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SCALING LAW IN THE STANDARD MAP CRITICAL FUNCTION. INTERPOLATING HAMILTONIAN AND FREQUENCY MAP ANALYSIS.

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ABSTRACT. We study the behaviour of the Standard map critical function in a neighbourhood of a fixed resonance, that is the scaling law at the fixed resonance.

We prove that for the fundamental resonance the scaling law is linear. We show numerical evidence that for the other resonances p/q, $q \geq 2$, $p \neq 0$ and p and q relatively prime, the scaling law follows a power–law with exponent $1/q$.

1. INTRODUCTION.

The standard map [15, 21] is the area–preserving twist map of the cylinder: $\mathbb{R} \,/\, (2\pi\mathbb{Z}) \times \mathbb{R}$ given by:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T_\epsilon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + \epsilon \sin x \\ y + \epsilon \sin x \end{pmatrix} \tag{1.1}$$

where ϵ is a real parameter. Note that for all values of $\epsilon \in \mathbb{R}$ one has $S \circ T_\epsilon \circ S = T_{-\epsilon}$ where S is the involution of the cylinder $S\binom{x}{y} = \binom{x+\pi}{y}$. This allows us to restrict the study of T_ϵ to ϵ non negative.

The dynamics of the Standard Map depends critically on the parameter ϵ. For $\epsilon = 0$ the curves $y_0 = \textit{constant}$ are preserved by the map and the motion on these curves is a rotation of frequency $\omega = y_0$. For $\epsilon > 0$ not all these curves persist. Given an initial condition (x_0, y_0) we define the *rotation number* ω_T by:

$$2\pi\omega_T\left(x_0, y_0, \epsilon\right) = \lim_{n\to\infty} \frac{\pi_1\left(T^n_\epsilon\left(x_0, y_0\right)\right) - x}{n}$$

when the limit exists, where π_1 is the projection on the first coordinate: $\pi_1(x, y) = x$. For $\epsilon > 0$ by Poincaré–Birkhoff's Theorem (see [44, 10, 11, 23] and [39] for a different proof) homotopically non–trivial invariant curves with rational rotation number do not exist: the invariant curves of rotation number $\omega_T(x, y, 0) = p/q$, for integer p, q, $q > 0$ relatively prime, split into $2q$ points when $\epsilon > 0$. This set of points is divided into two subsets of points each one composed of q points: the *elliptic* points and the *hyperbolic* ones.

Let us define

$$\begin{aligned}\epsilon_{crit}(\omega) = \sup\{\epsilon > 0 : \forall\epsilon' \in [0, \epsilon]\,, &\text{there exists a } \mathcal{C}^1\text{, homotopically} \\ &\text{non–trivial, invariant circle of } T_{\epsilon'} \text{ whose rotation number is } \omega\}\end{aligned}$$

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and $\epsilon_{crit}(\omega) = 0$ if $\omega \in \mathbb{Q}$. The function $\omega \mapsto \epsilon_{crit}(\omega)$ is called the *critical function* for the standard map. The definition of $\epsilon_{crit}(\omega)$ actually depends on the smoothness required for the invariant curve. However, whatever is the regularity of the curve (say $\mathcal{C}^1$, $\mathcal{C}^k$, $\mathcal{C}^\infty$ or $\mathcal{C}^\omega$) this function is very irregular: it is almost everywhere discontinuous and zero on a dense G_δ–set. Moreover J.Mather proved [38] that $\sup_{\omega\in\mathbb{R}} \epsilon_{crit}(\omega) \leq \frac{4}{3}$ and it is conjectured that $\sup_{\omega\in\mathbb{R}} \epsilon_{crit}(\omega) = \epsilon_{crit}\left(\frac{\sqrt{5}-1}{2}\right) = 0.971635\ldots$ [21, 29, 28].

For suitable irrational ω, KAM theory [24, 2, 40] assures that for sufficiently small $|\epsilon|$ there exists an analytic invariant circle upon which the dynamics of T_ϵ, restricted on this curve, is analytically conjugated to the rotation $R_\omega(\theta) = \theta + 2\pi\omega$. This means that there exists an analytic (w.r.t. θ on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $|\epsilon| < \epsilon_0$, for some $\epsilon_0 > 0$) function $u(\theta,\omega,\epsilon)$ such that

$$x = \theta + u(\theta,\omega,\epsilon), \quad 1 + \frac{\partial u}{\partial \theta}(\theta,\omega,\epsilon) > 0 \quad \forall \theta \in \mathbb{T}$$

and $\theta' = \theta + 2\pi\omega$, namely u reparametrizes the invariant curve. Using perturbation theory we can expand this function into a power series w.r.t. ϵ and Fourier' series w.r.t. θ

$$u(\theta,\omega,\epsilon) = \sum_{n\geq 1} \epsilon^n u_n(\theta,\omega) = \sum_{n\geq 1} \epsilon^n \sum_{k\in\mathbb{Z},|k|\leq n} \hat{u}_n^{(k)}(\omega)\, e^{ik\theta} \tag{1.2}$$

which gives the so called *Lindstedt series* [43]. One may define the *critical radius of convergence* of such series as

$$\rho_{crit}(\omega) = \left[\sup_{\theta\in\mathbb{T}} \limsup_{n\to\infty} |u_n(\theta,\omega)|^{\frac{1}{n}}\right]^{-1}. \tag{1.3}$$

It is clear that $\rho_{crit}(\omega) \leq \epsilon_{crit}(\omega)$, but it is an interesting open question to decide whether the two functions are equal or not on some set of frequencies. See [6, 5] for a positive numerical answer and [17] for a negative one. This will imply that the function defined for $\epsilon \in \mathbb{D}_{\rho_{crit}(\omega)}$ as the sum of the series (1.2) can be analytically continued outside its disk of convergence.

In [36] Marmi and Stark studied the relation between the $\epsilon_{crit}(\omega)$ and a "universal function" depending only on the frequency: the Brjuno function $B(\omega)$ (see [12, 37, 47]). Using the Greene residue criterion ([21]), Marmi and Stark find *numerically* evidence that it exists $\beta > 0$ and a positive constant C, such that

$$\left| \ln \epsilon_{crit}(\omega) + \beta\, B(\omega) \right| \leq C \tag{1.4}$$

uniformly in ω, as ω ranges in some set of quadratic irrationals, with β approximately equal 0.9. Comparing their results with similar problems in [35, 47] they expected the value $\beta = 2$.

In [9] and more recently in [7] the behaviour of $\rho_{crit}(\omega)$ when ω goes to a rational value is studied; in both cases *complex frequencies* were used. It is well known that if the imaginary part of ω is not zero the series (1.2) converges because there are no small divisors. In these papers the limit $\omega \to \frac{p}{q}$ of $\rho_{crit}(\omega)$ when $\omega = \frac{p}{q} + i\eta$ for some real $\eta \to 0$ was studied. The result is the following scaling law

$$\rho_{crit}(\omega) \sim C_{p/q} \left|\omega - \frac{p}{q}\right|^{\frac{2}{q}} \tag{1.5}$$

for some positive constant $C_{p/q}$ depending on the resonance.

Finally very recently Berretti and Gentile [8] proved that

$$\left| \ln \rho_{crit}(\omega) + 2B(\omega) \right| \leq C' \tag{1.6}$$

for some positive constant C', uniformly w.r.t. $\omega \in \mathbb{R}$. This implies that the series given in (1.2) converges if and only if ω is a Brjuno number (i.e. $B(\omega) < \infty$).

The scaling law is related to this result which makes a link between rotation number and Brjuno function. One can prove ([14]) that

$$\liminf_{\substack{\omega \to p/q \\ \omega \text{ is a Brjuno's number}}} \left(B(\omega) + \frac{1}{q} \ln \left| \omega - \frac{p}{q} \right| \right) = c'_{p/q}$$

with $c'_{p/q}$ some non negative finite constant depending on the involved resonance[1], then the result of Berretti and Gentile [8] implies the value $\beta = 2$.

To end this short review about standard map critical functions and their scaling laws we cite the very recent work of Treschev and Zubelevich [46], where they studied (quite general) area preserving twist maps of the standard cylinder into itself, ϵ–close to an integrable ones. They proved that under some assumptions: given any rational number $p/q \in \mathbb{Q}$, there exist two positive constants c and ϵ_0 such that for all $|\epsilon| < \epsilon_0$, there exist two invariant homotopically non–trivial invariant $\mathcal{C}^1$ curves with frequencies ω_l and ω_b and

$$\omega_l < \frac{p}{q} < \omega_b \quad \text{and} \quad |\omega_b - \omega_l| < c\epsilon.$$

The Theorem they prove (Theorem 3 p.76 [46]) is based on a strong hypothesis (in particular assumption 2., which depends on the involved resonance) which is not verified by the standard map, except for the resonance $0/1$.

We consider the fundamental resonance $0/1$, we fix $\epsilon > 0$ and we define

$$\begin{aligned}\omega^+(\epsilon) = \inf\{\omega > 0 : T_\epsilon &\text{ has a } \mathcal{C}^1\text{, homotopically non–trivial,} \\ &\text{invariant curve of rotation number } \omega\}\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}\omega^-(\epsilon) = \sup\{\omega < 0 : T_\epsilon &\text{ has a } \mathcal{C}^1\text{, homotopically non–trivial,} \\ &\text{invariant curve of rotation number } \omega\}\end{aligned} \tag{1.8}$$

note that $\omega^+(\epsilon) = -\omega^-(\epsilon)$. These definitions are similar to the one of $\mu(\omega)$ given by R.S. Mac Kay [34] in the study of the Hamiltonian of Escande. The existence of a stochastic layer around the hyperbolic fixed point $x = y = 0$, implies $\omega^+(\epsilon) \neq 0$, namely there is a "gap" in the curve which associates to each invariant circle its rotation number (frequency curve). The main result of the first part of this paper is the following Theorem[2].

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[1] We point out that the previous result is false if we replace lim inf with lim. In fact considering the fundamental resonance for simplicity and taking the sequence of Brjuno's numbers $(\omega_n)_{n\geq 0}$, $\omega_n^{-1} = a_1(n) + \frac{1}{a_2(n)+\xi}$, with $a_i : \mathbb{N} \to \mathbb{N}^*$ strictly monotone increasing functions for $i = 1, 2$ and ξ any Brjuno's number. Then we have $\lim_{n\to+\infty} B(\omega_n) + \ln \omega_n = 0$ if $\ln a_2(n) = o(a_1(n))$ for $n \to +\infty$, whereas if $a_1(n) = o(\ln a_2(n))$ for $n \to +\infty$, then $\lim_{n\to+\infty} B(\omega_n) + \ln \omega_n = +\infty$.

[2] We point out that already in [34] for a one degree of freedom Hamiltonian system depending periodically on time, logarithmic singularities were found for the breakup of invariant tori with frequencies closer and closer to rational ones.

Theorem 1.1. *There exist two positive constant $\Omega_{0/1}$ and $c_{0/1}$ such that for all $|\omega| < \Omega_{0/1}$ then*

$$\Big| \ln \epsilon_{crit}(\omega) - \ln|\omega| \Big| \leq c_{0/1} .$$

Following [46], to prove our Theorem we need a upper bound for $\ln \epsilon_{crit}(\omega) - \ln|\omega|$. We obtain this bound combining ideas taken from two different domains: the exponential smallness of the splitting of the separatrices and the interpolation of diffeomorphisms by vector fields.

The splitting of the separatrices for the standard map has been widely studied. Using the results of Lazutkin and Gelfreich [19, 31, 32, 33] we get a lower bound to the action corresponding to the last invariant torus. More precisely for a fixed ϵ we define $y^+(\epsilon)$, respectively $y^-(\epsilon)$, as the intersection point of the homotopically invariant $\mathcal{C}^1$ curve of frequency $\omega^+(\epsilon)$, respectively $\omega^-(\epsilon)$, with the y–axis, note that $y^+(\epsilon) = -y^-(\epsilon)$. Then we prove (Proposition 2.2) that

$$y^+(\epsilon) \geq c e^{-\frac{\pi^2}{2\sqrt{\epsilon}}}$$

for some positive constant c and ϵ small enough.

Using a result due to Benettin and Giorgilli [4], we construct an Hamiltonian system whose time–1 flow interpolates the Standard map; this allows us to exhibit an asymptotic development for the Standard map rotation number (Proposition 2.11) which, joint with the bound on $y^+(\epsilon)$, leads to the desired upper bound. We note that to prove our result it will be sufficient to find an interpolating Hamiltonian system $\epsilon^{\frac{3}{2}}$–close to the standard map; we nevertheless give the asymptotic development because to the best of our knowledge this result was not know and, secondly, the construction of an interpolating vector field exponentially close needs the same amount of difficulty as the construction of an ϵ^k–close vector field, for any power k.

In the second part (section 3) we present the *Frequency Map Analysis* (FMA) method of Laskar [18, 26, 25, 29] and its application to the numerical investigation of the Standard map critical function.

For the fundamental resonance $0/1$, we find numerical results in agreement with the analytical ones presented in section 2. For the other resonances we find the behaviour

$$\epsilon_{crit}(\omega) \sim c_{p/q} \Big|\omega - \frac{p}{q}\Big|^{\frac{1}{q}}$$

for ω in a neighbourhood of p/q and $c_{p/q} > 0$. We remark that using FMA we deal with *real frequencies*.

Using the FMA tool we can do investigations directly on the critical function and go deeply in the resonant region. Actually we deal with frequencies that are only $6 \cdot 10^{-4}$ far from the resonant value (in [36] the use of the Greene's residue criterion gives frequencies that are 10^{-2} close to the resonant value). The result we show in this paper for the critical function is in agreement with the one of [36].

Supported by considerations on both methods (analytical and numerical) we make the following conjecture:

> **Conjecture.** For all rational p/q, we can find positive constant $c_{p/q}$, such that when ω is in a small *real neighbourhood* of p/q, then
>
> $$\epsilon_{crit}(\omega) \sim c_{p/q} \Big|\omega - \frac{p}{q}\Big|^{\frac{1}{q}} .$$

Comparing this conjecture with [8, 9] we see that the domain of analyticity with respect to $\epsilon \in \mathbb{C}$ of the function defined by the series (1.2) is quite different from a circle as ω tends to a resonance [6, 5]. It is an open question to understand completely its geometry: it could be an ellipse whose major semi–axis is in the direction of real ϵ with a length proportional to the square root of the minor semi–axis, or probably a more complicated curve with the nearest singularity to the origin placed on the imaginary axis. We don't think our result and our conjecture are in disagreement with the one of [8, 9] in fact we think there is a difference when dealing with real or complex frequencies.

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2. Scaling law for the fundamental resonance.

The goal of this section is to prove our main result: Theorem 1.1. We note that from the definitions of $\epsilon_{crit}(\omega)$ and $\omega^{\pm}(\epsilon)$ we have $\epsilon_{crit}(\omega^{\pm}(\epsilon)) = \epsilon$ then Theorem 1.1 is equivalent to

Theorem 2.1. *Let $\omega^{\pm}(\epsilon)$ be the functions of ϵ defined in* (1.7) *and* (1.8)*. Then there exist $\epsilon_0' > 0$ and $c_0' > 0$ such that for all $0 < \epsilon < \epsilon_0'$ we have*

$$\Big| \ln|\omega^{\pm}(\epsilon)| - \ln \epsilon \Big| \leq c_0'.$$

Using Theorem 3. page 76 of [46] we can prove that there exist $c > 0$ and $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ there exist two invariant homotopically non–trivial $\mathcal{C}^1$ curves with frequencies ω_l and ω_b and

$$\omega_l < 0 < \omega_b \quad \text{and} \quad |\omega_b - \omega_l| < c\epsilon.$$

Recalling (1.7) and (1.8) we get the lower bound

$$\ln \epsilon - \ln|\omega^{\pm}(\epsilon)| > \ln 2 - \ln c$$

for ϵ small enough. Thus the proof of Theorem 2.1 reduces to prove the following bound

$$\ln \epsilon - \ln|\omega^{\pm}(\epsilon)| < c'' \tag{2.1}$$

for some $c'' > 0$ and ϵ small enough. This will be done in Proposition 2.4. Now we present the two results we need to prove Proposition 2.4.

Let us fix $\epsilon > 0$ and recall the definitions of $y^{\pm}(\epsilon)$: $y^{+}(\epsilon)$, respectively $y^{-}(\epsilon)$, are the intersection point of the homotopically invariant $\mathcal{C}^1$ curve of frequency $\omega^{+}(\epsilon)$, respectively $\omega^{-}(\epsilon)$, with the y–axis. Then from the exponentially smallness of the splitting of the separatrices we deduce

Proposition 2.2. *There exist $c > 0$ and $\epsilon_0'' > 0$ such that for all $0 < \epsilon < \epsilon_0''$ we have*

$$|y^{\pm}(\epsilon)| \geq c e^{-\frac{\pi^2}{2\sqrt{\epsilon}}}. \tag{2.2}$$

We leave the proof of this Proposition to subsection 2.1 and we introduce the second element we need to prove our result. We rewrite the Standard map as perturbation of the identity map, namely

$$S_\mu \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix} + \mu \begin{pmatrix} z \\ \sin x \end{pmatrix} + \mu^2 \begin{pmatrix} \sin x \\ 0 \end{pmatrix} \tag{2.3}$$

we can pass from (2.3) to (1.1) putting $\mu = \sqrt{\epsilon}$ and $\mu z = y$. And we define ω_S to be its rotation number, namely

$$\omega_S (x, z, \mu) = \lim_{n\to\infty} \frac{\pi_1 \left(S^n_\mu (x, z) \right) - x}{n}$$

when this limit exists. The relation between ω_S and ω_T is

$$\omega_T (x, y, \epsilon) = \sqrt{\epsilon}\omega_S \left(x, \frac{y}{\sqrt{\epsilon}}, \sqrt{\epsilon} \right). \tag{2.4}$$

Following [4] we construct (Proposition 2.8 and Proposition 2.9) an integrable Hamiltonian system whose time–1 flow differs from the standard map in the formulation (2.3) for an exponentially small term: $\mathcal{O}\left(e^{-\frac{1}{\sqrt{\epsilon}}}\right)$ for $\epsilon \to 0^+$. This allows us to prove (Proposition 2.11) the existence of an asymptotic development of the standard map rotation number in terms of $\sqrt{\epsilon}$, whose first order term is the pendulum frequency for rotation orbits. In fact the best interpolating Hamiltonian system is an integrable perturbation of order $\mathcal{O}(\sqrt{\epsilon})$ of the pendulum $H_0(x, z) = \frac{z^2}{2} + \cos x$. This implies the following Lemma

Lemma 2.3. *Let h_0 be a real number strictly greater than 1. Let ω_T be the standard map rotation number and let ω_{pend} be the frequency of a rotation orbit of energy h_0 for the pendulum $H_p(x, z) = \frac{z^2}{2} + \cos x$. For $\epsilon > 0$ we consider the invariant circle of the standard map of initial conditions $(x_0, y_0) = \left(x_0, \frac{y_0}{\sqrt{\epsilon}}\right)$, then for ϵ sufficiently small we have:*

$$\left| \omega_T (x_0, y_0, \epsilon) - \sqrt{\epsilon}\omega_{pend} \left(x_0, \frac{y_0}{\sqrt{\epsilon}} \right) \right| = \mathcal{O}\left(|\epsilon|^{\frac{3}{2}} \right).$$

We can now prove the bound (2.1).

Proposition 2.4 (Upper bound.). *Let $\omega^\pm(\epsilon)$ be the functions of ϵ defined in (1.7) and (1.8). Then there exist $\bar{\epsilon} > 0$ and $c'' > 0$ such that for all $0 < \epsilon < \bar{\epsilon}$ we have*

$$\ln \epsilon - \ln|\omega^\pm(\epsilon)| \le c''.$$

Proof. Lemma 2.3 and its refinement (Corollary 2.10) say that the standard map rotation number is "well approximated" by the pendulum frequency. Consider a rotation orbit of the pendulum starting at $(0, z_0)$. Its frequency is given by

$$\omega_{pend}(h_0) = \frac{\pi}{k} \frac{1}{\mathcal{K}(k)}$$

where $k^2 = \left(1 + \frac{z_0^2}{4}\right)^{-1}$, $h_0 = 1 + \frac{z_0^2}{2}$ and

$$\mathcal{K}(k) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}}$$

is the complete Elliptic integral of first kind. For $k=1$ (that is $z_0=0$) the integral diverges logarithmically and (see [1], formula 17.3.26 p.591)

$$\lim_{z_0\to 0}\left[\mathcal{K}-\frac{1}{2}\ln\frac{4+z_0^2}{z_0^2}\right]=2\ln 2.$$

Therefore in the limit $z_0\to 0$, ω_{pend} behaves as

$$\left|\omega_{pend}-\frac{\pi}{\ln\frac{2}{|z_0|}}\right|=\mathcal{O}\left(\frac{z_0^2}{\ln|z_0|}\right). \tag{2.5}$$

Assume now that $\omega_T\left(0,y_0,\epsilon\right)$ exists for ϵ small enough, where $y_0=z_0\mu$. By (2.4), Lemma 2.3 and (2.5) one has

$$\left|\omega_T\left(0,y_0,\epsilon\right)\right|\geq\sqrt{\epsilon}\frac{\pi}{\ln\frac{8\sqrt{\epsilon}}{|y_0|}}+\mathcal{O}\left(|\epsilon|^{\frac{3}{2}}\right). \tag{2.6}$$

If y_0 is the intersection of the last invariant torus with the axis $x=0$, namely $y_0=y^+\left(\epsilon\right)$ and $\omega_T\left(0,y_0,\epsilon\right)=\omega^+\left(\epsilon\right)$, then the bound (2.2) gives

$$\left|\omega^+\left(\epsilon\right)\right|\geq\sqrt{\epsilon}\frac{\pi}{C+\frac{1}{2}\ln\epsilon+\frac{\pi^2}{2\sqrt{\epsilon}}}+\mathcal{O}\left(|\epsilon|^{\frac{3}{2}}\right)$$

for some positive constant C. Then for ϵ small enough we conclude that there exist another positive constant c such that

$$\left|\omega^+\left(\epsilon\right)\right|\geq c\epsilon.$$

Recalling that $\omega^-\left(\epsilon\right)=-\omega^+\left(\epsilon\right)$, the Proposition is proved. □

2.1. **Splitting of the separatrices.** The aim of this subsection is to introduce some results about the splitting of separatrices for the Standard Map necessary to prove Proposition 2.2. The splitting of the separatrices has been studied by many authors, here we use the notations and the presentation given in [19].

For $\epsilon=0$ the circle $y=0$ is formed by fixed points of the Standard Map, for $\epsilon>0$ only two fixed points survive: $(0,0)$ and $(\pi,0)$. The first one is hyperbolic (for these values of ϵ) and the other one elliptic. The linearised map at the origin is

$$\begin{pmatrix}1+\epsilon & 1\\ \epsilon & 1\end{pmatrix}$$

and its eigenvalues are λ and λ^{-1} where

$$\lambda=1+\frac{\epsilon}{2}+\sqrt{\epsilon}\sqrt{1+\frac{\epsilon}{4}}.$$

The stable, $\mathcal{W}^s$, and unstable, $\mathcal{W}^u$, manifolds of this fixed point are analytic curves passing through $(0,0)$. We introduce the parameter $\delta=\ln\lambda$ and we note that $\epsilon\sim\delta^2$ for small ϵ. We parametrise the part of the unstable, $\mathcal{W}_1^u$ separatrix which grows upwards from the origin, with $\left(x^-\left(t\right),y^-\left(t\right)\right)$, imposing the boundary conditions[3]

$$\lim_{t\to-\infty}x^-\left(t\right)=0,\quad x^-\left(0\right)=\pi.$$

[3]The $y\left(t\right)$ can be found using $y\left(t+\delta\right)=x\left(t+\delta\right)-x\left(t\right)$.

We assume that $t = 0$ is the first intersection of $\mathcal{W}_1^u$ with the vertical line $x = \pi$. The parametrisation of $\mathcal{W}_1^s$ is given by

$$\left(x^+ (t), y^+ (t)\right) = \left(2\pi - x^- (-t), y^- (-t) + \epsilon \sin x^- (-t)\right).$$

One can show that

$$\lim_{t\to+\infty} x^+ (t) = 0, \quad x^+ (0) = \pi,$$

i.e. $t = 0$ corresponds to a homoclinic point. Lazutkin [32] proposed to study the *homoclinic invariant* defined by

$$\Omega = \det \begin{pmatrix} \dot{x}^- (0) & \dot{x}^+ (0) \\ \dot{y}^- (0) & \dot{y}^+ (0) \end{pmatrix}.$$

The homoclinic invariant has the same value for all points of one homoclinic trajectory and it is invariant with respect to symplectic coordinate changes. Gelfreich in [19], Proposition 2.1, gives an asymptotic expansion in power of δ of the parametrisation $x^{\pm} (t)$. Its first terms are

$$x_0 (t) + x_1 (t) + x_2 (t) + \dots =$$

$$= 4 \arctan e^t + \delta^2 \frac{1}{4} \frac{\sinh t}{(\cosh t)^2} + \delta^4 \left[-\frac{41}{1728} \frac{\sinh t}{(\cosh t)^2} + \frac{91}{864} \frac{\sinh t}{(\cosh t)^4} \right] + \dots. \tag{2.7}$$

Of course the zero order term is nothing but the parametrisation of the separatrix of the pendulum.

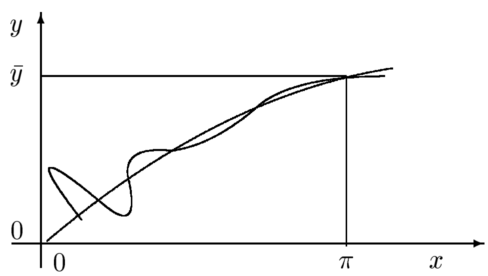

FIGURE 1. The stable and unstable manifolds for the Standard Map with some of their infinitely many intersection points.

The main result of [19] is $\Omega \geq \frac{c}{\delta^2} e^{-\frac{\pi^2}{\delta}}$ for some positive constant c and δ small enough. Using this result we obtain an estimation of $y^{\pm} (\epsilon)$ in Proposition 2.2.

Proposition 2.2. *There exist $c > 0$ and $\epsilon_0'' > 0$ such that for all $\epsilon < \epsilon_0''$ we have*

$$|y^{\pm} (\epsilon)| \geq c e^{-\frac{\pi^2}{2\sqrt{\epsilon}}}.$$

Remark 2.5. *Compare with Figure 11 where we plot $\ln|y^+ (\epsilon_{crit}) - y^- (\epsilon_{crit})|$ against ϵ_{crit}, with data obtained numerically using the FMA.*

Proof. For a fixed value of $\epsilon > 0$, the stable and unstable manifolds intersect firstly at $(\pi, \bar{y}(\epsilon))$ and then infinitely many times. We call *lobe* the arc of manifold between two successive intersections and *lobe area* the area bounded by the lobe. Each lobe will be distinguished by an integer: the number of iterates (of T_ϵ) needed to put the point $(\pi, \bar{y})$ onto the right point base of the lobe.

The hyperbolic fixed point $(0,0)$ stretches the unstable manifold along the dilating direction (corresponding to the eigenvalue λ), so there exists an integer $\hat{k}$ such that the $\hat{k}$-th lobe intersect the y–axis at some point $y_{int}(\epsilon)$. Because the invariant circles of the Standard Map are at least Lipschitz graph, we cannot have homotopically non–trivial invariant circles, passing by $(0, \bar{y}')$ with $\bar{y}' < y_{int}(\epsilon)$, thus $y^+(\epsilon) \geq y_{int}(\epsilon)$. The lobe area is conserved under iteration of the Standard Map and by [19], Corollary 1.3, this area is[4] for δ small enough

$$A_{lobe} \sim \frac{2\omega_0}{\pi} e^{-\frac{\pi^2}{\delta}}$$

where $\omega_0 = 1118.827706\cdots$ and $\delta = \ln\lambda \sim \sqrt{\epsilon}$ for ϵ small.

In Figure 2 we represent the geometry of the generic k–th lobe: the points $A_k = (x_{-k}, y_{-k})$ and $B_k = \left(x_{-(k-1)}, y_{-(k-1)}\right)$ are the iterates of the standard map

$$\begin{cases} x_{-(k-1)} = x_{-k} + y_{-(k-1)} \\ y_{-(k-1)} = y_{-k} + \delta^2 \sin x_{-k} \end{cases}$$

the segment of line $\overline{A_k C_k}$ is tangent to the unstable manifold at A_k, while $\overline{A_k B_k}$ is tangent to the stable manifold at A_k.

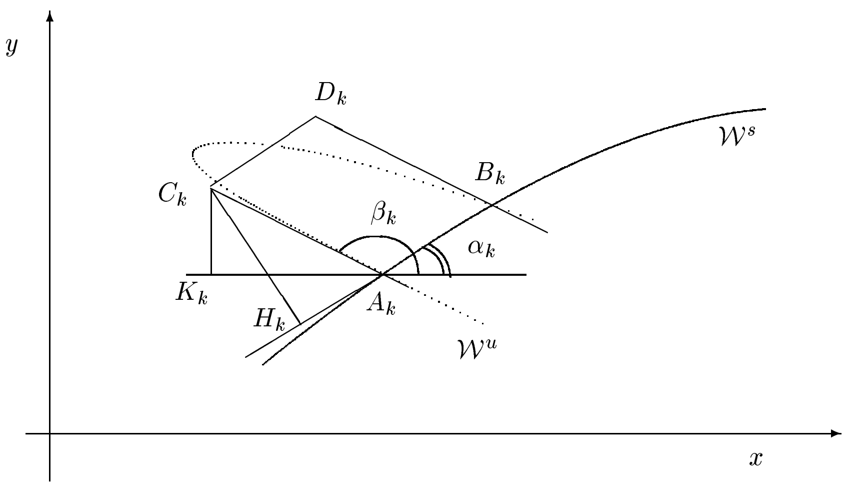

FIGURE 2. Two intersection points of the stable, $\mathcal{W}^s$, and unstable, $\mathcal{W}^u$ (dotted curve), manifolds, producing the k–th lobe.

[4]In the follow we will use often the standard notation (see [20] n.4 chap. II) $f(t) \sim g(t)$ for $t \to 0$, this means that f and g are equivalent when t is close enough to 0. We can restate it by saying: $f(t) = g(t) + o(g(t))$ when $t \to 0$.

The angle α_k is defined by

$$\tan\alpha_k = \frac{y_{-(k-1)} - y_{-k}}{x_{-(k-1)} - x_{-k}} = \frac{\delta^2 \sin x_{-k}}{\Delta x_{-k}}.$$

Let us consider the lobe which intersects for the first time the y–axis and let us call y_{int} the intersection point. We don't know exactly the analytical form of the arc of unstable manifold joining $A_{\hat{k}} = \left(x_{-\hat{k}}, y_{-\hat{k}}\right)$ to $(0, y_{int})$, so we can not calculate exactly y_{int}. Then we approximate this arc with its tangent at $A_{\hat{k}}$:

$$y = y_{-\hat{k}} + \left(\tan\beta_{\hat{k}}\right)\left(x - x_{-\hat{k}}\right). \tag{2.8}$$

This straight line intersect the y–axis in some point $\tilde{y}$ and $|\tilde{y} - y_{int}| = \mathcal{O}\left(x^2_{-\hat{k}}\right)$.

We approximate the lobe with a parallelogram of sides A_kB_k and A_kC_k where:

1. the straight line through A_k and C_k is tangent to the lobe at A_k;
2. the area, $\overline{A_kB_k} \cdot \overline{A_kC_k} \sin\left(\beta_k - \alpha_k\right)$, of the parallelogram is equal to A_{lobe}.

The condition of intersection with the y–axis is now $\overline{A_{\hat{k}}K_{\hat{k}}} = x_{-\hat{k}}$: this defines implicitly $\hat{k}$.

Let H_k denote the orthogonal projection of C_k on the straight line through A_k and B_k. Then $A_{lobe} = \overline{A_kB_k} \cdot \overline{C_kH_k} = \frac{\Delta x_{-k}}{\cos\alpha_k}\overline{C_kA_k}\sin\left(\beta_k - \alpha_k\right)$. Since $\overline{A_kK_k} = -\overline{C_kA_k}\cos\beta_k$, the intersection condition becomes

$$\overline{A_{\hat{k}}K_{\hat{k}}} = \frac{-\cos\alpha_{\hat{k}}\cos\beta_{\hat{k}}}{\Delta x_{-\hat{k}}\sin\left(\beta_{\hat{k}} - \alpha_{\hat{k}}\right)} A_{lobe} = x_{-\hat{k}}. \tag{2.9}$$

From (2.7), using $t = -\hat{k}\delta$ one gets

$$x_{\hat{k}} \sim 4e^{-\hat{k}\delta}, \quad \Delta x_{-\hat{k}} \sim 4\delta e^{-\hat{k}\delta},$$

thus

$$\tan\alpha_{\hat{k}} \sim \delta\left(1 + e^{-2\hat{k}\delta}\right), \quad \sin\alpha_{\hat{k}} \sim \delta\left(1 + e^{-2\hat{k}\delta}\right), \quad \cos\alpha_{\hat{k}} \sim 1.$$

To estimate $\beta_{\hat{k}}$ we use again the fact that the intersection will happen very close to the origin: the tangent to the unstable manifold at $A_{\hat{k}}$ is very close to the expanding direction at the origin, so we get

$$\tan\beta_{\hat{k}} \sim -\delta\left(1 + \frac{\delta}{2}\right), \quad \cos\beta_{\hat{k}} \sim 1 - \delta^2.$$

Using these approximations we can rewrite (2.9) as

$$\left(1 - \delta^2\right)\frac{2\omega_0}{\pi}e^{-\frac{\pi^2}{\delta}} = 4e^{-\hat{k}\delta}\frac{4\delta e^{-\hat{k}\delta}}{1 + e^{-2\hat{k}\delta}}2\delta + o\left(\delta^2 e^{-2\hat{k}\delta}\right),$$

from which we obtain

$$\delta^2 e^{-2\hat{k}\delta} = C_1 e^{-\frac{\pi^2}{2\delta}} + o\left(e^{-\frac{\pi^2}{\delta}}\right)$$

for some positive constant C_1. This allows to compute $\hat{k}$ explicitly. To determine $\tilde{y}$, we recall (2.8) and using the previous value of $\hat{k}$ we get

$$\tilde{y} = y_{-\hat{k}} - x_{-\hat{k}}\tan\beta_{\hat{k}} \sim 4\delta e^{-\hat{k}\delta} + 4e^{-\hat{k}\delta}\delta\left(1 + \frac{\delta}{2}\right) \sim C_1' e^{-\frac{\pi^2}{2\delta}},$$

for some new positive constant C_1'. Finally, since $|\tilde{y}-y_{int}| = \mathcal{O}\left(x^2_{-\hat{k}}\right) = \mathcal{O}\left(e^{-\frac{\pi^2}{\sqrt{\epsilon}}}\right)$, we conclude

$$-y^-(\epsilon) = y^+(\epsilon) \geq c e^{-\frac{\pi^2}{2\sqrt{\epsilon}}},$$

for some positive constant c. $\square$

2.2. **Interpolating vector fields.** Once we have determined a lower bound for $y^+(\epsilon)$ in terms of the intersection of the lobe with the y–axis, we need to compute a lower bound for the rotation number of the corresponding invariant circle. To this purpose following [4], we consider the problem of the construction of a vector field interpolating a given diffeomorphism. With this subsection and the following we prove the existence of an interpolating Hamiltonian system exponentially close to the map (2.3) (Proposition 2.8 and Proposition 2.9). These results will be used in section 2.4 to prove the existence of an asymptotic development for the standard map rotation number (Proposition 2.11) from which Lemma 2.3 immediately follows.

Let D be a bounded subset of $\mathbb{R}^m$ for $m \in \mathbb{N}^*$, I a real interval containing the origin and let us consider the one–parameter family of diffeomorphisms Ψ

$$\begin{aligned}\Psi : & D \times I \to \mathbb{R}^m \\ & (x,\mu) \mapsto \Psi(x,\mu)\end{aligned}$$

such that for all $x \in D$ and $\mu \in I$ it has the form

$$\Psi(x,\mu) = x + \sum_{n\geq 1} \Psi_n(x)\mu^n. \tag{2.10}$$

The problem consists in finding a one–parameter family of $F : D \times I \to \mathbb{R}^m$ vector fields such that the time–1 flow of the differential equation

$$\frac{dx}{dt} = F(x,\mu)$$

coincides with the diffeomorphism Ψ.

We assume that $\Psi_n(x)$, defined in (2.10), are real analytic functions of x in some complex neighbourhood $\mathcal{D}_\rho$ of D, defined as follows. Given $\rho = (\rho_1, \dots, \rho_m) \in \mathbb{R}^m_+$ then

$$\mathcal{D}_\rho = \bigcup_{x\in D} \{z \in \mathbb{C}^m : |z_i - x_i| < \rho_i, 1 \leq i \leq m\}.$$

We introduce standard norms on complex analytic functions. Let f be a scalar analytic function on some domain $D \subset \mathbb{R}^m$, let $\rho \in \mathbb{R}^m_+$ and consider its complex extension $\mathcal{D}_\rho$, then

$$||f||_\rho = \sup_{z\in\mathcal{D}_\rho} |f(z)| \tag{2.11}$$

defines a norm. If F is vector valued analytic function $F = (f_1, \dots, f_m)$ on $\mathcal{D}_\rho$, for some $\rho = (\rho_1, \dots, \rho_m) \in \mathbb{R}^m_+$, then we define a norm (noted as in the scalar case)

$$||F||_\rho = \max_{1\leq i\leq m} ||f_i||_\rho.$$

If Ψ is a formal μ–series we can look for a formal interpolating vector field $F(x,\mu) = \sum_{n\geq 1} F_n(x)\mu^n$ whose time–1 flow coincides with the diffeomorphism, as we show with the following Proposition.

Proposition 2.6. *Let $\Psi(x,\mu)$ be a diffeomorphism of the form* (2.10) *and assume $\Psi_n(x)$ be analytic in $\mathcal{D}_\rho$ for all $n \geq 1$ and for some $\rho \in R^m_+$. Then there exists a formal vector field F_μ of the form*

$$F_\mu(x) = \sum_{n\geq 1} F_n(x)\mu^n \tag{2.12}$$

such that

$$e^{L_{F_\mu}}(x) = \Psi(x,\mu) \tag{2.13}$$

for all $x \in \mathcal{D}_\rho$.

Proof. The proof is a very natural application of the Lie series. We give it for the sake of completeness.

By definition of exponential of the Lie derivative

$$e^{L_{F_\mu}}(x) = x + \sum_{n\geq 1}\frac{1}{n!}L^n_{F_\mu}(x).$$

Looking for F_μ in the form (2.12) and using the linearity of the Lie operator we have

$$L_{F_\mu} = \sum_{n\geq 1}\mu^n L_{F_n}.$$

For all integer $n \geq 1$ and $2 \leq m \leq n$ we define

$$\mathcal{L}_{1,n}(x) = L_{F_n}(x) = F_n(x) \qquad \mathcal{L}_{m,n}(x) = \sum_{j=1}^{n-m+1} L_{F_j}\mathcal{L}_{m-1,n-j}(x)$$

and it is easy to prove that

$$L^m_{F_\mu} = \sum_{n\geq m}\mu^n\mathcal{L}_{m,n}.$$

Then by (2.13) and (2.10) we conclude that for all $k \geq 1$

$$\sum_{n=1}^{k}\frac{1}{n!}\mathcal{L}_{n,k}(x) = \Psi_k(x),$$

namely

$$L_{F_k}(x) = \Psi_k(x) - \sum_{n=2}^{k}\frac{1}{n!}\mathcal{L}_{n,k}(x). \tag{2.14}$$

Let $\Psi_k = \left(\Psi_k^{(1)},\dots,\Psi_k^{(m)}\right)$ and let $G = \sum_{n=2}^{k}\frac{1}{n!}\mathcal{L}_{n,k}(x) = (G_1,\dots,G_m)$. Note that G depends on Ψ_j and F_{j-1} for $j \leq k$. Using (2.14) we have for all $1 \leq l \leq m$

$$F_k^{(l)}(x) = \Psi_k^{(l)} + G_l,$$

from which we obtain $F_k(x) = \left(F_k^{(1)}(x),\dots,F_k^{(m)}(x)\right)$. □

If Ψ is analytic w.r.t μ then we have the following result([4] Proposition 1 p.1122)

Theorem 2.7. *Let $\Psi(x,\mu)$ be a diffeomorphism of the form (2.10) with $\Psi_n(x)$ real analytic in some complex domain $\mathcal{D}_\rho$ for $\rho \in \mathbb{R}^m_+$, and let F_μ be the formal interpolating vector field obtained in Proposition 2.6. Assume that for all k*

$$||\Psi_k||_\rho \leq \Gamma\gamma^{k-1},$$

for some positive constants Γ and γ then

$$||F_1||_\rho \leq \Gamma, \quad \forall k \geq 2 \quad ||F_k||_{\frac{\rho}{2}} < \frac{1}{2}k^{k-1}\beta^{k-1}\Gamma,$$

with $\beta = 4\max\{\gamma, \Gamma\}$.

Moreover if $\Psi(x,\mu)$ is symplectic then for all n $F_n(x)$ are locally Hamiltonian.

In the form (2.3) the Standard map verifies the hypothesis of Theorem 2.7, but to get optimal estimates we state an Hamiltonian version of Theorem 2.7 adapted to the Standard map.

Proposition 2.8. *Let $\bar{z} > 0$ and consider the interval $I = (-\bar{z}, \bar{z})$. Fixed $R_0 > 0$ and $S_0 > 0$ we define the complex cylinder*

$$\mathcal{D}_{(R_0,S_0)} = \{x \in \mathbb{C}/2\pi\mathbb{Z} : |\Im x| < S_0\} \times \bigcup_{z\in I}\{z' \in \mathbb{C} : |z - z'| < R_0\}.$$

Then we can find a formal Hamiltonian $H(x,z,\mu) = \sum_{n=1}^\infty H_{n-1}(x,z)\mu^n$ such that

$$\forall (x,z) \in \mathcal{D}_{(R_0,S_0)} \quad e^{L_H}(x,z) = S_\mu(x,z).$$

Moreover for all $0 < d < 1$ there exist $A = 2S_0||H_0||_{(R_0,S_0)}$ and $B = \frac{20}{3}\frac{||H_0||_{(R_0,S_0)}}{R_0 d^2}$ such that

$$\forall k \quad ||H_k||_{(1-d)(R_0,S_0)} \leq AB^k(k+1)!,$$

namely H is 1–Gevrey w.r.t. μ.

With norm defined in 2.11 it results that: $||H_0||_{(R_0,S_0)} = \frac{R_0^2}{2} + e^{S_0}$.

The proof of Proposition 2.8 is an adaptation of the proof of Theorem 2.7 and we omit it.

We write down the first few terms of H

$$H(x,z,\mu) = \frac{z^2}{2} + \cos x + \mu\frac{z}{2}\sin x + \mathcal{O}\left(\mu^2\right) \tag{2.15}$$

and as we stated this is a perturbation of order $\mathcal{O}(\sqrt{\epsilon})$ of the pendulum.

2.3. **Exponentially small estimates.** Proposition 2.8 gives us an Hamiltonian function represented as a (maybe divergent) series. It is then interesting to define a truncation of this series, namely take $N \in \mathbb{N}^*$ define $H^{(N)}(x,z,\mu)$ by

$$H^{(N)}(x,z,\mu) = \sum_{n=1}^N H_{n-1}(x,z)\mu^n$$

and to ask for the relations between $S_\mu(x,z)$ and $\Psi^{(N)}(x,z,\mu)$, where the latter is defined by

$$\Psi^{(N)}(x,z,\mu) = e^{L_{H^{(N)}}}(x,z)$$

We can prove that we can chose N such that $S_\mu(x,z)$ and $\Psi^{(N)}(x,z,\mu)$ are exponentially close. More exactly:

Proposition 2.9. *Let $H^{(N)}(x,z,\mu)$ be the truncation of the 1–Gevrey interpolating Hamiltonian system given by Proposition 2.8 , and let $\Psi^{(N)}(x,z,\mu)$ be its time–1 flow. Then we can find a positive integer N^* such that for all $0<d<\frac{2}{3}$ and for all $|\mu|\leq \frac{eR_0S_0d^2}{32A}$*

$$||S_\mu(x,z)-\Psi^{(N^*)}(x,z,\mu)||_{\left(1-\frac{3}{2}d\right)(R_0,S_0)}\leq |\mu|De^{-\frac{D'}{|\mu|}},$$

with $D=M+\frac{16Am}{eR_0S_0d^2}$, $D'=(2eB)^{-1}$, $M=\max\{R_0+|\mu|e^{S_0},e^{S_0}\}$ and $m=\max\{R_0,e^{S_0}\}$.

This Proposition is an adaptation of the corollary 1 p.1124 of [4] and we don't prove it.

We will note in the follow $H^*(x,z,\mu)=H^{(N^*)}(x,z,\mu)$: the "best" truncation of the interpolating Hamiltonian.

Let us introduce the frequency of the rotation orbit of H^*.

$$2\pi\omega_*(x,z,\mu)=\lim_{n\to\infty}\frac{\pi_1\left(e^{nL_{H^*}}(x,z)\right)-x}{n}.$$

Then from Proposition 2.9 we deduce the following corollary.

Corollary 2.10. *Let ω_S be the rotation number of the map (2.3). Let H^* be best truncation of the interpolating Hamiltonian system given by Proposition 2.9 and let $2\pi\omega_*$ be its frequency. Then we have*

$$|\omega_S(x,z,\mu)-\omega_*(x,z,\mu)|\leq D|\mu|e^{-\frac{D'}{|\mu|}}. \tag{2.16}$$

Proof. We can prove that for all positive k and for initial data (x,z) on some domain such that we can iterate $e^{L_{H^*}}$ and S_μ at least k times, we have

$$S^k_\mu(x,z)-e^{kL_{H^*}}(x,z)=\sum_{l=0}^{k-1}e^{(k-l-1)L_{H^*}}\left(S_\mu-e^{L_{H^*}}\right)S^l_\mu(x,z).$$

For real x, z and μ, the standard map is real valued so $\xi_l=S^l_\mu(x,z)$ is in $2\pi\mathbb{T}\times\mathbb{R}$ for all $l\geq 0$.

Let us define $\xi'_l=\left(S_\mu-e^{L_{H^*}}\right)(\xi_l)$, then Proposition 2.9 implies $|\xi'_l|\leq D|\mu|e^{-\frac{D'}{|\mu|}}$ for $(x,z)\in\mathcal{D}_{\left(1-\frac{3}{2}d\right)(R_0,S_0)}$. Such a D depends on the domain troughs d, for some fixed S_0 and R_0, but using the previous remark on the reality of ξ_l, we can fix once for all a domain, and so a constant D, in such a way that ξ_l belongs for all l to this domain. For all n: $|e^{nL_{H^*}}(\xi'_l)|\leq D|\mu|e^{-\frac{D'}{|\mu|}}$, because $e^{nL_{H^*}}(\xi'_l)=\xi'_l+\mathcal{O}(\xi'_l)$.

We conclude that

$$\left|\left[\pi_1\left(S^k_\mu(x,z)\right)-x\right]-\left[\pi_1\left(e^{kL_{H^*}}(x,z)\right)-x\right]\right|\leq k|\mu|De^{-\frac{D'}{|\mu|}},$$

from which (2.16) follows. □

2.4. **Asymptotic development.** We can now prove the existence of an asymptotic development for the standard map rotation number in a neighbourhood of $\epsilon=0$.

The Hamiltonian H^* defines an integrable (it has only one degree of freedom) Hamiltonian system whose period is given by

$$T=\frac{2\pi}{\omega_*}=\int_0^{2\pi}\frac{dx}{\left.\frac{\partial H^*}{\partial z}\right|_{h_0}}, \tag{2.17}$$

where h_0 is an energy level corresponding to rotation orbits.

Let us fix $h_0 > 1$. We are interested in finding curves on the energy level $H^*(x,z,\mu) = h_0$. Take $z_0 > 0$ and consider the point $P_0 = (x_0, z_0, 0)$ such that $P_0 \in {H^*}^{-1}\{h_0\}$, from (2.15) it follows that: $\frac{\partial H^*}{\partial z}(P_0) = z_0 > 0$. We can then apply the implicit function Theorem which assures that it exists a function $\hat{z} = \hat{z}(x,\mu)$ such that locally it satisfies

$$H^*(x, \hat{z}, \mu) = h_0 \tag{2.18}$$

moreover this function is smooth. We compute the first terms and we get: $\hat{z}(x,\mu) = \sqrt{2(h_0 - \cos x)} - \frac{\mu}{2}\sin x + \mathcal{O}\left(|\mu|^2\right)$ and

$$\begin{aligned} &\frac{\partial H^*}{\partial z}(x,\hat{z},\mu) - \sqrt{2(h_0-\cos x)} \\ &= \hat{z} + \frac{\mu}{2}\sin x + \sum_{n=2}^{N^*} \frac{\partial H_n}{\partial z}(x,\hat{z})\,\mu^n - \sqrt{2(h_0-\cos x)} = \mathcal{O}\left(|\mu|^2\right). \end{aligned} \tag{2.19}$$

Recalling 2.4 we prove the following Proposition

Proposition 2.11. *Let $\omega_T(x,y,\epsilon)$ be the rotation number of the standard map in the formulation* (1.1)*, then it exists a formal $\sqrt{\epsilon}$–power series $\sum_{n=0}^{\infty} B_n\left(\sqrt{\epsilon}\right)^n$ asymptotic to $\omega_T(x,y,\epsilon)$ for $\epsilon \to 0$. Namely for all positive integer N we have*

$$\lim_{\epsilon\to 0}\left(\sqrt{\epsilon}\right)^{-(N+1)}\left|\,\omega_T - \sum_{n=0}^{N} B_n\left(\sqrt{\epsilon}\right)^{n+1}\right| = 0. \tag{2.20}$$

Proof. The solution, $\hat{z}$, of (2.18) given by the implicit function Theorem is analytic w.r.t $x \in \mathbb{T}$ and μ in a neighbourhood of 0, $\left.\frac{\partial H^*}{\partial z}\right|_{h_0}(x,0)$ is positive for $h_0 > 1$, then for μ close enough to the origin $\left.\frac{\partial H^*}{\partial z}\right|_{h_0}(x,\mu)$ is positive and analytic w.r.t to $x \in \mathbb{T}$ and μ. We can then write the following Taylor development

$$\left(\left.\frac{\partial H^*}{\partial z}\right|_{h_0}(x,\mu)\right)^{-1} = \sum_{n\geq 0} K_n(x,h_0)\,\mu^n, \tag{2.21}$$

where $(K_n)_{n\geq 0}$ are some known functions of (x, h_0). Using (2.19) we get

$$K_0(x,h_0) = \frac{1}{\sqrt{2(h_0-\cos x)}} \text{ and } K_1(x,h_0) = 0, \tag{2.22}$$

we left to remark 2.12 an iterative schema to calculate K_n.

For all $k \geq 0$ we can integrate the $K_n(x,h_0)$ and we rewrite (2.17) as

$$\frac{2\pi}{\omega_*} = 2\pi \sum_{n\geq 0} C_n(h_0)\,\mu^n,$$

from (2.22) we get

$$C_0(h_0) = \frac{1}{\omega_{pend}(h_0)} = \int_0^{2\pi} \frac{dx}{\sqrt{2(h_0-\cos x)}} \text{ and } C_1(h_0) = 0$$

where $\omega_{pend}(h_0)$ is the frequency of a rotation orbit of energy h_0 of the pendulum $H_p(x,z) = \frac{z^2}{2} + \cos x$.

Clearly for all positive integer N we have

$$\lim_{\mu\to 0}\mu^{-N}\Big|\frac{1}{\omega_*\left(h_0\right)}-\sum_{n=0}^{N}C_n\left(h_0\right)\mu^n\Big|=0. \tag{2.23}$$

We show now that we can obtain a similar estimate for ω_*. Let us set, for all positive integer N, $S_N\left(\mu,h_0\right)=\sum_{n=0}^{N}C_n\left(h_0\right)\mu^n$. Because $S_N\left(0,h_0\right)\neq 0$ we can express $1/S_N$ as a μ–power series

$$\begin{aligned}\frac{1}{S_n\left(\mu,h_0\right)}&=\frac{1}{C_0\left(h_0\right)}+\sum_{n\geq 2}\mu^n\sum_{m=\lfloor\frac{n}{N}\rfloor+1}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^m}{C_0^{m+1}\left(h_0\right)}\sum_{\substack{n_1+\ldots+n_m=n\\2\leq n_i\leq N}}C_{n_1}\left(h_0\right)\ldots C_{n_m}\left(h_0\right)\\&=\sum_{n\geq 0}B_n\left(h_0\right)\mu^n,\end{aligned} \tag{2.24}$$

where the second line is the definition of the B_n's, $B_0\left(h_0\right)=C_0\left(h_0\right)^{-1}$ and $B_1\left(h_0\right)=0$. Recalling (2.23) and (2.24) we can then conclude that for all positive integer N

$$\Big||\mu|^{-N}\left(\omega_*-\sum_{n=0}^{N}\mu^nB_n\left(h_0\right)\right)-|\mu|^{-N}\sum_{n\geq N+1}\mu^nB_n\left(h_0\right)\Big|=\frac{\omega_*}{S_N\left(\mu,h_0\right)}o\big(|\mu|^N\big)$$

and

$$\lim_{\mu\to 0}|\mu|^{-N}\Big|\omega_*-\sum_{n=0}^{N}\mu^nB_n\left(h_0\right)\Big|=0.$$

From 2.4 and corollary 2.10 we get for all positive integer N

$$\Big|\omega_T-\sum_{n=0}^{N}B_n\left(h_0\right)\left(\sqrt{\epsilon}\right)^{n+1}\Big|\leq\epsilon De^{-\frac{D'}{\sqrt{\epsilon}}}+\sqrt{\epsilon}\,o\big(\sqrt{\epsilon}\big)^N,$$

dividing by $(\sqrt{\epsilon})^{N+1}$ and passing to the limit $\epsilon\to 0$ we obtain (2.20). □

From this Proposition Lemma 2.3 follows immediately.

Remark 2.12. *The calculation of the K_n's defined by* (2.21) *is a little cumbersome but easy, it needs only some algebraic manipulations of power series. We assume known the solution, $\hat z\left(x,\mu,h_0\right)$, of* (2.18) *given by the implicit function Theorem, and using the analyticity hypothesis we write it as a μ–power series*

$$\hat z\left(x,\mu,h_0\right)=\hat z_0\left(x,h_0\right)+\sum_{n\geq 1}\hat z_n\left(x,h_0\right)\mu^n.$$

Starting from H^* given by (2.15)*, we calculate its partial derivative w.r.t. z, then we substitute $z=\hat z$ and we reorder the powers of μ to get*

$$\begin{aligned}\frac{\partial H^*}{\partial z}\left(x,\hat z,\mu\right)=\hat z_0\left(x,h_0\right)+\sum_{l\geq 2}\hat z_l\mu^l+\sum_{l\geq 3}\mu^l\sum_{\substack{n+m=l\\m\geq 1,2\leq n\leq N}}\sum_{k=1}^{m}\frac{1}{k!}\frac{\partial^{k+1}H_n}{\partial z}\left(x,\hat z_0\right)\cdot\\\sum_{\substack{m_1+\ldots+m_k=m\\m_i\geq 1}}\hat z_{m_1}\ldots\hat z_{m_k}=\hat z_0\left(x,h_0\right)+\sum_{l\geq 2}\mathcal{H}_l\left(x,h_0\right)\mu^l.\end{aligned}$$

This series is invertible in the field of the formal μ–power series, then (2.21) *defines correctly a μ–power series, which is given by*

$$\begin{aligned}\left(\frac{\partial H^*}{\partial z}(x,\hat{z},\mu)\right)^{-1} &= \frac{1}{\hat{z}_0} + \sum_{l\geq 2}\mu^l \sum_{m=1}^{\lfloor \frac{l}{2}\rfloor} \frac{(-1)^m}{\hat{z}_0^{m+1}} \sum_{\substack{l_1+\ldots+l_m=l \\ l_i\geq 2}} \mathcal{H}_{l_1}\ldots\mathcal{H}_{l_m} \\ &= K_0(x,h_0) + \sum_{l\geq 2} K_l(x,h_0)\mu^l .\end{aligned}$$

3. Numerical Analysis of the scaling law at the resonance p/q.

3.1. **Frequency Map Analysis.** The Frequency Map Analysis of Laskar (FMA) is a numerical method which allows to obtain a global view of the dynamics of Hamiltonian systems by studying the properties of the frequency map, numerically defined from the action like variables to the frequency space using adapted Fourier techniques. Thanks to its precision it was used for the study of stability questions and/or diffusion properties of a large class of dynamical systems : Solar System [26], Particles Accelerator [30], Galactic dynamics [41, 42], Standard Map [25, 29]. We present here the outlines of the method, following [27].

3.1.1. *Frequency Maps.* Considering a n–degrees of freedom quasi–integrable Hamiltonian system in the form

$$H(J,\theta;\epsilon) = H_0(J) + \epsilon H_1(J,\theta) \tag{3.1}$$

where H is real analytic for $(J,\theta)\in B\times\mathbb{T}^n$, being B an open domain in $\mathbb{R}^n$, and ϵ is a real "small" parameter. For $\epsilon = 0$ this system reduces to an integrable one. The motion takes place on invariant tori $J_j = J_j(0)$ described at constant velocity $\nu_j(J) = \left.\frac{\partial H_0}{\partial J_j}\right|_{J(0)}$, for $j = 1,\ldots,n$. Assuming a non–degenerate condition on H_0 the Frequency Map $F : B \to \mathbb{R}^n$

$$F : J \mapsto F(J) = \nu$$

is a diffeomorphism on its image Ω. In this case KAM theory [24, 2, 40] insures that for sufficiently small values of ϵ, there exists a Cantor set $\Omega_\epsilon \subset \Omega$ of frequency vectors satisfying a diophantine condition

$$|< k,\nu >| > \frac{C_\epsilon}{|k|^m}$$

for some positive constants C_ϵ and m, for which the quasi–integrable system (3.1) still possess smooth invariant tori, ϵ–close to the tori of the umpertubed system, with linear flow $t\mapsto \nu_j t + \theta_j(0) \mod 2\pi$ for $j = 1,\ldots,n$. Moreover, according to Pöschel [45] there exists a diffeomorphism $\Psi : \mathbb{T}^n\times\Omega \to \mathbb{T}^n\times B$

$$\Psi : (\phi,\nu) \mapsto (\theta, J)$$

which is analytic with respect to ϕ in $\mathbb{T}^n$ and $\mathcal{C}^\infty$ w.r.t ν in Ω_ϵ, and which transforms the Hamiltonian system (3.1) into

$$\begin{cases}\frac{d\nu_j}{dt}(t) = 0 \\ \frac{d\phi_j}{dt}(t) = \nu_j\end{cases}$$

For frequency vectors $\nu \in \Omega_\epsilon$ the invariant torus can be represented in the complex variables $(z_j = J_j e^{i\theta_j})_{j=1,n}$ by a quasiperiodic expression

$$z_j(t) = z_j(0) e^{i\nu_j t} + \sum_m a_{j,m}(\nu) e^{i<m,\nu>t} \tag{3.2}$$

If we take the section $\theta = \theta_0$, for some $\theta_0 \in \mathbb{T}^n$, of the phase space, we obtain a frequency map $F_{\theta_0} : B \to \Omega$

$$F_{\theta_0} : J \mapsto \pi_2\left(\Psi^{-1}(\theta_0, J)\right) \tag{3.3}$$

where $\pi_2(\psi, \nu) = \nu$ is the projection on Ω. For sufficiently small ϵ the non–degeneracy condition ensure that F_{θ_0} is a smooth diffeomorphism.

3.1.2. *Quasiperiodic approximations.* If we have the numerical values of a complex signal over a finite time span $[-T, T]$ and we think that it has a quasiperiodic structure represented by a quasiperiodic function $f(t)$, we can construct a quasi periodic approximation, $f'(t)$, as follows. We represent the given signal as:

$$f(t) = e^{i\nu_1 t} + \sum_{k \in \mathbb{Z}^n \setminus (1,0,\dots,0)} a_k e^{i<k,\nu>t} \quad a_k \in \mathbb{C}, \tag{3.4}$$

and the signal reconstructed by FMA as $f'(t) = \sum_{k=1}^N a'_k e^{i\omega'_k t}$. Frequencies ω'_k and amplitudes a'_k are determined with an iterative scheme. We first determine the first frequency ω'_1 as the value of σ which maximise:

$$\phi(\sigma) = \left| \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\sigma t} \chi(t)\, dt \right|,$$

where $\chi(t)$ is some weight function[5]. Once the first frequency is found the associated complex amplitude a'_1 is obtained by orthogonal projection. We then iterate the scheme to the function: $f_1(t) = f(t) - a'_1 e^{i\omega'_1 t}$.

Given the signal (3.4) over a time span $[-T, T]$, if we assume some arithmetic properties of the frequencies (Diophantine numbers for example), then one can prove ([27]) that, using the Hanning filter of order p: $\chi_p(t) = \frac{2^p (p!)^2}{(2p)!}(1 + \cos \pi t)^p$, FMA converges towards the first frequency ν_1^T with the asymptotic expression for $T \to +\infty$:

$$\nu - \nu_1^T = \frac{(-1)^{p+1} \pi^{2p} (p!)^2}{A_p T^{2p+2}} \sum_k \frac{\Re a_k}{\Omega_k^{2p+1}} \cos(\Omega_k T) + o\left(\frac{1}{T^{2p+2}}\right), \tag{3.5}$$

where $\Omega_k = < k, \nu > -\nu_1$ and $A_p = -\frac{2}{\pi^2}\left(\frac{\pi^2}{6} - \sum_{k=1}^p \frac{1}{k^2}\right)$. We can thus note that the accuracy of the determination of the frequency is $O(1/T^{2p+2})$, which will usually be several order of magnitude better that with usual FFT ($O(1/T)$).

[5] A weight function is a positive and even function such that:

$$\frac{1}{2T} \int_{-T}^{T} \chi(t)\, dt = 1.$$

In all the computations we will take the Hanning filter: $\chi_1(t) = 1 + \cos \frac{\pi t}{T}$.

3.1.3. *Frequency Map Analysis.* With the previous algorithm, it is possible to construct numerically a frequency map in the following way : a) Fix all values of the angles $\theta_j = \theta_{j0}$. b) for any value J_0 of the action variables, integrate numerically the trajectories of initial condition (J_0, θ_0) over the time span T. c) Search for a quasiperiodic approximation of the trajectory with the algorithm of the previous section. d) Identify the fundamental frequencies ν of this quasiperiodic approximation. The frequency map is then

$$F^T_{\theta_0} : J \mapsto \nu \tag{3.6}$$

and from the previous section, we know that for $T \longrightarrow +\infty$, on the set of regular KAM solutions, $F^T_{\theta_0} \longrightarrow F_{\theta_0}$. In particular, $F^T_{\theta_0}$ should converge towards a smooth function of the set of invariant KAM curves. The destruction of these invariant curves can thus be identified by the non regularity of the frequency map $F^T_{\theta_0}$.

3.2. **Application to the Standard Map.** We are interested in the study of homotopically non–trivial invariant orbits of the Standard map, in particular in the relation between the rotation number ω and the critical function $\epsilon_{crit}(\omega)$. Given a couple of initial conditions (x_0, y_0) the FMA gives us approximate amplitudes and frequencies of the resulting orbit. For $\epsilon = 0$ all invariant curves are tranverse to section $x = constant$, for ϵ sufficiently small this fact still holds, that is the tranversality property. For $\epsilon > 0$, "curves" of rational frequency do not persit and are replaced by "set of periodic fixed points" such that if $\omega(x_0, y_0) = p/q$ then this set contains $2q$ periodic points, an half of which are hyperbolic (HFP) and the other elliptic (EFP).

Near a HFP appears a stochastic layer, that is there are not invariant curves with frequency "very close" to the frequency of the HFP. This is shown in Figure 3 where the stochactic layer is revealed by the nonregularity of the frequency map around the frequency value 1/3.

For a fixed value of ϵ and a HFP of rotation number p/q, we study the frequency curve $y \mapsto \omega(x, y)$ for y in some neighbourhood, $\mathcal{I}$, of y_{HFP}, and $x = x_{HFP}$ fixed. We generalise in an obvious way the definitions (1.7) and (1.8) of $\omega^+(\epsilon)$ and $\omega^-(\epsilon)$ defined in the vicinity of rotation number 0 for invariant curves in the vicinity of a general HPF of rotation number p/q as $\omega^+_{p/q}(\epsilon)$, $\omega^-_{p/q}(\epsilon)$ and $\Delta\omega_{p/q} = \omega^+_{p/q} - \omega^-_{p/q}$. Similarly we extend the definitions of $y^{\pm}(\epsilon)$ as $y^{\pm}_{p/q}(\epsilon)$.

Numerically $y^+_{p/q}(\epsilon)$ is determined as the smallest value of $y(0)$, larger than y_{HFP}, for which the frequency curve is still regular (see appendix A), and similarly for $y^-_{p/q}(\epsilon)$. Then we assume $\omega^{\pm}_{p/q}(\epsilon) = \omega\left(x_{HFP}, y^{\pm}_{p/q}(\epsilon)\right)$. Studying $\Delta\omega_{p/q}(\epsilon)$ as a function of ϵ we are then able to reconstruct the curve $\epsilon_{crit}\left(\Delta\omega_{p/q}\right)$ for a fixed resonance. In fact the non-existence of invariant curves, for a fixed value of ϵ, with frequency $\tilde{\omega}$, between $\omega^+_{p/q}$ and $\omega^-_{p/q}$ means that $\epsilon_{crit}(\tilde{\omega}) > \epsilon$ and $\epsilon_{crit}\left(\omega^{\pm}_{p/q}\right) = \epsilon$.

3.3. **Presentation of the numerical results.** Using the FMA we studied the behaviour of $\epsilon_{crit}\left(\Delta\omega_{p/q}\right)$ in neighbourhood of the resonances $\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{2}{5}, \frac{1}{3}, \frac{1}{2}$. The results are presented in Table 1 and in Figure 4, where the plot of $\ln \epsilon_{crit}\left(\Delta\omega_{p/q}\right)$

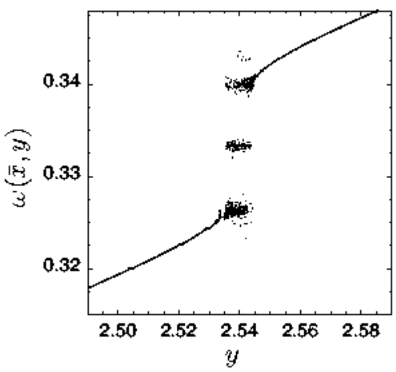

FIGURE 3. Particular of a nearby zone of a HFP, section $\bar{x} = 4.408$ and $\epsilon = 0.7$ (neighbourhood of the 1/3 resonance). Note that for $y \in [2.535, 2.545]$ the frequency curve looks very irregular.

as a function of $\ln \Delta\omega_{p/q}$ is shown. All numerical calculations are done with 100 digits accuracy using the Fortran 90 multi–precision package *mpfun* [3].

These results suggest the following geometry of the ϵ_{crit}–ω plane. Consider only the resonance p/q, then for ω close to p/q we have:

$$\epsilon_{crit}(\omega) \sim C'_{p/q} \left|\omega - \frac{p}{q}\right|^{\frac{1}{q}} \tag{3.7}$$

Because of the density of the rational numbers on the real line, close to p/q there are infinitely many rational numbers p'/q', with q' bigger and bigger as p'/q' is

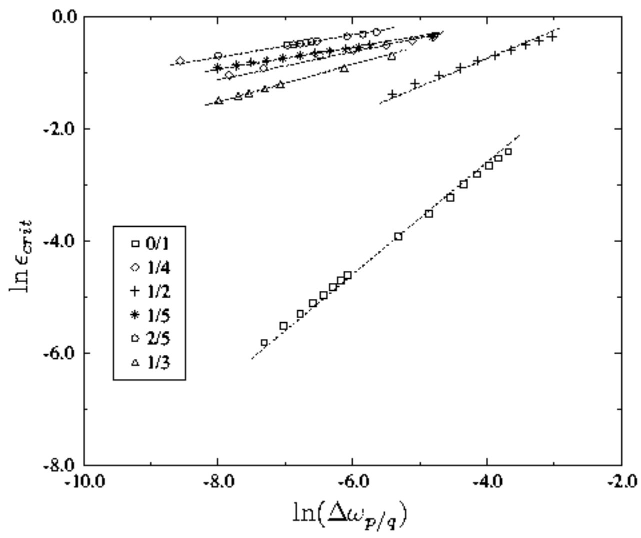

FIGURE 4. Plot of $\ln \epsilon_{crit}$ vs $\ln\left(\Delta\omega_{p/q}\right)$ for $p/q \in \{0/1, 1/5, 1/4, 2/5, 1/3, 1/2\}$.

closer to p/q. The relation between ϵ_{crit} and ω is not the simple one given by (3.7), even in a very small neighbourhood of p/q, but we must add the contributions of all p'/q' near p/q. These contributions are nevertheless negligible because for very big q' the right hand side of (3.7) is almost equal to one. Then the local description given by the (3.7) should be a very good approximation of the relation between ϵ_{crit} and ω (see Figure 5).

4. CONCLUSIONS.

We have studied both analytically and numerically the Standard Map critical function in a neighbourhood of a fixed resonance. For the fundamental resonance

resonance p/q	$A_{p/q}$	$C_{p/q}$
$0/1$	0.937	1.066
$1/5$	0.175	0.489
$1/4$	0.223	0.723
$2/5$	0.175	0.714
$1/3$	0.312	1.000
$1/2$	0.427	0.957

TABLE 1. $\ln \epsilon_{crit} = A_{p/q} \ln \Delta\omega_{p/q} + C_{p/q}$, where the coeffcients $A_{p/q}$ and $C_{p/q}$ are determined by a linear least square fit to the FMA results.

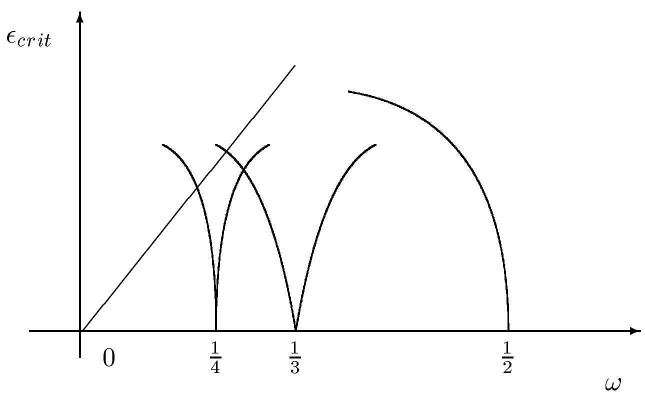

FIGURE 5. The resonant domains in the ϵ_{crit}–ω plane.

we find a linear scaling law, whereas in [7] and [8] a quadratic scaling law was found. We think these two results agree, in fact we think there is a difference in the scaling law for real or complex frequencies. This shows in particular that for diophantine values of the rotation number ω, sufficiently close to zero, the critical function $\epsilon_{crit}(\omega)$ and the critical radius of convergence $\rho_{crit}(\omega)$ are different. More precisely $\epsilon_{crit}(\omega) > \rho_{crit}(\omega) > 0$

We stress again that this result implies that the conjugating function u defined as the sum of the Standard Map Lindstedt series (1.2) for $|\epsilon| < \rho_{crit}(\omega)$, can be analytically continued outside its disk of convergence at least for ϵ real. This implies that in the plane ϵ–complex the domain of convergence of the Lindstedt series is not a disk; it will be interesting to study its geometry, in particular its boundary: it could be a fractal curve as in the case of the Schröder–Siegel center problem [13], we only know that the nearest singularity to the origin is on the imaginary axis.

The proof of Theorem 1.1 is different from the one of [7]. The FMA method doesn't allow us (at least for the moment) to use complex frequencies so we couldn't verify the results of [7] directly. It would be interesting to investigate the scaling law for the generic resonance p/q with our methods, but the proof of Theorem 1.1 is specific to the fundamental resonance: we don't have a bound of the initial data for the last invariant torus for a generic resonance as we proved in Proposition 2.2

for the resonance 0/1. Supported by the numerical results of section 3, we can nevertheless make the following conjecture

Conjecture 1. *For all rational p/q, we can find a positive constant $c_{p/q}$, such that for all ω in a small* real neighbourhood *of p/q*

$$\epsilon_{crit}(\omega) \sim c_{p/q}\left|\omega - \frac{p}{q}\right|^{\frac{1}{q}}. \tag{4.1}$$

We can make a link with the Brjuno function. The remark in footnote 1 with Theorem 1.1 imply that there exists a neighbourhood of the origin where $B(\omega) + \ln \epsilon_{crit}(\omega)$ is uniformly bounded. If we are able to prove the previous conjecture the same bound will hold in the neighbourhood of every rational, that is we could prove $B(\omega) + \ln \epsilon_{crit}(\omega)$ is uniformly bounded for all frequencies.

In this paper we were interested in finding the optimal scaling law, namely the optimality of the exponent $1/q$ in (4.1), neglecting the behaviour of the proportionality constants $c_{p/q}$ as function of p/q. It will be interesting to investigate (at least numerically) this function, in particular to test its continuity, and to search if the function $\epsilon_{crit}(\Delta\omega)/\Delta\omega^{\frac{1}{q}}$ is continuous for all rational numbers.

Appendix A. Numerical precision of the results.

Here we analyse more precisely the numerical results, pointing out some important remarks concerning the precision of the calculations.

The destruction of invariant curve is identified by the nonregularity of the frequency map. In general, this nonregularity is clearly visible (fig 3), and the destruction of invariant curves is determined with no ambiguity. But when one searches for very fine details, which correspond to small values of the perturbing coefficient ϵ, one needs to be able to distinguish between nonregularity of the frequency map and the apparent nonregularity resulting from numerical errors in the computations.

One way to work around this problem is to search always for a regular part of the frequency curve. The existence of this regular part is the indication that we have not yet reached the threshold for numerical roundoff error. An illustration is given in figure 6, where the frequency curve is plotted for $\epsilon = 0.1, T = 90000$, for both double and quadruple precision computations. It is clear that with double precision computations, we are no longer able to distinguish precisely between regular and non regular behavior of the frequency curve, while it is still possible with quadruple precision artihmetics.

As we know that on the regular curves, the FMA algorithm converges towards the true frequency as $1/T^4$ with the Hanning window of order 1 ($p = 1$ in Eq.3.5), we can check the errors of the FMA method by increasing T. If the variation of frequencies increases instead of decreasing, and still present a regular behavior as in Figure 6.a, it should come from numerical noise, and needs to be checked further.

A.1. **False interpretation of a first result.** In fact, the numerical accuracy for the iteration of the map is very important when one searches for very small values of ϵ, and not taking it properly into account can lead to false results. Indeed, in figures 7, we show sections $x = x_{HFP}$ of the Standard Map with $\epsilon = 0.1$ for double precision (Fig. 7.a) and quadruple precision (Fig. 7.b) calculation. One can note

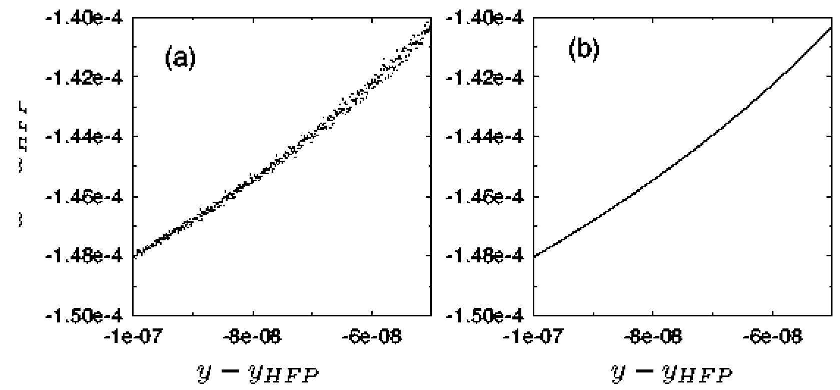

FIGURE 6. Frequency curve calculated in double precision (a) and quadruple precision (b). Section $x = 4.408$ for the Standard Map for $\epsilon = 0.1$, integration time $T = 90000$. The regular set $\mathcal{I}_{reg}$ is represented.

a very large change in the y–range, which implies that the values of $y^-_{p/q}$ and $y^+_{p/q}$ obtained using double precision are largely over–estimated. In this case, we must use at least the quadruple calculation, while for the final results of the present paper, we had to use the 100–digits multi-precision using the Fortran 90 package *mpfun*.

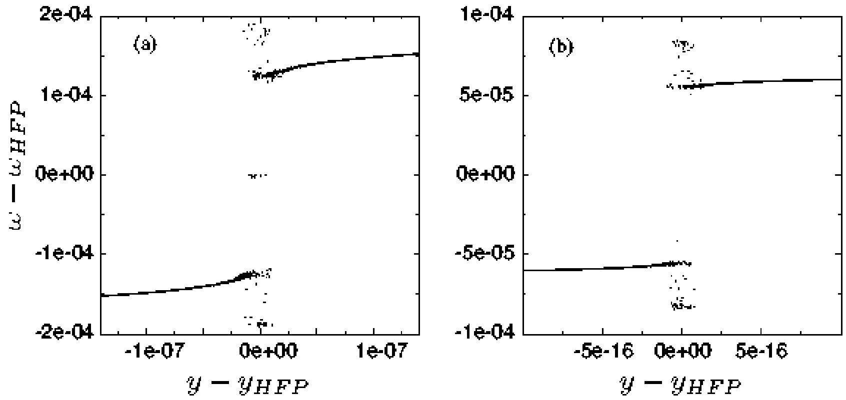

FIGURE 7. Double precision calculations (a) and Quadruple precision calculations (b) for the section $x = x_{HFP}$ for the Standard map with $\epsilon = 0.1$.

Because of the finite precision arithmetic there is a threshold x_{thr} such that for all x below the threshold then $\sin x = x$. When we start with initial condition $(x_0, y_0) = o(x_{thr})$ then instead of iterating the standard map (1.1) we iterate the linear map T_ϵ^{lin}

$$T_\epsilon^{lin} : \begin{cases} y' &= y + \epsilon x \\ x' &= x + y' \qquad \mod 2\pi. \end{cases}$$

The iteration of T_ϵ^{lin} produces a strange result which is misleading. Using data below the threshold value x_{thr}, the iteration of T_ϵ^{lin} and then the numerical analysis of the signal gives the following scaling law: There exists $0 < \epsilon^* < 1$ such that:

1. if $\epsilon_{crit}(\omega) > \epsilon^*$ then: $\ln \epsilon_{crit}\left(\Delta\omega_{\frac{p}{q}}\right) \sim \frac{1}{q} \ln \Delta\omega_{\frac{p}{q}} + c_1$,
2. if $\epsilon_{crit}(\omega) < \epsilon^*$ then: $\ln \epsilon_{crit}\left(\Delta\omega_{\frac{p}{q}}\right) \sim \frac{2}{q} \ln \Delta\omega_{\frac{p}{q}} + c_2$;

namely it seems to be a discontinuity in the scaling law as Figure 8 shows.

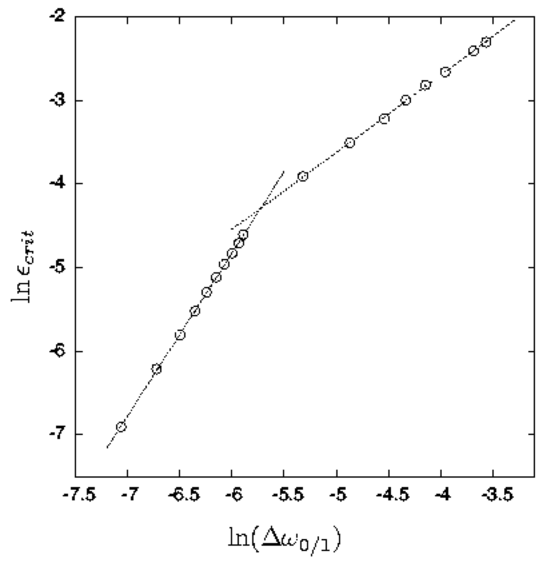

FIGURE 8. Change of slope phenomenon for the resonance 0/1. $\ln \epsilon_{crit}$ is plotted versus $\ln \Delta\omega_{0/1}$ using quadruple precision. The linear fit $\ln \epsilon_{crit} = 1.948 \ln \Delta\omega_{0/1} + 6.863$ and $\ln \epsilon_{crit} = 0.92 \ln \Delta\omega_{0/1} + 0.998$ are also plotted.

Looking at [7] and comparing with the results of [36] at first time, using the double precision, we interpreted the result of Figure 8 as follows:

> *The scaling law at resonances for the Standard Map is the one presented in* [7]. *The different result of* [36] *can be justified saying that they didn't go deep enough inside the resonant region and the non linearity of the model added some distortion . We are in presence of two behaviours: close to the resonant region we find the proper law, far away the distorted one.*

In studying further the reasons for this transition, we convinced ourselves that this interpretation was not correct. We understood that this behaviour was due to the finite precision arithmetic of the iteration of the map. We also observed that for values of ϵ smaller than a critical value ϵ_k, there was a saturation of the measured value of $\Delta y_{0/1} = y^+(\epsilon) - y^-(\epsilon)$, which was also the sign of numerical problems (Fig. 9). This was confirmed by the analytic result of Theorem 1.1 for the resonance 0/1.

A.2. **New interpretation.** With Lemma 2.3 we proved that the rotation number of the standard map is very well approximated by the pendulum frequency. Rewriting (2.6) as follows

$$\ln \epsilon_{crit} \sim 2 \ln \Delta\omega_{0/1} + C + 2 \ln \ln \frac{8\sqrt{\epsilon_{crit}}}{\Delta y_{0/1}} \tag{A.1}$$

and using the previous remark on $\Delta y_{0/1}$, it follows that the term $\ln \ln \frac{8\sqrt{\epsilon_{crit}}}{\Delta y_{0/1}}$ in (A.1) is almost constant in the interval of variation of ϵ, say for example $\epsilon \in [0.001, 0.01]$. A similar result holds for every resonance p/q; then for a fixed numerical precision we obtain the following scaling law

$$\ln \epsilon_{crit} \sim \frac{2}{q} \ln \Delta\omega_{p/q} + c$$

To prove that this change of slope is "not real" we increased the precision of the calculations (i.e. passing from double precision to quadruple and then to multi-precision–100), obtaining that the threshold ϵ^* decreases (see Figure 10), showing that in the limit of exact arithmetic this threshold doesn't exist.

Figure 11 shows $\Delta y_{0/1}$ as a function of ϵ_{crit} for different machine precision. Also in this case it is clear that the threshold value under which y is nearly constant goes to zero when we increase the numerical precision.

For the quadruple precision the threshold is $x_{thr} \sim 10^{-12}$ and it goes down to order 10^{-32} using the multi-precision package 100–digits. This value is sufficiently small to do precise calculations in reasonable CPU–times. In fact the threshold value for the multi-precision package 1000–digits is order 10^{-333} but the CPU–time for the calculations increases several order of magnitude, and this was not necessary to get good confidence with our present results.

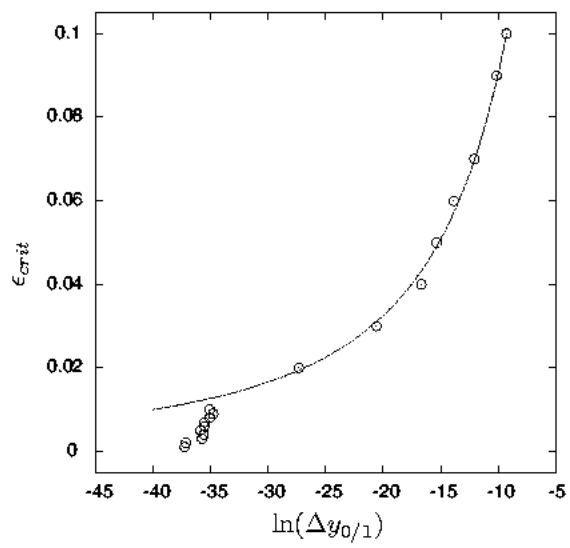

FIGURE 9. $\ln \Delta y_{0/1}$ as function of ϵ_{crit} for the fundamental resonance using the quadruple precision. It is evident that for $\epsilon \leq 0.01$ then $\ln \Delta y_{0/1} \sim -35$.

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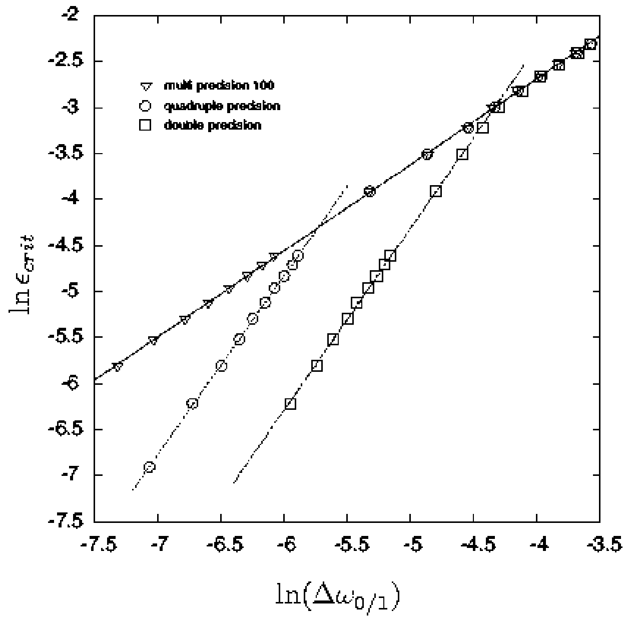

FIGURE 10. Plot of $\ln \epsilon_{crit}$ as a function of $\Delta\omega_{0/1}$ using the multiple precision 100, quadruple precision and double precision.

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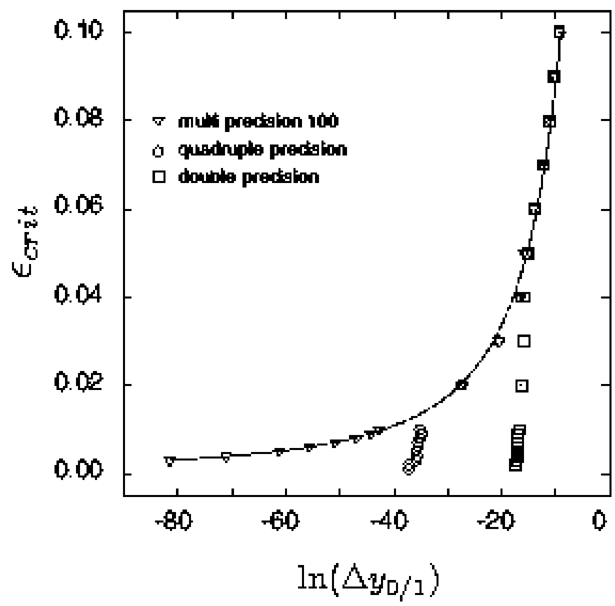

FIGURE 11. Plot of ϵ_{crit} vs $\ln \Delta y_{0/1}$ for different precisions. The solid curve is the fitting of the multiprecision–100 data: $\ln \Delta y_{0/1} = a + \frac{b}{\sqrt{\epsilon_{crit}}}$, with $a = 6.177$ and $b = -4.804$. To compare with the analytical result of Proposition 2.2 which gives $\ln \Delta y_{0/1} \geq C - \frac{\pi^2}{2\sqrt{\epsilon_{crit}}}$. We don't have a good estimation for C but we can compare b with $-\pi^2/2 \sim -4.935$, exhibiting a good agreement between the numerical results and the analytical one.