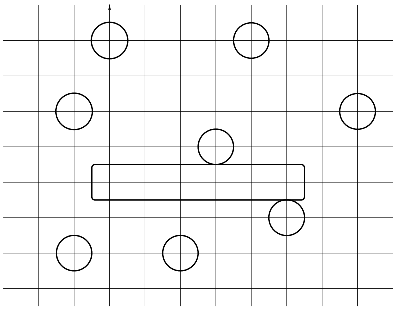

Fig.1: A polymer between monomers

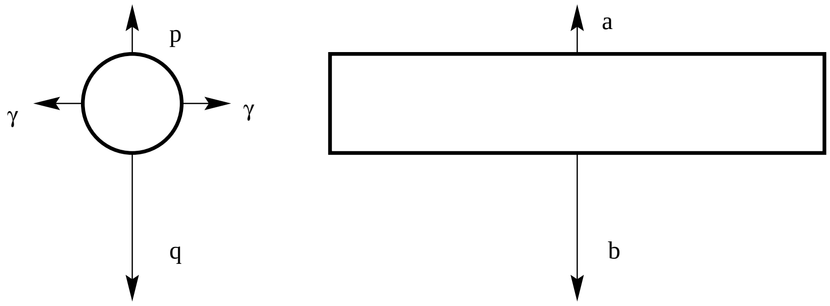

Fig.2: Rates of jumping for monomer and polymer where $\gamma_2 = \gamma$ and $\gamma_1 = 1$.

Here comes the formal definition of the generators of these motions. Let f be a local function on Ω (i.e., a function that depends on the configuration in a finite region of $\mathbb{Z}^2$). The first part of the generator represents horizontal monomer-jumping:

$$\mathcal{L}_{\mathrm{h}}(\eta, y) = \frac{1}{2} \sum_{\langle ij \rangle : i_2 = j_2} I[\langle ij \rangle \cap A_N(y) = \emptyset][f(\eta^{i,j}, y) - f(\eta, y)] \tag{2.1}$$

where $\eta^{i,j}(k) = \eta(i)$ if $k = j, = \eta(j)$ if $k = i$ and $= \eta(k)$ otherwise; the summation is over nearest neighbor pairs $\langle ij \rangle$ with the same vertical coordinate $(i_2 = j_2)$. The notation $I[\cdot]$ will always stand for the indicator function.
Second comes the vertical monomer-jumping: with $p < q$,

$$\begin{aligned} \mathcal{L}_{\mathrm{v}} f(\eta, y) \quad = \quad & \sum_i \Big\{ p\, \eta(i)\, (1 - \eta(i_1, i_2 + 1))\, I[(i_1, i_2 + 1) \notin A_N(y)] \\ & \qquad \times [f(\eta^{i,(i_1,i_2+1)}, y) - f(\eta, y)] \\ & + q\, \eta(i)\, (1 - \eta(i_1, i_2 - 1))\, I[(i_1, i_2 - 1) \notin A_N(y)] \\ & \qquad \times [f(\eta^{i,(i_1,i_2-1)}, y) - f(\eta, y)] \Big\}. \end{aligned} \tag{2.2}$$

Finally, there is the polymer-jumping: with $a, b \in \mathbb{R}^+$,

$$\begin{aligned} \mathcal{L}_{\mathrm{poly}} f(\eta, y) \quad = \quad & a\, I[\eta(i) = 0, \forall i \in A_N(y + 1)][f(\eta, y + 1) - f(\eta, y)] \\ & + b\, I[\eta(i) = 0, \forall i \in A_N(y - 1)]\, [f(\eta, y - 1) - f(\eta, y)] \end{aligned} \tag{2.3}$$

We will then choose $p/q, a/b < 1$ to represent an external field in the vertical direction driving all particles, big and small, downward. E.g. in the case of a gravitational field, we could have $p/q = \exp(-mg/kT), a/b = \exp(-Mg/kT)$, where m, M denotes the mass of a monomer, resp. polymer.
The formal generator $\mathcal{L}$ of the full dynamics consists of three pieces:

$$\mathcal{L} = \gamma_1 \mathcal{L}_{\mathrm{h}} + \gamma_2 \mathcal{L}_{\mathrm{v}} + \mathcal{L}_{\mathrm{poly}}, \tag{2.4}$$

where $\gamma_1, \gamma_2 > 0$ are additional parameters governing the rates of the monomer-jumping. Notice that $\mathcal{L}_{\mathrm{mono}} = \gamma_1 \mathcal{L}_{\mathrm{h}} + \gamma_2 \mathcal{L}_{\mathrm{v}}$ works on the configuration of monomers only (for fixed rod position), while $\mathcal{L}_{\mathrm{poly}}$ works on the polymer configuration (for fixed monomers). The only interaction is by excluded volume. The generator (2.4) can be rewritten in the form:

$$\mathcal{L} f(\eta, y) = \mathcal{L}_\eta f_\eta(y) + \mathcal{L}_y f_y(\eta) \tag{2.5}$$

where $f_\eta(\cdot) = f(\eta, \cdot)$ and $f_y(\cdot) = f(\cdot, y)$.

2.1.3 Initial distribution and extra remarks.

At time 0 (starting time) we put the polymer at the origin: $Y_{t=0} = 0$. Then fix a real parameter κ and distribute the monomers independently with density

$$\rho(i) = \frac{\kappa(p/q)^{i_2}}{1 + \kappa(p/q)^{i_2}} \tag{2.6}$$

varying in the vertical direction (constant in the horizontal direction), conditioned on $\eta(i) = 0, \forall i \in A_N(0)$.
More precisely, we let ν_ρ denote the product measure on $\{0,1\}^{\mathbb{Z}^2}$ with density

$$\nu_\rho(\eta(i)) = \rho(i), \tag{2.7}$$

defined by (2.6). This measure is reversible for each of the monomer generators process without polymer —i.e. the generators defined by (2.1) and (2.2) but without the indicator functions prohibiting jumps. The proof of this fact is a simple computation. The one-dimensional analogue is well known, see [8].

For any given $y \in \mathbb{Z}$, we write

$$\nu_\rho^y = \nu_\rho(\cdot | \eta(i) = 0, \forall i \in A_N(y)), \tag{2.8}$$

At time 0, we put the distribution μ_0 on Ω defined by

$$\mu_0(d\eta, y) = \delta_{y,0}\, \nu_\rho^y(d\eta), \tag{2.9}$$

where $\delta_{y,0}$ stands for the Kronecker-delta.

From the initial condition described above and the dynamics defined via (2.4) the process (η_t, Y_t) is generated. The measure at time $t \geq 0$ is denoted by μ_t. Of course this depends on the choice of parameters p, q, a, b, γ_1 and γ_2 and we will sometimes make this explicit in the notation.
A useful way to imagine the process is by associating two exponential clocks

(at rate a respectively b) to the polymer: one clock gives rise to the trial times for the polymer to jump up, the other indicates the trials for the polymer to jump down. If, just before the trial time τ, say for jumping up, there are no monomers right above the polymer ($\eta_{\tau^-}(i) = 0, \forall i \in A_N(Y_{\tau^-} + 1)$), then the jump is performed and at time τ the polymer is at height $Y_\tau = Y_{\tau^-} + 1$, otherwise it stays where it was. Between the trial times of the polymer, only the monomers move. The dynamics for the monomers for a fixed position of the polymer (say at y) is generated by

$$\mathcal{L}^y_{\text{mono}} f(\eta) = \gamma_1 \mathcal{L}^y_{\text{h}} + \gamma_2 \mathcal{L}^y_{\text{v}} \tag{2.10}$$

which can be read off from (2.1) and (2.2). The associated semigroup is denoted by $S^y(t)$. Now, the important thing where the 'equilibrium' in the title of this paper refers to, is that ν^y_ρ is a reversible measure for $S^y(t)$. This will be proven as Lemma 3.1 in Section 3.

2.2 Results.

2.2.1 Limiting random walk.

In the limit $\gamma_1 \uparrow +\infty$ the motion of the rod will decouple from the monomer dynamics. It will be a random walk. We first introduce this limiting rod motion. For $a, b \in \mathbb{R}^+$ consider the continuous time random walk on $\mathbb{Z}$ with generator

$$\mathcal{L}^{\text{RW}} f(y) = a[1-\rho(y+1)]^N[f(y+1) - f(y)] + b[1-\rho(y-1)]^N[f(y-1) - f(y)], \tag{2.11}$$

where the density profile ρ is obtained from (2.6). Remark that, in the notation of (2.5),

$$\mathcal{L}^{\text{RW}} f(y) = \int \nu^y_\rho(d\eta) \mathcal{L}^y_\eta f(y) \tag{2.12}$$

$\mathcal{L}^{\text{RW}}$ generates a continuous time random walk Y^{RW}_t which we start at $Y^{\text{RW}}_{t=0} = 0$ and with rate for moving one step upward $a[1-\rho(y+1)]^N$ and rate moving one step downward equal to $b[1-\rho(y-1)]^N$. We fix the initial state to be 0 for the sake of definiteness. Our results hold for any other initial (deterministic or random) state.

Proposition 2.1 *If $a/b > (p/q)^N$, then the random walk with generator (2.11) defined above has a unique reversible probability measure m on $\mathbb{Z}$, which is given by*

$$m(y) = \frac{1}{Z} \frac{(a/b)^y}{(1 + (p/q)^y)^N}, \tag{2.13}$$

where Z is a normalizing constant. In particular, the random walk is positive recurrent.

Proof: Reversibility of $m(y)$ is immediate, and the condition $a/b > (p/q)^N$ guarantees that $m(y)$ can be normalized (i.e. $Z < \infty$). Positive recurrence follows immediately from the existence of a reversible probability measure. ∎

Remark that the condition $a/b > (p/q)^N$ in the case of a gravitational field just means $M/N < m$, i.e. the density of the polymer is smaller than the density of the monomer-fluid. It is thus very natural that in this case the polymer will drift up and will float at a height where the fluid density is proportional to $1/N$, see (2.20) and [3].

In order to study some global properties of the limiting random walk, in particular its behavior for large N, we replace the discrete distribution $m(y)$ on $\mathbb{Z}$ by a continuous distribution:

$$m(dx) := \frac{\exp(-\alpha x)}{(1+\exp(-\beta x))^N} \frac{1}{Z(\alpha,\beta,N)} dx. \tag{2.14}$$

Here

$$Z(\alpha,\beta,N) = \int_{-\infty}^{\infty} dx \frac{\exp(-\alpha x)}{(1+\exp(-\beta x))^N} = \frac{1}{\beta}\frac{\Gamma(\alpha/\beta)\Gamma(N-\alpha/\beta)}{\Gamma(L)}, \tag{2.15}$$

and $e^{-\alpha} = a/b$, $e^{-\beta} = p/q$. From (2.15) we can calculate the cumulants of the continuous distribution $m(dx)$: in particular

$$\int x\, m(dx) = \frac{1}{\beta}\left(\psi(\frac{\alpha}{\beta}) - \psi(L-\frac{\alpha}{\beta})\right), \tag{2.16}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. Using the asymtotic expansion

$$\psi(z) = \log z - \frac{1}{2z} - \frac{1}{12z^2} + \ldots \tag{2.17}$$

we obtain

$$\int x\, m(dx) = \frac{1}{\beta}\log N + O(1), \text{ as } N \to \infty \tag{2.18}$$

and all higher order cumulants are of order 1 as N tends to infinity. The modus of m (the position where $m(x)$ reaches its maximum) is

$$\begin{aligned} \mathrm{Mo}(m) &= (-\log(p/q))^{-1} \log\left(\frac{\log(p/q)^N}{\log(a/b)} - 1\right) \\ &\simeq \frac{1}{\beta}\log N\,, \text{ as } N \to \infty. \end{aligned} \tag{2.19}$$

2.2.2 Main result.

Our main result states

Theorem 2.1 *Let $0 \le p < q < \infty$ and $a, b \in \mathbb{R}^+$ and consider the joint monomer-polymer process with generator (2.4). For any finite time-interval K, the marginal law of the polymer motion $(Y_t^{\gamma_1} : t \in K)$ converges, as $\gamma_1 \to \infty$, to the law of the random walk $(Y_t^{\mathrm{RW}} : t \in K)$ defined by (2.11).*

2.2.3 Discussion.

Since for $a/b > (p/q)^N$ the limiting motion is an ergodic random walk in a countable state space, the process starting from any initial distribution will converge to the (unique) invariant measure. Hence, by (2.18), the polymer will rise from the zero level to a level at height proportional to $\frac{1}{\beta}\log N$. If it starts in equilibrium, then it will perform a random walk around this position. This is exactly what we would expect from general hydrodynamics, see [3]. After all, the fluid density at height $\frac{1}{\beta}\log N$ is precisely, cf (2.6):

$$\rho(\frac{1}{\beta}\log N) = \frac{\kappa/N}{1+\kappa/N} \sim \frac{\kappa}{N} \tag{2.20}$$

confirming Archimedes' law in this model of granular matter.

3 Proofs.

We start this section with an easy lemma on reversible Markov processes.

Lemma 3.1 *Let $\{\eta_t : t \geq 0\}$ be a Markov process on Ω with generator $\mathcal{L}$ and let μ be a reversible measure for $\mathcal{L}$. Suppose $A \subset \Omega$ such that $\mu(A) > 0$ and such that 1_A is in the domain of the generator. Consider the process with generator*

$$\mathcal{L}_A f = 1_A \mathcal{L}(1_A f) - (1_A \mathcal{L} 1_A) f \tag{3.21}$$

That is, $\mathcal{L}_A$ corresponds to a process with "forbidden region" A^c (i.e., jumps from A to A^c are suppressed). Then the measure $\mu_A := \mu(\cdot|A)$ is reversible for $\mathcal{L}_A$.

Proof: Because the second term in the right hand side of (3.21) is just multiplication with the function $1_A\mathcal{L}(1_A)$, it suffices to show that $\tilde{\mathcal{L}}_A f := 1_A\mathcal{L}(1_A f)$ defines a symmetric operator on $L^2(A, \mu_A)$. Let f, g be in the domain of $\tilde{\mathcal{L}}_A$. Since $d\mu_A = (1/\mu(A))1_A\, d\mu$, we get, using the symmetry of $\mathcal{L}$ in $L^2(\mu)$:

$$\begin{aligned} \int g(\tilde{\mathcal{L}}_A f) d\mu_A &= \frac{1}{\mu(A)} \int 1_A\, g \tilde{\mathcal{L}}_A f\, d\mu \\ &= \frac{1}{\mu(A)} \int \mathcal{L}(g 1_A) 1_A f\, d\mu \\ &= \int \tilde{\mathcal{L}}_A g f\, d\mu_A. \end{aligned} \tag{3.22}$$

■

Note that reversibility is crucial in the proof of this lemma. Indeed if μ is only stationary, then we cannot conclude in general that μ_A will be stationary for the process with generator $\mathcal{L}_A$. Indeed, one easily computes

$$\int \mathcal{L}_A f d\mu_A = \frac{1}{\mu(A)} \int (1_A \mathcal{L}^* 1_A - 1_A \mathcal{L} 1_A) f d\mu, \tag{3.23}$$

i.e., μ_A will be stationary iff $1_A\mathcal{L}^*1_A - 1_A\mathcal{L}1_A = 0$ μ-a.s. Since the profile measures are reversible for the exclusion process of the monomers without polymer, we can apply lemma 3.1 for $\mu = \nu_\rho$, $A = \{\eta \in \{0,1\}^{\mathbb{Z}^2} : \sum_{z\in A(y,N)} \eta(z) \neq 0\}$, i.e., those monomer configurations which are excluded when the polymer is at vertical position y. This yields:

Corollary 3.1 *For fixed polymer position at $y \in \mathbb{Z}$ the measure ν_ρ^y is reversible for the monomer dynamics with semigroup $S^y(t)$.*

Lemma 3.2 *Fix $y \in \mathbb{Z}$. Let f be a local function on $\{0,1\}^{\mathbb{Z}^2}$ which only depends on the monomer configuration in the layers at height $y+1$ and $y-1$. Suppose that*

$$\nu_\rho^y(f) = 0.$$

Then, for any $t > 0$,

$$\lim_{\gamma_1 \uparrow +\infty} \|S^y(t)f\|_{L^2(\nu_\rho^y)} = 0.$$

Proof: Abbreviate $\mu := \nu_\rho^y$ and consider the case $f_y(\eta) = 1_A(\eta) - \mu(A)$ for a set A in the space of configurations depending only on a finite number of coordinates in labels $y-1$ and $y+1$. The extension to general local f is straightforward. Denote

$$D_A := \{x \in \mathbb{Z}^2 : 1_A(\eta) \neq 1_A(\eta^x) \text{ for some } \eta\}, \tag{3.24}$$

the dependence set of A. By reversibility:

$$\begin{aligned} \int (S^y(t)f_y)^2 d\mu &= \int (S^y(2t)f_y) f_y d\mu. \\ &= \mu(A)\left(\mathbb{E}^y_{\mu(\cdot|A)}(1_A(\eta_t)) - \mathbb{E}^y_\mu(1_A(\eta_t))\right). \end{aligned} \tag{3.25}$$

To compute the difference of the expectations in the above expression we realize the processes with initial configurations η and ζ in the same probability space (coupling).

To construct this coupling we first associate two Poisson clocks to each site of $\mathbb{Z}$ with parameters $\gamma_2 p$ and $\gamma_2 q$ respectively and use them to decide the times of the vertical attempted jumps. A jump from (i_1, i_2) to (i_1, i_2+1) is performed at time t if an event of the Poisson process of rate p occurs at that time, a particle is present at (i_1, i_2) and no particle is present at (i_1, i_2+1) at time $t-$. Similarly, a jump from (i_1, i_2) to (i_1, i_2-1) is performed at time t if an event of the Poisson process of rate q occurs at that time, a particle is present at (i_1, i_2) and no particle is present at (i_1, i_2-1) at time $t-$. Jumps either to or from sites occupied by the rod are suppressed. This takes care of the vertical jumps. See Ferrari (1992) for details of this construction. For the horizontal jumps we associate Poisson clocks with rate γ_1 to pairs of horizontal nearest-neighbor sites. When the clock associated with sites (i_1, i_2) and (i_1+1, i_2) rings, the contents of those sites are interchanged. Also here, if at least one of the sites is occupied by the rod, the jump is suppressed. The horizontal motion is also called *stirring* process. See Arratia (1986) for details of this construction. More rigourosly, let $(N_t(i,j) : i = (i_1, i_2) \in \mathbb{Z}, j = (i_1+1, i_2))$, $(N_t^+(i) : i \in \mathbb{Z})$ and

$(N_t^-(i) : i \in \mathbb{Z})$ three independent families of independent Poisson processes of rates $\gamma_1/2$, $p\gamma_2$ and $q\gamma_2$, respectively —the Poisson clocks. Use the notation $\mathrm{d}N_t(\cdot) = 1$ if there is an event of the Poisson process $(N_t(\cdot))$ at time t, otherwise it is zero. The motion is defined by

$$\begin{aligned}
\mathrm{d}f(\eta_t) \quad = \quad & \sum_{\langle ij\rangle : i_2=j_2} \mathrm{d}N_t(i,j)\, I[\langle ij\rangle \cap A_N(y) = \emptyset]\,[f(\eta^{i,j}, y) - f(\eta, y)] \\
& + \sum_i \Big\{ \mathrm{d}N_t^+(i)\, \eta(i)\,(1 - \eta(i_1, i_2+1))\, I[(i_1, i_2+1) \notin A_N(y)] \\
& \qquad \times [f(\eta^{i,(i_1,i_2+1)}, y) - f(\eta, y)] \\
& + \mathrm{d}N_t^-(i)\, \eta(i)\,(1 - \eta(i_1, i_2-1))\, I[(i_1, i_2-1) \notin A_N(y)] \\
& \qquad \times [f(\eta^{i,(i_1,i_2-1)}, y) - f(\eta, y)] \Big\}.
\end{aligned} \tag{3.26}$$

Standard arguments, see for instance Durrett (1993) show that (3.26) defines a process $\eta_t = \Phi(\eta_0; N[0,t])$, with initial configuration η_0, where Φ is the function induced by (3.26) and $N[0,t] := (N_s(\cdot,\cdot),\, N_s^+(\cdot),\, N_s^-(\cdot) : 0 \le s \le t)$; furthermore it is immediate to see that η_t has generator $\mathcal{L}_y$. Given two initial configurations η and ζ, the coupling of their evolutions is constructed using the same Poisson processes: define

$$(\eta_t, \zeta_t) := (\Phi(\eta; N[0,t]), \Phi(\zeta; N[0,t])).$$

Let $\mathbb{E}^y_{(\eta,\zeta)}$ denote expectation in the coupling starting with (η, ζ). We need also to couple the initial configurations. Let $\widetilde{\mu}_A$ be the law of a pair of configurations (η, ζ) with marginal distributions μ_A and μ and such that $\eta(x) = \zeta(x)$ for all $x \in \mathbb{Z}^2 \setminus D_A$. It is possible to construct a measure with these properties because μ is a product measure. We then have

$$\mathbb{E}^y_{\mu(\cdot|A)}(1_A(\eta_t)) - \mathbb{E}^y_\mu(1_A(\eta_t)) = \int \widetilde{\mu}_A(d(\eta,\zeta))\, \mathbb{E}_{(\eta,\zeta)}[1_A(\eta_t) - 1_A(\zeta_t)] \tag{3.27}$$

The number of initial discrepancies is finite, that is, $\sum_x I(\eta(x) \ne \zeta(x)) \le |D_A| < \infty$. At each site x of $\mathbb{Z}$ we have one of three possibilities: $(\eta(x) - \zeta(x)) = 0$, no discrepancies; $(\eta(x) - \zeta(x))^+ > 0$, positive discrepancies; or $(\eta(x) - \zeta(x))^- > 0$, negative discrepancies. Following the evolution of the particles and the discrepancies we notice that if a positive discrepancy jumps over a negative one, then both discrepancies collide, giving place to a coupled particle and a hole; if a coupled particle attempts to jump to a discrepancy, the jumps occur and then the discrepancy must jump to the site previously occupied by the coupled particle. These two behaviors only occur when vertical jumps are involved. In the horizontal jumps, discrepancies and coupled particles just interchange positions according to the Poisson horizontal (stirring) clocks.

We say that there is a *first class particles* at site i at time t when $\xi_t(i) = \eta_t(i)\zeta_t(i) = 1$, a *positive second class particles* when $(\eta\zeta)_t(i) = \eta_t(i) - \zeta_t(i) = 1$ and a *negative second class particles* when $(\zeta\eta)_t(i) = \zeta_t(i) - \eta_t(i) = 1$. The first class particles occupy initially those sites i occupied by both η and ζ. Locally in time, the motion of the first class particles is the one given by generator $\mathcal{L}^y$

but superposed to it there is a pure birth process of first class particles: with rate

$$p\,(\eta\zeta)_t(i_1,i_2-1)\,(\zeta\eta)_t(i_1,i_2)$$

the second class particles at (i_1,i_2-1) and (i_1,i_2) annihilate each other and a first class particle appears at (i_1,i_2) and an empty site appears at (i_1,i_2-1). Similarly, at rate

$$q\,(\eta\zeta)_t(i_1,i_2+1)\,(\zeta\eta)_t(i_1,i_2)$$

the second class particles at (i_1,i_2+1) and (i_1,i_2) annihilate each other and a first class particle appears at (i_1,i_2) and an empty site appears at (i_1,i_2-1).

The marginal distribution of a second class particle between two vertical jumps (or between a jump and an annihilation) corresponds to the law of a nearest neighbor symmetric random walk —with reflection at the rod when at level y— in the horizontal direction. In the vertical direction the motion is not Markovian —it depends on the configuration of the first and second class particles at the instants of attempted jumps— and either there is an annihilation as described above or the second class particles just change horizontal line. For instance, at time t, jumps of a $(\eta\zeta)$ second class particle from site (i_1,i_2) to site (i_1,i_2+1) occur with rate

$$p\,(\eta\zeta)_t(i_1,i_2)\,(1-\xi_t(i_1,i_2+1))+q\,(\eta\zeta)_t(i_1,i_2)\,\xi_t(i_1,i_2+1)$$

and similarly for the other cases. The first term corresponds to the jump over an empty site and the second one to the interchange of positions with a first class particle.

This coupling has the property

$$\mathbb{P}^y_{(\eta,\zeta)}\Big\{\sum_x I(\eta(x)\neq\zeta(x))\geq\sum_x I(\eta_t(x)\neq\zeta_t(x))\Big\}=1, \tag{3.28}$$

i.e., the number of discrepancies cannot increase.

Since by construction the discrepancies between η and ζ are all located at D_A, we have the estimate

$$\mathbb{E}^y_{(\eta,\zeta)}[1_A(\eta_t)-1_A(\zeta_t)]\leq\sum_{i\in D_A}\sum_{z\in D_A}\mathbb{P}(X^i(t)=z), \tag{3.29}$$

where $X^i(t)$ is the position of a second class particle initially at i. If at site i there were no discrepancy we use the convention $X^i_t\notin\mathbb{Z}^2$ (and hence $\neq z$, for all $z\in D_A$). If particles i and j were discrepancies of different sign and collided before time t, we also set $X^i_t,X^j_t\notin\mathbb{Z}^2$. The process X^i_t has rate γ_1 to move symmetrically in the horizontal direction. If the rod were not present, we could dominate $\mathbb{P}(X^i(t)=y)$ by $\mathbb{P}((X^i(t))_1=y_1)$, where $(X^i(t))_1$ is the first coordinate of the walk. Since without the rod the first coordinate makes just a symmetric random walk at rate γ_1, that probability would be dominated by $\gamma_1^{-1/2}$ times a constant. But with the rod we have to work a bit more. The process $X^i(t)$ has rate at most $\gamma_2(p+q)$ to move in the vertical direction. This implies that the time elapsed between the last vertical jump and t is dominated by the minimum between an exponential time of rate $\gamma_2(p+q)$ and t. With this in hand it is not difficult to prove that also in this case $\mathbb{P}(X^i(t) = y)$ is bounded above by $\gamma_1^{-1/2}$ times a(nother) constant. Here we use that γ_2 remains bounded when γ_1 goes to infinity. We conclude that for any pair i, y in D_A:

$$\lim_{\gamma_1 \uparrow \infty} \mathbb{P}(X^i(t) = y) \;=\; 0 \tag{3.30}$$

Therefore we conclude, combining (3.25), (3.27), (3.29) and (3.30) and the fact that D_A is a finite set:

$$\lim_{\gamma_1 \uparrow \infty} \int d\mu \, f_y \, S^y(2t) f_y \;=\; 0. \tag{3.31}$$

■

Remark: We postpone until Section 4, Remark 3, an alternative more general proof of Lemma 3.2 which works equally well for a broader class of exclusion dynamics (e.g. with speed change) provided the projection of the invariant measure on horizontal layers is ergodic for the horizontal dynamics.

We now prove an intermediate result which is important for the proof of Theorem 2.1.

Proposition 3.1 *Let f_y be a function depending only on the configuration values at the N sites of $A_N(y-1)$ or $A_N(y+1)$. Then we have for all $t > 0$:*

$$\limsup_{\gamma_1 \uparrow \infty} \mathbb{E}^{(\gamma_1,\gamma_2)}_{\nu^0_\rho \times \delta_0} \left(f_{Y_t}(\eta_t) - \int f_{Y_t}(\eta)\, \nu^{Y_t}_\rho(d\eta) \right) = 0. \tag{3.32}$$

Proof: We first want to condition on a sequence $\underline{T}^\epsilon := (T_1^{\epsilon_1}, \ldots, T_n^{\epsilon_n})$ of marked trial jumps before t. Here $\epsilon \in \{-1,+1\}$ is the mark of the jump: $+1$ for up, -1 for down. Next we consider $\alpha_1, \ldots, \alpha_n \in \{0,1\}$ with interpretation $\alpha_i = 1$ if i-th marked trial jump succeeds, $\alpha_i = 0$ if not. Given $(T_1^{\epsilon_1}, \ldots, T_n^{\epsilon_n})$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$, we define

$$Y_k^\alpha = \sum_{j=1}^k \epsilon_j \alpha_j. \tag{3.33}$$

This corresponds to the position of the polymer at time $T_k^{\epsilon_k}$, given succeeded and failed jumps $(\alpha_1, \ldots, \alpha_k)$. Finally we denote by $V_p^{\alpha,\epsilon} := V^\alpha_{\underline{T}^\epsilon, \alpha_p}(\eta)$ the event that the polymer in Y^α_{p-1} can (for $\alpha_p = 1$) or cannot (for $\alpha_p = 0$) perform the jump to $Y^\alpha_{p-1} + \epsilon_p$. With this notation, we can write

$$\begin{aligned}
&\mathbb{E}^{(\gamma_1,\gamma_2)}_{\nu^0_\rho \times \delta_0} \left(f_{Y_t}(\eta_t) - \int f(\eta) \nu^{Y_t}_\rho(d\eta) \;\middle|\; T_1^{\epsilon_1}, \ldots, T_n^{\epsilon_n} \,;\, T_n^{\epsilon_n} < t < T_{n+1}^{\epsilon_{n+1}} \right) \\
&\quad = \sum_{\alpha \in \{0,1\}^{\{1,\ldots,n\}}} \mathbb{P}^{(\gamma_1,\gamma_2)}_{\nu^0_\rho \times \delta_0}(\alpha) \\
&\qquad \times \int d\mu^{(\gamma_1,\gamma_2)}_{\alpha, Y_n^\alpha} S^{Y_n^\alpha}_{(\gamma_1,\gamma_2)}(t - T_n^{\epsilon_n}) \left(f_{Y_n^\alpha} - \int f_{Y_n^\alpha}(\eta) \nu^{Y_n^\alpha}_\rho(d\eta) \right).
\end{aligned} \tag{3.34}$$

Here $\mu^{(\gamma_1,\gamma_2)}_{\alpha,Y^\alpha_n}$ denotes the monomer distribution at time $s = (T^{\epsilon_n}_n)^+$, given the successes $(\alpha_1, \ldots, \alpha_n)$, and $\mathbb{P}^{(\gamma_1,\gamma_2)}_{\nu^0_\rho \times \delta_0}(\alpha)$ denotes the probability of the sequence of succeeded and failed jumps prescribed by α at the times $\underline{T}^\epsilon$. The crucial thing to realize at this point is that the probability measure $\mu^{(\gamma_1,\gamma_2)}_{\alpha,Y^\alpha_n}$ is absolutely continuous with respect to the conditioned Bernoulli measure $\nu_\rho^{Y^\alpha_n}$. In Lemma (3.3) below we shall give a uniform bound on the density

$$\Psi^{(\gamma_1,\gamma_2)}_{\alpha,n} := \frac{d\mu^{(\gamma_1,\gamma_2)}_{\alpha,Y^\alpha_n}}{d\nu_\rho^{Y^\alpha_n}}. \tag{3.35}$$

By dominated convergence, the proof of the proposition is reduced to showing that for any $\alpha \in \{0,1\}^{\{1,\ldots,n\}}$ and any $\delta > 0$:

$$\lim_{\gamma_1 \uparrow \infty} \int d\mu^{(\gamma_1,\gamma_2)}_{\alpha,Y^\alpha_n} S^{Y^\alpha_n}_{(\gamma_1,\gamma_2)}(\delta) \left(f_{Y^\alpha_n} - \int f(\eta) \nu_\rho^{Y^\alpha_n}(d\eta) \right) = 0. \tag{3.36}$$

The expression inside the limit in the left hand side of (3.36) is bounded by

$$\|\Psi^{(\gamma_1,\gamma_2)}_{\alpha,n}\|_\infty \left\| S^{Y^\alpha_n}_{(\gamma_1,\gamma_2)}(\delta) \left(f - \int f(\eta)\, \nu_\rho^{Y^\alpha_n}(d\eta) \right) \right\|_{L^2\left(\nu_\rho^{Y^\alpha_n}\right)}. \tag{3.37}$$

Therefore, (3.36) is a consequence of Lemma 3.2 and the following estimate on the density $\Psi^{(\gamma_1,\gamma_2)}_{\alpha,n}$.

Lemma 3.3 *Put $c(\rho, x) := [\rho(x+1) \wedge \rho(x-1) \wedge (1 - \rho(x))]^{-N}$. For any $\alpha \in \{0,1\}^{\mathbb{N}}$ and for any $n \in \mathbb{N}$, we have the estimate:*

$$\limsup_{\gamma_1 \uparrow \infty} \|\Psi^{(\gamma_1,\gamma_2)}_{\alpha,n}\| \le \prod_{p=0}^{n-1} c(\rho, Y^\alpha_p) \tag{3.38}$$

Proof: We fix α and proceed by induction in n. First put $n = 1$. By stationarity of ν^0_ρ under the evolution $S^0_{(\gamma_1,\gamma_2)}$, we have

$$\mu^{(\gamma_1,\gamma_2)}_{\alpha_1,Y^{\alpha_1}_1} = \nu^0_\rho[\,\cdot\,|V^\alpha_1]. \tag{3.39}$$

First consider $\alpha_1 = 1$, i.e., the jump succeeds. Denote $V(x)$ the event that the set $A_N(x)$ contains no monomers. Then we can write:

$$\int f(\eta) \nu^0_\rho[d\eta | V^{\alpha_1}_1] = \int d\nu_\rho^{Y^{\alpha_1}_1} [f I(V(0))] \frac{\nu_\rho(V(Y^{\alpha_1}_1))}{\nu_\rho(V(0))}. \tag{3.40}$$

Hence, we conclude

$$\Psi^{(\gamma_1,\gamma_2)}_{\alpha_1} = I(V(0)) \frac{\nu_\rho(V(Y^{\alpha_1}_1))}{\nu_\rho(V(0))}. \tag{3.41}$$

And we can estimate

$$\|\Psi^{(\gamma_1,\gamma_2)}_{\alpha_1}\| \le \frac{1}{\nu_\rho(V(0))} \le c(\rho, 0). \tag{3.42}$$

Next consider $\alpha_1 = 0$, i.e., the jump fails (and thus $Y_1^{\alpha_1} = 0$). We write

$$\int f(\eta)\nu_\rho^0[d\eta|V_1^{\alpha_1}] = \int \nu_\rho^{Y_1^{\alpha_1}}(d\eta)[fI(V_1^{\alpha_1}(\eta))]\frac{1}{\nu_\rho^{Y_1^{\alpha_1}}(V_1^{\alpha_1})}. \tag{3.43}$$

Hence,

$$\Psi_{\alpha_1}^{(\gamma_1,\gamma_2)} = \frac{I[V_1^{\alpha_1}]}{\nu_\rho^0(V_1^{\alpha_1})} \tag{3.44}$$

So also in that case we have the estimate

$$\|\Psi_{\alpha_1}^{(\gamma_1,\gamma_2)}\| \le \frac{1}{\nu_\rho^0(V_1^{\alpha_1})} \le c(\rho, 0). \tag{3.45}$$

This proves the claim for $n = 1$. Suppose the claim is true for $n = 1, \dots, p-1$. Put $\alpha_p = 1$, the case $\alpha_p = 0$ can be treated analogously. In order to simplify the notation, we make some further abbreviations:

1. $\mu_{\alpha, Y_p^\alpha}^{(\gamma_1,\gamma_2)} := \mu_p$.
2. $\nu_\rho^{Y_p^{\alpha_1,\dots,\alpha_p}} := \nu_\rho^p$
3. $\Psi_{\alpha_1,\dots,\alpha_p}^{(\gamma_1,\gamma_2)} := \Psi_p^{(\gamma_1,\gamma_2)}$
4. $S_{(\gamma_1,\gamma_2)}^{Y_p^{\alpha_1,\dots,\alpha_p}}(t) := S_p^{(\gamma_1,\gamma_2)}(t)$
5. $T_p^{\epsilon_p} - T_{p-1}^{\epsilon_{p-1}} := \tau_p$
6. $V_p^{\alpha_1,\dots,\alpha_p} := V_p$

We compute $\Psi_p^{(\gamma_1,\gamma_2)}$:

$$\begin{aligned}
\mu_p(f) &= \left(\mu_{p-1}S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)\right)[f|V_p] \\
&= \frac{\int d\mu_{p-1}\ S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)(f1_{V_p})}{\int d\mu_{p-1}\ S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)(1_{V_p})} \\
&= \frac{\int d\nu_\rho^{p-1}\left(S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)(\Psi_{p-1}^{(\gamma_1,\gamma_2)})\ f1_{V_p}\right)}{\int d\nu_\rho^{p-1}\left(S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)(\Psi_{p-1}^{(\gamma_1,\gamma_2)})\ 1_{V_p}\right)} \\
&= \frac{\int d\nu_\rho^p\left(1_{V_{p-1}}f\ S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)(\Psi_{p-1}^{(\gamma_1,\gamma_2)})\right)}{\int d\nu_\rho^p\left(1_{V_{p-1}}S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)(\Psi_{p-1}^{(\gamma_1,\gamma_2)})\right)},
\end{aligned} \tag{3.46}$$

where in the third step we used reversibility of ν_ρ^{p-1}. From (3.46) we read off the density:

$$\Psi_p^{(\gamma_1,\gamma_2)} = \frac{1_{V_{p-1}}\ S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)(\Psi_{p-1}^{(\gamma_1,\gamma_2)})}{\int d\nu_\rho^p\left(1_{V_{p-1}}\ S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)\Psi_{p-1}^{(\gamma_1,\gamma_2)}\right)}. \tag{3.47}$$

We first estimate the nominator of the rhs of (3.47):

$$\begin{aligned}
&\int d\nu_\rho^p \; 1_{V_{p-1}} \; S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)(\Psi_{p-1}^{(\gamma_1,\gamma_2)}) \\
= \;& \int d\nu_\rho^{p-1} \left(1_{V_p} S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)(\Psi_{p-1}^{(\gamma_1,\gamma_2)}) \right) \frac{\nu_\rho(V_{p-1})}{\nu_\rho(V_p)} \\
= \;& \int d\nu_\rho^{p-1} \left(S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)(1_{V_p}) \; \Psi_{p-1}^{(\gamma_1,\gamma_2)} \right) \frac{\nu_\rho(V_{p-1})}{\nu_\rho(V_p)} \\
\geq \;& \nu_\rho(V_{p-1}) \\
- \;& \frac{\nu_\rho(V_{p-1})}{\nu_\rho(V_p)} \|\Psi_{p-1}^{(\gamma_1,\gamma_2)}\|_\infty \|S_{p-1}^{(\gamma_1,\gamma_2)}(\tau_p)[1_{V_p} - \nu_\rho^{p-1}(V_p)]\|_{L^2(\nu_\rho^{p-1})} \\
\geq \;& \frac{1}{c(\rho, Y_{p-1}^\alpha)} - o(\gamma_1), \qquad (3.48)
\end{aligned}$$

where $o(\gamma_1)$ tends to zero as $\gamma_1 \uparrow \infty$ by Lemma 3.2. By the induction hypothesis, we obtain from (3.47), (3.48):

$$\begin{aligned}
\limsup_{\gamma_1 \uparrow \infty} \|\Psi_p^{(\gamma_1,\gamma_2)}\|_\infty &\leq \limsup_{\gamma_1 \uparrow \infty} \|\Psi_{p-1}^{(\gamma_1,\gamma_2)}\|_\infty c(\rho, Y_{p-1}^\alpha) \\
&\leq \prod_{k=0}^{p-1} c(\rho, Y_k^\alpha). \qquad (3.49)
\end{aligned}$$

This finishes the proof of Lemma 3.3 and Proposition 3.1. ■

As a first application we obtain convergence of the one-point marginales of the processes $\{Y_t^{(\gamma_1,\gamma_2)} : t \geq 0\}$. For $f : \mathbb{Z} \to \mathbb{R}$ a bounded function, we have, using the notation of (2.5).

$$\mathbb{E}_{\nu_\rho^0 \times \delta_0}^{(\gamma_1,\gamma_2)} \left(f(Y_t) - f(Y_0) - \int_0^t ds \; (\mathcal{L}_{\eta_s} f)(Y_s) \right) = 0. \qquad (3.50)$$

By Proposition 3.1 we obtain in the limit $\gamma_1 \uparrow \infty$:

$$\begin{aligned}
&\lim_{\gamma_1 \uparrow \infty} \mathbb{E}_{\nu_\rho^0 \times \delta_0}^{(\gamma_1,\gamma_2)} \left(f(Y_t) - f(Y_0) - \int_0^t ds \; [(\mathcal{L}_\eta) \nu_\rho^{Y_s}(d\eta)] f(Y_s) \right) \\
= \;& \lim_{\gamma_1 \uparrow \infty} \mathbb{E}_{\nu_\rho^0 \times \delta_0}^{(\gamma_1,\gamma_2)} \left(f(Y_t) - f(Y_0) - \int_0^t ds \; (\mathcal{L}^{\mathrm{RW}} f)(Y_s) \right) = 0. \qquad (3.51)
\end{aligned}$$

This implies in particular that

$$\lim_{\gamma_1 \uparrow \infty} \mathbb{E}_{\nu_\rho^x \times \delta_x}^{(\gamma_1,\gamma_2)} f(Y_t) = \mathbb{E}_x^{\mathrm{RW}} f(Y_t^{\mathrm{RW}}). \qquad (3.52)$$

In order to prove that the *processes* $\{Y_t^{(\gamma_1,\gamma_2)} : t \geq 0\}$ converge weakly in the Skorohod space to the random walk $\{Y_t^{\mathrm{RW}} : t \geq 0\}$, i.e. the content of Theorem 2.1, it suffices to show that the process $\{Y_t^{(\gamma_1,\gamma_2)} : t \geq 0\}$ is asymptotically Markovian. Indeed, then it is uniquely determined by its one-point marginals which are those of the random walk $\{Y_t^{\mathrm{RW}} : t \geq 0\}$. More precisely it is sufficient to prove the following lemma:

Lemma 3.4 *Let $\{\mathcal{F}_t : t \geq 0\}$ denote the canonical filtration of the process $\{(\eta_t, Y_t) : t \geq 0\}$. We have*

$$\lim_{\gamma_1 \uparrow \infty} \mathbb{E}^{(\gamma_1,\gamma_2)}_{\nu^0_\rho \times \delta_0} \left| \mathbb{E}^{(\gamma_1,\gamma_2)}_{\nu^0_\rho \times \delta_0} \left(f(Y_t) | \mathcal{F}_s \right) - \sum_y p^{\mathrm{RW}}_{t-s}(Y_s, y) f(y) \right| = 0 \tag{3.53}$$

Proof: By the Markov property of the process $\{(\eta_t, Y_t) : t \geq 0\}$,

$$\begin{aligned} \mathbb{E}^{(\gamma_1,\gamma_2)}_{\nu^0_\rho \times \delta_0} \left(f(Y_t) | \mathcal{F}_s \right) &= \mathbb{E}^{(\gamma_1,\gamma_2)}_{\eta_s \times \delta_{Y_s}} \left(f(Y_{t-s}) \right) \\ &= f(Y_s) + \mathbb{E}^{(\gamma_1,\gamma_2)}_{\eta_s \times \delta_{Y_s}} \int_0^{t-s} \mathcal{L}_{\eta_r} f(Y_r) dr. \end{aligned} \tag{3.54}$$

Therefore, it suffices to show that

$$\lim_{\gamma_1 \uparrow \infty} \mathbb{E}^{(\gamma_1,\gamma_2)}_{\eta_s \times \delta_{Y_s}} \left(\int_0^{t-s} \mathcal{L}_{\eta_r} f(Y_r) dr - \int_0^{t-s} dr \int \mathcal{L}_\eta f(Y_r) \nu^{Y_r}_\rho (d\eta) \right) = 0 \tag{3.55}$$

Since the trial jumps of the polymer are on the event times of a Poisson process with rate independent of (γ_1, γ_2), we can write

$$\int_0^{t-s} \mathcal{L}_{\eta_r} f(Y_r) dr = \int_0^{t-s} \frac{1}{\epsilon} \int_r^{r+\epsilon} \mathcal{L}_{\eta_{r'}} f(Y_{r'}) dr' + o(\epsilon), \tag{3.56}$$

where $o(\epsilon)$ goes to zero in $L^2(\mathbb{P}^{(\gamma_1,\gamma_2)}_{\nu^0_\rho \times \delta_0})$, uniformly in (γ_1, γ_2), when ϵ tends to zero. Therefore, it is sufficient to show that

$$\lim_{\gamma_1 \uparrow \infty} \left(\mathbb{E}^{(\gamma_1,\gamma_2)}_{\nu^0_\rho \times \delta_0} \mathbb{E}^{(\gamma_1,\gamma_2)}_{\eta_s \times \delta_{Y_s}} \left| \frac{1}{\epsilon} \int_0^\epsilon f_{Y_r}(\eta_r) dr - \int \nu^{Y_s}_\rho (d\eta) f_{Y_s}(\eta) \right| \right) = 0 \tag{3.57}$$

Following the same strategy as in the proof of Proposition 3.1, i.e., by estimates on the density of the monomer distribution with respect to the appropriate conditioned Bernoulli measure, this reduces to showing that for any $\epsilon > 0$, for any $y \in \mathbb{Z}$ and for f_y depending on layer $y+1$ or $y-1$:

$$\lim_{\gamma_1 \uparrow \infty} \mathbb{E}^{(\gamma_1,\gamma_2),y}_{\nu^y_\rho} \left(\frac{1}{\epsilon} \int_0^\epsilon ds\ f_y(\eta_s) - \int \nu^y_\rho (d\eta) f_y(\eta) \right)^2 = 0. \tag{3.58}$$

Putting $\tilde{f}_y := f_y - \nu^y_\rho(f)$, the expression inside the limit in the left hand side of (3.58) can be rewritten as

$$\begin{aligned} &\int \nu^y_\rho (d\eta) \frac{1}{\epsilon^2} \int_0^\epsilon ds \int_0^\epsilon dr \tilde{f}_y S^y_{(\gamma_1,\gamma_2)}(|r-s|) \tilde{f}_y \\ \leq\ & \frac{1}{\epsilon^2} \int_0^\epsilon ds \int_0^\epsilon dr \|\tilde{f}_y\|_{L^2} \|S^y_{(\gamma_1,\gamma_2)}(|r-s|)\tilde{f}_y\|_{L^2}. \end{aligned} \tag{3.59}$$

Hence we obtain (3.58) as an application of Lemma 3.2. ■

Arrived at this point, we know that any weak limit point of the processes $\{Y^{(\gamma_1,\gamma_2)}_t : t \geq 0\}$ equals in distribution the random walk $\{Y^{\mathrm{RW}}_t : t \geq 0\}$. Hence, to finish the proof of Theorem 2.1, it is sufficient to see that such a weak limit point actually exists. This is an easy task:

Lemma 3.5 *The sequence of processes* $\{Y_t^{(\gamma_1,\gamma_2)} : t \in [0,T],\}_{(\gamma_1,\gamma_2)}$ *is tight.*

Proof: Since the number of jumps the polymer makes in [0,T] is bounded by a mean one Poisson process, we have

$$\mathbb{P}\big(\sup_{0\le t\le T} |Y_s^{(\gamma_1,\gamma_2)}| \ge M\big) \le \frac{2T}{M}, \tag{3.60}$$

and also

$$\lim_{\delta\downarrow 0} \mathbb{P}\big(\sup_{s,t\in[0,T],|s-t|\le\delta} |Y_s^{(\gamma_1,\gamma_2)} - Y_t^{(\gamma_1,\gamma_2)}| > \epsilon\big) = 0. \tag{3.61}$$

This proves tightness (cf. Theorem 1.3 p.51 of [7]). ■

4 Additional remarks

Remark 1: What happens when the system is out of equilibrium? For instance, start the monomers in a homogeneous product measure. When the density is constant and equal to $\rho \in [0,1]$ (no p, q, i_2- dependence in (2.6), the measure ν_ρ^y is no longer invariant for the monomer dynamics $S^y(t)$ at fixed rod position y. However in the limit $\gamma_1 \to \infty$ the polymer will perform a continuous time random walk with rates $a(1-\rho)^N$ and $b(1-\rho)^N$ for up and down jumps respectively. Significant corrections in the case $\gamma_1 < \infty$ can be expected, cf. [2]. Another problem is obtained if we start the monomers from a sharp density profile. That is, above the polymer the fluid density is constant ρ_1 and under the polymer the density is also homogeneous equal to ρ_2. In this case the vertical density will follow a discrete space noiseless Burgers equation: $\rho(i,t) \in [0,1]$, $t \in \mathbb{R}$, $i \in \mathbb{Z}$

$$\begin{aligned}\frac{\partial \rho(i,t)}{\partial t} &= -p\,\rho(i,t)\,(1-\rho(i+1,t)) - q\,\rho(i,t)\,(1-\rho(i-1,t)) \\ &\quad + p\,\rho(i-1,t)\,(1-\rho(i,t)) + q\,\rho(i+1,t)\,(1-\rho(i,t))\end{aligned}$$

with initial condition $\rho(i,0) = \rho_2 I(i \le 0) + \rho_1 I(i > 0)$. The limiting motion of the rod will be a non-homogeneous (in time) Markov process described by

$$\begin{aligned}\frac{d\mathbb{E}(f(Y_t)\,|\,\mathcal{F}_t)}{dt} &= a\,[1-\rho(Y_t+1,t)]^N\,[f(Y_t+1) - f(Y_t)] \\ &\quad + b\,[1-\rho(Y_t-1,t)]^N\,[f(Y_t-1) - f(Y_t)].\end{aligned}$$

where $\mathcal{F}_t$ is the sigma field generated by $\{Y_s : s \le t\}$. These results can be obtained with the techniques we used to prove Theorem 2.1 and will be the content of a future publication, cf. [9].

Remark 2: One may wonder how general the results are. As an illustration of this we consider the following somewhat abstract modification of Lemma 3.2. Suppose that μ is a reversible measure on $\{0,1\}^{\mathbb{Z}^2}$ both for a monomer dynamics with generator $\mathcal{L}_1$ and one with generator $\mathcal{L}_2$. As an example, we could keep in mind the case where $\mathcal{L}_{12} = \mathcal{L}_1 + \mathcal{L}_2$ is a Kawasaki dynamics (exclusion process with speed change) at finite temperature with $\mathcal{L}_1$ generating the horizontal and $\mathcal{L}_2$ generating the vertical jumps; μ is the corresponding Gibbs measure. The measure μ is then also reversible for $\mathcal{L}_{12}^{\gamma} = \gamma\mathcal{L}_1 + \mathcal{L}_2$. Now we insert the polymer and we fix it at some position $y \in \mathbb{Z}$. The dynamics of the monomers is now conditioned on having no monomers in the excluded volume $A_N(y)$: $\eta_t(i) = 0, \forall i \in A_N(y), \forall t \geq 0$. The new generator is $\mathcal{L}_{12}^{y;\gamma} = \gamma\mathcal{L}_1^y + \mathcal{L}_2^y$ obtained by setting all of the original rates equal to zero for all updating that would create a monomer in the region $A_N(y)$ (the direct analogue of what was done in (2.1) and (2.2)). It follows then from Lemma 3.1 that $\mu^y = \mu(\cdot|\eta(i) = 0, \forall i \in A_N(y))$ is reversible for $\mathcal{L}_{12}^y$. We finally denote by μ_x^y the restriction of μ^y to the layer at height x (i.e., the set $\{i \in \mathbb{Z}^2, i_2 = x\}$). This measure is reversible for $\mathcal{L}_1^y$. We have the following result:

Proposition 4.1 *Denote by $S_y^{\gamma}(t)$ the semigroup with generator $\mathcal{L}_{12}^{y;\gamma}$. Assume that for all x μ_x^y is ergodic for $\mathcal{L}_1^y$. Let f_x be a function in $L^2(\mu^y)$ with dependence set on layer $x \neq y$. We have:*

$$\lim_{\gamma\uparrow\infty} \|S_y^{\gamma}(t)f_x - \int d\mu_x^y \; f_x\|_{L^2(\mu^y)} = 0. \tag{4.62}$$

Proof: By ergodicity $\mathcal{L}_1^y$ has simple eigenvalue 0 with corresponding eigenspace the constant functions. Hence by the spectral theorem,

$$(\int d\mu_x^y f_x)^2 = \mathbb{E}_{f_x,f_x}^{\mathcal{L}_1^y}(\{0\}), \tag{4.63}$$

where $\mathbb{E}_{f_x,f_x}^{\mathcal{L}_1^y}$ denotes the spectral measure of the selfadjoint operator $\mathcal{L}_1^y$. Therefore, we have to show that if f_x is a function on layer x such that

$$\mathbb{E}_{f_x,f_x}^{\mathcal{L}_1^y}(\{0\}) = 0, \tag{4.64}$$

then

$$\lim_{\gamma\uparrow\infty} \|S_y^{\gamma}(t)\|_{L^2(\mu^y)} = 0. \tag{4.65}$$

For every φ in the domain of $\mathcal{L}_{12}^{y;\gamma}$,

$$\lim_{\gamma\uparrow\infty} \frac{1}{\gamma}(\mathcal{L}_{12}^{y;\gamma}\varphi) = \mathcal{L}_1^y\varphi. \tag{4.66}$$

Hence the spectral measures $\mathbb{E}_{f_x,f_x}^{\frac{1}{\gamma}\mathcal{L}_{12}^{y,\gamma}}$ converges weakly to the spectral measure $\mathbb{E}_{f_x,f_x}^{\mathcal{L}_1^y}$. Therefore, we can estimate

$$\begin{aligned} \|S_y^{\gamma}(t)f_x\|_{L^2(\mu^y)}^2 &= \int_0^{\infty} e^{-\gamma t\lambda}\mathbb{E}_{f_x,f_x}^{\frac{1}{\gamma}\mathcal{L}_{12}^{y,\gamma}}(d\lambda) \\ &\leq \int_0^{\delta} e^{-\gamma t\lambda}\mathbb{E}_{f_x,f_x}^{\frac{1}{\gamma}\mathcal{L}_{12}^{y,\gamma}}(d\lambda) + e^{-\gamma t\delta}\|f_x\|_{L^2(\mu^y)} \\ &\leq \mathbb{E}_{f_x,f_x}^{\frac{1}{\gamma}\mathcal{L}_{12}^{y,\gamma}}([0,\delta]) + e^{-\gamma t\delta}\|f_x\|_{L^2(\mu^y)}. \end{aligned} \tag{4.67}$$

Letting γ tend to infinity, and then δ to zero, using (4.64), we obtain (4.65). ■

Remark 3: Proposition 4.1 is general but has a strong hypothesis: the ergodicity of the one-layer horizontal dynamics. This is known only in a few cases, in particular in the symmetric simple exclusion process we treated in Lemma 3.2. It is however expected to be true at least for high temperature Kawasaki dynamics.

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