Existence and properties of p-tupling fixed points

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Abstract

We prove the existence of fixed points of p-tupling renormalization operators for interval and circle mappings having a critical point of arbitrary real degree $r > 1$. Some properties of the resulting maps are studied: analyticity, univalence, behavior as r tends to infinity.

1 Introduction

Two problems have a strong resemblance, and have both found their origin in the theory of period doubling for maps of the interval [F1, F2, CT]. The first is to prove the existence and properties of solutions of the $(p+1)$-Cvitanović-Feigenbaum functional equation, i.e. fixed points of the $(p+1)$-tupling operator $\mathcal{R}_{p+1}$:

$$g(x) = (\mathcal{R}_{p+1}g)(x) = -\frac{1}{\lambda}g^{p+1}(-\lambda x), \qquad g(0) = 1. \tag{1.1}$$

Here g is required to be an even, $\mathcal{C}^1$ map of $[-1,\ 1]$ into itself, strictly decreasing on $[0,\ 1]$ and $\lambda = -g^{p+1}(0)$ is required to be in $(0,\ 1)$. More precisely, the restrictions g_+ and g_- to $[0,\ 1]$ and $[-1,\ 0]$, respectively, must satisfy

$$g_+ = -\frac{1}{\lambda}g_-^{p-1} \circ g_+^2 \circ (\lambda), \qquad g(0) = 1. \tag{1.2}$$

Denoting u the inverse function of g_+, and $\check{u}(z) = u(-z)$, this can be reexpressed as

$$u = \frac{1}{\lambda}u \circ \check{u}^p \circ \lambda\ . \tag{1.3}$$

We shall also require g_+ to have the form

$$g_+(x) = f(x^r) \quad \forall x \in [0,\ 1], \tag{1.4}$$

where $r > 1$ is a real number, and f is real-analytic on $[0,\ 1]$, with $f'(x) < 0$ on this closed interval. The class of those g having the property (1.4) for a fixed r is left invariant by $\mathcal{R}_{p+1}$.

The second problem is to prove the existence and properties of solutions of the system

$$\begin{aligned} \eta &= -\tfrac{1}{\lambda}\eta^p \circ \xi \circ (-\lambda)\ , \\ \xi &= -\tfrac{1}{\lambda}\eta \circ (-\lambda), \quad \xi(0) = 1\ , \end{aligned} \qquad \lambda = -\eta(0) \in (0,\ 1). \tag{1.5}$$

Here ξ is a real $\mathcal{C}^1$, strictly increasing function defined on a certain interval $[-L,\ 0]$ of the negative real axis ($L > 1$) and satisfies $\xi(x) > x$ on this interval. Again ξ is required to be of the form

$$\xi(x) = f(|x|^r) \quad \forall x \in [-L,\ 0]\ , \tag{1.6}$$

where $r > 1$ is a real number, and f is real analytic without critical points on $[0,\ L^r]$. Let $-u$ be the inverse function of ξ, and $\check{u}(z) = u(-z)$. Then (1.5) implies

$$u = \frac{1}{\lambda^2} u \circ \lambda \circ \check{u}^p \circ \lambda\ . \tag{1.7}$$

The system (1.5) is part of the theory, initiated in [FKS] and [ORSS], of critical circle mappings whose rotation number has the continued fraction expansion $[p,\ p,\ \ldots, p,\ \ldots]$. It is natural to attempt a unified treatment of the two functional equations (1.3) and (1.7) by introducing an interpolating parameter $\nu \in [1,\ 2]$ and considering the functional equation

$$u = \frac{1}{\lambda^\nu} u \circ \lambda^{\nu-1} \circ \check{u}^p \circ \lambda\ . \tag{1.8}$$

As a device for avoiding repetitions, this works rather well for $p = 1$, ([EE, E2, E3]). It is much less effective, as we shall see, for $p > 1$. It is also of some interest to consider the case when $\nu < 1$. The history of this subject is long, even if restricted to rigorous results (see e.g. [L1, L2]), and the literature has experienced a veritable explosion in recent times. For the case of interval maps, the paper of M. Lyubich [Ly] (a kind of culminating point) contains a historical note and references to which I refer the reader. For the case of circle maps, the reader is referred to the paper of M. Yampolsky [Y] and to references therein. However the literature has tended to concentrate on the case of integer r, with notable exceptions such as [CEL, JR, M, MO]. Another, most important exception is the whole theory of "real a priori bounds" (see [dMvS, S1, S2, Sw, dFdM] and other references given in [Ly, Y]). In this paper, we look for solutions of the functional equations (1.8), for arbitrary real r, which are subjected to some additional constraints (see Section 3). All the available theoretical and numerical evidence indicates that, for each $\nu \in [1,\ 2]$, each $r > 1$, and each $p \geq 1$, there is one and only one solution obeying all the constraints. This suggests that the solution (and in particular λ) must depend analytically on the parameters ν, r, and p. For $p = 1$, it has been proved in [E3] that solutions exist for all $\nu \in [1,\ 2]$ and all $r > 1$, and the proof extends without any change to the case $\nu \in (0,\ 1]$ provided $r\nu > 1$. In the case $\nu = 1$, the existence of solutions for all p and all $r > 1$ has been proved by M. Martens [M], whose results go much farther since they include all possible periodic points and kneading sequences. In this paper, the existence of solutions will be proved, by another method, in the case $1 \leq \nu \leq 2$, for all $r > 1$ and all (integer) $p \geq 1$. It will be seen that in the case $0 < \nu \leq 1$, the condition $r\nu - 1 - (p-1)(1-\nu) > 0$ is necessary and sufficient for the existence of solutions. This work had remained unfinished for a long time[1] when I belatedly became aware of the paper of B. Mestel and A. Osbaldestin [MO], devoted to the proof of the existence of a period 2 point of the doubling operator for non-even maps (with arbitrary $r > 1$). One of the ideas in that paper allowed me to finish the proof of existence in the case $\nu > 1$ (see Subsection 6.2).

Section 2 collects some notations and well-known or straightforward facts (see [D, V]). Sections 3-6 contain the proofs of existence. In Section 7 some properties of the solutions are derived (univalence, boundedness). In Section 8 it is shown that for $\nu \in (0,\ 1]$, when r tends to infinity the solutions behave similarly to those of the case $p = 1$ (see [EW, E1, EE]).

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2 Notations and preliminaries

1. We denote $\mathbf{C}_+ = -\mathbf{C}_- = \{z \in \mathbf{C} : \ \text{Im } z > 0\}$. A function f is a Herglotz or Pick function [D] (and $-f$ is an anti-Herglotz function) if it is holomorphic in $\mathbf{C}_+ \cup \mathbf{C}_-$, $f(z^*) = f(z)^*$, and f maps $\mathbf{C}_+$ (resp. $\mathbf{C}_-$) into its closure, $f(\mathbf{C}_\pm) \subset \overline{\mathbf{C}_\pm}$. If f is also holomorphic on a real non-empty open segment $(a,\ b)$, then, for each $x \in (a,\ b)$, and each $N \in \mathbf{N}$ the $N \times N$ matrix M with components $M_{jk} = D^{j+k+1} f(x)/(j+k+1)!$, $(0 \leq j,\ k < N)$, is positive. This follows immediately from the Herglotz integral representation theorem ([D], pp. 20 ff.). The case $N = 2$ shows that if f is not a constant, then $f'(x) > 0\ \forall x \in (a,\ b)$, and f has non-negative Schwarzian derivative $Sf = (f''/f')' - (f''/f')^2/2$ in $(a,\ b)$. Denote $v = f''/f'$ and suppose $a < x < y < b$. If v does not vanish in $[x,\ y]$,

$$\frac{1}{v(x)} - \frac{1}{v(y)} \geq \frac{y-x}{2}. \tag{2.1}$$

If $v(x) > 0$ then $v(y) > 0$ since v is increasing, hence $v(x) \leq 2/(y-x)$, which also holds if $v(x) \leq 0$. Similarly $v(y) \geq -2/(y-x)$. Letting y tend to b or x tend to a, we find:

$$-\frac{2}{z-a} \leq \frac{f''(z)}{f'(z)} \leq \frac{2}{b-z} \qquad \forall z \in (a,\ b). \tag{2.2}$$

If $f((a,\ b))$ has a finite upper bound, then $f|(a,\ b)$ extends continuously to $(a,\ b]$ with $f(b) = \sup f((a,\ b))$, and similarly if there is a finite lower bound. If f maps $[a,\ b]$ into $(a,\ b)$, it has a fixed point in $(a,\ b)$ which (by Schwarz's lemma) is unique and attractive; in this case every subinterval of $(a,\ b)$ which contains the fixed point is mapped into itself by f. If F is an increasing function with non-negative Schwarzian on $(0,\ +\infty)$ (in particular if F is a Herglotz function holomorphic in $\mathbf{C}_+ \cup \mathbf{C}_- \cup (0,\ +\infty)$), then its restriction to $(0,\ +\infty)$ is concave as a special case of (2.2). The following corollary will be needed later:

[1]The case $\nu = 1$, all r and p, was presented at the Meeting on new developments in Mathematical Physics and Neuroscience (Hunziker-Hepp Fest), ETH, Zurich, 21-23/9/1995.

Lemma 1 *Let $A < a < b < B$ be real numbers, and f be a Herglotz function holomorphic in $\mathbf{C}_+ \cup \mathbf{C}_- \cup (A,\ B)$. Then, for each $z \in (a,\ b)$,*

$$f(z) \geq \frac{(B-b)(z-a)f(b) + (B-a)(b-z)f(a)}{(b-a)(B-z)}\ . \tag{2.3}$$

Proof. We define $F(z) = f(B - z^{-1})$, i.e. $f(z) = F(1/(B-z))$. Then F is a Herglotz function holomorphic in $\mathbf{C}_+ \cup \mathbf{C}_- \cup (1/(B-A),\ +\infty)$. Setting now $a' = 1/(B-a)$, $b' = 1/(B-b)$ and $z' = 1/(B-z)$, with $z \in (a,\ b)$, the concavity of F implies

$$F(z') \geq \frac{z'-a'}{b'-a'}F(b') + \frac{b'-z'}{b'-a'}F(a')\ , \tag{2.4}$$

which translates back into (2.3).
2. Let $A,\ B,\ A',\ B'$ be strictly positive real numbers. Then the homographic function

$$z \mapsto m(z\ ;\ A, B, A', B') = \frac{z\left(\frac{1}{A} + \frac{1}{B}\right)}{z\left(\frac{1}{AB'} - \frac{1}{BA'}\right) + \frac{1}{A'} + \frac{1}{B'}} \tag{2.5}$$

is a bijection of $\mathbf{C}_+ \cup \mathbf{C}_- \cup [-A,\ B]$ onto $\mathbf{C}_+ \cup \mathbf{C}_- \cup [-A',\ B']$, and fixes 0. Its derivative at 0 is

$$m'(0\ ;\ A, B, A', B') = \frac{A'B'(A+B)}{AB(A'+B')}\ . \tag{2.6}$$

Hence if f is a holomorphic map of $\mathbf{C}_+ \cup \mathbf{C}_- \cup (-A,\ B)$ into $\mathbf{C}_+ \cup \mathbf{C}_- \cup (-A',\ B')$ which fixes 0, it follows from Schwarz's lemma applied to $m^{-1} \circ f$, that $|f'(0)| \leq m'(0\ ;\ A, B, A', B')$.
3. Let $b,\ s$ be real numbers, with $0 < b < 1$, and $s > 1$. Then the homographic function

$$z \mapsto \chi_{b,s}(z) = 1 + b^{s-1}\,\frac{b(1+b)(z-1)}{1+b-b^2(z-1)} \tag{2.7}$$

is holomorphic in $\Omega(-1/b,\ 1/b^2)$, Herglotzian, and

$$\chi_{b,s}(-1/b) > 0, \quad \chi_{b,s}(1/b^2) < 1/b^2, \quad \chi_{b,s}(1) = 1, \quad \chi'_{b,s}(1) = b^s\ . \tag{2.8}$$

4. In the sequel, if s and $t > s$ are real numbers, we shall denote $\Omega(s,\ t)$ the domain

$$\Omega(s,\ t) = \mathbf{C}_+ \cup \mathbf{C}_- \cup (s,\ t). \tag{2.9}$$

If u_- and u_+ are two real numbers in $(0,\ 1)$, we denote $\mathbf{E}_0(u_-,\ u_+)$ the space of functions ψ, holomorphic and anti-Herglotzian in $\Omega(-1/u_-,\ 1/u_+)$, and such that $\psi(0) = 1$, $\psi(1) = 0$. Such a function has an integral representation

$$\log\psi(z) = \int_{\mathbf{R}\setminus(-1/u_-,\ 1)} \sigma(t)\left[\frac{1}{t} - \frac{1}{t-z}\right] dt\ , \quad \forall z \in \Omega(-1/u_-,\ 1)\ . \tag{2.10}$$

Here σ is an L^∞ function with $0 \leq \sigma \leq 1$ and $\sigma(t) = 1$ for all $t \in [1,\ 1/u_+]$. It follows that ψ satisfies the following inequalities:

$$\frac{\psi(z)(1-u_+z)}{1-z} \leq 1 \leq \frac{\psi(z)(1+u_-z)}{1-z} \qquad \text{for all}\ \ z \in (0,\ 1/u_+)\setminus\{1\},$$
$$\text{reversed for } z \in (-1/u_-,\ 0), \tag{2.11}$$

$$\frac{1-u_+}{(1-z)(1-u_+z)} \;\le\; -\frac{\psi'(z)}{\psi(z)} \;\le \frac{1+u_-}{(1-z)(1+u_-z)} \qquad \text{for all } \; z\in(-1/u_-,\ 1/u_+)\setminus\{1\}. \tag{2.12}$$

Suppose now that $\psi \in \mathbf{E}_0(u_-,\ u_+)$ has a finite upper bound M on $(u_-,\ u_+)$. (By (2.11), M must satisfy $M \ge \psi(-1/u_-) \ge (1+u_-)/(u_+ + u_-)$.) Then

$$\int_{\mathbf{R}\setminus(-1/u_-,\ 1)} \frac{\sigma(t)\,dt}{t(1+u_-t)} \;\le\; \log M\ , \tag{2.13}$$

Therefore, if $-1/u_- < z < 0$,

$$\begin{aligned} -\frac{\psi'(z)}{\psi(z)} &= \int_{\mathbf{R}\setminus(-1/u_-,\ 1)} \frac{\sigma(t)}{t(1+u_-t)}\ \frac{t(1+u_-t)}{(t-z)^2}\,dt \\ &\le (\log M) \max_{t\notin(-1/u_-,\ 1)} \frac{t(1+u_-t)}{(t-z)^2} \;\le\; \frac{\log M}{(-4z)(1+u_-z)} \end{aligned} \tag{2.14}$$

3 More precise statement of the problem

We begin with a few heuristic considerations. It is easy to see that if u is a solution of (1.8), it will analytically extend to the real interval $(-1/\lambda,\ 1)$. Moreover, since the function f is analytic without critical point in an open real interval containing 0, its inverse function, denoted U, will be analytic, with strictly negative derivative, in an open interval containing 1. The functions u and U are related by $U(z) = u(z)^r$. Thus

$$U(z) = \frac{1}{\lambda^{r\nu}} U(\lambda^{\nu-1}\check{u}^p(\lambda z)) \tag{3.1}$$

should hold wherever both sides are analytic. The main condition which we impose on the solutions we seek is that u and U be anti-Herglotz functions. It is in fact sufficient to impose this condition on u. Indeed denote

$$\varphi = \lambda^{\nu-1}\check{u}^p \circ \lambda\ . \tag{3.2}$$

Then φ is Herglotzian and the equation (3.1), rewritten as

$$U(z) = \frac{1}{\lambda^{r\nu}} U(\varphi(z)) \tag{3.3}$$

shows first that $\varphi(1)$ must be a zero of U, i.e. that $\varphi(1) = 1$, and then that U is a linearizer of φ at 1, the multiplier $\varphi'(1)$ being equal to $\lambda^{r\nu} < 1$. Therefore U is also anti-Herglotzian, and is holomorphic in the basin of attraction of 1 for φ. The reasons for imposing the Herglotz condition have been given e.g. in [EE, E2]. It is more convenient to work with $\psi = U/U(0)$ rather than with U, and we denote $z_1/\lambda^{\nu-1}$ the quantity $u(0)$. This implies $z_1 = \lambda^{\nu-1}\check{u}(0) \le \varphi(0) < 1$. We thus adopt the following definition:

Given two real numbers $\nu \in (0,\ 2]$, $r > 1$, and an integer $p \ge 1$, a solution associated with these values consists of two functions ψ and u and two real numbers $\lambda \in (0,\ 1)$ and $z_1 \in (0,\ 1)$ with the following properties:

(1) ψ is an anti-Herglotzian function holomorphic in $\Omega(-1/\lambda,\ 1/a)$ for some $a \in (0,\ 1)$ with $\psi(1) = 0$ and $\psi(0) = 1$.

(2) u is holomorphic and anti-Herglotzian in $\Omega(-1/\lambda,\ 1)$, and is given there by

$$u(z) = \frac{z_1}{\lambda^{\nu-1}}\,\psi(z)^{1/r}\ . \tag{3.4}$$

(3) The identity

$$\psi(z) = \frac{1}{\lambda^{r\nu}}\psi(\lambda^{\nu-1}\check{u}^p(\lambda z)) \tag{3.5}$$

holds for all $z \in \Omega(-1/\lambda,\ 1/a)$, where again $\check{u}(z) = u(-z)$.

The following theorem will be proved.

Theorem 1 *(i) For any integer $p \geq 1$ and any real $\nu \in (0,\ 1]$, there exist solutions if and only if r satisfies*

$$r\nu - 1 - (p-1)(1-\nu) > 0\ . \tag{3.6}$$

(ii) For any integer $p \geq 1$ and any real $\nu \in [1,\ 2]$, there exist solutions for all real $r > 1$.

The necessity of the condition (3.6) will be shown in the next section, but it is easy to see that the conditions we have imposed require $r\nu > 1$. It suffices to consider the case $0 < \nu \leq 1$. In this case, we must have

$$\psi(-1/\lambda) = \frac{1}{\lambda^{r\nu}}\psi(\varphi(-1/\lambda)) \leq \frac{1}{\lambda^{r\nu}} \quad \Rightarrow \quad u(-1/\lambda) \leq \frac{z_1}{\lambda^{2\nu-1}} < \frac{1}{\lambda}\ , \tag{3.7}$$

from which it follows that $\varphi = \lambda^{\nu-1}\check{u}^p \circ \lambda$ is analytic in $\Omega(-1/\lambda,\ 1/\lambda^2)$ and that:

$$\varphi(-1/\lambda) \geq 0, \quad \varphi(1) = 1, \quad \varphi(1/\lambda^2) \leq \frac{z_1}{\lambda^\nu}\ . \tag{3.8}$$

We can apply Remark 2 of Section 2, i.e. Schwarz's lemma as in (2.6) to bound $\varphi'(1)$, with $A = 1 + 1/\lambda$, $B = 1/\lambda^2 - 1$, $A' = 1$, $B' = \lambda^{-\nu} - 1$ and find

$$\varphi'(1) = \lambda^{r\nu} \leq \frac{\lambda\,(1-\lambda^\nu)}{(1-\lambda^2)} \leq \frac{\lambda}{1+\lambda}\ . \tag{3.9}$$

This implies $\lambda^{r\nu-1} < 1$, i.e. $r\nu > 1$.

In the case $\nu > 1$, it is well-known (see [JR]), and easy to verify, that for $r = 1$, $p \geq 1$, there is a solution such that $\psi(z) = 1 - z$, all functions u, φ, etc. are affine, $z_1 = \lambda^{\nu-1}$, and λ is the unique solution in $(0,\ 1)$ of

$$\lambda^\nu + p\lambda^{\nu-1} - 1 = 0\ . \tag{3.10}$$

Moreover it is proved in [JR] that (in the case $p = 1$, $\nu = 2$) there exist solutions for all sufficiently small $r - 1 > 0$.

The proof of Theorem 1 will occupy Sections 4-6. Many repetitions occur in these sections, since variations of the same method apply to several cases. But avoiding the repetitions would produce more obscurity than brevity.

4 Existence for $r > 1$, $p \geq 1$ and $\nu \in (0,\ 1]$

In this section, $r > 1$ and $\nu \in (0,\ 1]$ are fixed real numbers such that $r\nu > 1$, and $p \geq 1$ is a fixed integer. The real number $b \in (0,\ 1)$ is also fixed, but its value will be chosen later (as a function of r). For any two $s,\ t \in (0,\ 1)$, we denote

$$h_{s,t}(z) = \frac{z(s+1)}{z(s-t)+1+t} \qquad \begin{cases} 0 & \mapsto & 0 \\ 1 & \mapsto & 1 \\ -1/s & \mapsto & -1/t \end{cases} \tag{4.1}$$

Obviously $h_{s,t} = h_{t,s}^{-1}$.

We denote $\mathcal{Q}_0(b,\ r\nu)$ the space of all functions Φ with the following properties:

(Q1) Φ is a Pick function holomorphic in the domain:

$$\Omega(-1/b,\ 1/b^2) = \mathbf{C}_+ \cup \mathbf{C}_- \cup \left(-\frac{1}{b},\ \frac{1}{b^2}\right)\ , \tag{4.2}$$

and maps this domain into itself.

(Q2) $\Phi(z) \geq 0$ for all $z \in (-1/b,\ 1/b^2)$,

(Q3) $\Phi(1) = 1$, and $0 < \Phi'(1) \leq b^{r\nu}$.

We regard $\mathcal{Q}_0(b,\ r\nu)$ as a subset of the real Fréchet space of the self-conjugated functions holomorphic in $\Omega(-1/b,\ 1/b^2)$, equipped with the topology of uniform convergence on compact subsets. We shall define a continuous operator $B(b,\ r,\ p,\ \nu)$ on this space by describing its action on an arbitrary $\Phi_0 \in \mathcal{Q}_0(b,\ r\nu)$.

Given $\Phi_0 \in \mathcal{Q}_0(b,\ r\nu)$, we denote $\lambda = \Phi_0'(1)^{1/r\nu}$. By (Q3), $0 < \lambda \leq b$. We define a function φ_0 by

$$\varphi_0 = h_{b,\lambda} \circ \Phi_0 \circ h_{b,\lambda}^{-1}\ . \tag{4.3}$$

If $\lambda = b$, $h_{b,\lambda}$ is the identity. Otherwise, since $\lambda < b$, its pole is below $-1/b$ and $h_{b,\lambda}$ maps $\Omega(-1/b,\ 1/b^2)$ onto $\Omega(-1/\lambda,\ 1/a_0(\lambda))$ where

$$\frac{1}{a_0(\lambda)} = h_{b,\lambda}\left(\frac{1}{b^2}\right)\ ,$$

$$b^2 \ \leq\ a_0(\lambda) = b - \lambda(1-b) \ <\ b. \tag{4.4}$$

The function φ_0 possesses the following properties:

(Q′1) φ_0 is a Pick function holomorphic in $\Omega(-1/\lambda, 1/a_0(\lambda))$, and maps this domain into itself.

(Q′2) $\varphi_0(z) \geq 0$ for all $z \in (-1/\lambda,\ 1/a_0(\lambda))$.

(Q′3) $\varphi_0(1) = 1$, and $\varphi_0'(1) = \lambda^{r\nu}$.

We denote ψ the linearizer of φ_0 normalized by the condition $\psi(0) = 1$, i.e. the unique function, holomorphic in $\Omega(-1/\lambda, 1/a_0(\lambda))$, such that

$$\psi(z) = \frac{1}{\lambda^{r\nu}}\psi(\varphi_0(z)) \quad \forall z \in \Omega(-1/\lambda, 1/a_0(\lambda)), \quad \psi(1) = 0, \quad \psi(0) = 1\ . \tag{4.5}$$

This is an anti-Herglotz function given, as it is well known ([Mi, V]), by

$$\psi(z) = h(z)/h(0), \quad h(z) = \lim_{n\to\infty} \frac{1}{\lambda^{nr\nu}}(\varphi_0^n(z) - 1) \ . \tag{4.6}$$

The limit converges uniformly on compact subsets of $\Omega(-1/\lambda,\ 1/a_0(\lambda))$, which is a basin of attraction of 1 for φ_0. It is easy to check that ψ depends continuously on φ_0. On $[-1/\lambda,\ 1/a_0(\lambda))$, ψ is strictly decreasing and, because $0 \le \varphi_0(-1/\lambda)) < 1$,

$$\psi(-1/\lambda) = \frac{1}{\lambda^{r\nu}}\psi(\varphi_0(-1/\lambda)) \quad \le \frac{1}{\lambda^{r\nu}} \ . \tag{4.7}$$

ψ satisfies the inequalities (2.11) and (2.12), with $u_- = \lambda$ and $u_+ = a_0(\lambda)$. We also define

$$v(z) = (\psi(-z))^{1/r} \quad \forall z \in \mathbf{C}_+ \cup \mathbf{C}_- \cup (-1,\ 1/\lambda) \ . \tag{4.8}$$

v is a Pick function which extends to a strictly increasing continuous function on $[-1,\ 1/\lambda]$. It satisfies:

$$v(-1) = 0, \quad v(0) = 1, \quad v(1/\lambda) \le \lambda^{-\nu} \ . \tag{4.9}$$

We now show that there is a unique $z_1 \in (0,\ 1)$ such that

$$\left(z_1\lambda^{1-\nu}v\right)^p(\lambda) = \lambda^{1-\nu} \ . \tag{4.10}$$

As a consequence of the inequality in (4.9), the function $(s\lambda^{1-\nu}v)^k$ is defined on $[-1,\ 1/\lambda]$ for every $s \in [0,\ 1]$ and every integer $k \ge 0$. The functions $s \mapsto x_k(s) = (s\lambda^{1-\nu}v)^k(\lambda)$ are thus defined, continuous, and strictly increasing in s for $k > 0$ and $s \in [0,\ 1]$. For $s_* = \lambda^\nu/v(\lambda)$, $x_k(s_*) = \lambda$ for all k. For $s > s_*$, $x_1(s) > x_1(s_*) = \lambda = x_0(s)$, hence $x_0(s) < x_1(s) < \ldots x_{p+1}(s)$. Since $x_p(1) > x_1(1) = \lambda^{1-\nu}v(\lambda) > \lambda^{1-\nu}$, there exists a unique $z_1 \in (s_*,\ 1)$ such that $x_p(z_1) = \lambda^{1-\nu}$. In the case $p = 1$, (4.10) reduces to $z_1 = 1/v(\lambda)$.

The derivative $x'_p(z_1)$ is strictly positive. Therefore, if v is allowed to change slightly, z_1 can be computed by a Cauchy integral along a small circle which remains fixed. Thus z_1 depends continuously (in fact analytically) on v, hence on φ_0.

We denote $\zeta_j = (z_1\lambda^{1-\nu}v)^j(\lambda)$, $(0 \le j \le p+1)$. By the preceding argument,

$$\lambda = \zeta_0 < \zeta_1 < \ldots < \zeta_p = \lambda^{1-\nu} < \zeta_{p+1} = z_1\lambda^{1-\nu}v(\lambda^{1-\nu}), \tag{4.11}$$

Since $v(\lambda^{1-\nu}) \le \lambda^{-\nu}$,

$$z_1 > \lambda^\nu \ , \quad z_1\lambda^{1-\nu} > \lambda \ . \tag{4.12}$$

Note also that, since $v(\lambda) > v(0) = 1$,

$$\zeta_1 > z_1\lambda^{1-\nu} \ . \tag{4.13}$$

The last inequality in (4.11), the upper bound on $\psi(-\lambda^{1-\nu})$ from (2.11), and $\nu \le 1$ give

$$z_1 \quad \ge \quad \left(\frac{1-\lambda^{2-\nu}}{1+\lambda^{1-\nu}}\right)^{\frac{1}{r}} \quad > \quad \frac{1-\lambda}{2} \tag{4.14}$$

We can now define a new function φ by:

$$\varphi(z) = \lambda^{\nu-1}\left(z_1\lambda^{1-\nu}v\right)^p(\lambda z)\ , \quad z \in \Omega(-1/\lambda,\ 1/\lambda^2)\ . \tag{4.15}$$

In this domain, φ is a Pick function, which extends continuously to the ends of its real interval of definition, and

$$\varphi(-1/\lambda)\ = 0 \quad \text{if} \quad p = 1, \tag{4.16}$$

$$\varphi(-1/\lambda)\ =\ \lambda^{\nu-1}(z_1\lambda^{1-\nu}v)^{p-1}(0)\ \geq\ z_1\ >\ \lambda^\nu \quad \text{if} \quad p \geq 2, \tag{4.17}$$

$$\varphi(1)\ =\ 1\ , \quad \varphi(1/\lambda^2) = \lambda^{\nu-1}(z_1\lambda^{1-\nu}v)^p(1/\lambda) \leq\ z_1/\lambda^\nu\ . \tag{4.18}$$

The domain $\Omega(-1/\lambda,\ 1/\lambda^2)$ is thus a basin of attraction of the fixed point 1 of φ. This domain contains $\Omega(-1/\lambda,\ 1/a_0(\lambda))$ since $a_0(\lambda) \geq \lambda^2$ (see (4.4)).

We now use Schwarz's lemma, as mentioned in Section 2, to obtain an upper bound for $\varphi'(1)$. If $p \geq 2$,

$$\varphi'(1) \leq \frac{A'B'(A+B)}{AB(A'+B')} \quad \text{with}$$

$$A = 1 + \frac{1}{\lambda}, \quad B = \frac{1}{\lambda^2} - 1\ , \quad A' = 1 - z_1, \quad B' = \frac{z_1}{\lambda^\nu} - 1\ .$$

This gives

$$\varphi'(1) \leq \frac{\lambda(1-z_1)(z_1-\lambda^\nu)}{z_1(1-\lambda^\nu)(1-\lambda^2)}\ . \tag{4.19}$$

When $z_1 \in (\lambda^\nu,\ 1)$ this expression is maximum at $z_1 = \lambda^{\nu/2}$, so that

$$\varphi'(1) \leq \frac{\lambda}{Z_1(\lambda)} = \frac{\lambda}{(1+\lambda)(1+\sqrt{\lambda})^2} < \frac{1}{8} \quad \text{if} \quad p \geq 2\ . \tag{4.20}$$

Therefore, if $p \geq 2$ and we choose $b \geq b_0(r\nu) = (1/8)^{1/r\nu}$, then $\varphi'(1) < b^{r\nu}$. For a slightly better choice of b, we note that $\lambda \mapsto \lambda/Z_1(\lambda)$ is increasing on $(0,\ 1)$, so that $\lambda \leq b \Rightarrow \varphi'(1) \leq b/Z_1(b)$. This will be less than $b^{r\nu}$ if $b \geq b_1(r\nu)$, where $s \mapsto b_1(s)$ is the solution of $b_1^s = b_1/Z_1(b_1)$, i.e. the inverse function (defined on $(1,\ \infty)$) of

$$b \mapsto 1 + \frac{\log Z_1(b)}{\log(1/b)} = 1 + \frac{\log((1+b)(1+\sqrt{b})^2)}{\log(1/b)}\ . \tag{4.21}$$

This last function is strictly increasing on $(0,\ 1)$, and tends to 1 as b tends to 0 and to $+\infty$ as b tends to 1. Obviously $b_1(s) \leq b_0(s)$. A useful inequality (proved in Appendix 1) is:

$$\frac{\log((1+b)(1+\sqrt{b})^2)}{\log(1/b)} \quad > \quad \frac{2b}{1-b} \quad \forall b \in (0,\ 1) \tag{4.22}$$

i.e.

$$b = b_1(s), \quad s > 1 \quad \Rightarrow s > \frac{1+b}{1-b}\ . \tag{4.23}$$

If $p = 1$,

$$\varphi'(1) \leq \frac{\lambda(1-\lambda^\nu)}{(1-\lambda^2)} \leq \frac{\lambda}{(1+\lambda)} \leq \frac{1}{2}, \quad (p = 1). \tag{4.24}$$

Thus, if b is chosen at least equal to $b_2(r\nu) = (1/2)^{1/r\nu}$ or to $b_3(r\nu)$, where b_3 is the inverse function of $b \mapsto 1 + \log(1+b)/\log(1/b)$, it follows from $\lambda \le b$ that $\varphi'(1) \le b^{r\nu}$.

We now define the action of the operator $B(b,\ r,\ p,\ \nu)$ on Φ_0 by

$$B(b,\ r,\ p,\ \nu)\,\Phi_0 = \Phi = h_{b,\lambda}^{-1} \circ \varphi \circ h_{b,\lambda}\ . \tag{4.25}$$

This definition implies that if Φ_0 is a fixed point of $B(b,\ r,\ p,\ \nu)$, i.e. $\Phi = \Phi_0$, then the functions φ_0 and φ constructed above coincide, and λ, z_1, ψ, and $u(z) = z_1\lambda^{1-\nu}\psi(z)^{1/r}$ provide a solution to the problem set in Section 3. Conversely, given a solution to the problem, the function Φ given by Eq. (4.25) (with $\lambda = \Phi'(1)^{1/r\nu}$) is a fixed point of $B(b,\ r,\ p,\ \nu)$ for any $b \in [\lambda,\ 1)$.

The preceding estimates show that if $p \ge 2$ and $b \ge b_0(r\nu)$ or $b \ge b_1(r\nu)$, or if $p = 1$ and $b \ge b_2(r\nu)$ or $b \ge b_3(r\nu)$, then

$$B(b,\ r,\ p,\ \nu)\,\mathcal{Q}_0(b,\ r\nu) \subset \mathcal{Q}_0(b,\ r\nu)\ . \tag{4.26}$$

The same estimates show that, for *any* solution of our problem, the inequalities $\lambda < b_j(r\nu)$ must hold ($j = 0,\ 1$ for $p \ge 2$, $j = 2,\ 3$ for $p = 1$).

The set $\mathcal{Q}_0(b,\ r\nu)$ is not compact: we have to guard against λ tending to zero, i.e. to find a reproducing lower bound for λ. This will be feasible only under certain restrictions on r, ν, and p. We first show that such restrictions are unavoidable. $\varphi'(1)$ is given by

$$\begin{aligned}\varphi'(1) = \lambda^\nu \prod_{j=0}^{p-1} z_1\lambda^{1-\nu}v'(\zeta_j) &= \lambda^\nu \prod_{j=0}^{p-1} \frac{\zeta_{j+1}v'(\zeta_j)}{v(\zeta_j)} \\ = \prod_{j=0}^{p-1} \frac{\zeta_j v'(\zeta_j)}{v(\zeta_j)} &= \prod_{j=0}^{p-1} \frac{-\zeta_j\psi'(-\zeta_j)}{r\,\psi(-\zeta_j)}\ .\end{aligned} \tag{4.27}$$

Here we have used $\zeta_p/\zeta_0 = \lambda^{-\nu}$. The upper bound in (2.12) (with $u_- = \lambda$) give

$$\begin{aligned}\varphi'(1) \le \prod_{j=0}^{p-1} \frac{\zeta_j(1+\lambda)}{r(1+\zeta_j)(1-\lambda\zeta_j)} &\le \frac{\lambda}{r(1-\lambda^2)}\left(\frac{\lambda^{1-\nu}(1+\lambda)}{r(1+\lambda^{1-\nu})(1-\lambda^{2-\nu})}\right)^{p-1} \\ &\le \lambda^{1+(p-1)(1-\nu)}\,(r\,(1-\lambda))^{-p}\ .\end{aligned} \tag{4.28}$$

If we suppose $p \ge 2$ and $\lambda \le b_1(r\nu)$, then $r(1-\lambda) \ge 1 + \lambda > 1$ by the inequality (4.23). Therefore a fixed point can exist only if

$$r\nu - 1 - (p-1)(1-\nu) > 0\ . \tag{4.29}$$

Using (4.27) and the lower bound in (2.12), with $u_+ = a_0(\lambda) < b$ gives

$$\varphi'(1) \ge \prod_{j=0}^{p-1} \frac{\zeta_j(1-b)}{r(1+\zeta_j)(1+b\zeta_j)}\ , \tag{4.30}$$

and using $1 \ge \zeta_j \ge z_1\lambda^{1-\nu}$ (for $j > 0$), and $z_1 \ge (1-b)/2$ (see (4.14)), we find

$$\varphi'(1) \ge c^p\lambda^{1+(p-1)(1-\nu)}, \qquad c = \frac{(1-b)^2}{4r(1+b)}\ . \tag{4.31}$$

Assume now that $\lambda \geq \lambda_0 > 0$. Then a sufficient condition for $\varphi'(1) \geq \lambda_0^{r\nu}$ to hold is that

$$\lambda_0^{r\nu-1-(p-1)(1-\nu)} \leq c^p \ . \tag{4.32}$$

If the condition (4.29) holds, we can take

$$\lambda_0 = \lambda_0(p,\ r,\ \nu) = \left(\frac{(1-b)^2}{4r(1+b)}\right)^{\frac{p}{r\nu-1-(p-1)(1-\nu)}} \in (0,\ 1). \tag{4.33}$$

Assume that the inequality (4.29) holds. Let, for definiteness, $b(r\nu) = b_1(r\nu)$ if $p \geq 2$, and $b(r\nu) = b_3(r\nu)$ if $p = 1$. We observe that

$$\mathcal{Q}_1(p,\ r,\ \nu) = \mathcal{Q}_0(b(r\nu),\ r\nu) \cap \{\Phi\ :\ \Phi'(1) \geq \lambda_0(p,\ r,\ \nu)^{r\nu}\} \tag{4.34}$$

is not empty. Indeed the function $\chi_{b,s}$ (see (2.8)) with $b = b(r\nu)$ and $s = r\nu$ belongs to $\mathcal{Q}_0(b(r\nu),\ r\nu)$ and $\chi'_{b,s} = b^{r\nu}$. Therefore $\Phi = B(b(r\nu),\ r,\ p,\ \nu)\,\chi_{b,s}$ also belongs to $\mathcal{Q}_0(b(r\nu),\ r\nu)$, and the preceding estimates show that $b^{r\nu} \geq \Phi'(1) \geq b^{1+(p-1)(1-\nu)}c^p$ with c as in (4.31), and $b = b(r\nu)$. Hence $b(r\nu) \geq \lambda_0(p,\ r,\ \nu)$, in particular $\chi_{b,s} \in \mathcal{Q}_1(p,\ r,\ \nu)$. (This is not really essential since we could have redefined $\lambda_0(p,\ r,\ \nu)$ to be less than $b(r\nu)$.) The continuous map $B(b(r\nu),\ r,\ p,\ \nu)$ maps the compact convex non-empty set $\mathcal{Q}_1(p,\ r,\ \nu)$ into itself. Therefore it has a fixed point there by the Schauder-Tikhonov theorem. As noted before, if $\Phi_0 = \Phi$ is such a fixed point, the functions φ_0 and φ constructed as above coincide, and ψ and $u(z) = z_1\lambda^{1-\nu}\psi(z)^{1/r}$ provide a solution to our problem. Note that here again *any* solution must satisfy $\lambda \geq \lambda_0(p,\ r,\ \nu)$, since it must satisfy $\varphi'(1) = \lambda^{r\nu} \geq c^p\lambda^{1+(p-1)(1-\nu)}$, with c as in (4.31). Thus any solution is associated to a fixed point of $B(b(r\nu),\ r,\ p,\ \nu)$ in $\mathcal{Q}_1(p,\ r,\ \nu)$.

5 Case $p = 1$ and $\nu \in [1,\ 2]$

This case has been dealt with in [E3]. It will be shown in this section that the method of the preceding section also applies to this case with minor modifications. Let $r > 1$ and $\nu \in [1,\ 2]$ be fixed reals. We define the space $\mathcal{Q}(b,\ r\nu)$ and the operator $B(b,\ r,\ 1,\ \nu)$ in the same way as in the preceding section. In particular, starting from $\Phi_0 \in \mathcal{Q}(b,\ r\nu)$, the functions φ_0, ψ and v are defined by the same formulae and have the same properties, in particular

$$v(-1) = 0, \quad v(0) = 1, \quad v(1/\lambda) \leq \lambda^{-\nu}\ , \tag{5.1}$$

but we note that now $\lambda^{-\nu} \geq 1/\lambda$. We define

$$z_1 = 1/v(\lambda) \in (0,\ 1). \tag{5.2}$$

It follows from (5.1) that

$$z_1 > \lambda^{\nu} \tag{5.3}$$

and from the upper bound (2.11) on $\psi(-\lambda)$ that

$$z_1 \geq (1-\lambda)^{1/r} > 1 - \lambda\ . \tag{5.4}$$

The function

$$z \mapsto \varphi(z) = z_1 v(\lambda z) \tag{5.5}$$

is again Herglotzian, holomorphic in $\Omega(-1/\lambda,\ 1/\lambda^2)$, continuous on $[-1/\lambda,\ 1/\lambda^2]$ with

$$\varphi(-1/\lambda)\ = 0, \quad \varphi(1) = 1, \quad \varphi(1/\lambda^2) = z_1 v(1/\lambda) \le z_1/\lambda^\nu < 1/\lambda^2\ . \tag{5.6}$$

Schwarz's lemma can be again applied as in the preceding section, but now with

$$A = 1 + \frac{1}{\lambda}, \quad B = B' = \frac{1}{\lambda^2} - 1, \quad A' = 1\ . \tag{5.7}$$

This gives

$$\varphi'(1) \le \lambda\ , \tag{5.8}$$

which is not sufficient for our purposes. We therefore use the bound (2.14) with $u_- = \lambda$ and $M = 1/\lambda^{r\nu}$, to get

$$\frac{zv'(z)}{v(z)} = -\frac{z\psi'(-z)}{r\,\psi(-z)} \le \frac{\nu\log(1/\lambda)}{4(1-\lambda z)} \quad \forall z \in (0,\ 1/\lambda)\ , \tag{5.9}$$

hence

$$\varphi'(1) = \frac{\lambda v'(\lambda)}{v(\lambda)} \le \frac{\log(1/\lambda)}{2(1-\lambda^2)}\ . \tag{5.10}$$

The r.h.s. of this inequality is a decreasing function of λ, tending to $+\infty$ when $\lambda \to 0$, and to $1/4$ when $\lambda \to 1$. For any choice of $b \in (0,\ 1)$, if $\lambda < b^{r\nu}$, then $\varphi'(1) < b^{r\nu}$ by (5.8). If $\lambda \ge b^{r\nu}$, then, by (5.10), a sufficient condition for $\varphi'(1) < b^{r\nu}$ is that $b^{r\nu} > \mu$, where μ is the unique zero, in $(0,\ 1)$, of the increasing function

$$x \mapsto x - \frac{\log(1/x)}{2(1-x^2)}\ . \tag{5.11}$$

One finds $\mu < 0.479$, and we choose, from now on, $b = b_5(r\nu) = (0.479)^{1/r\nu}$. Note that for *any* solution, $\varphi'(1) = \lambda^{r\nu}$ must satisfy (5.10), and since the rhs of this inequality is decreasing, the function defined in (5.11) must be negative at $x = \lambda^{r\nu}$, i.e. $\lambda \le b_5(r\nu)$.

The lower bound in (2.12), with $u_+ = a_0(\lambda) < b$, gives

$$\varphi'(1) = -\frac{\lambda\psi'(-\lambda)}{r\psi(-\lambda)} \ge \frac{\lambda(1-a_0(\lambda))}{r\,(1+\lambda)(1+\lambda a_0(\lambda))} \ge \frac{\lambda(1-b)}{r\,(1+b)(1+b^2)}\ . \tag{5.12}$$

If $\lambda \ge \lambda_0 > 0$ then $\varphi'(1) \ge \lambda_0^{r\nu}$ provided $\lambda_0 \le \lambda_0(r,\ \nu)$ with

$$\lambda_0(r,\ \nu) = \left(\frac{(1-b)}{r\,(1+b)(1+b^2)}\right)^{1/(r\nu-1)}\ , \qquad b = b_5(r\nu)\ . \tag{5.13}$$

Therefore the operator $B(b_5(r\nu),\ r,\ 1,\ \nu)$ preserves the compact convex set

$$\mathcal{Q}_0(b_5(r\nu),\ r\nu) \cap \{\Phi\ :\ \Phi'(1) \ge \lambda_0(r,\ \nu)\}. \tag{5.14}$$

Again *any* solution must satisfy $\lambda \ge \lambda_0(r,\ \nu)$.

6 Case $p \geq 2$ and $\nu \in [1,\ 2]$

In this section, $r > 1$ and $\nu \in [1,\ 2]$ are fixed real numbers, and $p \geq 2$ is a fixed integer. The real number $b \in [1/2,\ 1)$ is also fixed, but its value will be chosen later (as a function of r). We shall need the function $a : [0,\ 1] \to [0,\ 1]$ given by

$$a(t) = \min\left\{\frac{2t}{1-t}\ ,\ \frac{1+t}{2}\right\} = \begin{cases} \dfrac{2t}{1-t} & \text{if} \quad 0 \leq t \leq \sqrt{5}-2\ , \\ \dfrac{1+t}{2} & \text{if} \quad \sqrt{5}-2 \leq t \leq 1\ . \end{cases} \tag{6.1}$$

$t \mapsto a(t)$ is continuous and strictly increasing in $[0,\ 1]$. (Note that $\sqrt{5}-2 \approx 0.236 < 1/4$.)

We denote $\widetilde{\mathcal{Q}}_0(b,\ \nu r)$ the space of all functions Φ with the following properties:

($\tilde{\mathrm{Q}}$1) Φ is a Pick function holomorphic in the domain:

$$\Omega(-1/b,\ 1/a(b)) = \mathbf{C}_+ \cup \mathbf{C}_- \cup \left(-\frac{1}{b},\ \frac{1}{a(b)}\right)\ , \quad a(b) = \frac{1+b}{2}\ , \tag{6.2}$$

and maps this domain into itself.

($\tilde{\mathrm{Q}}$2) $\Phi(z) \geq 0$ for all $z \in (-1/b,\ 1/a(b))$,

($\tilde{\mathrm{Q}}$3) $\Phi(1) = 1$, and $0 < \Phi'(1) \leq b^{\nu r}$.

$\widetilde{\mathcal{Q}}_0(b,\ \nu r)$ is a convex subset of the real Fréchet space of all self-conjugated functions holomorphic in $\Omega(-1/b,\ 1/a(b))$. It is not empty since it contains the function $\chi_{b,\nu r}$ (see Section 2).

6.1 The operator $B(b,\ r,\ p,\ \nu)$

We shall define a continuous operator $B(b,\ r,\ p,\ \nu)$ on the space $\widetilde{\mathcal{Q}}_0(b,\ \nu r)$ by describing its action on an arbitrary element Φ_0.

Given $\Phi_0 \in \widetilde{\mathcal{Q}}_0(b,\ \nu r)$, we denote $\lambda = \Phi_0'(1)^{1/r\nu}$. Note that $\lambda \leq b$. We define a function φ_0 by

$$\varphi_0 = h_{b,\lambda} \circ \Phi_0 \circ h_{b,\lambda}^{-1}\ . \tag{6.3}$$

Here $h_{b,\lambda}$ is the homographic function defined in Section 4 (see (4.1)). It maps the domain $\Omega(-1/b,\ 1/a(b))$ onto $\Omega(-1/\lambda,\ 1/a_1(\lambda))$, where $1/a_1(\lambda) = h_{b,\lambda}(1/a(b))$, i.e.

$$a_1(\lambda) = \frac{1+\lambda}{2} + \frac{b-\lambda}{1+b} \ \geq a(b) \ \geq \ a(\lambda), \qquad a_1(\lambda) \ \leq \ a_1(0) = \frac{1+3b}{2(1+b)}\ . \tag{6.4}$$

The function φ_0 possesses the following properties:

($\tilde{\mathrm{Q}}'$1) φ_0 is a Pick function holomorphic in $\Omega(-1/\lambda, 1/a_1(\lambda))$, and maps this domain into itself.

($\tilde{\mathrm{Q}}'$2) $\varphi_0(z) \geq 0$ for all $z \in (-1/\lambda,\ 1/a_1(\lambda))$.

($\tilde{\mathrm{Q}}'$3) $\varphi_0(1) = 1$, and $\varphi_0'(1) = \lambda^{\nu r}$.

As in previous sections we denote ψ the linearizer of φ_0, normalized by the condition $\psi(0) = 1$, i.e.

$$\psi(z) = \frac{1}{\lambda^{\nu r}}\psi(\varphi_0(z)) \quad \forall z \in \Omega(-1/\lambda, 1/a_1(\lambda)), \quad \psi(1) = 0, \quad \psi(0) = 1 \ . \tag{6.5}$$

ψ is anti-Herglotz, holomorphic in $\Omega(-1/\lambda, 1/a_1(\lambda))$, and satisfies the inequalities (2.11) and (2.12), with $u_- = \lambda$ and $u_+ = a_1(\lambda)$. Also,

$$\psi(-1/\lambda) = \frac{1}{\lambda^{\nu r}}\psi(\varphi_0(-1/\lambda)) \quad \le \frac{1}{\lambda^{\nu r}} \ . \tag{6.6}$$

We again define $v(z) = (\psi(-z))^{1/r}$ fo all $z \in \Omega(-1,\ 1/\lambda)$. This is a Pick function which extends to a strictly increasing continuous function on $[-1,\ 1/\lambda]$. It satisfies:

$$v(-1) = 0, \quad v(0) = 1, \quad v(1/\lambda) \le \lambda^{-\nu} \ . \tag{6.7}$$

We now show that there is a unique $z_1 \in (0,\ 1)$ such that

$$\left(\frac{z_1}{\lambda^{\nu-1}}v\right)^p(\lambda) = \frac{1}{\lambda^{\nu-1}} \ . \tag{6.8}$$

For real $s \ge 0$ let $x_0(s) = \lambda$, $x_1(s) = s\lambda^{1-\nu}v(\lambda)$. The function $s \mapsto x_1(s)$ is strictly increasing on $\mathbf{R}_+$ and takes the values λ at $s_* = \lambda^\nu/v(\lambda)$ and $1/\lambda$ at $s_1 = \lambda^{\nu-2}/v(\lambda)$. By induction we can construct a strictly decreasing infinite sequence $s_1 > \ldots > s_j > \ldots > s_*$ such that, for $j \ge 2$, $s \mapsto x_j(s) = (s\lambda^{1-\nu}v)^j(\lambda)$ is continuous and strictly increasing on $[s_*,\ s_{j-1}]$, $x_0(s) < \ldots < x_j(s)$ in $(s_*,\ s_{j-1}]$, $x_j(s_*) = \lambda$, and $x_j(s_j) = 1/\lambda$. Indeed it follows that $x_{j+1}(s) = s\lambda^{1-\nu}v(x_j(s))$ is defined, continuous, and strictly increasing on $[s_*,\ s_j]$ and $x_{j+1}(s) > s\lambda^{1-\nu}v(x_{j-1}(s)) = x_j(s)$ for all $s \in (s_*,\ s_j]$. Since $x_{j+1}(s_j) > x_j(s_j) = 1/\lambda$ and $x_{j+1}(s_*) = \lambda$, s_{j+1} exists in $(s_*,\ s_j)$. In particular $x_p(s_{p-1}) > 1/\lambda$. Therefore there is a unique $z_1 \in (s_*,\ s_{p-1})$ such that $x_p(z_1) = \lambda^{1-\nu}$. It must satisfy $z_1 < 1$ since $z_1 v(x_{p-1}(z_1)) = 1$, and $v(x_{p-1}(z_1)) > 1$. Note also that for $s \in (s_*,\ s_{p-1})$, there exists a unique $x_{-1}(s) < x_0(s) = \lambda$ such that $s\lambda^{1-\nu}v(x_{-1}(s)) = \lambda$. The function $z \mapsto s_*\lambda^{1-\nu}v(z)$ maps $\Omega(-1,\ 1/\lambda)$ into $\Omega(0,\ \lambda^{1-\nu}/v(\lambda))$, so that it has a unique and attractive fixed point at λ by Schwarz's lemma. Hence $s_*\lambda^{1-\nu}v(x) \ge x$ for all $x \in [-1,\ \lambda]$. When $s > s_*$, $s\lambda^{1-\nu}v(x) > x$ for all $x \in [-1,\ \lambda]$. Since this includes $[x_{-1}(s),\ x_0(s)]$, it follows that $s\lambda^{1-\nu}v(x) > x$ for all $x \in [-1, x_p(s)]$, for all $s \in (s_*,\ z_1]$. The function x_p is analytic, with a strictly positive derivative, on $(s_*,\ s_{p-1})$. Therefore z_1 depends continuously on v, hence on φ_0.

We denote $\zeta_j = (z_1\lambda^{1-\nu}v)^j(\lambda)$, $(0 \le j \le p+1)$:

$$\lambda = \zeta_0 < \zeta_1 < \ldots \zeta_p = \frac{1}{\lambda^{\nu-1}} < \zeta_{p+1} = \frac{z_1}{\lambda^{\nu-1}}v(1/\lambda^{\nu-1}), \tag{6.9}$$

Since $v(1/\lambda^{\nu-1}) \le \lambda^{-\nu}$,

$$z_1 > \lambda^\nu \ , \qquad \frac{z_1}{\lambda^{\nu-1}} > \lambda \ , \tag{6.10}$$

and since $v(\lambda) > v(0) = 1$,

$$\zeta_1 > \frac{z_1}{\lambda^{\nu-1}} \ . \tag{6.11}$$

We have seen above that

$$\frac{z_1}{\lambda^{\nu-1}} v(x) > x \quad \forall x \in [-1,\ 1/\lambda^{\nu-1}] \ . \tag{6.12}$$

Applying this to $x = 1$ gives $z_1/\lambda^{\nu-1} \geq 1/v(1)$, and, using (2.11),

$$\frac{z_1}{\lambda^{\nu-1}} \ \geq \ \left(\frac{1-\lambda}{2}\right)^{\frac{1}{r}} \ > \ \frac{1-\lambda}{2} \ . \tag{6.13}$$

The function φ is defined by:

$$\varphi(z) = \lambda^{\nu-1} \left(\frac{z_1}{\lambda^{\nu-1}} v\right)^p (\lambda z) \ , \quad z \in \mathbf{C}_+ \cup \mathbf{C}_- \cup (-1/\lambda,\ \zeta_1/\lambda) \ . \tag{6.14}$$

In this domain, φ is a Pick function, which extends continuously to the ends of its real interval of definition, and

$$\begin{aligned} \varphi(-1/\lambda) \ &= \ \lambda^{\nu-1} \left(\frac{z_1}{\lambda^{\nu-1}} v\right)^{p-1} (0) \ \geq \ z_1 \ \geq \ \lambda^\nu \ \geq \ \lambda^2 \ , \\ \varphi(1) \ &= \ 1 \ , \quad \varphi(\zeta_1/\lambda) \ = \ z_1 v(1/\lambda^{\nu-1}) \ \leq \ \frac{z_1}{\lambda^\nu} \ < \ \zeta_1/\lambda \ . \end{aligned} \tag{6.15}$$

(Note that the first inequality in (6.15) has used $p \geq 2$.) The domain $\Omega(-1/\lambda,\ \zeta_1/\lambda)$ is a basin of attraction of the fixed point 1 of φ, hence $\varphi'(1) < 1$ by Schwarz's lemma. For a better upper bound on this derivative, we shall need a better lower bound for ζ_1/λ. This is provided by

Lemma 2 *The inequality*

$$\frac{z_1}{\lambda^{\nu-1}} v(z) \geq \frac{z(1-2\lambda)+\lambda}{1-\lambda z} \tag{6.16}$$

holds for all $z \in [0,\ 1]$.

Proof. This is simply the result of applying Lemma 1 of Section 2, with $f = z_1 \lambda^{1-\nu} v$, and $a = 0$, $b = 1$, $B = 1/\lambda$. This function satisfies $f(0) = z_1 \lambda^{1-\nu} \geq \lambda$ by (6.10), and $f(1) \geq 1$ by (6.12).

For $z = \zeta_0 = \lambda$, this gives $\zeta_1 \geq 2\lambda/(1+\lambda)$. Since we also have the lower bounds (6.11) and (6.13),

$$\frac{\zeta_1}{\lambda} \geq \max\left\{\frac{2}{1+\lambda},\ \frac{1-\lambda}{2\lambda}\right\} \ = \ \frac{1}{a(\lambda)} \ . \tag{6.17}$$

This is the reason for our original definition of the function a in (6.1). We conclude that the domain $\Omega(-1/\lambda,\ \zeta_1/\lambda)$ where φ is holomorphic, and which it maps into itself, certainly contains the domain of analyticity $\Omega(-1/\lambda,\ 1/a_1(\lambda))$ of φ_0 in view of (6.4). We now use Schwarz's lemma, as mentioned in Section 2, to obtain an upper bound for $\varphi'(1)$:

$$\varphi'(1) \leq \frac{A'B'(A+B)}{AB(A'+B')} \quad \text{with}$$
$$A = 1 + \frac{1}{\lambda} \quad , \quad B = B' = \frac{1}{a(\lambda)} - 1, \quad A' = 1 - \lambda^2 \ . \tag{6.18}$$

This gives

$$\begin{aligned}
\varphi'(1) &\le \frac{(1-\lambda^2)(a(\lambda)+\lambda)}{(1+\lambda)(1-a(\lambda)\lambda^2)} \\
&\le Z(\lambda) = \frac{1+3\lambda}{2+2\lambda+\lambda^2} \\
&\le Z_{\max} = 9/(4+2\sqrt{13}) < 0.803. \qquad (6.19)
\end{aligned}$$

Therefore choosing $b \ge b_6(r\nu) = m_0^{1/r\nu}$, $m_0 = 0.803$ ensures that $\varphi'(1) < b^{r\nu}$.
We define the operator $B(b,\ r,\ p,\ \nu)$ by

$$\Phi = B(b,\ r,\ p,\ \nu)\Phi_0 = h_{b,\lambda}^{-1} \circ \varphi \circ h_{b,\lambda}\ . \qquad (6.20)$$

It then follows from the preceding estimates that, if $b \ge b_6(r\nu)$,

$$B(b,\ r,\ p,\ \nu)\, \widetilde{\mathcal{Q}}_0(b,\ \nu r) \subset \widetilde{\mathcal{Q}}_0(b,\ \nu r)\ . \qquad (6.21)$$

In the remainder of this section, it will always be understood that $b = b_6(r\nu)$.

6.1.1 Lower bound for $\varphi'(1)$.

We use

$$\varphi'(1) = \prod_{j=0}^{p-1} \frac{-\zeta_j \psi'(-\zeta_j)}{r\,\psi(-\zeta_j)}\ . \qquad (6.22)$$

The lower bound in (2.12) gives, for $\zeta \in [0,\ 1/\lambda)$,

$$\frac{-\zeta\psi'(-\zeta)}{\psi(-\zeta)} \ge \frac{\zeta(1-c)}{(1+\zeta)(1+c\zeta)}\ , \qquad (6.23)$$

Here $c = a_1(\lambda)$, where $a_1(\lambda)$ is given by (6.4) and satisfies

$$a(\lambda) \le a(b) \le a_1(\lambda) < a_1(0) = \frac{1+3b}{2(1+b)}\ . \qquad (6.24)$$

However it will be convenient to suppose only, at first, that (6.23) holds for a certain c satisfying $a(\lambda) \le c < 1$. For $j > 0$, $\zeta_j \ge \zeta_1 \ge \lambda/a(\lambda)$ hence $\zeta_j \in [\lambda/c,\ 1/\lambda]$. When ζ varies in this interval, the second expression in (6.23) is minimum at $\zeta = 1/\lambda$. Therefore

$$\varphi'(1) \ge \lambda^p \left(\frac{1-c}{r\,(1+\lambda)(1+c\lambda)}\right)\left(\frac{1-c}{r\,(1+\lambda)(\lambda+c)}\right)^{p-1}\ . \qquad (6.25)$$

It is easy to verify that the rhs of this inequality is decreasing in c and increasing in λ provided $c \ge \lambda^2$ (note that $a(\lambda) > \lambda$). Setting now $c = a_1(\lambda)$, and using the inequalities (6.24) and $\lambda \le b$, this gives

$$\varphi'(1) \ge \lambda^p \left(\frac{1-b}{16r}\right)^p\ . \qquad (6.26)$$

Supposing $\lambda \ge \lambda_0 > 0$, the last inequality will imply $\varphi'(1) \ge \lambda_0^{r\nu}$ if λ_0 satisfies

$$\lambda_0^{r\nu-p} \le \left(\frac{1-b}{16r}\right)^p\ , \qquad (6.27)$$

and this is possible only if $r\nu - p > 0$. In this case we can choose

$$\lambda_0 = \lambda_0(r,\ \nu) = \left(\frac{1-b}{16r}\right)^{\frac{p}{r\nu-p}}\ , \tag{6.28}$$

and obtain the existence of a fixed point in the same way as in the preceding sections. Recall that in these formulae, b stands for $b_6(r\nu) = m_0^{1/r\nu}$. It is easy to verify that $\lambda_0(r,\ \nu) \to 1$ when $r \to \infty$.

The condition $r\nu - p > 0$ is just a limitation of the present method. The inadequacy of the estimate (6.26) is due to the fact that $a_1(\lambda)$ does not tend to 0 as λ tends to 0. By contrast, in the case of fixed points, the lower bound on λ can be improved. Indeed, since ψ and φ are holomorphic in $\Omega(-1/\lambda,\ 1/a(\lambda))$, the bound (6.23) and consequently (6.25) hold with c replaced by $a(\lambda)$ (instead of $a_1(\lambda)$). Assume $\lambda \le 1/7$. We can then set $c = 2\lambda/(1-\lambda)$ in (6.25) and obtain

$$\varphi'(1) \ge \lambda \left(\frac{1-3\lambda}{r\,(1+\lambda)(1-\lambda+2\lambda^2)}\right)\left(\frac{1-3\lambda}{r\,(1+\lambda)(3-\lambda)}\right)^{p-1} > \lambda\,(6r)^{-p}\ . \tag{6.29}$$

Therefore the lower bound

$$\lambda\ =\ \varphi'(1)^{1/r\nu} \ge \min\{1/7,\ (6r)^{-p/(r\nu-1)}\} \tag{6.30}$$

holds for all fixed points.

This fact suggests the use of another operator instead of $B(b,\ r,\ p,\ \nu)$, and this will be done in the next subsection.

6.2 The operator $N(b,\ r,\ p,\ \nu,\ \lambda_1)$

In this subsection, we define a new operator $N(b,\ r,\ p,\ \nu,\ \lambda_1)$ on the space $\widetilde{\mathcal{Q}}_0(b,\ \nu r)$. This construction closely follows an idea of Mestel and Osbaldestin [MO]. It consists in replacing the operator $B(b,\ r,\ p,\ \nu)$ (which is analytic on $\widetilde{\mathcal{Q}}_0(b,\ \nu r)$) by a "truncated version" $N(b,\ r,\ p,\ \nu,\ \lambda_1)$ which is only continuous, but maps $\widetilde{\mathcal{Q}}_0(b,\ \nu r)$ into a compact subset. This operator depends on an additional real parameter $\lambda_1 \in (0,\ 1/2)$. It will be later shown later that for small values of this parameter, any fixed point of $N(b,\ r,\ p,\ \nu,\ \lambda_1)$ is a fixed point of $B(b,\ r,\ p,\ \nu)$.

The notations are the same as in the preceding subsection unless explicitly mentioned. In particular $\nu \in [1,\ 2]$ and $r > 1$ are fixed and b will stand for $b_6(r\nu) = m_0^{1/r\nu}$, $m_0 = 0.803$. We denote $\tau_1 = \lambda_1^r$.

We define $N(b,\ r,\ p,\ \nu,\ \lambda_1)$ by its action on an arbitrary element Φ_0 of $\widetilde{\mathcal{Q}}_0(b,\ \nu r)$. Let $\sigma^\nu = \Phi_0'(1)$. Recall that, by the definition of $\widetilde{\mathcal{Q}}_0(b,\ \nu r)$, $\sigma^\nu \le b^{r\nu}$.

If $\sigma^\nu \ge \tau_1^\nu$, we define

$$N(b,\ r,\ p,\ \nu,\ \lambda_1)\Phi_0 = B(b,\ r,\ p,\ \nu)\Phi_0, \quad (\sigma^\nu \ge \tau_1^\nu). \tag{6.31}$$

If $\sigma^\nu < \tau_1^\nu$, we define $\lambda = \lambda_1$ (so that $\lambda \le b$), and define φ_0, as before, by

$$\varphi_0 = h_{b,\lambda} \circ \Phi_0 \circ h_{b,\lambda}^{-1}\ . \tag{6.32}$$

The function φ_0 is holomorphic and Herglotzian in the domain $\Omega(-1/\lambda,\ 1/a_1(\lambda))$, which it maps into itself. Here $a_1(\lambda) = a_1(\lambda_1)$ is given by (6.4). φ_0 possesses the same properties as in Subsection 6.1, except for

$$\varphi_0'(1) = \sigma^\nu\ . \tag{6.33}$$

The linearizer ψ_1 is the unique function holomorphic in $\Omega(-1/\lambda,\ 1/a_1(\lambda))$ such that

$$\psi_1(z) = \frac{1}{\sigma^\nu}\psi_1(\varphi_0(z)) \quad \forall z \in \Omega(-1/\lambda,\ 1/a_1(\lambda)), \quad \psi_1(0) = 1,\ \ \psi_1(1) = 0\ . \tag{6.34}$$

It is anti-Herglotzian and satisfies

$$\psi_1(-1/\lambda) = \frac{1}{\sigma^\nu}\psi_1(\varphi_0(-1/\lambda)) \quad \le \frac{1}{\sigma^\nu}\ . \tag{6.35}$$

In the preceding subsection, much depended on the bound $\psi(-1/\lambda) \le \lambda^{-r\nu}$. To restore an analogous situation we define a new function ψ as

$$\psi = \theta_{\sigma^{-\nu},\tau_1^{-\nu}} \circ \psi_1\ , \tag{6.36}$$

where $\theta_{\sigma^{-\nu},\tau_1^{-\nu}}$ denotes the homographic function which fixes 0 and 1, and sends $\sigma^{-\nu}$ to $\tau_1^{-\nu}$:

$$\theta_{\sigma^{-\nu},\tau_1^{-\nu}}(z) = \frac{z(1-\sigma^\nu)}{z(\tau_1^\nu - \sigma^\nu) + 1 - \tau_1^\nu}\ . \tag{6.37}$$

This function is Herglotzian and has a pole at a negative value temporarily denoted k. As a consequence ψ is holomorphic and anti-Herglotzian in $\Omega(-1/\lambda,\ 1/a_2)$, where $1/a_2 = \psi_1^{-1}(k)$ if $k \in \psi_1((1,\ 1/a_1(\lambda)))$, and $1/a_2 = 1/a_1(\lambda)$ otherwise. For $z > 1$, $\psi_1(z) < 0$ and (using the inequalities (2.11)),

$$\psi_1(z) \ge \psi_2(z) = \frac{1-z}{1-a_1(\lambda)z} \tag{6.38}$$

If $y = \psi_1^{-1}(k) < 1/a_1$, we have, since ψ_2 is decreasing,

$$k = \psi_1(y) \ge \psi_2(y), \quad \psi_2^{-1}(k) \le y\ . \tag{6.39}$$

Thus ψ is holomorphic in $\Omega(-1/\lambda,\ 1/\ell)$, where $1/\ell = \psi_2^{-1}(k) = \psi_2(k)$. This gives:

$$\ell = \frac{\tau_1^\nu - \sigma^\nu + (1-\tau_1^\nu)a_1(\lambda)}{1-\sigma^\nu}, \tag{6.40}$$

$$a(\lambda) \quad \le \quad a_1(\lambda) \quad \le \quad \ell \quad < \quad a_3(\lambda) \quad = \quad \tau_1^\nu + (1-\tau_1^\nu)a_1(\lambda)\ . \tag{6.41}$$

The function ψ has been defined so as to satisfy $\psi(-1/\lambda) \le \lambda^{-r\nu}$. We now proceed to define v, z_1, φ etc. exactly as in the preceding subsection and obtain the same inequalities with the single exception that, in the lower bound (6.25), c must be replaced by $a_3(\lambda)$. Since $\lambda = \lambda_1$, we find

$$\varphi'(1) \ge l(\lambda_1) = \lambda_1^p \left(\frac{1-a_3(\lambda_1)}{r\,(1+\lambda_1)(1+a_3(\lambda_1)\lambda_1)}\right)\left(\frac{1-a_3(\lambda_1)}{r\,(1+\lambda_1)(\lambda_1+a_3(\lambda_1))}\right)^{p-1}. \tag{6.42}$$

Recall that φ is holomorphic in $\Omega(-1/\lambda,\ 1/a(\lambda))$ and maps this domain into itself, with $a(\lambda)$ given by (6.1). The bound (6.42) also holds in the cases when $\lambda > \lambda_1$ since then $a_1(\lambda) < a_3(\lambda_1)$.

Finally we define

$$N(b,\ r,\ p,\ \nu,\ \lambda_1)\Phi_0 = h_{b,\lambda}^{-1} \circ \varphi \circ h_{b,\lambda}\ . \tag{6.43}$$

The operator $N(b,\ r,\ p,\ \nu,\ \lambda_1)$ maps the domain $\widetilde{\mathcal{Q}}_0(b,\ \nu r)$ into $\widetilde{\mathcal{Q}}_0(b,\ \nu r)\cap\{\Phi\ :\ \Phi'(1)\geq l(\lambda_1)\}$, which is compact and convex, hence it has fixed points there.

Our task is now to prove that if λ_1 has been chosen sufficiently small, any fixed point of $N(b,\ r,\ p,\ \nu,\ \lambda_1)$ is actually a fixed point of $B(b,\ r,\ p,\ \nu)$. We assume, from now on, that $\lambda_1\leq 1/8$. Let Φ_0 be a fixed point of $N(b,\ r,\ p,\ \nu,\ \lambda_1)$. If $\sigma^\nu=\Phi_0'(1)\geq\tau_1^\nu$, there is nothing to prove. Otherwise, we have $\lambda=\lambda_1$ and $\varphi_0=\varphi$, so that φ_0 and ψ_1 are now holomorphic in $\Omega(-1/\lambda_1,\ 1/a(\lambda_1))$. Thus ψ is now holomorphic in $\Omega(-1/\lambda_1,\ 1/a_4(\lambda_1))$, with

$$a(\lambda_1)\quad<\quad a_4(\lambda_1)=\frac{\tau_1^\nu-\sigma^\nu+(1-\tau_1^\nu)a(\lambda_1)}{1-\sigma^\nu}\quad<\quad\tau_1^\nu+(1-\tau_1^\nu)a(\lambda_1)\ . \qquad (6.44)$$

Recalling that $\lambda_1\leq 1/8$, we find

$$a_4(\lambda_1)\quad\leq\quad\lambda_1^{r\nu}+2\lambda_1\frac{1-\lambda_1^{r\nu}}{1-\lambda_1}\quad\leq\quad\frac{3\lambda_1}{1-\lambda_1} \qquad (6.45)$$

Inserting this in the lower bound obtained by setting $\lambda=\lambda_1$ and $c=a_4(\lambda_1)$ in (6.25) gives

$$\varphi'(1)\geq\lambda_1\left(\frac{1-4\lambda_1}{r\,(1+\lambda_1)(1-\lambda_1+3\lambda_1^2)}\right)\left(\frac{1-4\lambda_1}{r\,(1+\lambda_1)(4-\lambda_1)}\right)^{p-1}\ , \qquad (6.46)$$

and, using $\lambda_1\leq 1/8$,

$$\varphi'(1)\geq\lambda_1(9r)^{-p}, \qquad (6.47)$$

and since $\lambda_1\geq(\varphi'(1))^{1/r\nu}$,

$$\varphi'(1)\geq(9r)^{-pr\nu/(r\nu-1)}\ . \qquad (6.48)$$

If we assume that λ_1 has been chosen so that

$$\lambda_1<(9r)^{-p/(r\nu-1)}\ , \qquad (6.49)$$

the inequality (6.48) contradicts our hypothesis that $\Phi_0'(1)<\lambda_1^{r\nu}$. Therefore Φ_0 is a fixed point of $B(b,\ r,\ p,\ \nu)$.

7 Properties of solutions

This section is devoted to some properties of the solutions, i.e. of functions ψ and u, and numbers $\nu\in(0,\ 1]$, $p\geq 2$, $r>1$, ($r\nu>1+(p-1)(1-\nu)$ if $\nu<1$), λ, z_1, satisfying the requirements of Section 3. These properties are extensions of those established for $p=1$ in [EE, EL, E2]. We do not consider the case $p=1$.

We denote $\varphi=\varphi_0,\ v,\ \zeta_0,\ldots,\zeta_{p+1}$, the objects constructed from ψ as in the definition of $B(b,\ r,\ p,\ \nu)$. We also denote $\tau=\lambda^r$, and

$$\begin{aligned}u(z)=\check{u}(-z)\quad&=\quad\frac{z_1}{\lambda^{\nu-1}}\,v(-z)\\&=\quad\frac{z_1}{\lambda^{\nu-1}}\,\psi(z)^{1/r}\quad=\quad U(z)^{1/r}\ ,\qquad z\in\Omega(-1/\lambda,\ 1)\ ,\\U(z)=\quad&\left(\frac{z_1}{\lambda^{\nu-1}}\right)^r\psi(z)\ ,\qquad z\in\Omega(-1/\lambda,\ \zeta_1/\lambda)\ .\end{aligned} \qquad (7.1)$$

Recall that it has been shown in Section 4 that

$$\frac{1}{\tau^\nu} \geq \frac{1}{\lambda}(1+\lambda)(1+\sqrt{\lambda})^2 > 8, \quad \lambda \leq b_1(r\nu), \quad \text{if } 0 < \nu \leq 1, \quad p \geq 2, \tag{7.2}$$

and

$$r\nu \geq \frac{1+\lambda}{1-\lambda}, \quad \lambda^{r\nu-1-(p-1)(1-\nu)} \leq (1+\lambda)^{-p} \quad \text{if } 0 < \nu \leq 1, \quad p \geq 2. \tag{7.3}$$

Moreover (4.33) and $r\nu \geq (1+b)/(1-b)$ give

$$\lambda \geq (4r^3\nu^2)^{-p/(r\nu-1-(p-1)(1-\nu))} \quad \text{if } 0 < \nu \leq 1, \quad p \geq 2. \tag{7.4}$$

For $1 < \nu \leq 2$, it was shown in Section 6 that

$$\lambda \leq b_6(r\nu) = m_0^{1/r\nu}, \quad m_0 = 0.803, \quad \frac{\zeta_1}{\lambda} \geq \frac{1}{a(\lambda)}, \quad (1 < \nu \leq 2, \ \ p \geq 2)\,, \tag{7.5}$$

where $a(\lambda)$ is defined in (6.1), and that

$$\lambda \ \geq \min\{1/7, \ (6r)^{-p/(r\nu-1)}\}, \quad (1 < \nu \leq 2, \quad p \geq 2)\ , \tag{7.6}$$

$$\lambda \ \geq \left(\frac{1-m_0^{1/r\nu}}{16r}\right)^{p/(r\nu-p)}, \quad (1 < \nu \leq 2, \quad 2 \leq p < r\nu)\ . \tag{7.7}$$

The function u has an angular derivative at infinity equal to zero (i.e. $u(z)/z$ tends to 0 as $z \to \infty$ in non real directions) because $u(z) = U(z)^{1/r}$, U is anti-Herglotzian, and $r > 1$. Similarly v and φ have zero angular derivative at infinity.

7.1 Analyticity

The function φ is holomorphic in $\Omega(-1/\lambda, \ \xi_{\max})$, where $\xi_{\max} = \lambda^{-2}$ if $\check{u}(1/\lambda) \leq 1/\lambda$ (as is the case for $\nu \leq 1$, since $\lambda\check{u}(1/\lambda) \leq z_1\lambda^{2-2\nu} < 1$). In this case,

$$\varphi(\xi_{\max}) \ = \ \varphi(\lambda^{-2}) \ = \ z_1 v(\check{u}^{p-1}(\lambda^{-1})) \ < \ \frac{z_1}{\lambda^\nu} \ < \lambda^{-2}\ . \tag{7.8}$$

If $\check{u}(1/\lambda) > 1/\lambda$, we denote $\xi_p = 1/\lambda^2$ and $\lambda\xi_{p-1} = \check{u}^{-1}(1/\lambda)$. We construct by a descending induction the strictly increasing sequence $\xi_1, \ldots, \ \xi_p$ satisfying $\check{u}^j(\lambda\xi_{p-j}) = \lambda\xi_p = 1/\lambda$. Supposing $\xi_{p-j} < \ldots \xi_p$ already constructed for a certain $j < p-1$, we have $\check{u}^{j+1}(\lambda\xi_{p-j}) = \check{u}(1/\lambda) > 1/\lambda$, while $\check{u}^{j+1}(\lambda) = \zeta_{j+1} < 1/\lambda$. Hence $\lambda\xi_{p-j-1} = \check{u}^{-(j+1)}(1/\lambda)$ exists in $(1, \ \xi_{p-j})$. We set $\xi_{\max} = \xi_1$ so that $\check{u}^{p-1}(\lambda\xi_{\max}) = 1/\lambda$. Recalling that $\check{u}^{p-1}(\zeta_1) = \lambda^{1-\nu}$, we find:

$$\frac{\zeta_1}{\lambda} \leq \xi_{\max} = \xi_1 < \xi_2 < \ldots < \xi_p = \lambda^{-2}\ . \tag{7.9}$$

The first inequality here is replaced by the equality $\xi_{\max} = \zeta_1/\lambda$ when $\nu = 2$ (and, of course, $p > 1$). More generally $\check{u}^{p-j}(\zeta_j) = \lambda^{1-\nu}$ implies $\zeta_j \leq \lambda\xi_j$ for all $j \in [1, \ p-1]$, equality holding when $\nu = 2$. Note (see (6.9) and (6.11)) that $z_1/\lambda^{\nu-1} < \zeta_1 < 1/\lambda^{\nu-1}$, and

$$\varphi(\xi_{\max}) \ = \ z_1 v(\lambda^{-1}) \ \leq \ \frac{z_1}{\lambda^\nu} \ < \ \zeta_1/\lambda\ . \tag{7.10}$$

In both cases the whole domain $\Omega(-1/\lambda,\ \xi_{\max})$ is a basin of attraction of the fixed point 1 of φ, hence the domain of ψ is also $\Omega(-1/\lambda,\ \xi_{\max})$, and

$$\begin{aligned} \psi(z) &= \frac{1}{\lambda^{\nu r}}\psi(\varphi(z)) \ , \\ \varphi(z) &= \lambda^{\nu-1}\check{u}^p(\lambda z) \ , \end{aligned} \tag{7.11}$$

hold for all $z \in \Omega(-1/\lambda,\ \xi_{\max})$. Also

$$u(z) = \frac{1}{\lambda^\nu}u(\lambda^{\nu-1}\check{u}^p(\lambda z)), \qquad z \in \Omega(-1/\lambda,\ 1)\ . \tag{7.12}$$

7.2 Univalence for $p \geq 2$

We prove in this subsection that ψ and φ are univalent in $\Omega(-1/\lambda,\ \xi_{\max})$. We temporarily denote

$$\begin{aligned} \phi_j(z) &= -\check{u}^j(\lambda z), \quad \text{for } 0 \leq j \leq p-1 \ , \\ \phi_p(z) &= \varphi(z) \ . \end{aligned} \tag{7.13}$$

Let c be fixed with $1 < c < \min\{1/\lambda,\ \zeta_1/\lambda\}$. We first verify that each ϕ_j, $0 \leq j \leq p$, maps the interval $(-1/\lambda,\ c)$ into an open interval X_j with closure contained in $(-1/\lambda,\ c)$. This is clear in the case $j = p$, since $\phi_p = \varphi$. For $j = 0$, $\phi_0(-1/\lambda) = 1$, and $\phi_0(c) = -\lambda c > -1$. If $1 \leq j \leq p-1$, ϕ_j is decreasing, $\phi_j(c) < \phi_j(-1/\lambda) \leq 0$ and $\phi_j(c) > -\check{u}^j(\zeta_1) = -\zeta_{j+1} \geq -1/\lambda$. Let X be the convex hull of $X_0 \cup\ \ldots\ \cup X_p$. This is an open interval with closure contained in $(-1/\lambda,\ c)$, such that, for all $j = 0,\ \ldots,\ p$, $\phi_j((-1/\lambda,\ c)) \subset X$.

Suppose that w' and w'' are distinct points in $\Omega(-1/\lambda,\ \xi_{\max})$ such that $\psi(w') = \psi(w'')$. This implies that w' and w'' are not real, and have imaginary parts of the same sign. We inductively construct a sequence of triples $\{w'_n,\ w''_n,\ j_n\}_{0\leq n<\infty}$, where $w'_0 = w'$, $w''_0 = w''$, and, for all $n \geq 0$, $w'_n \neq w''_n$ are non-real, $\psi(w'_n) = \psi(w''_n)$, and $0 \leq j_n \leq p$ is such that $w'_{n+1} = \phi_{j_n}(w'_n)$, $w''_{n+1} = \phi_{j_n}(w''_n)$. Assuming that w'_n and w''_n have already been constructed, it follows from (7.11) and the definition (7.13) of the functions ϕ_j that there is a unique j_n in $[0,\ p]$ such that $\phi_{j_n}(w'_n) \neq \phi_{j_n}(w''_n)$ and either $j_n = p$ or $\phi_{j_n+1}(w'_n) = \phi_{j_n+1}(w''_n)$. This implies that $\psi(\phi_{j_n}(w'_n)) = \psi(\phi_{j_n}(w''_n))$, and we take $w'_{n+1} = \phi_{j_n}(w'_n)$, $w''_{n+1} = \phi_{j_n}(w''_n)$. It is easy to see (as e.g. in [E2]) that, as n tends to infinity, the Poincaré distances, relative to $\mathbf{C}_+ \cup \mathbf{C}_- \cup X$, of w'_n and w''_n to the segment X tend to 0. Therefore as n becomes sufficiently large, the points w'_n and w''_n enter a complex neighborhood of the real segment X so thin that ψ is injective there, producing a contradiction. Thus ψ, and therefore also u and φ are univalent in their respective domains.

7.3 Boundary values of u

We show in this subsection that the restriction of u to the upper half-plane $\mathbf{C}_+$ extends to a continuous bounded injective function on the closed upper half-plane $\overline{\mathbf{C}_+}$. The same, of course, holds in the lower half-plane, since $u(z) = u^*(z^*)$.

We rewrite (7.12) as

$$u(z) = F(u(-\lambda z)), \tag{7.14}$$

$$F(z) = \frac{1}{\lambda^\nu} u(\lambda^{\nu-1} \check{u}^{p-1}(z)) \ . \tag{7.15}$$

The function F is anti-Herglotzian, holomorphic and univalent in $\Omega(-1,\ \zeta_1)$ and vanishes at $\zeta_1 = u(-\lambda)$. It has a fixed point at $u(0) = z_1 \lambda^{1-\nu}$ with $F'(z_1\lambda^{1-\nu}) = -1/\lambda$, and (since it is strictly decreasing) no other fixed point in the real interval $[-1,\ \zeta_1]$. The equation (7.14) can be rewritten as $F = u \circ (-\lambda^{-1}) \circ u^{-1}$ on the intersection of the domain of F with the range of u. This range includes the real segment $(0,\ u(-\lambda^{-1}))$ and hence $(0,\ \zeta_1]$, since $u(-\lambda^{-1}) > u(-\lambda) = \zeta_1$. Any periodic orbit of F in $[-1,\ \zeta_1]$ must be contained in $(0,\ \zeta_1]$, so that $\{u(0)\}$ is the only such orbit. Therefore the Herglotz function F^2 also has $u(0)$ as its unique real fixed point, with $F^{2\prime}(u(0)) = \lambda^{-2}$. Both F and F^2 have zero angular derivative at ∞ since $z \mapsto F(z)^r$ is anti-Herglotzian and $z \mapsto F^2(z)^r$ is Herglotzian. As in [EL, E2], the theory of iterations of maps of $\mathbf{C}_+$ into itself (Wolff-Denjoy-Valiron Theorem, see [V, Mi]) shows that, uniformly on every compact subset of $\mathbf{C}_+$, F^{2n} tends to a finite constant c as $n \to \infty$. In particular for all z in a fixed compact subset of $\mathbf{C}_-$, $u(\lambda^{-2n}z) = F^{2n}(u(z))$ tends uniformly to c, i.e. $c = u(-i\infty)$.

For $n = 0, 1, 2, ...$, we denote I_n the closed real interval

$$I_n = (-\lambda)^{-n}[1,\ \lambda^{-2}]. \tag{7.16}$$

We first consider the case when $\xi_{\max} = \lambda^{-2}$ (in particular the case $0 < \nu \leq 1$), for which the argument of the case $p = 1$ [EL, E2] can be repeated almost verbatim. Let z follow $I_0 - i0$. Then $u(z)$ follows the segment

$$\tau_0 = e^{i\pi/r}[0,\ |U(\lambda^{-2})|^{1/r}]. \tag{7.17}$$

If z crosses the interior of I_0 into $\mathbf{C}_+$, u gets continued by $v_0 \equiv e^{2\pi i/r}u$. The image of $\mathbf{C}_+$ given by v_0 is contained in $\{z\ :\ \pi/r < \arg z < 2\pi/r\}$. It is contained in $\mathbf{C}_+$ if and only if $r \geq 2$ (in particular if r is an integer). Let $\mathcal{V}_0$ denote the open set $\{z \in \mathbf{C}_+\ :\ v_0(z) \in \mathbf{C}_+\}$, and, for $n \geq 1$, $\mathcal{V}_n = -\lambda^{-1}\mathcal{V}_{n-1}^*$ (so that $\mathcal{V}_n = \mathbf{C}_+$ when $r \geq 2$). If z follows $I_1 - i0$, then $-\lambda z$ follows $I_0 + i0$ and, by (7.14), $u(z)$ follows the analytic arc $\tau_1 = F(\tau_0^*)$ which lies entirely in $\mathbf{C}_+$ except for its starting point, $u(-\lambda^{-1}) = F(0)$. An easy induction shows that when z follows $I_n - i0$, $(n \geq 1)$, $u(z)$ follows an analytic arc τ_n lying entirely in $\mathbf{C}_+$ for $n > 1$, and u can be continued across the interior of I_n into $\mathbf{C}_+$ by a function v_n holomorphic in $\mathcal{V}_n$, with $v_n(\mathcal{V}_n) \subset \mathbf{C}_+$ and

$$\begin{aligned} \tau_n &= F(\tau_{n-1}^*) \ , \\ v_n(z) &= F(v_{n-1}(-\lambda z^*)^*) \ . \end{aligned} \tag{7.18}$$

The starting point of τ_{n+2} is the end of τ_n, and $\tau_{n+2} = F^2(\tau_n)$. Hence the arcs τ_n tend to the point c. Thus $u|\mathbf{C}_+$ extends to a continuous bounded function on $\overline{\mathbf{C}_+}$. This function is injective. Indeed at each step of its inductive construction, a new extension is obtained by composing copies of the previously constructed extension and scalars.

We now consider the case when $\xi_{\max} < \lambda^{-2}$ which occurs if $\check{u}(\lambda^{-1}) > \lambda^{-1}$. (In particular for $\nu = 2$, $\zeta_p = \lambda^{-1} < \zeta_{p+1} = \check{u}(\lambda^{-1})$.) Recall that we denote ξ_j, for $1 \leq j \leq p$, the unique number in $[\zeta_1/\lambda,\ \lambda^{-2}]$ such that

$$\check{u}^{p-j}(\lambda\, \xi_j) = \lambda^{-1} \ , \tag{7.19}$$

and that

$$\begin{aligned} \xi_{\max} = \xi_1 < \cdots < \xi_p = \lambda^{-2} \ , \\ \zeta_j \ \leq \ \lambda \xi_j \quad \forall j \in [1,\ p] \ . \end{aligned} \tag{7.20}$$

Suppose z follows $[1,\ \xi_{\max}] \mp i0$. Then $u(z) = e^{\pm i\pi/r}|U(z)|^{1/r}$ follows the segment $e^{\pm i\pi/r}[0,\ |U(\xi_{\max})|^{1/r}]$. Hence if z follows $(-1/\lambda)[1,\ \xi_{\max}] - i0$, $u(z)$ is given by (7.14), and follows an analytic arc entirely contained in $\mathbf{C}_+$ except for its starting point $\check{u}(1/\lambda)$. Thus $z \mapsto u(z - i0)$ now has a continuous, non-real extension to $[-\xi_{\max}/\lambda,\ -1/\lambda)$. The extension thus obtained of $u|\mathbf{C}_-$ to $\mathbf{C}_- \cup [-\xi_1/\lambda,\ \xi_1]$ is also obviously injective, as well as the conjugate extension of $u|\mathbf{C}_+$. Recall also that

$$1/\lambda < \check{u}(1/\lambda) \leq z_1 \lambda^{1-2\nu} < \zeta_1/\lambda^\nu \leq \xi_{\max}/\lambda \ . \tag{7.21}$$

Hence

$$\check{u}(1/\lambda) \in (1/\lambda,\ \xi_{\max}/\lambda) \ , \tag{7.22}$$

and

$$1 \leq \lambda^{\nu-2} < \lambda^{\nu-1}\check{u}(1/\lambda) < \zeta_1/\lambda \ . \tag{7.23}$$

This shows that $\lambda^{\nu-1}\check{u}(1/\lambda)$ is in the domain of analyticity of U and U is negative there.

We assume inductively that there exists, for a certain $j \in [1,\ p-1]$, a continuous injective extension, temporarily denoted $u_{(j)}$, of $u|\mathbf{C}_-$ to $\mathbf{C}_- \cup [-\xi_j/\lambda,\ \xi_j]$, such that $u_{(j)}(z - i0) \in \mathbf{C}_+$ if $z \in [-\xi_j/\lambda,\ -1/\lambda)$ or if $z \in (1,\ \xi_j]$. This implies of course a symmetrical situation for $u|\mathbf{C}_+$. By abuse of notation we also denote $u_{(j)}(z + i0) = u_{(j)}(z^* - i0)^*$. We also assume that (7.12) holds with u replaced by $u_{(j)}$ in the domain of the latter. In order to prove the same for $j+1$, we denote

$$u_{(j+1/2)}(z \mp i0) = \lambda^{-\nu} u_{(j)}(\lambda^{\nu-1}\check{u}^j_{(j)}(\check{u}^{p-j}_{(j)}(\lambda z \mp i0))) \ . \tag{7.24}$$

Note that the rhs of the above equation is equal to $u_{(j)}(z \mp i0)$ in the domain of $u_{(j)}$ by the induction hypothesis. Thus $u_{(j+1/2)}$ is a new extension of $u|\mathbf{C}_-$, which is injective wherever it is defined. If z increases along $(\xi_j,\ \xi_{j+1}]$, $\check{u}^{p-j}_{(j)}(\lambda z - i0)$ moves along $(1/\lambda,\ \check{u}(1/\lambda)] - i0$. This, by the induction hypothesis and (7.22), is within the domain of the already constructed $\check{u}_{(j)}$, and $\check{u}^j_{(j)}(\check{u}^{p-j}_{(j)}(\lambda z - i0))$ moves along an arc entirely contained in $\mathbf{C}_-$, and $u_{(j+1/2)}(z - i0)$ moves along an arc entirely contained in $\mathbf{C}_+$. A little more detail is needed, since $\check{u}(1/\lambda)$ is real, when z moves in a small real interval containing ξ_j so that $\check{u}^{p-j}(\lambda z - i0)$ moves along a small real interval containing $1/\lambda$. If $j = 1$, then $u_{(1)}$ is continuous and non-real at $\lambda^{\nu-1}\check{u}(1/\lambda) \pm i0$ since, as noted above, this is a point of analyticity of U. If $j > 1$, $\check{u}^2_{(j)}(1/\lambda \pm i0)$ is defined and non-real by (7.22). Denote now $u_{(j+1)}(z - i0) = \lambda^{-\nu} u_{(j+1/2)}(\lambda^{\nu-1}\check{u}^p_{(j+1/2)}(\lambda z - i0))$. This is a continuous injective extension of $u|\mathbf{C}_-$ to $\mathbf{C}_- \cup [-\xi_{j+1}/\lambda,\ \xi_{j+1}]$ since the arc $u_{(j+1/2)}([\xi_j,\ \xi_{j+1}] \mp i0)$ is entirely contained in $\mathbf{C}_\pm$. The construction makes it obvious that (7.12) holds with u replaced by $u_{(j+1)}$ in the domain of the latter.

We conclude that $u|\mathbf{C}_-$ has a continuous injective extension to $\mathbf{C}_- \cup [-\lambda^{-3},\ \lambda^{-2}]$ which takes real values only on $[-1/\lambda,\ 1]$. It maps $I_0 - i0$ onto a union τ_0 of p consecutive arcs contained in $\mathbf{C}_+$ except for the point $u(1) = 0$: $\tau_0 = \tau_{00} \cup \ldots \tau_{0(p-1)}$, with $\tau_{00} = u([1,\ \xi_1] - i0)$ and $\tau_{0j} = u([\xi_j,\ \xi_{j+1}] - i0)$ for $1 \leq j < p$. It maps $I_1 - i0$ onto another finite union $\tau_1 = \tau_{10} \cup \ldots \tau_{1(p-1)}$, with $\tau_{1j} = F(\tau_{0j}^*)$, contained in $\mathbf{C}_+$ except for the point $u(-1/\lambda)$. As in the previous case, $u|\mathbf{C}_-$ extends to a continuous injective function on $\overline{\mathbf{C}_-}$. The images $\tau_n = u(I_n - i0)$ all lie in $\mathbf{C}_+$ for $n > 1$. The sequence of the $\tau_n = F(\tau_{n-1}^*) = F^2(\tau_{n-2})$ tends to the point c.

Let $I_{00} = [1,\ \xi_1]$, $I_{0j} = [\xi_j,\ \xi_{j+1}]$ for $1 \le j < p$, and $I_{nj} = (-\lambda)^{-n} I_{0j}$ for $n \in \mathbf{N}$, $0 \le j < p$. If z crosses $(1,\ \xi_{\max})$ from $\mathbf{C}_-$ into $\mathbf{C}_+$, $u(z)$ gets continued by $v_{00}(z) = e^{2i\pi/r}u(z)$. If z crosses $(-\xi_{\max}/\lambda,\ -1/\lambda)$ from $\mathbf{C}_-$ into $\mathbf{C}_+$, $u(z)$ gets continued by $v_{10}(z) = F(v_{00}(-\lambda z^*)^*)$, holomorphic in $\mathcal{V}_{10} = \mathcal{V}_1$. If z crosses $(\xi_j,\ \xi_{j+1})$ from $\mathbf{C}_-$ into $\mathbf{C}_+$ (with $1 \le j < p$), $u(z)$ gets continued by $v_{0j}(z) = \lambda^{-\nu}u(\lambda^{\nu-1}\check{u}^{j-1}(v_{10}(-\check{u}^{p-j}(\lambda z^*))^*))$. If z crosses $(-\xi_{j+1}/\lambda,\ -\xi_j/\lambda)$ from $\mathbf{C}_-$ into $\mathbf{C}_+$, $u(z)$ gets continued by $v_{1j}(z) = F(v_{0j}(-\lambda z^*)^*)$. If z crosses the interior of I_{nj} from $\mathbf{C}_-$ into $\mathbf{C}_+$, $u(z)$ gets continued by $v_{nj}(z) = F(v_{(n-1)j}(-\lambda z^*)^*)$. If $r \ge 2$, all the functions v_{nj} are holomorphic in $\mathbf{C}_+$ and map it into itself.

Note that, in all cases, the extension of u to $\overline{\mathbf{C}_-}$ (resp. $\overline{\mathbf{C}_+}$) takes real values only on $[-\lambda^{-1},\ 1]$. The function $F|\mathbf{C}_+$ (resp. $F|\mathbf{C}_-$) also has a continuous injective extension to $\overline{\mathbf{C}_+}$ (resp. $\overline{\mathbf{C}_-}$) which takes real values only on the real segment $[-1,\ \zeta_1]$. The point c cannot be real. Indeed if we suppose it is and let $w_0 \in \mathbf{C}_+$, $w_n = F^{2n}(w_0)$ for $n \in \mathbf{N}$, the sequences $\{w_n\}$ and $\{F^2(w_n)\}$ both tend to c, so that, by the continuity of the extensions of F to $\overline{\mathbf{C}_\pm}$, $F^2(c+i0) = c$. Since this is real, $F(c+i0)$, hence also c, must belong to $[-1,\ \zeta_1]$, and c is a fixed point of F^2, i.e. c coincides with $z_1\lambda^{1-\nu}$. But the latter is repulsive, contradicting the attractive property of c. Hence c is in $\mathbf{C}_+$ and is a fixed point of F^2. It is attractive and unique by Schwarz's lemma applied to $F^2|\mathbf{C}_+$. Therefore the compact sets τ_n converge geometrically to c. It follows that the functions u, φ, ψ are all bounded.

7.4 Commutativity for $\nu = 2$

The following is a transcription into the notations of this paper of (a special case of) a result due to O. Lanford. This will prove that the properties of u and ψ recalled at the beginning of this section suffice to imply, in the case $\nu = 2$, a form of commutativity for the functions ξ and η given by

$$\xi = (-u)^{-1}, \quad \eta = -\lambda\xi \circ (-1/\lambda), \quad \eta^{-1} = \lambda\check{u} \circ (1/\lambda)\ . \tag{7.25}$$

Recall that the functional equations (7.11), (7.12), and $\nu = 2$, imply that ξ and η satisfy the system (1.5). With the notations of the beginning of this section, we have

Lemma 3 (Lanford) *For every solution with $\nu = 2$,*

$$\psi(\lambda u(-z/\lambda)) = -\lambda^r \psi(u(z)/\lambda) \quad \text{for all } z \in \Omega(-\lambda,\ 1)\ . \tag{7.26}$$

Proof. The domains of the two anti-Herglotzian functions

$$\begin{aligned} F_1 &= \psi \circ \lambda \circ u \circ (-1/\lambda)\ , \\ F_2 &= -\lambda^r \psi \circ (1/\lambda) \circ u\ , \end{aligned} \tag{7.27}$$

are equal to $\Omega(-\lambda,\ 1)$. Indeed the function $z \mapsto \lambda u(-z/\lambda)$ has this domain and maps $-\lambda$ to 0, and 1 to $z_1 v(1/\lambda) \le z_1/\lambda^2 < \zeta_1/\lambda$, hence it maps $\Omega(-\lambda,\ 1)$ into the domain of ψ. The function $(1/\lambda)u$ is holomorphic in $\Omega(-1/\lambda,\ 1)$. It maps

1 to 0 and $-\lambda$ to ζ_1/λ, hence it also maps $\Omega(-\lambda,\ 1)$ into the domain of ψ (and F_2 has a branch point at $-\lambda$ since ψ has one at ζ_1/λ). We now substitute for u, in the equation for F_1, the r.h.s. of (7.12), and substitute for ψ, in the equation for F_2, the r.h.s. of the first equation in (7.11). This gives

$$F_1 = -\frac{1}{\lambda^r} F_2 \circ G_0 \ , \qquad F_2 = -\frac{1}{\lambda^r} F_1 \circ G_0 \ , \tag{7.28}$$

$$G_0(z) = \lambda \breve{u}^p(-z) = \varphi(-z/\lambda) \ . \tag{7.29}$$

Since the functional equations (7.12) and (7.11) hold with domains, so does the system (7.28). In fact the anti-Herglotzian function G_0 maps the domain $\Omega(-\lambda,\ 1)$ into itself, since G_0 is holomorphic in $\Omega(-\zeta_1,\ 1)$ and satisfies:

$$G_0(1) = \varphi(-1/\lambda) \geq 0 \ , \quad G_0(0) = \varphi(0) < 1, \quad G_0(-\lambda) = \varphi(1) = 1 \ . \tag{7.30}$$

Since G_0 is strictly decreasing on $[-\lambda,\ 1]$, it has there a unique fixed point $\bar{x} \in (0,\ 1)$ which, by Schwarz's lemma, is attractive and has $\Omega(-\lambda,\ 1)$ as a basin of attraction. Let $\kappa = -G_0'(\bar{x}) \in (0,\ 1)$, and let h be the linearizer of G_0 at $\bar{x}$, normalized by $h'(\bar{x}) = 1$. This is a function holomorphic in $\Omega(-\lambda,\ 1)$, and satisfying, in this domain, $h = (-1/\kappa) h \circ G_0$ (in particular $h(\bar{x}) = 0$). The point $\bar{x}$ is also the unique fixed point of the function G_0^2 in $\Omega(-\lambda,\ 1)$ and its normalized linearizer is also h . On the other hand, because G_0 maps $\Omega(-\lambda,\ 1)$ into itself, the equation obtained by substituting the second equation in (7.28) into the first,

$$F_1 = \frac{1}{\lambda^{2r}} F_1 \circ G_0^2 \tag{7.31}$$

holds in $\Omega(-\lambda,\ 1)$. Therefore $F_1 = c_1 h$ with $c_1 = F_1'(\bar{x})$, and $\kappa = \lambda^r$. The second equation in (7.28) now reads $F_2 = (-c_1/\kappa) h \circ G_0 = c_1 h$. Therefore F_1 and F_2 coincide, which is the assertion of the lemma.

In particular, for $z = -\lambda$, this gives $1 = \psi(0) = -\lambda^r \psi(\zeta_1/\lambda)$, i.e. $\psi(\zeta_1/\lambda) = -1/\lambda^r$. Both sides of (7.26) must vanish at $\bar{x}$, hence $\breve{u}(\bar{x}/\lambda) = 1/\lambda$, $u(\bar{x}) = \lambda$ so that $\bar{x} = \lambda\zeta_{p-1} = -\zeta_{-1}$. Since $F_2(1) = -\lambda^r$, the common range of $F_1(z)$ and $F_2(z)$ as z varies in $[-\lambda,\ 1]$ is $[-\lambda^r,\ 1]$. The identity (7.26) continues to hold if ψ is replaced with $U = (z_1/\lambda)^r \psi$ on both sides. In order to translate this identity in terms of ξ and η, we denote

$$q(z) = |z|^r \mathrm{sign}(z) \ \ \forall z \in \mathbf{R}, \quad \widehat{u}(z) = q^{-1} \circ U(z) \ \ \forall z \in [-1/\lambda,\ \zeta_1/\lambda]. \tag{7.32}$$

The function $\widehat{u}$ is strictly decreasing, with range containing $[-z_1/\lambda^2,\ 1/\lambda]$, coincides with u on $[-1/\lambda,\ 1]$, and satisfies

$$\begin{aligned} \widehat{u} = (1/\lambda^2)\, \widehat{u} \circ \lambda \breve{u}^p \circ \lambda \qquad & \text{on } [-1/\lambda,\ \zeta_1/\lambda] \ , \\ \widehat{u} \circ \lambda u \circ (-1/\lambda) \ = \ -\lambda \widehat{u} \circ (1/\lambda)\, u \qquad & \text{on } [-\lambda,\ 1] \ . \end{aligned} \tag{7.33}$$

Let

$$\widehat{\xi} = (-\widehat{u})^{-1}, \qquad \widehat{\eta} = -\lambda \widehat{\xi} \circ (-1/\lambda) \ . \tag{7.34}$$

Then $\widehat{\xi}$ is an extension of ξ to an interval containing $(-1/\lambda,\ z_1/\lambda^2)$, $\widehat{\eta}$ is an extension of η, and

$$\begin{aligned} \widehat{\xi} = (1/\lambda^2)\, \eta^p \circ \widehat{\xi} \circ \lambda^2 \ , \quad \widehat{\xi} = (-1/\lambda)\, \widehat{\eta} \circ (-\lambda) \ , \quad & \text{on } (-1/\lambda,\ z_1/\lambda^2) \ , \\ \eta \circ \widehat{\xi} = \xi \circ \widehat{\eta} \qquad & \text{on } (-z_1/\lambda,\ z_1) \ . \end{aligned} \tag{7.35}$$

8 Behavior of fixed points as $r \to \infty$, $0 < \nu \le 1$

In this section we consider only the cases when $0 < \nu \le 1$ and $p \ge 2$ (and, of course, $r\nu - 1 - (p-1)(1-\nu) > 0$). In the case $p = 1$, the behavior of solutions as $r \to \infty$ was first elucidated by Eckmann and Wittwer in [EW], and also studied in [E1] (for $\nu = 1$) and [EE] (for $1 \le \nu \le 2$), and the method of [E1, EE] extends trivially to $0 < \nu \le 1$. The case $p \ge 2$ requires some additional work.

8.1 The functions V and W

The functional equation implies

$$\psi(z) = V(\psi(-\lambda z)) = W(\psi(\lambda^2 z)), \qquad \forall z \in \mathbf{C}_+ \cup \mathbf{C}_- \cup (-1/\lambda,\ 1/\lambda^2)\ , \quad (8.1)$$

where

$$\begin{aligned} V(\zeta) &= f(\zeta^{1/r}), \\ f(z) &= \frac{1}{\lambda^{r\nu}} \psi(\lambda^{\nu-1}\, \check{u}^{p-1}(z_1\lambda^{1-\nu}\, z))\ , \\ W &= V \circ V\ . \end{aligned} \qquad (8.2)$$

Recall that

$$\check{u}(z) = z_1\lambda^{1-\nu}\, \psi(-z)^{1/r}\ . \qquad (8.3)$$

The function f is anti-Herglotzian and holomorphic in $\Omega(-1/z_1\lambda^{1-\nu},\ 1/z_1\lambda^{2-\nu})$. We denote $\zeta_{\max} = (1/z_1\lambda^{2-\nu})^r$. These functions satisfy

$$V(1) = f(1) = 1, \qquad V'(1) = -\frac{1}{\lambda}, \qquad f'(1) = -\frac{r}{\lambda}\ . \qquad (8.4)$$

Since $\psi(1) = 0$, V vanishes at $\alpha = \psi(-\lambda)$, and f vanishes at $v(\lambda) = (z_1\lambda^{1-\nu})^{-1}\zeta_1$. We also define

$$\widehat{V}(\zeta) = 1 - V(1-\zeta), \qquad \widehat{W} = \widehat{V} \circ \widehat{V}\ . \qquad (8.5)$$

Since the functional equations (8.1) hold for all z in the domain of ψ, the real ranges of V and W contain that of ψ. The following estimates follow [EE] and [E1]. In the domain of V,

$$-\frac{V''(\zeta)}{V'(\zeta)} = \frac{1}{r\zeta}\left(r - 1 - \frac{z\, f''(z)}{f'(z)}\right), \qquad z = \zeta^{1/r}\ . \qquad (8.6)$$

For real $\zeta \in (0,\ \zeta_{\max})$,

$$\begin{aligned} -\frac{V''(\zeta)}{V'(\zeta)} &\ge \frac{1}{r\zeta}\left(r - 1 - \frac{2z}{1/\lambda^{2-\nu}z_1 - z}\right) \\ &= \frac{1}{r\zeta}\left(r - \frac{1 + \lambda^{2-\nu}z_1\, z}{1 - \lambda^{2-\nu}z_1\, z}\right)\ . \end{aligned} \qquad (8.7)$$

Recalling the bound $r\nu \ge (1+\lambda)/(1-\lambda)$, we find that

$$-\frac{V''(\zeta)}{V'(\zeta)} \ge \frac{1-\nu}{\zeta} \qquad \text{for } 0 < \zeta \le (z_1\lambda^{1-\nu})^{-r} \qquad (8.8)$$

This is in particular satisfied if $\zeta = \alpha = ((z_1\lambda^{1-\nu})^{-1}\zeta_1)^r$, since $\zeta_1 \le \lambda^{1-\nu} \le 1$. Integrating the inequality (8.8) from 1 to $\zeta > 1$, using $V'(1) = -1/\lambda$ and $V(1) = 1$, gives

$$V(\zeta) > 1 - \frac{1}{\lambda\nu}(\zeta^\nu - 1) \qquad \Rightarrow \alpha > (1+\lambda\nu)^{1/\nu} \ge (1+\lambda). \tag{8.9}$$

It follows similarly from (8.6) that

$$-\frac{V''(\zeta)}{V'(\zeta)} \le \frac{1}{r\zeta}\left(r - 1 + \frac{2z}{z+\lambda^{\nu-1}/z_1}\right) \quad = \frac{1}{\zeta}\left(1 - \frac{1 - z_1\lambda^{1-\nu}\, z}{r(1+z_1\lambda^{1-\nu}\, z)}\right) \ , \tag{8.10}$$

so that

$$-\frac{V''(\zeta)}{V'(\zeta)} \le \frac{1}{\zeta} \qquad \forall\zeta \in (0,\ \alpha)\ . \tag{8.11}$$

Integrating this from 1 to $\zeta > 1$ gives

$$-V'(\zeta) \ge \frac{1}{\lambda\zeta} \qquad \forall\zeta \in (1,\ \alpha)\ , \tag{8.12}$$

$$V(\zeta) \le 1 - \frac{1}{\lambda}\log\zeta \quad \forall\zeta \in (1,\ \alpha) \qquad \Rightarrow \alpha \le e^\lambda\ . \tag{8.13}$$

Since $V = f \circ q^{-1}$ where $q^{-1}(\zeta) = \zeta^{1/r}$, the Schwarzian derivative SV of V satisfies, for real ζ in the domain of V,

$$SV(\zeta) \ge Sq^{-1}(\zeta) = \frac{1 - r^{-2}}{2\zeta^2}\ . \tag{8.14}$$

The function W is Herglotzian and holomorphic in $\Omega(0,\ \alpha)$, where $\alpha = \psi(-\lambda) = V^{-1}(0)$ (since $V(0) \le \lambda^{-r\nu} < \zeta_{\max}$). It has a repelling fixed point at 1 with multiplier λ^{-2}. $\widehat{W}$ is Herglotzian and holomorphic in $\Omega(1-\alpha,\ 1)$ and has a fixed point at 0. By (8.14),

$$SW(\zeta) \ge \frac{(1-r^{-2})}{2}\left(\frac{V'(\zeta)^2}{V(\zeta)^2} + \frac{1}{\zeta^2}\right)\ . \tag{8.15}$$

Lower bound for $\widehat{W}$ in $[0,\ 1]$.

For $0 < \zeta < 1$, the convexity of V implies:

$$-V'(\zeta) \ge \frac{V(\zeta) - 1}{1 - \zeta}, \tag{8.16}$$

hence

$$-\frac{V(\zeta)}{V'(\zeta)} \le 1 - \zeta - \frac{1}{V'(\zeta)} \le 1 - \zeta + \lambda\ , \tag{8.17}$$

It follows that

$$2SW(\zeta) \ge (1-r^{-2})\left[\frac{1}{(1-\zeta+\lambda)^2} + \frac{1}{\zeta^2}\right], \tag{8.18}$$

and hence

$$2S\widehat{W}(\zeta) \ge (1-r^{-2})\left[\frac{1}{(\zeta+\lambda)^2} + \frac{1}{(1-\zeta)^2}\right]. \tag{8.19}$$

In (0, 1), the r.h.s. has a minimum at $\zeta = (1-\lambda)/2$, and, using the bound on $r \geq (1+\lambda)/(1-\lambda)$, we get

$$\frac{d}{d\zeta}\frac{\widehat{W}''(\zeta)}{\widehat{W}'(\zeta)} \geq S\widehat{W}(\zeta) \geq s(\lambda) \equiv \frac{16\lambda}{(1+\lambda)^4}. \tag{8.20}$$

By (8.11) and

$$\frac{\widehat{W}''(\zeta)}{\widehat{W}'(\zeta)} = \frac{\widehat{V}''(\widehat{V}(\zeta))}{\widehat{V}'(\widehat{V}(\zeta))}\widehat{V}'(\zeta) + \frac{\widehat{V}''(\zeta)}{\widehat{V}'(\zeta)}\ , \tag{8.21}$$

it follows that

$$\frac{\widehat{W}''(0)}{\widehat{W}'(0)} = -\left(\frac{1}{\lambda}-1\right)\frac{\widehat{V}''(0)}{\widehat{V}'(0)} = \left(\frac{1}{\lambda}-1\right)\frac{V''(1)}{V'(1)} \geq -\left(\frac{1}{\lambda}-1\right)\ . \tag{8.22}$$

Hence,

$$\frac{\widehat{W}''(\zeta)}{\widehat{W}'(\zeta)} \geq \frac{\widehat{W}''(0)}{\widehat{W}'(0)} + s(\lambda)\zeta \geq -\left(\frac{1}{\lambda}-1\right) + s(\lambda)\zeta\ , \tag{8.23}$$

$$\begin{aligned}\log \widehat{W}'(\zeta) &\geq 2\log(1/\lambda) - \left(\frac{1}{\lambda}-1\right)\zeta + s(\lambda)\zeta^2/2 \\ &\geq \left[2\log(1/\lambda) - \left(\frac{1}{\lambda}-1\right)\right] + s(\lambda)\zeta^2/2\ .\end{aligned} \tag{8.24}$$

As a function of λ in (0, 1), the first bracket in the last expression has a unique maximum at 1/2 and vanishes at 1. Since it is positive at e^{-1}, it is non negative in $[e^{-1},\ 1]$. Hence, for $\lambda \geq e^{-1}$ and $0 \leq \zeta < 1$,

$$\widehat{W}'(\zeta) \geq 1 + s(\lambda)\zeta^2/2\ , \tag{8.25}$$

$$\widehat{W}(\zeta) \geq \zeta\left(1 + \frac{s(\lambda)}{6}\zeta^2\right)\ , \tag{8.26}$$

and we note that, for $\lambda \geq 1/4$, $s(\lambda) \geq 1$.

On the other hand $\widehat{W}$ is Pick with 0 angular derivative at infinity in $\mathbf{C}_+ \cup \mathbf{C}_- \cup (1-\alpha,\ 1)$, and vanishes at 0. Hence there is a positive measure ρ with support in $\mathbf{R} \setminus (1-\alpha,\ 1)$ such that

$$\widehat{W}(\zeta) = \int_{\mathbf{R}\setminus(1-\alpha,\ 1)} \left(\frac{1}{t-\zeta} - \frac{1}{t}\right) d\rho(t)\ , \qquad \int_{\mathbf{R}\setminus(1-\alpha,\ 1)} \frac{d\rho(t)}{t^2} = \frac{1}{\lambda^2}\ . \tag{8.27}$$

Hence, for $0 \leq \zeta < 1$,

$$\begin{aligned}\widehat{W}(\zeta) \geq \frac{\zeta}{\lambda^2} \inf_{t\notin(1-\alpha,\ 1)} \frac{t}{t-\zeta} &= \frac{\zeta(\alpha-1)}{\lambda^2(\alpha-1+\zeta)} \\ &\geq \frac{\zeta}{\lambda(1+\lambda)}\ .\end{aligned} \tag{8.28}$$

Here we have used the lower bound (4.14) for α. For $\lambda \leq 1/2$, this implies $\widehat{W}(\zeta) \geq 4\zeta/3 \geq \zeta(1+\zeta^2/6)$, so that, for all λ and all $\zeta \in (0,\ 1)$,

$$\widehat{W}(\zeta) \geq \zeta(1+c'\zeta^2), \qquad c' = 1/6. \tag{8.29}$$

Remark 8.1 *Let ζ, y, a', and m be strictly positive real numbers such that*

$$0 \quad < \quad \zeta(1+a'\zeta^2) \quad \leq \quad y \quad \leq \quad m \ . \tag{8.30}$$

Then

$$\zeta \leq y(1-ay^2), \qquad a = \frac{a'}{1+3a'm^2} \ . \tag{8.31}$$

Indeed, note first that $am^2 \leq 1/3 < 1$. Moreover $\zeta \leq z$ for any z such that $a'z^3 + z - y \geq 0$, and inserting $z = y(1-ay^2)$ in this expression gives

$$y^3[a'(1-ay^2)^3 - a] \quad \geq \quad y^3[a'(1-3am^2) - a] = 0 \ .$$

This remark (with $m = 1$) and the lower bound (8.29) imply that $\widehat{W}^{-1}$ is defined on $[0,\ 1)$, and that, for all $y \in [0,\ 1)$,

$$\widehat{W}^{-1}(y) \leq y(1-cy^2), \qquad c = \frac{c'}{1+3c'} = 1/9 \ . \tag{8.32}$$

Lower bound for W in $[1,\ \alpha]$.

For $1 \leq \zeta \leq \alpha$, the inequalities (8.15) and (8.12), together with $0 \leq V(\zeta) \leq 1$, give

$$SW(\zeta) \geq \frac{1}{2\zeta^2}(1-r^{-2})(\lambda^{-2}+1) \geq \frac{1}{\zeta^2}\frac{2}{(1+\lambda)^2}(\lambda^{-1}+\lambda) \ . \tag{8.33}$$

The last inequality follows from the lower bound on r already used above. The last expression is decreasing in λ, so that, finally,

$$SW(\zeta) \geq \frac{1}{\zeta^2} \qquad \forall \zeta \in (1,\ \alpha) \ . \tag{8.34}$$

Since $W''(1)/W'(1) = -\widehat{W}''(0)/\widehat{W}'(0) \geq 0$ (see (8.22)), for $1 \leq \zeta \leq \alpha$,

$$\frac{W''(\zeta)}{W'(\zeta)} \geq \int_1^\zeta t^{-2}dt = (\zeta-1)/\zeta \geq (\zeta-1)/e \ , \tag{8.35}$$

by using the upper bound $\alpha \leq e$, and hence

$$\begin{aligned} W'(\zeta) \quad &\geq \quad \lambda^{-2}(1+(\zeta-1)^2/2e) \ , \\ W(\zeta)-1 \quad &\geq \quad (\zeta-1)(1+k'(\zeta-1)^2), \qquad k' = 1/6e \qquad \forall \zeta \in (1,\ \alpha) \ . \end{aligned} \tag{8.36}$$

The function $\underline{W}(\zeta) = W(\zeta+1)-1$ is thus defined on $[0, \alpha-1]$, where it satisfies

$$\underline{W}(\zeta) \geq \zeta(1+k'\zeta^2) \ . \tag{8.37}$$

We note that $W(\alpha) = W(\psi(-\lambda)) = \psi(-\lambda^{-1})$, hence the range of $W|(1,\ \alpha)$ contains in particular $\psi(-1)$. We wish to apply Remark 8.1 to the inverse function $\underline{W}^{-1}$ restricted to $[0,\ \psi(-1)-1]$, and we first obtain an upper bound for $\psi(-1)$. We use the representation (2.10) :

$$\begin{aligned} \log\psi(-1) - \log\psi(-\lambda) \quad &= \quad \int_{\mathbf{R}\setminus(-\lambda^{-1},\ 1)} \sigma(t)\left(\frac{1}{t+\lambda} - \frac{1}{t+1}\right) dt \\ &\leq \quad \int_{\mathbf{R}\setminus(-\lambda^{-1},\ 1)} \left(\frac{1}{t+\lambda} - \frac{1}{t+1}\right) dt \quad = \ \log 2 \end{aligned} \tag{8.38}$$

which yields (using (8.13))

$$\psi(-1) \leq 2\psi(-\lambda) \leq 2e^{\lambda} < 2e \ . \tag{8.39}$$

Thus (8.37) and Remark 8.1, with $m = 2e - 1$, show that

$$\underline{W}^{-1}(y) \leq y(1-ky^2), \qquad k = 1/(6e+3(2e-1)^2), \qquad \forall y \in [0,\ \psi(-1)-1] \ . \tag{8.40}$$

Note that we have obtained the following bounds:

$$1+\lambda \leq \psi(-\lambda) \leq e^{\lambda}, \qquad \psi(-1) \leq 2\psi(-\lambda) \leq 2e^{\lambda} \ . \tag{8.41}$$

This provides upper and lower bounds for $y_0 = z_1^r$. Indeed from $\zeta_1 = z_1\lambda^{1-\nu}\, v(\lambda) \leq \lambda^{1-\nu} = \zeta_p$, and $z_1\lambda^{1-\nu}\, v(1) \geq z_1\lambda^{1-\nu}\, v(\zeta_p) \geq \zeta_p$, it follows $z_1 \leq 1/v(\lambda)$ and $z_1 \geq 1/v(1)$, hence

$$(2e)^{-1} \leq y_0 \leq (1+\lambda)^{-1} \ . \tag{8.42}$$

8.2 The functions $H_\pm$

We define

$$H_\pm(w) = \psi(\pm e^{\beta w}), \qquad \beta = \log(1/\lambda) \ , \qquad \widehat{H}_\pm = 1 - H_\pm \ . \tag{8.43}$$

H_+ is holomorphic in rhe cut strip

$$\Delta_+(\lambda) = \{w \in \mathbf{C}\ :\ |\mathrm{Im}\, w| < \pi/\beta\} \setminus (2+\mathbf{R}_+). \tag{8.44}$$

It maps points in $\mathbf{C}_\pm$ into $\mathbf{C}_\mp$. It is decreasing on the reals, tends to 1 at $-\infty$, and vanishes at 0. H_- is holomorphic in the cut strip

$$\Delta_-(\lambda) = \{w \in \mathbf{C}\ :\ |\mathrm{Im}\, w| < \pi/\beta\} \setminus (1+\mathbf{R}_+), \tag{8.45}$$

maps points in $\mathbf{C}_\pm$ into $\mathbf{C}_\pm$, is increasing on the reals and tends to 1 at $-\infty$. They satisfy

$$\begin{aligned} H_\pm(w) &= V(H_\mp(w-1)) = W(H_\pm(w-2)) \ , \\ \widehat{H}_\pm(w) &= \widehat{V}(\widehat{H}_\mp(w-1)) = \widehat{W}(\widehat{H}_\pm(w-2)) \ . \end{aligned} \tag{8.46}$$

Moreover

$$\frac{H''_\pm(w)}{H'_\pm(w)} = \frac{\widehat{H}''_\pm(w)}{\widehat{H}'_\pm(w)} = \beta\left(1 + \frac{z\psi''(z)}{\psi'(z)}\right), \qquad z = \pm e^{\beta w} \ . \tag{8.47}$$

Since (for $0 < \nu \leq 1$) ψ is anti-Herglotzian in $\mathbf{C}_+ \cup \mathbf{C}_- \cup (-1/\lambda,\ 1/\lambda^2)$, the inequalities (2.2) imply, for $0 < z = e^{\beta w} < 1/\lambda$, i.e. for all $w \in (-\infty,\ 1)$,

$$\frac{H''_+(w)}{H'_+(w)} \geq \beta\left(1 - \frac{2\lambda z}{1+\lambda z}\right) \geq 0 \ . \tag{8.48}$$

For $0 < -z = e^{\beta w} < 1/\lambda$, i.e. again for all $w \in (-\infty,\ 1)$, we find similarly that

$$\frac{H''_-(w)}{H'_-(w)} \geq \beta\left(1 + \frac{2\lambda^2 z}{1-\lambda^2 z}\right) \geq 0 \ . \tag{8.49}$$

In other words, H_- and $\widehat{H}_+$ are increasing and convex, H_+ is decreasing and concave. In particular, for $w<0$, using (8.32),

$$\begin{aligned} 2\widehat{H}_+'(w) &\geq \widehat{H}_+(w)-\widehat{H}_+(w-2) \\ &= \widehat{H}_+(w)-\widehat{W}^{-1}(\widehat{H}_+(w)) \geq c\widehat{H}_+(w)^3 \ . \end{aligned} \tag{8.50}$$

Integrating this with the initial condition $\widehat{H}_+(0)=1$ gives

$$\begin{aligned} \widehat{H}_+(w) &\leq (1-cw)^{-1/2}, \\ H_+(w) &\geq 1-(1-cw)^{-1/2} \qquad \forall w\in \mathbf{R}_- \quad (c=1/9). \end{aligned} \tag{8.51}$$

Similarly, defining $\underline{H}_-(w)=H_-(w)-1$, recalling that $H_-(0)=\psi(-1)$, $H_-(-1)=\psi(-\lambda)$, we obtain, using (8.40),

$$\underline{H}_-'(w)\geq k\underline{H}_-(w)^3/2 \ ,$$

$$H_-'(w)\geq k(H_-(w)-1)^3/2 \quad \forall w\in\mathbf{R}_- \quad (k=1/(6e+3(2e-1)^2)) \ . \tag{8.52}$$

We will need a lower bound for $H_-'(w)/H_-(w)$ in the interval $w\in[-1,\ 0]$. This is provided by the lower bound $H_-(-1)=\psi(-\lambda)\geq 1+\lambda$, and by (8.52) :

$$\begin{aligned} \frac{H_-'(w)}{H_-(w)} &\geq \frac{k(H_-(w)-1)^3}{2H_-(w)} \\ &\geq \frac{k(H_-(-1)-1)^3}{2H_-(-1)} \geq \frac{k\lambda^3}{2(1+\lambda)} \qquad \forall w\in[-1,\ 0] \ . \end{aligned} \tag{8.53}$$

8.3 Lower bound on τ

Recall that the function φ satisfies

$$\begin{aligned} \varphi(z) &= \lambda^{\nu-1}\,\check{u}^p(\lambda z), \qquad \forall z\in\mathbf{C}_+\cup\mathbf{C}_-\cup(-1/\lambda,\ 1/\lambda^2), \qquad \varphi(1)=1, \\ \varphi'(1) &= \tau^\nu \ = \lambda^\nu \prod_{j=0}^{p-1}\check{u}'(\zeta_j), \quad \lambda\leq\zeta_j=\check{u}^j(\lambda)\leq\lambda^{1-\nu} \ . \end{aligned} \tag{8.54}$$

Let $T(w)=e^{\beta\, w}$, $\beta=\log(1/\lambda)$. Then the function

$$X=T^{-1}\circ\varphi\circ T \tag{8.55}$$

is given by

$$\begin{aligned} X(w) &= -\nu+1+Y^p(w-1) \qquad \forall w\in(-\infty,\ 2), \\ Y(w) &= T^{-1}\circ\check{u}\circ T(w) \\ &= \frac{\log y_0}{\log(1/\tau)}+\nu-1+\frac{1}{\log(1/\tau)}\log H_-(w) \qquad \forall w\in(-\infty,\ 1) \end{aligned} \tag{8.56}$$

It satisfies $X(0)=0$ and

$$X'(0)=\tau^\nu=\prod_{j=0}^{p-1}Y'(w_j)=\prod_{j=0}^{p-1}\frac{1}{\log(1/\tau)}\frac{H_-'(w_j)}{H_-(w_j)} \ , \tag{8.57}$$

where

$$-1 \le w_j = \frac{\log \zeta_j}{\log(1/\lambda)} \le \nu - 1 \; . \qquad (8.58)$$

Hence by (8.53),

$$\tau^{\nu/p - 3/r} \log(1/\tau) \ge \frac{k}{4} \; , \qquad k = 1/(6e + 3(2e-1)^2) \; . \qquad (8.59)$$

When $r > 3p/\nu$, this provides a lower bound for τ. We may e.g. rewrite (8.59) as

$$y \log(1/y) \ge (\nu/p - 3/r)k/4, \qquad y = \tau^{\nu/p-3/r} \; . \qquad (8.60)$$

8.4 Limiting fixed points

The preceding subsections have shown that, for any solution, the associated functions have the following properties:

(1) The function V is holomorphic and anti-Herglotzian in $\mathbf{C}_+ \cup \mathbf{C}_- \cup (0,\ \zeta_{\max})$, where $\zeta_{\max} \ge \tau^{\nu-2} \ge 8^{(2-\nu)/\nu}$. It satisfies $V(1) = 1$ and $V'(1) = -1/\lambda$.

(2) The function $W = V \circ V$ is holomorphic and Herglotzian in $\mathbf{C}_+ \cup \mathbf{C}_- \cup (0,\ \alpha)$, where $(1 + \lambda) \le \alpha = V^{-1}(0) \le e^{\lambda}$.

(3) The function H_+ is holomorphic in the cut strip $\Delta_+(\lambda)$ (see (8.44)), maps points in $\mathbf{C}_\pm$ into $\mathbf{C}_\mp$, vanishes at 0, and satisfies the bound (8.51).

(4) The function H_- is holomorphic in the cut strip $\Delta_-(\lambda)$ (see (8.45)), maps points in $\mathbf{C}_\pm$ into $\mathbf{C}_\pm$, and satisfies $H_-(-1) = \alpha$ and the bounds (8.52) and (8.53).

(5) $\tau = \lambda^r$ is bounded above by $\tau \le 8^{-1/\nu}$. For sufficiently large r, its is bounded below by (8.59), and for all r by $\lambda_0(p,\ r,\ \nu)^r$ (see (4.33)).

(6) $y_0 = z_1^r$ satisfies (8.42).

As a consequence, from every infinite sequence of solutions, with fixed ν and p, such that $r \to \infty$, it is possible to extract a subsequence such that τ and y_0 have limits in $(0,\ 1)$, and that the functions V, W, $H_\pm$ tend, uniformly over compact sets, to non-constant functions, holomorphic in cut planes. Meanwhile, λ and $z_1 > \lambda^\nu$ tend to 1 (see (7.4)), ψ and u tend to 1, uniformly over compact subsets of $\mathbf{C}_+ \cup \mathbf{C}_- \cup (-1,\ 1)$ (see (8.51)). However the functions

$$S_\pm(\zeta) = U(\pm\zeta^{1/r}) = y_0 \tau^{1-\nu} H_\pm \left(\frac{\log \zeta}{\log(1/\tau)} \right) \qquad (8.61)$$

have non-trivial limits and obey the functional equation:

$$S_\pm(\zeta) = \frac{1}{\tau^\nu} S_+(\tau^{\nu-1}\, S_-^{p-1} \circ S_\mp(\tau\zeta)) \; . \qquad (8.62)$$

A Appendix 1. Proof of the inequality (4.22)

This inequality is equivalent to

$$(1-x^2)\log((1+x^2)(1+x)^2)+4x^2\log(x)>0 \quad \forall x\in(0,\ 1)\ , \tag{A.1}$$

or to

$$f_1(x)-4xf_2(x)>0 \quad \forall x\in(0,\ 1)\ , \tag{A.2}$$

where

$$f_1(x)=\log((1+x^2)(1+x)^2), \qquad f_2(x)=\frac{x\log(1/x)}{1-x^2}\ . \tag{A.3}$$

The derivative $f_2'(x)$ has the sign of

$$-2\log(x)-2\left(\frac{1-x^2}{1+x^2}\right)=-\log(t)-\frac{4}{1+t}+2, \quad t=x^2\ . \tag{A.4}$$

The last expression vanishes at 1 and has negative derivative in t on $(0,\ 1)$. Hence f_2 is increasing on $(0,\ 1)$. It tends to $1/2$ as x tends to 1, so that $f_2<1/2$ on $(0,\ 1)$. It now suffices to prove that $f_1(x)-2x>0$ for all $x\in(0,\ 1)$. This quantity vanishes for $x=0$, and

$$f_1'(x)-2 \;=\; \frac{2x^2(1-x)}{(1+x^2)(1+x)}>0 \quad \forall x\in(0,\ 1)\ . \tag{A.5}$$

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