PERIODIC SOLUTIONS OF THE FORCED

PENDULUM EQUATION WITH FRICTION

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ABSTRACT: *This paper is devoted to the study of the general forced pendulum equation in the presence of friction,*

$$u'' + a(t)u' + b(t)\sin\, u = f(t)$$

with $a, b \in C([0,T])$ *and* $f \in L^2(0,T)$. *We'll show that* T*-periodic solutions may be obtained as zeroes of a* 2π*-periodic continuous real function. Furthermore, the existence of infinitely many solutions is proved under appropiate conditions on* a, b *and* f.

Introduction

The periodic problem for the forced pendulum equation has been studied by different authors. In 1922 Hamel [H] found that the equation

$$u'' + b\sin\, u = \beta\sin\,(t)$$

with b, β constants admits a 2π-periodic solution that can be obtained as a minimum in $C_{per}[0,2\pi]$ of the functional

$$J(u) = \int_0^{2\pi} \frac{u'^2}{2} + b\cos\, u + u\beta\sin\, t$$

and the same argument can be extended for T-periodic solutions of the equation $u'' + b\sin\, u = f(t)$ where f is T-periodic and orthogonal to constants (see [M1-2]).

If we allow the presence of friction, namely the equation

$$u'' + au' + b\sin\, u = f(t) \tag{1}$$

(where a is a positive constant) then variational methods are not appliable to the periodic problem. In fact, examples of T-periodic functions f such that $f \perp 1$ and (1) is not solvable are given in [A] and [O], under some conditions on the parameters a, b and T. In a recent work a rather general class of counterexamples is constructed for arbitrary a, b, T [OST]. Periodic type conditions for a general system are studied in [AM] by fixed point methods.

The purpose of this work is to give conditions for the existence of T-periodic solutions of the general equation

$$u'' + a(t)u' + b(t)\sin u = f(t) \tag{2}$$

In the first section we'll show that the Dirichlet problem for (2) is solvable for any boundary data, and uniquely solvable under an appropiate condition on b.

In the second section we prove that the T-periodic solutions of (2) are the zeroes of a 2π-periodic continuous real function.

Finally, in the third section we give explicit conditions on a, b and f in order to obtain T-periodic solutions of (2).

1. Unique solvability of the Dirichlet problem

The existence of solutions of (2) for arbitrary Dirichlet boundary data φ is easily obtained from classical results available in the literature (see e.g. [M3]). We sketch a simple proof here which makes use of Schauder fixed point theorem. For any $\overline{u} \in L^2(0,T)$ let $K\overline{u} = u$ be the unique solution of the linear problem

$$\begin{cases} u'' + a(t)u' = f - b(t)\sin(\overline{u}) \\ u|_{\partial I} = \varphi \end{cases}$$

As $\sin \overline{u} \in L^2(0,T)$ then $K : L^2 \to L^2$ is well defined and compact. Moreover, using a standard apriori bound for the operator $Lu = u'' + a(t)u'$, we obtain:

$$\|K\overline{u} - K0\|_2 \le c\|L(K\overline{u} - K0)\|_2 = c\|b\sin \overline{u}\|_2 \le c\|b\|_2$$

Hence, taking $M = c\|b\|_2$ we obtain that $R(K) \subset B_M(K0)$ and the result follows.

A sufficient condition for the uniqueness is: $\|b\|_\infty < \frac{1}{c}$. However, for further applications we'll state a slightly more precise uniqueness result providing also an apriori bound for the nonlinear operator $Qu := u'' + au' + b\sin u$.

Lemma 1

Let p be a positive solution of the equation $p' = a(t)p - k(t)$ for $k \in H^1(0,T)$ with $k' \geq 0$ a.e. and λ_p the first eigenvalue of the problem $-(pu')' = \lambda u$. Then, if $\|bp\|_\infty < \lambda_p$,

$$\|u - v\|_2 \leq \frac{1}{\lambda_p - \|pb\|_\infty} \|p(Qu - Qv)\|_2$$

and

$$\left(\int_0^T p[(u - v)']^2 \right)^{1/2} \leq \frac{\sqrt{\lambda_p}}{\lambda_p - \|pb\|_\infty} \|p(Qu - Qv)\|_2$$

for any $u, v \in H^2(0,T)$ such that $u = v$ on ∂I.

In particular, Dirichlet problem is uniquely solvable for any boundary data φ.

<u>Proof</u>

Let $w = u - v$, then

$$\|p(Qu - Qv)\|_2 \|w\|_2 \geq -\int_0^T p(Qu - Qv).w = \int p(w')^2 - kw'w - pb(\sin u - \sin v)w$$

As $-\int kw'w = \int k' \frac{w^2}{2} \geq 0$, and $|\int pb(\sin u - \sin v)w| \leq \|pb\|_\infty \|w\|_2^2$, it follows that

$$\|p(Qu - Qv)\|_2 \|w\|_2 \geq \int p(w')^2 - \|pb\|_\infty \|w\|_2^2$$

and the result holds since $\|w\|_2^2 \leq \frac{1}{\lambda_p} \int p(w')^2$.

Remark:

If $a \in H^1(0,T)$ with $a' \geq 0$ a.e., we may take $p \equiv 1$, and the result holds for $\|b\|_\infty < (\frac{\pi}{T})^2$.

2. An embedded curve in $H^1(0,T)$

In this section we'll assume that b satisfies the assumptions of lemma 1 for some $\widetilde{p} > 0$, and prove that all the possible T-periodic solutions of (2) belong to a continuous curve $\mathcal{C} \subset H^1(0,T)$. More precisely, if the functional $I : H^1(0,T) \to \mathbb{R}$ is given by $I(u) = \int_0^T a(t)u' + b(t)\sin u$, then the set of T-periodic solutions of (2) is $\mathcal{C} \cap I^{-1}(\int_0^T f)$. Indeed, we may prove directly the following

THEOREM 2

Let $\psi : \mathbb{R} \to \mathbb{R}$ *given by* $\psi(s) = I(u_s) - \int_0^T f$ *, where* u_s *is the unique solution of the problem*

$$\begin{cases} u'' + a(t)u' + b(t)\sin\,(u) = f \\ u|_{\partial I} = s \end{cases}$$

Then ψ *is continuous and* 2π *-periodic. Moreover, the set of* T *-periodic solutions of (2) can be characterized as*

$$\{u_s : \psi(s) = 0\}$$

<u>*Proof*</u>

Let $s \to s_0$, and define $w_s = u_s - u_{s_0}$. In order to prove that $I(u_s) \to I(u_{s_0})$, by dominated convergence it will suffice to show that $w_s, w_s' \to 0$ a.e. As

$$0 = Qu_s - Qu_{s_0} = w_s'' + aw_s' + b(\sin\,u_s - \sin\,u_{s_0}) = w_s'' + aw_s' + b\cos\,\xi w_s$$

it follows that $\|w_s\|_{1,2} \le c|s - s_0|$ for some constant c. Hence, ψ is continuous.

Moreover, by uniqueness $u_{s+2\pi} = u_s + 2\pi$, and by definition of I we conclude that ψ is 2π-periodic. Finally, u is a T-periodic solution of (2) if and only if $u \in \mathcal{C} = \{u_s : s \in \mathbb{R}\}$, and $0 = u'(T) - u'(0) = \int_0^T f - (au' + b\sin\,u)$. Hence, the proof is complete.

REMARKS:

i) With the notation of the preceeding theorem, it's immediate that $\mathcal{C}$ *is an embedded curve of* $H^1(0,T)$ *.*

ii) A more general version of Theorem 2 may be obtained if we define $p > 0$ *verifying:*

(P) $$p' = a(t)p - k(t), \qquad p(0) = p(T)$$

and

$$\psi_p(s) = \int_0^T p(t)(b(t)\sin\,u_s - f) + k(t)u_s' := I_p(u_s) - \int_0^T pf$$

iii) By periodicity, extremals s_{min} *and* s_{max} *of* ψ_p *are achieved in* $[0, 2\pi]$ *. Using dominated convergence, it can be shown that if* $w = \frac{\partial}{\partial s}u_s$ *then satisfies the linear equation*

$$\begin{cases} (pw')' + kw' + b(t)\cos\,(u_s)w = 0 \\ w|_{\partial I} = 1 \end{cases}$$

For a solution of the equation $\psi_p'(s) = 0$, if k is constant then w is periodic.

As an immediate consequence of theorem 2 we have:

COROLLARY 3

Let $p > 0$ verify (P) and assume that $I_p(u_{s_-}) \leq \int_0^T pf \leq I_p(u_{s_+})$ for some $s_- \leq s_+ \in \mathbb{R}$. Then there exist $s_1 \in [s_-, s_+]$, $s_2 \in [s_+, s_- + 2\pi]$ such that $u_{s_i} + 2n\pi$ is a T-periodic solution of (2) for any integer n and $i = 1, 2$.

3. Sufficient conditions for the existence of T-periodic solutions of (2)

In this section we'll obtain T-periodic solutions of (2) under rather explicit conditions on a, b and f.

For a straightforward application of corollary 3 we first note that a positive p satisfying (P) for some constant k may be constructed in a unique way up to a constant factor. Indeed, from the equation $p' = ap - k$ we obtain that

$$p(t) = \left(c_0 - k \int_0^t e^{-\int_0^s a} ds \right) e^{\int_0^t a}$$

Without loss of generality we may assume that $c_0 = 1$, and as $p(0) = p(T)$ we deduce that

$$k = \frac{e^{\int_0^T a} - 1}{\int_0^T e^{\int_s^T a} ds}$$

Now, if $k \leq 0$ it's immediate that $p > 0$. On the other hand, if $k > 0$ and p vanishes in $(0, T)$, since $p(0) = p(T)$ there exists $t_0 \in (0, T)$ such that $p(t_0) = 0$ and $p'(t_0) \geq 0$. Then $k = -p'(t_0) \leq 0$, a contradiction.

REMARK: *in particular, if $a \perp 1$ then $p(t) = e^{\int_0^t a}$, and if a is a constant then $p \equiv 1$.*

In the following p will be considered as in the previous construction, and we'll assume that $\|\widetilde{p}b\|_\infty < \lambda_{\widetilde{p}}$ for some $\widetilde{p}$ such that lemma 1 holds. As $Qu_s - Qs = f - b\sin s$, we may denote by δ the best possible constant such that

$$\|u_s - s\|_\infty \leq \delta \|p(f - b\sin s)\|_2$$

A simple computation shows that if $\|pb\|_\infty < \lambda_p$, then

$$\delta \leq \left(\frac{T\lambda_p}{\inf_t p} \right)^{1/2} \frac{1}{\lambda_p - \|pb\|_\infty}$$

and if $\|pb\|_2 \leq \|pb\|_\infty$ (for example, when $T \leq 1$) then $\|pb\|_\infty$ may be replaced in the formula by $\|pb\|_2$.

We obtain the following

THEOREM 4

Let us assume that b does not vanish in $[0,T]$, and

i) $\|\widetilde{p}b\|_\infty < \lambda_{\widetilde{p}}$ for some $\widetilde{p}$ satisfying the hypothesis of lemma 1

ii) $\|p(f \pm b)\|_2 \leq \frac{c}{\delta}$ for some constant $c < \frac{\pi}{2}$.

iii) $|\int_0^T pf| \leq \|pb\|_1 \cos(c)$

Then (2) has infinitely many T-periodic solutions. More precisely, there exist $s_1 \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, $s_2 \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ such that $\{u_{s_i} + 2n\pi : n \in Z\}$ is a family of T-periodic solutions of (2).

<u>*Proof*</u>

With the previous notations, for $s_n = \frac{(2n-1)\pi}{2}$ we obtain that $\|u_{s_n} - s_n\|_\infty \leq c$. Furthermore, as

$$I_p(u_{s_n}) = \int_0^T ku'_{s_n} + pb\sin u_{s_n} = (-1)^n \int_0^T pb\cos(u_{s_n} - s_n),$$

taking n such that $(-1)^n b > 0$ we obtain:

$$I_p(u_{s_n}) \geq \|pb\|_1 \cos(c)$$

and in the same way

$$I(u_{s_{n\pm1}}) \leq -\|pb\|_1 \cos(c)$$

Hence, by iii)

$$I_p(u_{s_{n\pm1}}) \leq \int_0^T pf \leq I_p(u_{s_n})$$

and from corollary 3 the result holds.

For constant a, conditions $i) - iii)$ can be written in a more explicit way:

COROLLARY 5

Let us assume that a is a constant, and b does not vanish in $[0,T]$. Then, if

i) $\|b\|_\infty < (\frac{\pi}{T})^2$

ii) $\|f \pm b\|_2 \leq c\frac{\pi^2 - \|b\|_\infty T^2}{T^{3/2}}$ for some constant $c < \frac{1}{2}$.

iii) $|\int_0^\alpha f| \leq \|b\|_1 \cos\,(c\pi)$

there exist $s_1 \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, $s_2 \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ *such that* $\{u_{s_i} + 2n\pi : n \in Z\}$ *is a family of* T*-periodic solutions of (2).*

REMARKS:

1) Condition ii) in Theorem 4 may be easily generalized in the following way:

ii') There exist s_0, s_1 *such that* $\delta\|p(f - b\sin\, s_i)\|_2 + |s_i - (-1)^i\frac{\pi}{2}| \leq c$ *for some constant* $c < \frac{\pi}{2}$.

Replacing ii) by ii') a family $\{u_s + 2n\pi\}$ *of* T*-periodic solutions of (2) is obtained for some* $s \in [s_1, s_0]$ *and also for some* $s \in [s_0, s_1 + 2\pi]$.

Furthermore, the existence of one of the numbers s_i *may be avoided if the following extra condition on* f *holds:*

$$\text{(F)} \qquad \|pb\|_2\|pf\|_2 \leq \frac{1}{\delta_2}|\int pf|$$

where δ_2 *is the best constant such that* $\|u_s - s\|_2 \leq \delta_2\|pf\|_2$. *Indeed, for* $s = 0, \pi$

$$|\int_0^T pb\sin\, u_s| = |\int_0^T pb\sin\,(u_s - s)| \leq \delta_2\|pb\|_2\|pf\|_2 \leq |\int_0^T pf|,$$

and one of the inequalities required in corollary 3 is proved.

For example, (F) holds for $sg(f)$ *constant, if* $0 \leq |f| \leq \frac{1}{p}$ *and* $|\int pf| \geq \delta_2\|pb\|_2$.

2) As a particular case, condition iii) in Theorem 4 (Corollary 5) holds for $f \perp p$ *(respectively:* $\int_0^T f = 0$ *).*

3) The results of this paper may be simplified and constants can be computed in a sharper way by the use of a weighted Hilbert space $L^2_p(0,T)$. *For example, if* $\|\cdot\|$ *denotes the* p*-weighted norm, and* μ_p *is the first eigenvalue of the problem* $(-pu')' = \mu pu$, *then the inequalities of lemma 1 can be transformed in:*

$$\|u - v\| \leq \frac{1}{\mu_p - \|b\|_\infty}\|(Qu - Qv)\|$$

and

$$\|(u - v)'\| \leq \frac{\sqrt{\mu_p}}{\mu_p - \|b\|_\infty}\|(Qu - Qv)\|$$

References

[A] Alonso, J.: Nonexistence of periodic solutions for a damped pendulum equation. Diff. and Integral Equations, 10 (1997), 1141-8.

[AM] Amster, P., Mariani, M.C.: Resolution of Semilinear Equations by Fixed Point Methods. To appear in the Bulletin of the Belgian Math. Society, Simon Stevin.

[H] Hamel, G.: Über erzwungene Schwingungen bei endlichen Amplituden. Math. Ann., 86 (1922), 1-13.

[M1] Mawhin, J.: Periodic oscillations of forced pendulum-like equations. Lecture Notes in Math., Springer, 964 (1982), 458-76.

[M2] Mawhin, J.: The forced pendulum: A paradigm for nonlinear analysis and dynamical systems. Expo. Math., 6 (1988), 271-87.

[M3] Mawhin, J.: Boudary value problems for nonlinear ordinary differential equations: from successive approximations to topology.

[O] Ortega, R.: A counterexample for the damped pendulum equation. Bull. Classe des Sciences, Ac.Roy. Belgique, LXXIII (1987), 405-9.

[OST] Ortega, R., Serra, E., Tarallo, M.: Non-continuation of the periodic oscillations of a forced pendulum in the presence of friction. To appear.

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