Lifschitz tail in a magnetic field: coexistence of classical and quantum behavior in the borderline case.

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Abstract

We establish the exact low-energy asymptotics of the integrated density of states (Lifschitz tail) in a homogeneous magnetic field and Poissonian impurities with a repulsive single-site potential of Gaussian decay. It has been known that the Gaussian potential tail discriminates between the so-called "classical" and "quantum" regimes, and precise asymptotics are known in these cases. For the borderline case, the coexistence of the classical and quantum regimes was conjectured. Here we settle this last remaining open case to complete the full picture of the magnetic Lifschitz tails.

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1 Introduction

The magnetic Lifschitz tail is the asymptotic behavior of the integrated density of states (IDS), $N(E)$, at energy E near the bottom of the spectrum of the two dimensional random Schrödinger operator with a constant magnetic field B. The random potential, V_ω, represents repulsive impurities that are modelled by a single-site potential profile $V^{(0)} \geq 0$ convolved with a homogeneous Poisson point process.

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The low energy asymptotics of the IDS exhibits two qualitatively different behaviors. For long range $V^{(0)}$, the asymptotics of $N(E)$ is solely determined by the potential, i.e., by classical effects, hence it is called *classical asymptotics* or *classical regime*. In this regime, the low energy behavior of $N(E)$ is sensitive to the details of the tail of $V^{(0)}$ and it is insensitive to the strength of the magnetic field. For short range potentials, the asymptotics of $N(E)$ is determined by the quantum kinetic energy (*quantum asymptotics* or *quantum regime*) and it is universal; it depends only on the strength of the magnetic field, but it is insensitive to the potential profile $V^{(0)}$.

It has been established in [2] that potentials $V^{(0)}$ with algebraically decaying tail of any finite order belong to the classical regime if $B \neq 0$. The strength of the magnetic field does not appear in the leading term asymptotics of $N(E)$. This result is in contrast to the nonmagnetic case, where it has been shown ([9], [3], [10]) that an algebraic decay, $V^{(0)}(x) \sim |x|^{-(d+2)}$, discriminates between the classical and quantum regimes in d dimensions.

Nevertheless, quantum regime does appear in the magnetic case as well, but the discriminating potential decay is much faster than algebraic; in fact it is Gaussian. The existence of the quantum regime for compactly supported potentials was proven in [5] and the Gaussian threshold was conjectured. This threshold has been verified in [7] relying on [5] for the most involved technical part. More precisely, it has been proven that *stretched-Gaussian* decay leads to the classical asymptotics, while *super-Gaussian* decay leads to the quantum asymptotics. Later, some refined results were obtained in [8].

The borderline case, when $V^{(0)}$ is asymptotically Gaussian, has not been settled conclusively, only two-sided estimates were given in [7]. The lower bound indicates a coexistence of the classical and quantum effects, as it is determined by $2\ell_B^2 + \lambda^2$. Here $\ell_B := B^{-1/2}$ is the magnetic lengthscale representing the kinetic energy contribution, and λ is the lengthscale of the Gaussian potential. The upper bound is determined by $\max\{2\ell_B^2, \lambda^2\}$, indicating no coexistence of the two regimes. The conjecture of [7] was that the lower bound is the true asymptotics.

The purpose of this paper is to show this conjecture. We emphasize that in the Gaussian borderline case *both* classical and quantum effects are important, hence none of them can be neglected along the proof. This is the main novelty of the present paper, which is a extension of our earlier work [5].

1.1 Definitions

We consider a nonnegative potential function

$$V^{(0)} \in L^2_{loc}(\mathbf{R}^2), \qquad V^{(0)} \geq 0 \,, \tag{1}$$

that is strictly positive on a non-empty open set, i.e.,

$$V^{(0)}(x) \geq v \cdot \mathbf{1}(|x - x_0| \leq a) \tag{2}$$

for some $v, a > 0$ and $x_0 \in \mathbf{R}^2$. Here $\mathbf{1}(\,\cdot\,)$ denotes the characteristic function. Let

$$V(x) = V_\omega(x) := \sum_i V^{(0)}(x - x_i(\omega)) \tag{3}$$

be a random potential, where $x_i(\omega)$ is the realization of the Poisson point process on $\mathbf{R}^2$ with a constant intensity ν (here ω refers to the randomness, but we shall usually omit it from the notations). The expectation with respect to this process is denoted by $\mathcal{E}$.

We consider the following magnetic Schrödinger operator with a random potential V_ω

$$H(B, V_\omega) = H_\omega = \frac{1}{2}\Big[(-i\nabla - A)^2 - B\Big] + V_\omega \qquad \text{on} \quad L^2(\mathbf{R}^2) \,, \tag{4}$$

where $A : \mathbf{R}^2 \to \mathbf{R}^2$ is a deterministic vector potential (gauge) generating the constant $B > 0$ magnetic field, i.e., curl $A = B$. The properties we are interested in are independent of the actual gauge choice, so, conveniently, we choose the standard gauge $A(x) := \frac{B}{2}\binom{-x_2}{x_1}$. Here $x = (x_1, x_2) \in \mathbf{R}^2$. We subtracted the constant $B/2$ term in the kinetic energy both for physical reasons (spin coupling) and for mathematical convenience. The spectrum of the free operator $H(B, V \equiv 0)$ is $\{nB \,:\, n = 0, 1, 2, \ldots\}$.

We also define $H_{Q,\omega} = H_Q(B, V_\omega)$ as the restriction of H_ω onto a domain $Q \subset \mathbf{R}^2$ (with Dirichlet boundary conditions). In this paper, by domain we mean an open, bounded subset of $\mathbf{R}^2$ with regular (piecewise C^1) boundary, which is not necessarily connected.

We shall always assume that $V^{(0)}$ has sufficient decay so that $V_\omega \in L^2_{loc}$ with probability one, i.e., these operators are almost surely selfadjoint. Moreover, in all cases we consider it is easy to show that

$$\inf \mathrm{Spec}\, H(B, V_\omega) = 0 \qquad \text{almost surely.} \tag{5}$$

We define the integrated density of states (IDS) as

$$N(E) := \lim_{Q \nearrow \mathbf{R}^2} \frac{1}{|Q|} \mathcal{E}\, \mathrm{Tr}\, P_E(H_{Q,\omega}), \tag{6}$$

where P_E is the spectral projection onto the half line $(-\infty, E]$, and $Q \nearrow \mathbf{R}^2$ is an increasing sequence of nested regular domains, say, squares or disks. The trace is over $L^2(Q)$. For the existence of this limit and equivalent definitions we refer to [2], [5] and references therein.

Following [7], we assume that $V^{(0)}$ has one of the following behaviors at infinity:

Sub-Gaussian decay:

$$\lim_{|x|\to\infty} \frac{|x|^2}{\log V^{(0)}(x)} = -\infty \ ;$$

Gaussian decay:

$$\lim_{|x|\to\infty} \frac{\log V^{(0)}(x)}{|x|^2} = -\frac{1}{\lambda^2} \tag{7}$$

for some $0 < \lambda < \infty$;

Super-Gaussian decay:

$$\lim_{|x|\to\infty} \frac{\log V^{(0)}(x)}{|x|^2} = -\infty \ .$$

The sub-Gaussian decay leads to the classical regime where the potential determines the Lifschitz tail. Hence precise results require more definite tail behavior in this case. The following definition is taken from [8].

Regular (F, α)-decay:

$$\lim_{|x|\to\infty} \frac{F(1/V^{(0)}(x))}{|x|} = 1$$

for some positive function F, which is regularly varying of index $1/\alpha \in [0, \infty]$ and is strictly increasing towards infinity. Recall that a positive measurable function F is said to be *regularly varying of index* γ if $\lim_{t\to\infty} F(ct)/F(t) = c^\gamma$ for all $c > 0$. Such class of functions is denoted by R_γ. Two important cases are:

Algebraic decay: $\lim_{|x|\to\infty} |x|^\alpha V^{(0)}(x) = \mu$ with some exponent $\alpha > 2$ and constant $0 < \mu < \infty$. This corresponds to $F(t) \sim (\mu t)^{1/\alpha}$.

Stretched Gaussian decay: $\lim_{|x|\to\infty} |x|^{-\alpha} \log V^{(0)}(x) = -\lambda^{-\alpha}$ for some $0 < \lambda < \infty$ and $0 < \alpha < 2$. This corresponds to $F(t) \sim \lambda(\log t)^{1/\alpha}$.

1.2 Results

The result of [2] for algebraically decaying potential $V^{(0)}$ is

$$\lim_{E\searrow 0} E^{2/(\alpha-2)} \log N(E) = -C(\alpha, \mu, \nu)$$

with an explicitly computed constant $C(\alpha, \mu, \nu)$.

The general regular (F, α)-decaying sub-Gaussian case was discussed in details in [8]. The Lifschitz tail is given by the de Bruijn conjugate $f^{\#}$ of the function $t \mapsto f(t) = [t^{-1/\alpha}F(t)]^{2\alpha/(2-\alpha)}$. Recall that the de Bruijn conjugate of a slowly varying function $f \in R_0$ is $f^{\#} \in R_0$ such that $f(t)f^{\#}(tf(t)) \to 1$ and $f^{\#}(t)f(tf^{\#}(t)) \to 1$ as $t \to \infty$. With this definition

$$\lim_{E \searrow 0} \frac{E^{2/(\alpha-2)} \log N(E)}{f^{\#}(E^{\alpha/(2-\alpha)})} = -C(\alpha, \nu)$$

with an explicit constant. In particular, for stretched-Gaussian potential $V^{(0)}$ this asymptotics is explicitly given as ([7])

$$\lim_{E \searrow 0} \frac{\log N(E)}{|\log E|^{2/\alpha}} = -\pi\nu\lambda^2 \ .$$

For the super-Gaussian case it is proven in [7] that

$$\lim_{E \searrow 0} \frac{\log N(E)}{|\log E|} = -2\pi\nu\ell_B^2 = -\frac{2\pi\nu}{B} \ , \tag{8}$$

with the additional assumption (2). In particular, the super-Gaussian decay includes compactly supported potentials; the case for which (8) was proven in [5]. In [8] a slightly more general definition of super-Gaussian decay was introduced:

$$\inf_{R>0} \operatorname*{ess\,sup}_{|x|>R} \frac{\log V^{(0)}(x)}{|x|^2} = -\infty \ , \tag{9}$$

and (8) was proven for such potentials (in addition to the condition (2)).

Finally, the following estimates were given in [7] for the Gaussian case (7)

$$-\pi\nu(\lambda^2 + 2\ell_B^2) \leq \liminf_{E \searrow 0} \frac{\log N(E)}{|\log E|} \leq \limsup_{E \searrow 0} \frac{\log N(E)}{|\log E|} \leq -\pi\nu \max\{\lambda^2, 2\ell_B^2\} \tag{10}$$

and the upper bound was weakened to $-2\pi\nu\ell_B^2$ in [8] if a more general definition of Gaussian decay is used that is analogous to (9). Our goal is to prove that the lower bound in (10) is the correct one as conjectured in [7].

Theorem 1.1 *Suppose that $V^{(0)}$ satisfies (1) and (7). Then*

$$\lim_{E \searrow 0} \frac{\log N(E)}{|\log E|} = -\pi\nu(\lambda^2 + 2\ell_B^2), \qquad \ell_B := B^{-1/2} \ . \tag{11}$$

Since the lower bound (10) has been proven in [7], we focus only on the upper bound. As usual, we define the Laplace transform of $N(E)$ as

$$L(t) := \int_0^\infty e^{-Et} dN(E) = \mathcal{E} e^{-tH_\omega}(x, x) \ .$$

Recall that the diagonal element of the averaged heat kernel is independent of x. For more details, see [5]. Using a standard Tauberian argument (see for example Appendix of [7] for details), the upper bound in (11) follows from

$$\limsup_{t\to\infty} \frac{\log L(t)}{\log t} \leq -\pi\nu(\lambda^2 + 2\ell_B^2) \ . \tag{12}$$

In the rest of the paper we prove (12). Several steps will be similar to [5], these will not be repeated in details. We give detailed proofs only for the new parts of the argument.

2 Localization

We use a two-step localization as in [5]. The first localization is identical to the upper bound in Proposition 3.2 of [5] and the proof is the same.

Proposition 2.1 *Let $M := [-m, m]^2$ be a square box, then*

$$L(t) \leq \liminf_{m\to\infty} \frac{1}{|M|} \mathcal{E}\, Tr\, e^{-tH_{M,\omega}} \ . \ \square$$

For the second, more refined localization we cannot neglect the tail of the impurity potentials. We will define effective *boundary potentials* that estimate the potential tails inside a domain Ω that come from impurities located outside of Ω.

To prove (12), it is enough to show that

$$\limsup_{t\to\infty} \frac{\log L(t)}{\log t} \leq -\pi\nu(L^2 + 2\ell_B^2) \tag{13}$$

for any $L < \lambda$. We fix two numbers, $0 < L < \overline{L} < \lambda$, for the rest of the proof and we omit the dependence on L and $\overline{L}$ of various quantities in the notation.

Using (7), there exists $R \geq 1$ such that

$$V^{(0)}(x) \geq e^{-|x|^2/\overline{L}^2} \qquad \text{for all} \quad |x| \geq R \ . \tag{14}$$

We also choose $R = R(\overline{L}, B)$ so large that $e^{-(2R)^2/\overline{L}^2} \leq B$. For any domain Ω we define the following boundary potentials ($\partial\Omega$ stands for the boundary of Ω):

$$\overline{V}_\Omega(x) := \exp\Big[-(\text{dist}(x, \partial\Omega))^2/\overline{L}^2 \Big] \cdot \mathbf{1}(x \in \Omega) \cdot \mathbf{1}(\text{dist}(x, \partial\Omega) \geq R) \ , \tag{15}$$

$$V_\Omega(x) := \exp\Big[-(\text{dist}(x, \partial\Omega))^2/L^2 \Big] \cdot \mathbf{1}(x \in \Omega) \ . \tag{16}$$

Similarly to Section 6 of [5] we fix parameters $0 < \beta < B/2$, $1 \leq s \leq m$ and let $M := [-m,m]^2$, $\widetilde{M} := [-m-s, m+s]^2$, $S := [-s,s]^2$, $\widetilde{S} := [-\frac{s}{2}, \frac{s}{2}]^2$ and $Q_z := Q + z$ for any square $Q \subset \mathbf{R}^2$ and $z \in \mathbf{R}^2$. Finally, let $\lambda^{(B+2\beta)}_{S_z,\omega}$ be the lowest eigenvalue of

$$\overline{H}^{(B+2\beta)}_{S_z,\omega} := \frac{1}{2}\Big\{ \Big[-i\nabla - \frac{B+2\beta}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \Big]^2 - (B+2\beta) \Big\} + V_\omega + \overline{V}_{S_z}$$

with Dirichlet boundary conditions on S_z. The magnetic field of $\overline{H}^{(B+2\beta)}_{S_z,\omega}$ is $B+2\beta$. Notice that this operator differs from its counterpart in Section 6 of [5] by the additional boundary potential $\overline{V}_{S_z}$. We have

Proposition 2.2 *Assume that $\beta < 1/(2\overline{L}^2)$, $\beta s^2 \geq 128$ and $s \geq 4R$. For any $z \in \mathbf{R}^2$ there exists a function η_z supported on S_z such that for any $f \in H^1_0(M)$*

$$\langle f, H_{M,\omega} f \rangle \geq \frac{\beta}{2\pi} \int_{\widetilde{M}} \mathrm{d}z \langle f\eta_z, \overline{H}^{(B+2\beta)}_{S_z,\omega} f\eta_z \rangle - 65 s^{-2} e^{-\beta s^2/8} \|f\|_{L^2(M)} .$$

Using this result, we obtain the following theorem from Proposition 2.1 exactly as Theorem 6.3 was proven in [5]:

Proposition 2.3 *Let $\ell(t) := 10(\log t/B)^{1/2}$, $s := n_0 \ell(t)$ and $S = [-s,s]^2$. For any fixed $0 < \beta < 1/(2\overline{L}^2)$ and $n_0 \geq (B/\beta)^{1/2}$, $n_0 \in \mathbf{Z}$*

$$\limsup_{t\to\infty} \frac{\log L(t)}{\log t} \leq \limsup_{t\to\infty} (\log t)^{-1} \log \mathcal{E} \exp\Big(-t\lambda^{(B+2\beta)}_{S,\omega} \Big) . \qquad \square \tag{17}$$

Proof of Proposition 2.2. Similarly to the proof of Proposition 6.1 in [5], we define

$$\varphi_z(x) := e^{-\beta(x-z)^2/2} e^{i\beta(x_2 z_1 - x_1 z_2)}$$

and $T_\beta := -i\partial_1 + \partial_2 + (B/2+\beta)x_2 - i(B/2+\beta)x_1$. We use the following identity to localize the kinetic energy for $f \in H^1_0(M)$

$$\langle f, H_{M,\omega} f \rangle = \frac{\beta}{\pi} \int \mathrm{d}z \int \Big\{ \tfrac{1}{2} |T_\beta(\varphi_z f)|^2 + V_\omega |\varphi_z f|^2 \Big\} ,$$

where we let $\int$ denote $\int_{\mathbf{R}^2}$. This magnetic localization principle was first used in [6].

Fix a smooth function $\theta(x)$ such that $\theta \equiv 1$ on $\widetilde{S}$, $\theta \equiv 0$ on $\mathbf{R}^2 \setminus [-\frac{3s}{4}, \frac{3s}{4}]^2$, $0 \leq \theta \leq 1$ and $\|\nabla\theta\|_\infty \leq 8s^{-1}$. Let $\theta_z(x) := \theta(x-z)$ and $\eta_z := \theta_z \varphi_z$. The function θ_z can be commuted with T_β at the expense of an error of size $\|\nabla\theta_z\|^2$ on the support of $\nabla\theta_z$. The result is the analogue of (6.13) in [5]

$$\langle f, H_{M,\omega} f \rangle \geq \frac{\beta}{2\pi} \Big[\int_{\widetilde{M}_z} \Big\{ \int \frac{1}{2} |T_\beta(\eta_z f)|^2 + \int V_\omega |\eta_z f|^2 \Big\} \mathrm{d}z - 128\pi\beta^{-1} s^{-2} e^{-\frac{\beta}{8}s^2} \Big] . \tag{18}$$

Finally, we estimate the boundary term $\int \mathrm{d}z \int \overline{V}_{S_z} |\eta_z f|^2$ in $\overline{H}^{(B+2\beta)}_{S_z,\omega}$ using $\overline{V}_{S_z}|\eta_z f|^2 \le e^{-(s/4\overline{L})^2}|\varphi_z|^2|f|^2$;

$$\frac{\beta}{2\pi}\int_{\widetilde{M}_z} \mathrm{d}z \int \overline{V}_{S_z}|\eta_z f|^2 \le e^{-s^2/(4\overline{L})^2}\int |f|^2 \ ,$$

which can be included into the error term in (18). $\Box$.

3 Enlargement of obstacles

We follow the basic strategy of Sznitman [10] and its magnetic version from [5] to estimate the lowest eigenvalue $\lambda^{(B+2\beta)}_{S,\omega}$ of $\overline{H}^{(B+2\beta)}_{S,\omega}$ by the lowest eigenvalue of a Hamiltonian with enlarged, hard-core obstacles. We need this argument for $\overline{H}^{(B+2\beta)}_{S,\omega}$, but the actual field does not play much role in this section, so for brevity we consider $\overline{H}^{(B)}_{S,\omega}$ and let $\lambda^{(B)}_{S,\omega}$ be its smallest eigenvalue.

The main novelty is that we cannot simply use Dirichlet boundary conditions for the "enlarged obstacle" Hamiltonian, since the potential tail penetrating into the clearing regimes does influence the lowest eigenvalue. We add an appropriate Gaussian boundary potential to the hard-core Dirichlet wall, and we also keep the Gaussian tail of the original potentials. The obstacle configuration ω is fixed throughout this section.

The "enlarged obstacle" Hamiltonian requires several definitions that were listed in Section 7.1 of [5]. Here we recall only that four parameters, ℓ, b, $\varepsilon > 0$ and $r > 0$ have to be fixed. With these parameters, one defines (Section 7.1 of [5]) the set of "good" points (their indices denoted by $\mathcal{G}$), clearing boxes and the set A^1, which is the ℓ-neighborhood of clearing boxes. Recall that a point x_i is "good" if it is not isolated from other points in a certain hierarchical sense. Clearing boxes are squares of size ℓ that contain a large regular set ("clearing") free of good points. Finally we define, for $s > b$,

$$\Omega := S \setminus \bigcup_{i\in\mathcal{G}} \Big[\overline{B}(x_i, 2R)\setminus B(x_i,R)\Big] \ , \qquad \Omega^b_+ := \Big([-s+b, s-b]^2 \cap A^1\Big) \setminus \bigcup_{i\in\mathcal{G}} \overline{B}(x_i, b) \ , \quad (19)$$

where $B(x,\rho)$ denotes the open ball of radius ρ about x. We choose $\delta = \frac{R}{100}$. Notice that Ω is defined by removing annuli around the good points, unlike in [5], where balls were removed. Ω^b_+ is the "clearing set", where the "enlarged obstacle" Hamiltonian will be defined.

We let

$$U(x) := e^{-|x|^2/\overline{L}^2}\cdot \mathbf{1}(|x|\ge R), \qquad \widetilde{V}_\omega(x) := \sum_{i\in\mathcal{G}} U(x - x_i(\omega)) \ ,$$

then we clearly have $V_\omega \ge \widetilde{V}_\omega$. This definition of $\widetilde{V}_\omega$ is different from (7.3) of [5]. The role of v in [5] will be played by the constant $e^{-(2R)^2/\overline{L}^2}$; this is a lower bound on the potential $\widetilde{V}_\omega$ in the annuli $\{R \le |x - x_i| \le 2R\}$ around the good points. The role of a in [5] is played by $2R$. The specific upper bound $a \le 1$ imposed in [5] will not be important.

We will estimate the lowest eigenvalue, $\widetilde{\lambda}$, of the Hamiltonian with potential $\widetilde{V}_\omega$ by $\widetilde{\lambda}_b$, the lowest eigenvalue with hard core potential on Ω_+^b. We add boundary potentials to both Hamiltonians. Since we work on multiply connected domains, we must take the gauge freedom into account as in Section 7.2 of [5]. Hence both eigenvalues are defined as the infimum over all gauges on the complementary domain of the obstacles. We recall that for any $\underline{\alpha} = \{\alpha_i\}_{i\in\mathcal{G}} \in [0, 2\pi)^{\mathcal{G}}$ we defined $B_{\underline{\alpha}}(x) := B + \sum_{i\in\mathcal{G}} \alpha_i B^*(x - x_i)$ and its radial gauge $A_{\underline{\alpha}}$, curl $A_{\underline{\alpha}} = B_{\underline{\alpha}}$, where $B^* := (4/\pi) \cdot \mathbf{1}_{B(0,1/2)}(x)$ (the definition of B^* in [5] missed a $\frac{1}{2\pi}$ factor). The magnetic field $B_{\underline{\alpha}}$ includes flux tubes of strength α_i around the good points. We define

$$\widetilde{\lambda} = \widetilde{\lambda}(B) := \inf_{\underline{\alpha}} \lambda_{\underline{\alpha}} , \qquad \lambda_{\underline{\alpha}} := \inf \mathrm{Spec}\left(\frac{1}{2}\Big[(-i\nabla - A_{\underline{\alpha}})^2 - B_{\underline{\alpha}}\Big] + \widetilde{V}_\omega + \overline{V}_S\right)_S , \tag{20}$$

where the subscript refers to Dirichlet boundary conditions on S. Clearly $\lambda_{S,\omega}^{(B)} \ge \widetilde{\lambda}$. Similarly,

$$\widetilde{\lambda}_b = \widetilde{\lambda}_b(B) := \inf_{\underline{\alpha}} \lambda_{b,\underline{\alpha}} , \qquad \lambda_{b,\underline{\alpha}} := \inf \mathrm{Spec}\left(\frac{1}{2}\Big[(-i\nabla - A_{\underline{\alpha}})^2 - B_{\underline{\alpha}}\Big] + V_{\Omega_+^b}\right)_{\Omega_+^b} , \tag{21}$$

again with Dirichlet boundary conditions on Ω_+^b. Notice that the decay of the boundary potential $V_{\Omega_b^+}$ is slightly stronger than that of $\widetilde{V}_\omega + \overline{V}_S$ since $L < \overline{L}$.

Let g_U denote the Green's function of any domain U, i.e., the solution to $\Delta g_U = -1$ on U and $g_U = 0$ on ∂U. We let $G_U := \max_{x\in\overline{U}} g_U(x)$.

The importance of these functions is that the lowest magnetic Dirichlet eigenvalue of a large domain U is essentially e^{-2BG_U} (a factor 2 was missing on page 349 of [5]), and the eigenfunction is roughly e^{Bg_U} with some cutoff near the boundary. Moreover, for "round" domains, g_U is roughly quadratic in the distance from the boundary. Hence, roughly,

$$\widetilde{\lambda}_b \sim \exp\left(-2BG_{\Omega_+^b}\right) + \int_{\Omega_+^b} \exp\Big(-\Big[\frac{\mathrm{dist}(x, \partial\Omega_+^b)}{L}\Big]^2\Big)\left|\frac{\exp Bg_{\Omega_+^b}(x)}{\|\exp Bg_{\Omega_+^b}\|}\right|^2 \mathrm{d}x \, .$$

Here the first term represents the kinetic energy due to localization in the clearing. The second term is the interaction of the "quantum" wavefunction with the "classical" effect; the effective contribution of the potential tails. It turns out that the second term dominates. The main contribution comes from the interplay between the Gaussian character of the magnetic eigenfunction $\approx \exp Bg_{\Omega_+^b}$ and the Gaussian potential.

The basic comparison result is the analogue of Corollary 7.3 in [5] (there are two misprints in (7.16) in [5]; $r \to \infty$ should be $r \to 0$ and a minus sign is missing in front of $\log K$).

Proposition 3.1 *For any fixed positive integer n_0 we let $s := n_0\ell$ and let $\varrho > 0$ be a positive number. For small enough r, ε, there exist $K = K(b, B, r, \ell, s, L, \overline{L}, \varepsilon, \varrho)$ and $w(r)$ with $\lim_{r\to 0} w(r) = 1$ such that $\widetilde{\lambda}_b^{w(r)} \le \widetilde{\lambda}/K$ if $\widetilde{\lambda} \le \min\{4K, e^{-\varrho BG_\Omega}\}$, and K satisfies*

$$\limsup_{r\to 0} \limsup_{\substack{b\to\infty \\ \varepsilon\to 0}} \limsup_{\ell\to\infty} \frac{-\log K}{\ell^2} = 0 \; . \tag{22}$$

The basic intuition behind this comparison is that if the lowest eigenvalue of $H_{S,\omega}$ is very small, then there must be a big clearing in the obstacle configuration, and the lowest eigenfunction is essentially supported in this clearing. Hence this eigenvalue can be estimated by the Dirichlet eigenvalue within the clearing even with enlarged obstacles. The inclusion of the boundary potential does not change this mechanism, but it changes both eigenvalues. The threshold for such eigenvalues is controlled by two different functions. The control given by K is analogous to [5]. The control $\widetilde{\lambda} \le e^{-\varrho BG_\Omega}$ is new.

The proof of Proposition 3.1 is similar to that of Theorem 7.2 in [5], but we have to include the boundary potential. We first show that the increase of the eigenvalue due to the enlargement is given by the size of the eigenfunction near the boundary (Lemma 3.2). Then, by applying a probabilistic argument, we show that $g_\Omega(x) \ll G_\Omega$ if Ω is large and x is close to the boundary (Lemma 3.3). In other words, the eigenvalue increases by at most a factor $e^{o(BG_\Omega)}$. For technical reasons we give these estimates for a slightly enlarged domain

$$\Theta := \Omega + B(0, 2\delta) \; .$$

In [5] (Lemma 7.7), we finally estimated G_Ω by the logarithm of the magnetic Dirichlet eigenvalue of Ω to show that $e^{o(BG_\Omega)} \le \widetilde{\lambda}^{-o(1)}$ and therefore $\widetilde{\lambda}_b \le \widetilde{\lambda}^{1-o(1)}$. The analogue of this estimate with a boundary potential is more complicated because it requires a control on g_Ω not only near the boundary. But fortunately we do not need this estimate with a precise constant since it is used only in the error factor $e^{o(BG_\Omega)}$. So we choose an alternative method that estimates G_Ω by the logarithm of the magnetic Dirichlet eigenvalue without boundary potential, exactly as in [5]. The new control $\widetilde{\lambda} \le e^{-\varrho BG_\Omega}$ stems from this modification.

We will state these lemmas precisely, but we give details of the proof only for the modifications compared with [5].

Lemma 3.2 *There exist positive constants, c_1, c_2, depending only on $B, L, \overline{L}, R, b$, such that*

$$\widetilde{\lambda}_b \leq c_1 \widetilde{\lambda} s^2 e^{2B\eta} \ , \tag{23}$$

whenever $\widetilde{\lambda} \leq c_2 s^{-2} e^{-2B\eta}$, $s > 2b \geq 40R$, where

$$\eta = \max\left\{ g_\Theta(z) \ : \ z \in \overline{\Theta} \setminus \Omega_+^{2b} \right\}, \qquad \Theta := \Omega + B(0, 2\delta) \ , \qquad \delta := \frac{R}{100} \ .$$

Proof. We fix $\underline{\alpha} \in [0, 2\pi)^{\mathcal{G}}$ and let $\varphi_{\underline{\alpha}}$ be the normalized eigenfunction belonging to $\lambda_{\underline{\alpha}}$. We can assume that $\lambda_{\underline{\alpha}} \leq c_2 s^{-2} e^{-2B\eta}$.

Let $T_{\underline{\alpha}} := -i\partial_1 + \partial_2 - (A_{\underline{\alpha}})_1 - i(A_{\underline{\alpha}})_2$, then by variational principle and integration by parts

$$\lambda_{b,\underline{\alpha}} = \inf_{\psi \in H_0^1(\Omega_+^b)} \frac{\int_{\Omega_+^b} \frac{1}{2} |T_{\underline{\alpha}} \psi|^2 + V_{\Omega_+^b} |\psi|^2}{\int_{\Omega_+^b} |\psi|^2} \ .$$

Let θ be a cutoff function such that $\theta \equiv 1$ on Ω_+^{2b}, $\theta \equiv 0$ on $\mathbf{R}^2 \setminus \Omega_+^b$, $0 \leq \theta \leq 1$ and $|\nabla\theta| \leq 4b^{-1}$. Then

$$\begin{aligned} \lambda_{\underline{\alpha}} &= \frac{1}{2} \int_S |T_{\underline{\alpha}} \varphi_{\underline{\alpha}}|^2 + \int_S (\widetilde{V}_\omega + \overline{V}_S) |\varphi_{\underline{\alpha}}|^2 \\ &\geq \frac{1}{4} \int_S |T_{\underline{\alpha}} (\theta\varphi_{\underline{\alpha}})|^2 - \|\nabla\theta\|_\infty^2 \int_{S \cap supp(\nabla\theta)} |\varphi_{\underline{\alpha}}|^2 + c_3^{-1} \int_S V_{\Omega_+^b} |\varphi_{\underline{\alpha}}|^2 \ , \end{aligned}$$

using the pointwise inequality $V_{\Omega_+^b}(x) \leq c_3[\widetilde{V}_\omega(x) + \overline{V}_S(x)]$ with some $c_3 = c_3(B, L, \overline{L}, b) \geq 1$. We use $\psi := \theta\varphi_{\underline{\alpha}}$ as a trial function to obtain

$$\lambda_{b,\underline{\alpha}} \leq \frac{2c_3 \lambda_{\underline{\alpha}} + 16b^{-2} \int_{S \setminus \Omega_+^{2b}} |\varphi_{\underline{\alpha}}|^2}{1 - \int_{S \setminus \Omega_+^{2b}} |\varphi_{\underline{\alpha}}|^2}$$

similarly to (7.19) in [5].

To complete the proof of (23), we need the upper estimate

$$\int_{S \setminus \Omega_+^{2b}} |\varphi_{\underline{\alpha}}|^2 \leq c_4 \lambda_{\underline{\alpha}} s^2 e^{2B\eta}$$

(with some $c_4 = c_4(B, \overline{L}, R, b)$) whose derivation is identical to the rest of the proof of Lemma 7.4 [5]. The only difference is that the balls $\overline{B}(x_i, a)$ are replaced with the annuli $\overline{B}(x_i, 2R) \setminus B(x_i, R)$, according to the new definition of Ω in (19), and the constant v is replaced with $e^{-(2R)^2/\overline{L}^2}$. In particular

$$S \setminus \Omega_+^{2b} \subset \left(\Omega \setminus \Omega_+^{2b} \right) \cup \bigcup_{i \in \mathcal{G}} \left[\overline{B}(x_i, 2R) \setminus B(x_i, R) \right] ,$$

and we estimate

$$\int_{\bigcup_{i\in\mathcal{G}}\left[\overline{B}(x_i,2R)\setminus B(x_i,R)\right]} |\varphi_{\underline{\alpha}}|^2 \le \lambda_{\underline{\alpha}} e^{(2R)^2/\overline{L}^2}$$

instead of the second inequality in (7.20) in [5]. The details are omitted. □

The next lemma is an analogue of Lemma 7.5 in [5]. It states that near the boundary of the enlarged obstacles the Green's function g_Θ is much smaller than its maximum G_Θ.

Lemma 3.3 *Let $20R \le b$, $40b \le \ell \le s$, $r < 1/4$ and $G_\Theta \ge c_5\ell$ with some $c_5 = c_5(R)$.*

(i) For small enough ε, there exist $\ell_0(\varepsilon, b) > 0$ and $0 < k = k(\varepsilon, b) \le 1/4$ such that

$$\sup_{x\in\Sigma\cap\Theta} g_\Theta(x) \le \Big(\frac{b}{\ell}\Big)^k G_\Theta \qquad \text{for} \quad \ell \ge \ell_0(\varepsilon, b) \tag{24}$$

with $\Sigma := \left(\overline{S}_-^{2\delta} \setminus S_+^{2b}\right) \cup \bigcup_{i\in\mathcal{G}} \overline{B}(x_i, 2b)$, where $S_\pm^c := (-s \pm c, -s \mp c)^2 \subset \mathbf{R}^2$.

(ii) There exists a positive number c_0 such that

$$g_\Theta(x) \le \Big[(1-c_0)^{1/r} G_\Theta + c_0^{-1} r^2\ell^2\Big] + \sup_{y\in\Sigma\cap\Theta} g_\Theta(y) \tag{25}$$

for all $x \in \Theta$, $x \notin A^1 \cap S$.

Proof. The proof is almost identical to that of Lemma 7.5 [5]. The only difference is that Ω is defined by removing annuli, hence the case $x \in B(x_i, R + 2\delta)$, $i \in \mathcal{G}$, needs a separate estimate. For such x, the exit time from Θ is at most the hitting time of the circle $\partial B(x_i, R + 2\delta)$, hence its expected value depends only on R. This estimate is taken into account in Lemma 3.3 by the extra requirement $G_\Theta \ge c_5\ell$. □

We use the following lemma to establish that the maximal expected hitting time is essentially the same for Ω and $\Theta = \Omega + B(0, 2\delta)$. The proof is identical to that of Lemma 7.6 [5]; the geometric condition used in [5] is satisfied for the new definition of Ω (19) as well.

Lemma 3.4 *There exists two constant c_6, c_7 depending on R such that*

$$G_\Omega \le G_\Theta \le c_6 G_\Omega + c_7 \; . \qquad \square \tag{26}$$

Now we are ready to prove Proposition 3.1. We note that $\overline{\Theta} \setminus \Omega_+^{2b} \subset \Sigma \cup (A^1)^c$. Hence for large enough ℓ, the combination of (24), (25) and (26) gives

$$\eta \le c_8\Big[(1-c_0)^{1/r} G_\Omega + c_0^{-1} r^2\ell^2\Big] \tag{27}$$

with some $c_8 = c_8(R)$, similarly to (7.43) in [5]. This means that $\eta \ll G_\Omega$ for small r if $G_\Omega \gg r^2\ell^2$. Since $\widetilde{\lambda} \le e^{-\varrho B G_\Omega}$, we see that $e^{2B\eta}$ is bounded by a small inverse power of $\widetilde{\lambda}$, so from (23) we get that $\widetilde{\lambda}_b \le \widetilde{\lambda}^{1-o(1)}$ as $r \to 0$. The case $G_\Omega = O(r^2\ell^2)$ can be included in the error factor K. The details are very similar to [5] and are left to the reader. □.

4 Proof of the upper bound in Theorem 1.1

We recall the definition of $\widetilde{\lambda}(B)$ and $\widetilde{\lambda}_b(B)$ from (20) and (21). Using the notations and results of Section 3 for B replaced with $B+2\beta$, we have $\widetilde{\lambda}(B+2\beta) \le \lambda_{S,\omega}^{(B+2\beta)}$ with $0 \le \beta \le B/2$, $s = n_0 \ell(t)$ and $\ell = \ell(t) = 10\sqrt{\frac{\log t}{B}}$. We will need Ω defined in (19) and we note that Ω depends on B, R, t, b, ε, r and n_0.

Combining Proposition 2.3 with Proposition 3.1, we see that

$$\limsup_{t\to\infty} \frac{\log L(t)}{\log t} \le \tag{28}$$

$$\limsup_{t\to\infty} (\log t)^{-1} \log \left[\mathcal{E} \exp(-tK\widetilde{\lambda}_b^{w(r)}) + \mathcal{E}\exp\Big(-te^{-\varrho(B+2\beta)G_\Omega}\Big) + \exp(-4Kt)\right]$$

with $\widetilde{\lambda}_b = \widetilde{\lambda}_b(B+2\beta)$. Since the $n_0 \to \infty$ limit will always be taken before $\beta \to 0$, the condition $n_0 \ge (B/\beta)^{1/2}$ of Proposition 2.3 is satisfied. The last term in (28) is negligible for small enough ε, r and large enough b, using (22). The estimates on the other two terms are given in the following propositions:

Proposition 4.1 *For any magnetic field $B > 0$*

$$\limsup_{n_0\to\infty} \limsup_{r\to 0} \limsup_{\substack{b\to\infty \\ \varepsilon\to 0}} \limsup_{t\to\infty} (\log t)^{-1} \log \mathcal{E}\exp\Big(-tK[\widetilde{\lambda}_b(B)]^{w(r)}\Big) = -\pi\nu(L^2 + 2B^{-1}) \; . \tag{29}$$

Proposition 4.2 *For any magnetic field $B > 0$*

$$\limsup_{n_0\to\infty} \limsup_{r\to 0} \limsup_{\substack{b\to\infty \\ \varepsilon\to 0}} \limsup_{t\to\infty} (\log t)^{-1} \log \mathcal{E}\exp\Big(-te^{-BG_\Omega}\Big) \le -\frac{2\pi\nu}{B} \; . \tag{30}$$

Theorem 1.1 follows from these propositions via (12) just by choosing $\varrho < 2/(2+BL^2)$, using Proposition 4.1 with a magnetic field $B+2\beta$ and Proposition 4.2 with a magnetic field $\varrho(B+2\beta)$, and finally letting $\beta \to 0$. $\square$

Proof of Proposition 4.1. Let Ω be an arbitrary domain and let $B > 0$ fixed. Let

$$\widehat{\lambda}^{(B)}(\Omega) := \inf\left\{ \inf \mathrm{Spec}\Big(\frac{1}{2}[(-i\nabla - \widehat{A})^2 - B] + V_\Omega\Big)_\Omega \;:\; \widehat{A} \in \mathcal{A}(\Omega)\cap C^\infty(\overline{\Omega}),\ \mathrm{curl}\ \widehat{A} = B \text{ on } \Omega\right\}$$

be the smallest eigenvalue of the magnetic Hamiltonian with boundary potential and Dirichlet boundary conditions on Ω. We also took the infimum over all possible gauges, which is unnecessary for simply connected Ω. Here $\mathcal{A}(\Omega)$ is the set of real analytic vectorfields on Ω.

In Section 5 we show the following estimate for $\widehat{\lambda}^{(B)}(\Omega)$:

Proposition 4.3 *For any $\kappa > 0$, $L > 0$, $B > 0$ and any domain Ω with volume $|\Omega| \geq C(\kappa, L, B)$, we have*

$$\widehat{\lambda}^{(B)}(\Omega) \geq \exp\Big[-\frac{|\Omega|}{\pi(L^2 + 2B^{-1})}(1+\kappa)\Big] . \tag{31}$$

Using this estimate and that $\widehat{\lambda}^{(B)}$ is a monotone function of the domain, the proof of Proposition 4.1 is identical to the argument in Section 8 of [5]. □

Proof of Proposition 4.2. We consider

$$\Omega^* := S \setminus \bigcup_{i\in\mathcal{G}} \overline{B}(x_i, a)$$

with some fixed $0 < a < R$, we let $\widetilde{V}^* := v \cdot \mathbf{1}\Big(x \in \bigcup_{i\in\mathcal{G}} \overline{B}(x_i, a)\Big)$ with $v = e^{-(2R)^2/L^2}$ and we let $\widetilde{\lambda}^* = \widetilde{\lambda}^*(B)$ be the infimum over $\underline{\alpha}$ of the lowest eigenvalue of $\frac{1}{2}[(-i\nabla - A_{\underline{\alpha}})^2 - B_{\underline{\alpha}}] + \widetilde{V}^*$. These are exactly the set Ω, the potential $\widetilde{V}$ and the eigenvalue $\widetilde{\lambda}$ in [5], but here we use the star superscript to distinguish them from their counterparts used in the present paper.

We claim that for any fixed $B > 0$, $n_0, R, L, r, \varepsilon, b$

$$\limsup_{t\to\infty} (\log t)^{-1} \log \mathcal{E} \exp\Big(-te^{-BG_\Omega}\Big) \leq \limsup_{t\to\infty} (\log t)^{-1} \log \mathcal{E} \exp\Big(-te^{-BG_{\Omega^*}}\Big) . \tag{32}$$

For the proof, we define

$$\Omega^\# := S \setminus \bigcup_{i\in\mathcal{G}} \overline{B}(x_i, 2R) .$$

Clearly $g_\Omega(x) = g_{\Omega^\#}(x)$ for any $x \notin \bigcup_{i\in\mathcal{G}} \overline{B}(x_i, R)$, while $g_\Omega(x) \leq c_9(R)$ if $x \in \overline{B}(x_i, R)$ for some $i \in \mathcal{G}$. Hence $G_\Omega \leq G_{\Omega^\#} + c_9(R) \leq G_{\Omega^*} + c_9(R)$, where the second inequality follows from $\Omega^\# \subset \Omega^*$.

We then recall that Section 8 of [5], from (8.1) through (8.8) actually gave the following bound (there B was replaced by $B + 2\beta$):

$$\limsup_{n_0\to\infty} \limsup_{r\to 0} \limsup_{\substack{b\to\infty \\ \varepsilon\to 0}} \limsup_{t\to\infty} (\log t)^{-1} \log \mathcal{E} e^{-tN\widetilde{\lambda}^*(B)} \leq -\frac{2\pi\nu}{B} . \tag{33}$$

for any function N satisfying

$$\limsup_{r\to 0} \limsup_{\substack{b\to\infty \\ \varepsilon\to 0}} \limsup_{t\to\infty} \frac{-\log N}{\log t} = 0 .$$

Using Lemma 7.7 of [5], stating that $\widetilde{\lambda}^*$ is smaller than $e^{-BG_{\Omega^*}}$ modulo negligible factors, we easily obtain (30) from (32) and (33). □

5 Estimate on the magnetic eigenvalue with a boundary potential

In this section we prove Proposition 4.3. Let D be the disk of radius $R_\Omega := \pi^{-1/2}|\Omega|^{1/2}$ centered at the origin. For any function $a(r)$ with

$$0 \le 2\pi a(r) r \le B\pi r^2 , \qquad \text{for all} \quad 0 \le r \le R_\Omega , \tag{34}$$

we define a radial gauge $A_{rad}(x) = a(r)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ (in polar coordinates, $x = re^{i\theta}$) that generates the radial magnetic field $B_{rad}(x) = \text{curl } A_{rad}(x) = a'(r) + r^{-1}a(r)$. Condition (34) requires the flux of the magnetic field B_{rad} to be not smaller than that of the constant B field on all concentric disks $B(0, r)$.

Let $\mathcal{H}$ and $\mathcal{H}_D$ be the Hilbert spaces of radially symmetric $H^1(\mathbf{R}^2)$ and $H^1_0(D)$ functions, respectively. Let

$$H(a) := \frac{1}{2}\Big[(-i\nabla - A_{rad})^2 - B_{rad}\Big]$$

be defined on $\mathcal{H}_D$, and let $\lambda(a)$ be its lowest eigenvalue. It is easy to see that the corresponding eigenfunction can be chosen nonnegative. In fact

$$\langle f, H(a)f\rangle \ge \langle |f|, H(a)|f|\rangle = \Big\langle |f|, \frac{1}{2}(-\Delta + a^2 - B_{rad})|f|\Big\rangle \qquad f \in \mathcal{H}_D . \tag{35}$$

Proposition 2.1 of [4] states that the lowest magnetic Dirichlet eigenvalue on Ω with a constant magnetic field B is minorized by $\lambda(a)$ for some $a(r)$ that satisfies (34).

For any nonnegative function ψ we denote its symmetric rearrangement by ψ^*, i.e., ψ^* is the unique radial function with the property that $|\{\psi \ge c\}| = |\{\psi^* \ge c\}|$ for any c. It is not stated explicitly in [4], but actually the proof of Proposition 2.1 in [4] gives the following result from which the comparison of the eigenvalues has been derived.

Proposition 5.1 *Let $\widehat{A} \in \mathcal{A}(\Omega) \cap C^\infty(\overline{\Omega})$, curl $\widehat{A} = B$ on Ω and let $\widehat{H} = \frac{1}{2}\Big[(-i\nabla - \widehat{A})^2 - B\Big]$. Then for any function $f \in H^1_0(\Omega)$ there exists a function $a(r)$ that satisfies (34) such that*

$$\langle f, \widehat{H} f\rangle \ge \langle |f|^*, H(a)|f|^*\rangle .$$

For the proof one only has to notice that the radial trial function $q(r)$ defined on p. 289 of [4] as $q(r) = \Lambda^{-1}(h^*(r))$ is actually the symmetric rearrangement of $\psi = |f|$ since $h = \Lambda(\psi)$ and Λ is strictly monotone. $\square$

Now we include the boundary potential V_Ω (see (16)). We replace V_Ω by

$$\widehat{V}_\Omega := \frac{1}{\pi L^2} \cdot \mathbf{1}_{\Omega^c} \star e^{-|\cdot|^2/L^2}$$

where $\mathbf{1}_{\Omega^c}$ is the characteristic function of Ω^c and $\star$ denotes the convolution. It is straightforward that $\widehat{V}_\Omega \le V_\Omega$ on Ω and $\widehat{V}_\Omega \le 1$ everywhere.

By the Riesz rearrangement inequality,

$$\langle f, (1-\widehat{V}_\Omega) f\rangle = \frac{1}{\pi L^2} \iint |f(x)|^2 e^{-|x-y|^2/L^2} \mathbf{1}_\Omega(y) \mathrm{d}x\mathrm{d}y \le \frac{1}{\pi L^2} \iint (|f|^2)^*(x) e^{-|x-y|^2/L^2} \mathbf{1}_D(y) \mathrm{d}x\mathrm{d}y \ ,$$

where the disk D is the symmetric rearrangement of the set Ω. A simple estimate yields

$$\langle f, \widehat{V}_\Omega f\rangle \ge \langle |f|^*, W_\eta |f|^*\rangle$$

for any $f \in L^2(\Omega)$ and any $\eta > 0$ with

$$W_\eta(x) := \frac{1}{2} \exp\Big(-(1+\eta^{-1}) - \frac{(R_\Omega - |x|)^2}{L^2}(1+\eta)\Big) \ , \qquad |x| \le R_\Omega \ .$$

From these estimates and Proposition 5.1 we conclude that there exists a radial function $a(r)$, satisfying (34), such that

$$\widehat{\lambda}^{(B)}(\Omega) \ge \inf \mathrm{Spec}\Big(H(a) + W_\eta\Big) \qquad \text{on} \quad \mathcal{H}_D \ .$$

Next we claim that if $a_1(r) \le a_2(r)$ satisfy (34), then

$$\inf \mathrm{Spec}\Big(H(a_1) + W_\eta\Big) \ge \inf \mathrm{Spec}\Big(H(a_2) + W_\eta\Big) \qquad \text{on} \ \ \mathcal{H}_D \ .$$

This is proven exactly as Lemma 3.1 in [4]. It is easy to check that the inclusion of a bounded nonnegative radial potential W_η does not alter the trial function argument.

Therefore $H(a) + W_\eta$ has the lowest eigenvalue if $a(r) = Br/2$, i.e., in case of the constant field. Using (35), this eigenvalue is the same as the lowest eigenvalue, λ_η, of

$$H_\eta := H_{osc} + W_\eta \ , \qquad \text{with} \quad H_{osc} := \frac{1}{2}\Big[-\Delta + \frac{Bx^2}{4} - B\Big] \qquad \text{on} \ \ \mathcal{H}_D \ .$$

Let $\varphi_0(x) = \sqrt{\frac{B}{2\pi}} e^{-Bx^2/4}$ span the kernel of the harmonic oscillator H_{osc} on $\mathcal{H}$, and let $P := |\varphi_0\rangle\langle\varphi_0|$ be the projection onto this kernel. It is well known that H_{osc} has a gap of size B above zero on $\mathcal{H}$. We can estimate λ_η by decomposing the eigenfunction $f \in \mathcal{H}_D$ as $f = Pf + (I-P)f$:

$$\lambda_\eta = \Big\langle (I-P)f, H_{osc}(I-P)f\Big\rangle + \langle f, W_\eta f\rangle \ge B\|(I-P)f\|^2 + \langle f, W_\eta f\rangle \ . \tag{36}$$

Furthermore,

$$\lambda_\eta \geq \int_D W_\eta |f|^2 \geq \frac{1}{2}\int_D W_\eta |Pf|^2 - 2\int_D W_\eta |(I-P)f|^2 \geq \frac{1}{2}\int_D W_\eta |Pf|^2 - \lambda_\eta B^{-1} ,$$

using (36) and $W_\eta \leq 1/2$. Hence

$$\lambda_\eta \geq \frac{B}{2(B+1)} \int_D W_\eta |Pf|^2 . \tag{37}$$

Since $\|Pf\|^2 + \|(I-P)f\|^2 = \|f\|^2 = 1$ and $\|(I-P)f\|^2 \leq \lambda_\eta B^{-1}$ from (36), we have $\|Pf\|^2 = |\langle f, \varphi_0\rangle|^2 \geq 1 - \lambda_\eta B^{-1}$. We can assume that $\lambda_\eta < B/2$, otherwise Proposition 4.3 is trivial. Hence

$$\lambda_\eta \geq \frac{B}{4(B+1)} \int_D W_\eta |\varphi_0|^2 = \frac{B^2}{16\pi(B+1)} e^{-(1+\eta^{-1})} \int_D e^{-(1+\eta)(R_\Omega - |x|)^2/L^2} e^{-Bx^2/2} \mathrm{d}x \tag{38}$$

$$\geq C(B,L) \exp\Big[-(1+\eta^{-1}) - \frac{R_\Omega^2}{L^2 + 2B^{-1}}(1+\eta)\Big] .$$

From this bound, Proposition 4.3 easily follows. □

Remark: From the integration (38) one can see the interplay between the Gaussian eigenfunction φ_0 and the Gaussian potential W_η. In particular, the main contribution comes from the intermediate regime around $|x| \approx \frac{2}{BL^2+2} R_\Omega$.

References

[1] Aharonov, Y., Casher, A.: Ground state of spin-1/2 charged particle in a two-dimensional magnetic field. Phys. Rev. **A19**, 2461-2462 (1979)

[2] Broderix, K., Hundertmark, D., Kirsch, W., Leschke, H.: The fate of Lifshits tails in magnetic fields. J. Stat. Phys. **80**, 1-22 (1995)

[3] Donsker, M. D., Varadhan, S. R. S.: Asymptotics for the Wiener sausage. Commun. Pure Appl. Math. **28**, 525-566 (1975), Erratum: **28**, 677 (1975)

[4] Erdős, L.: Rayleigh type isoperimetric inequality with a homogeneous magnetic field. Calc. Var. and PDE. **4**, 283-292 (1996)

[5] Erdős, L.: Lifschitz tail in a magnetic field: the nonclassical regime. Probab. Theory Relat. Fields **112**, 321-371 (1998).

[6] Erdős, L., Solovej, J. P.: Semiclassical eigenvalue estimates for the Pauli operator with strong non-homogeneous magnetic fields. II. Leading order asymptotic estimates. Commun. Math. Phys. **188**, 599-656 (1997)

[7] Hupfer, T., Leschke, H., Warzel, S.: Poissonian obstacles with Gaussian walls discriminate between classical and quantum Lifshits tailing in magnetic fields. To appear in J. Stat. Phys.

[8] Hupfer, T., Leschke, H., Warzel, S.: The multiformity of Lifshits tails caused by random Landau Hamiltonians with repulsive impurity potentials of different decay at infinity. To appear in Proceedings of the 1999 UAB-GIT International Conference on Differential Equations and Mathematical Physics, Birmingham, AL. `http://xxx.lanl.gov/abs/math-ph/9910034` (1999)

[9] Pastur, L., Figotin, A.: Spectra of Random and Almost-periodic Operators. Berlin-Heidelberg-New York: Springer 1992.

[10] Sznitman, A. S.: Lifschitz tail and Wiener sausage, I. J. Func. Anal. **94**, 223-246 (1990)