CRITICAL LARGE DEVIATIONS IN HARMONIC CRYSTALS WITH LONG RANGE INTERACTIONS

P. CAPUTO AND J.-D. DEUSCHEL

Abstract. We continue our study of large deviations of the empirical measures of a massless Gaussian field on $\mathbb{Z}^d$, whose covariance is given by the Green function of a long range random walk, [7]. In this paper we extend techniques and results of [4] to the *non-local* case of a random walk in the domain of attraction of the symmetric α-stable law, with $\alpha \in (0, 2 \wedge d)$. In particular, we show that critical large deviations occur at the capacity scale $N^{d-\alpha}$, with a rate function given by the Dirichlet form of the embedded α-stable process. We also prove that if we impose zero boundary conditions, the rate function is given by the Dirichlet form of the killed α-stable process.

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1. Introduction and Results

We consider a field of random heights $\phi_x \in \mathbb{R}$ at sites $x \in \mathbb{Z}^d$ of the d-dimensional integer lattice. The distribution P of the field is Gaussian, with mean zero, and covariance given by the Green function of a lattice random walk, i.e.

$$P(\phi_x \phi_y) = G(x, y) = \sum_{n=0}^{\infty} p^n(x, y)\,, \tag{1.1}$$

where $p(x, y)$ is the probability to jump from x to y, p^n is the corresponding n-th iterate, and the random walk is assumed to be homogeneous, symmetric and transient. If $p(\cdot, \cdot)$ is the simple random walk in dimension $d \geqslant 3$, P is known as the massless (lattice) free field. In general, we refer to P as the harmonic crystal with random walk p. We are going to investigate large deviations of the empirical field of P.

1.1. **Non-critical large deviations.** Let R_N denote the empirical field in a cubic box Λ_N of side $N \in \mathbb{Z}_+$

$$R_N(\phi) = N^{-d} \sum_{x \in \Lambda_N} \delta_{\theta_x \phi}\,, \quad \phi \in \Omega\,, \tag{1.2}$$

where $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ denotes the set of height configurations equipped with the product topology, δ_ξ stands for the Dirac measure at $\xi \in \Omega$ and $\theta_x \phi$ denotes the shifted configuration $\phi_{\cdot + x}$. Let also $\mathcal{M}_1(\Omega)$ denote the set of probability measures on $(\Omega, \mathcal{B})$ where $\mathcal{B}$ is the Borel σ-field on Ω, and $\mathcal{M}_1^S(\Omega)$ the set of translation invariant elements of $\mathcal{M}_1(\Omega)$, i.e. invariant under the action of $\{\theta_x, x \in \mathbb{Z}^d\}$. $\mathcal{M}_1(\Omega)$ and $\mathcal{M}_1^S(\Omega)$ are equipped with the weak topology. If R_N is seen as a random element of $\mathcal{M}_1(\Omega)$, by the ergodic theorem $P(R_N \in \Gamma) \to 1$ as $N \to \infty$ for any open set $\Gamma \subset \mathcal{M}_1(\Omega)$ with $P \in \Gamma$. We have shown in

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[7] that R_N satisfies a weak large deviation principle at the volume scale with rate given by the specific relative entropy

$$h(\mu|P) = \begin{cases} \lim_{N\to\infty} N^{-d} H_N(\mu|P) & \mu \in \mathcal{M}_1^S(\Omega) \\ +\infty & \mu \in \mathcal{M}_1(\Omega) \setminus \mathcal{M}_1^S(\Omega) \end{cases} \tag{1.3}$$

where $H_N(\mu|P)$ denotes the usual relative entropy functional on the box Λ_N. Then, for any compact $F \subset \mathcal{M}_1^S(\Omega)$ such that $\sup_{\mu\in F} \mu(\phi_x^2) < \infty$,

$$\limsup_{N\to\infty} N^{-d} \log P\big(R_N \in F\big) \leqslant -\inf_{\mu\in F} h(\mu|P)\,, \tag{1.4}$$

and for any open set $G \subset \mathcal{M}_1^S(\Omega)$

$$\liminf_{N\to\infty} N^{-d} \log P\big(R_N \in G\big) \geqslant -\inf_{\mu\in G} h(\mu|P)\,. \tag{1.5}$$

The (nonnegative) function $h(\cdot|P)$ was also shown to satisfy the variational principle

$$h(\mu|P) = 0 \iff \mu \in \mathcal{G}_2^S \tag{1.6}$$

where $\mathcal{G}_2^S$ denotes the set of Gibbs measures associated to the formal Hamiltonian

$$\mathcal{H}(\phi) = \frac{1}{4} \sum_{x,y} p(x,y)\,(\phi_x - \phi_y)^2\,, \tag{1.7}$$

which are translation invariant and have finite second moments. The lower bound (1.5) together with (1.6) shows that deviations within the set $\mathcal{G}_2^S$ are of order $o(N^d)$ on the exponential scale. We then speak of critical large deviations.

As discussed in [7], we may replace P by P_N^0, the (Gaussian) Gibbs measure on Λ_N with zero boundary conditions, and the bounds (1.4) and (1.5) remain unchanged. In particular, the rate function is the same. Our main results will show that this is not the case for critical large deviations, i.e. boundary conditions play an important role here.

1.2. **Critical large deviations.** Critical large deviations have been studied for the massless free field, and more generally for any harmonic crystal arising from a finite range irreducible random walk, in the paper [4]. The transience assumption for such random walks requires $d \geqslant 3$, and by a classical invariance principle capacities scale as N^{d-2}. The authors prove a strong large deviation principle for R_N at the scale N^{d-2}, with rate function given by the Dirichlet form of an embedded brownian motion. In this paper we extend the analysis of [4] to the case of random walks in the domain of attraction of a stable law. In particular, we consider massless fields P_α, $\alpha \in (0, 2\wedge d)$ and $d \geqslant 1$, defined by (1.1) with $p = p_\alpha$ an homogeneous, isotropic random walk satisfying

$$\lim_{|x|\to\infty} |x|^{d+\alpha} p_\alpha(0,x) = v_{\alpha,d}\,, \tag{$H1$}$$

for some constant $v_{\alpha,d} \in (0,\infty)$, where $|x|$ denotes the euclidean norm of $x \in \mathbb{Z}^d$. We will sometimes omit the subscript α and simply write P for P_α. By (H1), the lattice process converges to the symmetric α-stable process on $\mathbb{R}^d$, after suitable rescaling. We also assume

$$\lim_{|x|\to\infty} |x|^{d-\alpha} G(0,x) = \omega_{\alpha,d}\,, \tag{$H2$}$$

with $\omega_{\alpha,d} \in (0,\infty)$. Note that (H2) is a kind of local CLT statement, which is known to hold for finite range irreducible random walks (with $\alpha = 2$), [23]. We show in Appendix

B how to construct examples of random walks satisfying both (H1) and (H2). As we shall see, in this setting the correct scale for critical large deviations is $N^{d-\alpha}$.

1.3. **Main results.** In analogy with [4], our first main result establishes a strong large deviation principle for the empirical field of P_α at the capacity scale $N^{d-\alpha}$, with rate function given by the Dirichlet form of the symmetric α-stable process in $\mathbb{R}^d$, embedded in the "macroscopic" box $V = \Lambda_N/N$. Let us describe the result in more detail. Using Theorem 2.1 of [7] we may characterize elements of $\mathcal{G}_2^S$ as mixtures of translates of P_α. In particular, following an idea introduced in [4], we may write $\mathcal{G}_2^S$ in the following convenient way. Let $V = [-\frac{1}{2}, \frac{1}{2}]^d$, and let $L^2(V)$ denote the space of real square integrable functions on V. For each $t \in \mathbb{R}$, let γ_t be the measure obtained from P_α by translating ϕ_x into $\phi_x + t$ for every $x \in \mathbb{Z}^d$, i.e. γ_t is the Gaussian field with constant mean t and covariance G. We then have

$$\mathcal{G}_2^S = \{\, \gamma(\varphi)\,,\ \ \varphi \in L^2(V) \}, \tag{1.8}$$

where

$$\gamma(\varphi) \equiv \int_V \gamma_{\varphi(x)}\,\mathrm{d}x\,. \tag{1.9}$$

We recall that, in the above decomposition, apart from the case of extremal Gibbs states ($\varphi =$ const. a.e.), the profile φ need not be unique, [4, 7].

The Dirichlet form of the symmetric α-stable process on $\mathbb{R}^d$ is given (up to a constant factor) by

$$\mathcal{E}(\psi,\psi) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(\psi(x) - \psi(y)\big)^2\, |x-y|^{-d-\alpha}\,\mathrm{d}x\,\mathrm{d}y\,. \tag{1.10}$$

The extended domain of $\mathcal{E}(\cdot,\cdot)$ is denoted $\mathcal{D}_{\mathcal{E}}$, see Appendix A for details. We shall use the notation $\psi 1_V$ to denote restriction of ψ to V. For $\mu \in \mathcal{M}_1(\Omega)$, the rate function is given by

$$\mathcal{C}_\alpha(\mu) = \frac{1}{2} \inf_{\substack{\psi\in\mathcal{D}_{\mathcal{E}}:\\ \mu=\gamma(\psi 1_V)}} \mathcal{E}(\psi,\psi)\,, \tag{1.11}$$

where the infimum over empty sets is set equal to $+\infty$. Note that, since the constraint on ψ is only relevant to the box V, the above variational problem is naturally linked to the problem of *balayage* on V. We refer to Appendix A for a thorough discussion, and a proof (see Theorem A.3) of the identification of so called embedded and balayaged Dirichlet forms. The latter is denoted

$$\mathcal{E}_V(\varphi,\varphi) = \inf_{\substack{\psi\in\mathcal{D}_{\mathcal{E}}:\\ \varphi=\psi 1_V \text{ a.e.}}} \mathcal{E}(\psi,\psi)\,, \qquad \varphi \in L^2(V)\,, \tag{1.12}$$

so that the rate function $\mathcal{C}_\alpha$ may also be written

$$\mathcal{C}_\alpha(\mu) = \frac{1}{2} \inf_{\substack{\varphi\in L^2(V):\\ \mu=\gamma(\varphi)}} \mathcal{E}_V(\varphi,\varphi)\,. \tag{1.13}$$

We prove the following strong large deviation principle for R_N.

Theorem 1.1. *$\mathcal{C}_\alpha$ is lower semi-continuous, it has compact level sets, and the following bounds hold*

- *for any closed set* $F \subset \mathcal{M}_1(\Omega)$,

$$\limsup_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(R_N \in F\big) \leqslant -\inf_{\mu\in F} \mathcal{C}_\alpha(\mu) \tag{1.14}$$

- *for any open set* $G \subset \mathcal{M}_1(\Omega)$,

$$\liminf_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(R_N \in G\big) \geqslant -\inf_{\mu\in G} \mathcal{C}_\alpha(\mu)\,. \tag{1.15}$$

Our next result is a strong large deviation principle for the empirical field of $P^0_{\alpha,N}$, the harmonic crystal on Λ_N with zero boundary conditions outside. This is the centered Gaussian field with covariance Γ_N, the Green function of the stable random walk, killed upon exiting Λ_N, see [7]. The rate function is now expressed in terms of the Dirichlet form of the symmetric α-stable process *killed* upon exiting the box V. The question of zero b.c. was not addressed in [4]. On the other hand, our result is easily seen to apply to the case of any irreducible, isotropic random walk with finite variance, replacing the stable process by brownian motion ($\alpha = 2$).

Let $\mathcal{D}^0_V$ denote the domain of the Dirichlet form for the killed process, i.e. the set of $\psi \in \mathcal{D}_\mathcal{E}$ such that $\psi 1_{V^c} = 0$ a.e. Then the rate function is given by

$$\mathcal{C}^0_\alpha(\mu) = \frac{1}{2} \inf_{\substack{\psi\in\mathcal{D}^0_V:\\ \mu=\gamma(\psi 1_V)}} \mathcal{E}(\psi,\psi) \tag{1.16}$$

Writing

$$\mathcal{E}^0_V(\varphi,\varphi) = \mathcal{E}(\psi,\psi)\,, \qquad \psi \in \mathcal{D}^0_V\,,\ \psi 1_V = \varphi \text{ a.e.}\,, \tag{1.17}$$

we may also express $\mathcal{C}^0_\alpha$ as

$$\mathcal{C}^0_\alpha(\mu) = \frac{1}{2} \inf_{\substack{\varphi\in L^2(V):\\ \mu=\gamma(\varphi)}} \mathcal{E}^0_V(\varphi,\varphi)\,. \tag{1.18}$$

Theorem 1.2. *$\mathcal{C}^0_\alpha$ is lower semi-continuous, it has compact level sets, and the following bounds hold*

- *for any closed set* $F \subset \mathcal{M}_1(\Omega)$,

$$\limsup_{N\to\infty} N^{\alpha-d} \log P^0_{\alpha,N}\big(R_N \in F\big) \leqslant -\inf_{\mu\in F} \mathcal{C}^0_\alpha(\mu) \tag{1.19}$$

- *for any open set* $G \subset \mathcal{M}_1(\Omega)$,

$$\liminf_{N\to\infty} N^{\alpha-d} \log P^0_{\alpha,N}\big(R_N \in G\big) \geqslant -\inf_{\mu\in G} \mathcal{C}^0_\alpha(\mu)\,. \tag{1.20}$$

Remark. The expressions (1.11) and (1.16) show that $\mathcal{C}^0_\alpha(\mu) \geqslant \mathcal{C}_\alpha(\mu)$ for any $\mu \in \mathcal{M}_1(\Omega)$. Strict inequality is easily seen to occur. For instance, let us compute the rate at which R_N concentrates around the extremal Gibbs states γ_t, $t \in \mathbb{R}$. For the infinite field problem we have

$$\mathcal{C}_\alpha(\gamma_t) = \frac{t^2}{2}\,\mathrm{cap}_\alpha(V)\,, \tag{1.21}$$

where $\mathrm{cap}_\alpha(V)$ is the *capacity* of the unit box V for the symmetric α-stable process. In particular, $\mathcal{C}_\alpha(\gamma_t) < \infty$. On the other hand we expect that the cost of such a deviation in the case of zero boundary condition is at least on the *surface* scale N^{d-1} for each $t \neq 0$.

This would imply $\mathcal{C}^0_\alpha(\gamma_t) = +\infty$ for $\alpha > 1$, $t \neq 0$. Indeed, if we replace V by a euclidean ball B we have (see Lemma A.4)

$$\mathcal{E}(1_B, 1_B) = \begin{cases} +\infty & \alpha \in [1,2)\,,\ d \geqslant 2 \\ < \infty & \alpha \in (0,1)\,,\ d \geqslant 1\,. \end{cases} \tag{1.22}$$

1.4. **Strategy of proofs.** We turn to a description of the main ideas involved in the proof of Theorem 1.1 and Theorem 1.2. This will lead us to the analysis of the large deviations of *profile measures*. The results about profiles, Theorem 1.3 and Theorem 1.4, can be considered of interest in their own right.

Lower bounds. The main idea of the lower bound is to use the convergence pointed out in Appendix B, Lemma B.5, which is seen to yield convergence of relative entropies on capacity scale. Namely, given a smooth profile $\psi \in \mathcal{C}^\infty(V)$, consider the Gaussian measure $\gamma^{\psi_N} \in \mathcal{M}_1(\mathbb{R}^{\Lambda_N})$, defined by

$$\begin{aligned} \gamma^{\psi_N}(\phi_x) &= \psi_N(x) = \psi(x/N)\,, \\ \gamma^{\psi_N}(\phi_x\phi_y) - \gamma^{\psi_N}(\phi_x)\gamma^{\psi_N}(\phi_y) &= G(x,y)\,, \quad x,y \in \Lambda_N\,. \end{aligned} \tag{1.23}$$

Its Radon-Nykodin derivative w.r.t. P_N, the marginal of P on the box Λ_N, is given by

$$\varphi_N(\phi) = \frac{d\gamma^{\psi_N}}{dP_N}(\phi) = \exp\big(-\frac{1}{2}\,\langle \psi_N, G_N^{-1}\psi_N\rangle_{\Lambda_N} + \langle \psi_N, G_N^{-1}\phi\rangle_{\Lambda_N}\big)\,, \tag{1.24}$$

where G_N denotes the Green function G restricted to Λ_N and $\langle\cdot,\cdot\rangle_{\Lambda_N}$ stands for the usual $l^2(\Lambda_N)$-scalar product. Therefore

$$H_N(\gamma^{\psi_N}|P) = \gamma^{\psi_N}(\log\varphi_N) = \frac{1}{2}\,\langle \psi_N, G_N^{-1}\psi_N\rangle_{\Lambda_N}\,, \tag{1.25}$$

and by Lemma B.5

$$\lim_{N\to\infty} N^{\alpha-d} H_N(\gamma^{\psi_N}|P) = \frac{1}{2}\,\mathcal{E}_V(\psi,\psi)\,. \tag{1.26}$$

The classical change of measure argument then shows that all we have to do to establish the lower bound is check that

$$\lim_{N\to\infty} \gamma^{\psi_N}\big(R_N \in G\big) = 1\,,$$

for any open set G containing the measure $\gamma(\psi)$.

Upper bounds: Profile measures. The proof of the upper bounds is based on the following analysis of ergodic properties of level 2 profiles. We define the profile measures

$$Y_N(\phi) = N^{-d} \sum_{x\in\Lambda_N} \delta_{x/N} \otimes \delta_{\phi_x}\,. \tag{1.27}$$

Y_N is a random variable with values in $\mathcal{M}_1(V\times\mathbb{R})$, the set of probability measures on the product $V\times\mathbb{R}$. We consider the rate function

$$\mathcal{I}_\alpha(\nu) = \begin{cases} \frac{1}{2}\,\mathcal{E}_V(\varphi,\varphi) & \text{if } \nu = \Phi(\varphi),\ \ \varphi \in L^2(V) \\ +\infty & \text{otherwise} \end{cases} \tag{1.28}$$

where $\nu \in \mathcal{M}_1(V\times\mathbb{R})$ and we have defined the map

$$\Phi(\varphi) = \,\mathrm{d}x \otimes \mathcal{N}(\varphi(x),\sigma^2)\,, \tag{1.29}$$

with $\mathcal{N}(m,\sigma^2)$ the normal distribution on $\mathbb{R}$ with mean m and variance $\sigma^2 \equiv G(0,0)$. We shall prove the following strong large deviations principle for Y_N under the infinite volume field P_α.

Theorem 1.3. *The function $\mathcal{I}_\alpha$ is lower semi-continuous and has compact level sets. Moreover, the profiles Y_N satisfy the estimates*

- *for any closed set $F \subset \mathcal{M}_1(V\times\mathbb{R})$,*

$$\limsup_{N\to\infty} N^{\alpha-d}\log P_\alpha\big(Y_N\in F\big) \leqslant -\inf_{\nu\in F}\mathcal{I}_\alpha(\nu) \tag{1.30}$$

- *for any open set $G \subset \mathcal{M}_1(V\times\mathbb{R})$,*

$$\liminf_{N\to\infty} N^{\alpha-d}\log P_\alpha\big(Y_N\in G\big) \geqslant -\inf_{\nu\in G}\mathcal{I}_\alpha(\nu)\,. \tag{1.31}$$

Convention. We shall sometimes abbreviate a statement like (1.30) and (1.31) above by saying that Y_N satisfies the strong $N^{d-\alpha}$-LDP (under the measure P) with rate function $\mathcal{I}_\alpha$. The LDP is said to be *weak* if the upper bound (1.30) is restricted to compact sets.

The idea behind the proof of Theorem 1.3 is the same as in [4]. We first consider the signed measures X_N on V, defined by

$$X_N(\phi) = N^{-d}\sum_{x\in\Lambda_N}\phi_x\delta_{x/N}\,. \tag{1.32}$$

Since X_N is a Gaussian variable, a large deviations principle under P is established in a standard way, by computing the logarithmic moment generating function

$$\log P\big(\exp N^{d-\alpha}\langle X_N,\psi\rangle\big) = \frac{1}{2}\,\mathrm{Var}_P(\langle X_N,\psi\rangle) = \frac{1}{2}\,N^{-2\alpha}\,\langle\psi_N, G_N\psi_N\rangle_{\Lambda_N}\,,$$

for $\psi\in\mathcal{C}(V)$, $\psi_N(x)=\psi(x/N)$, $x\in\Lambda_N$. Thanks to Lemma B.4, we obtain

$$\lim_{N\to\infty} N^{\alpha-d}\log P\big(\exp N^{d-\alpha}\langle X_N,\psi\rangle\big) = \frac{1}{2}\,(\psi,\mathcal{G}_V\psi)_V\,, \tag{1.33}$$

wher $\mathcal{G}_V$ is the integral operator associated to the Riesz kernel (see Appendix A). The Legendre transform of (1.33), which is precisely the Dirichlet form $\mathcal{E}_V$, is then the good rate function of the (level 1) $N^{d-\alpha}$-LDP for X_N, see Proposition 2.3.

The main point of the argument is to lift this level 1 LDP to a level 2 LDP for Y_N: we make a continuous transformation which sends X_N into a smooth profile $\phi_{N,\varepsilon}\in L^2(V)$, $\varepsilon>0$, given by

$$\phi_{N,\varepsilon}(x) = N^{-d}\sum_{z\in\Lambda_N}\rho_\varepsilon(x-z/N)\phi_z \tag{1.34}$$

where $\rho_\varepsilon\in\mathcal{C}^\infty(\mathbb{R}^d)$ is a smooth probability density which tends to a delta function as $\varepsilon\to 0$. We then define the level 2 profile measures

$$\Phi_\varepsilon(X_N) = \mathrm{d}x\otimes\mathcal{N}(\phi_{N,\varepsilon}(x),\sigma^2)\in\mathcal{M}_1(V\times\mathbb{R})\,. \tag{1.35}$$

A large deviations principle for $\Phi_\varepsilon(X_N)$ is obtained via contraction principle on the variables X_N, for each fixed $\varepsilon>0$. The crucial point is to show that as ε goes to zero the variables $\Phi_\varepsilon(X_N)$ and Y_N are exponentially well approximated on the scale $N^{d-\alpha}$. The main ingredient in the approximation is a conditioning argument which fully exploits the

Gaussian nature of the problem. Following [4], we choose a sub-lattice $\mathcal{L} = L\mathbb{Z}^d$, with scale $L > 0$ and consider the Gaussian fields

$$\zeta_x = P(\phi_x|\mathcal{F}_{\mathcal{L}}), \quad \tau_x = \phi_x - \zeta_x, \quad x \in \mathbb{Z}^d, \tag{1.36}$$

where $\mathcal{F}_{\mathcal{L}}$ denotes the σ-algebra generated by the variables ϕ_x, $x \in \mathcal{L}$. The fields τ and ζ are independent. Moreover, we have a random walk representation for the τ-field:

$$P(\tau_x\tau_y) = G^L(x,y), \quad x, y \in \mathbb{Z}^d, \tag{1.37}$$

where G^L is the Green function of the random walk which is killed upon hitting the sub-lattice $\mathcal{L}$. In particular, τ is a weakly coupled field, see Lemma C.2. We then choose the scale $L = \log N$ and introduce the auxiliary profiles

$$Z_N(\phi) = N^{-d} \sum_{x\in\Lambda_N} \delta_{x/N} \otimes \mathcal{N}(\zeta_x, \sigma^2). \tag{1.38}$$

Using the weak coupling property of the field τ we prove that Z_N approximates exponentially both the profiles Y_N and $\Phi_\varepsilon(X_N)$, see Proposition 2.2 and Proposition 2.1.

Let us finally discuss the profiles Y_N under the measure $P^0_{\alpha,N}$, the zero boundary conditions field. We define the rate function

$$\mathcal{I}^0_\alpha(\nu) = \begin{cases} \frac{1}{2}\,\mathcal{E}^0_V(\varphi,\varphi) & \text{if } \nu = \Phi(\varphi),\ \varphi \in L^2(V) \\ +\infty & \text{otherwise} \end{cases}. \tag{1.39}$$

Theorem 1.4. *The function $\mathcal{I}^0_\alpha$ is lower semi-continuous, it has compact level sets, and*

- *for any closed set $F \subset \mathcal{M}_1(V \times \mathbb{R})$,*

$$\limsup_{N\to\infty} N^{\alpha-d} \log P^0_{\alpha,N}\big(Y_N \in F\big) \leqslant -\inf_{\nu\in F} \mathcal{I}^0_\alpha(\nu) \tag{1.40}$$

- *for any open set $G \subset \mathcal{M}_1(V \times \mathbb{R})$,*

$$\liminf_{N\to\infty} N^{\alpha-d} \log P^0_{\alpha,N}\big(Y_N \in G\big) \geqslant -\inf_{\nu\in G} \mathcal{I}^0_\alpha(\nu). \tag{1.41}$$

The proof of Theorem 1.4 will be accomplished through the following main steps. We introduce the measures $P^{N,L}_k$, $L > 0$ and $k > 1$, defined as the harmonic crystal corresponding to a random walk with *soft* killing proportional to L/N^α in the region $W_{k,N} = \Lambda_{kN}\setminus\Lambda_N$, i.e.

$$\frac{\mathrm{d}P^{N,L}_k}{\mathrm{d}P}(\phi) = \big(Z^L_{k,N}\big)^{-1} \exp\Big(-\frac{1}{2}\, LN^{-\alpha} \sum_{x\in W_{k,N}} \phi_x^2\Big).$$

We denote $G^{N,L}_k$ the corresponding Green function. The first step then consists in proving a large deviations principle for Y_N under $P^{N,L}_k$, for each fixed $L > 0$ and $k > 1$. This will be obtained essentially as an application of Varadhan's lemma, cf. Proposition 3.1. Note that the choice of the scaling $LN^{-\alpha}$ is precisely the one required. The second step is the exponential approximation between $P^{N,L}_k \circ Y_N^{-1}$ and $P^{N,\infty}_k \circ Y_N^{-1}$ when L goes to infinity. Here $P^{N,\infty}_k$ stands for the harmonic crystal in the large box Λ_{kN} with zero boundary conditions on $W_{k,N}$[1]. The approximation is achieved by means of a probabilistic estimate

[1]Note that in the local case $P^{N,\infty}_k$ is not distinguished from P^0_N as soon as $k > 1$.

on the top eigenvalue of the difference of covariances $\Gamma_k^{N,L} \equiv G_k^{N,L} - \Gamma_{k,N}$, cf. Proposition 3.2. We have

$$\Gamma_k^{N,L}(x,y) = \mathbb{E}_x \left[\sum_{n=\tau_{k,N}}^{\infty} 1_{\{y\}}(\xi_n) \exp\Big(-\log(1+LN^{-\alpha}) \sum_{l=0}^{n} 1_{W_{k,N}}(\xi_l) \Big) \right],$$

where $\tau_{k,N}$ is the hitting time

$$\tau_{k,N} = \inf\{n \geqslant 1\,,\ \xi_n \in W_{k,N}\}\ .$$

We use in our proof very explicitly the random walk, in particular Levy's inequality and the arc-sine law. Finally, we shall prove that, as $k \to \infty$, the field $P_k^{N,\infty} \circ Y_N^{-1}$ is exponentially equivalent to $P_N^0 \circ Y_N^{-1}$.

1.5. **A remark on conditional limit theorems.** We conclude this introduction with an application of Theorem 1.3 to conditional limit laws. Let us define the level 3 empirical field

$$\mathcal{Y}_N(\phi) = N^{-d} \sum_{x \in \Lambda_N} \delta_{x/N} \otimes \delta_{\theta_x \phi}.$$

This is a random measure with values in $\mathcal{M}_1(V \times \Omega)$. Note that

$$\mathcal{Y}_N(\phi) \circ \pi_0^{-1} = Y_N(\phi),$$

where $\pi_0 : \Omega \to \mathbb{R}$ is the canonical projection onto the origin, $\pi_0\phi = \phi_0$, and, for all $F \in C_b(\Omega)$,

$$\big\langle R_N(\phi), F \big\rangle = \big\langle \mathcal{Y}_N(\phi), 1_V \otimes F \big\rangle.$$

Next, define the rate function

$$\mathcal{J}_\alpha(Q) = \begin{cases} \frac{1}{2}\mathcal{E}_V(\phi,\phi) & \text{if } Q = \Psi(\phi),\ \phi \in L^2(V) \\ +\infty & \text{otherwise} \end{cases}$$

where

$$\Psi(\phi) = \mathrm{d}x \otimes \gamma_{\phi(x)}.$$

Now in view of Theorem 1.3 and using a projective limit argument, it is not too difficult to show that $\mathcal{Y}_N$ satisfies the strong $N^{d-\alpha}$- LDP (under P_α) with rate function $\mathcal{J}_\alpha$.

Let us present a simple application of this result. For $m, s^2 > 0$, define

$$A(m) = \{Q \in \mathcal{M}_1(V \times \Omega) : \big\langle Q \circ \pi_0^{-1}, 1_V \otimes \phi_0 \big\rangle \geq m\},$$

$$C(s^2) = \{Q \in \mathcal{M}_1(V \times \Omega) : \big\langle Q \circ \pi_0^{-1}, 1_V \otimes \phi_0^2 \big\rangle \geq s^2\},$$

and set

$$A_N(m) = \{\mathcal{Y}_N(\phi) \in A(m)\} = \{\phi : N^{-d} \sum_{x \in \Lambda_N} \phi_x \geq m\},$$

$$C_N(s^2) = \{\mathcal{Y}_N(\phi) \in C(s^2)\} = \{\phi : N^{-d} \sum_{x \in \Lambda_N} \phi_x^2 \geq s^2\}.$$

The asymptotic of the conditional measures $P_\alpha(\,\cdot\,\big| A_N(m))$ and $P_\alpha(\,\cdot\,\big| C_N(s^2))$ as $N \to \infty$ are characterized by solutions of the following variational problems:

$$\inf\{\mathcal{E}_V(\psi,\psi) : \psi \in \mathcal{D}_\mathcal{E},\ \int_V \psi(x)\,\mathrm{d}x = m\}, \tag{1.42}$$

and

$$\inf\{\mathcal{E}_V(\psi,\psi) : \psi \in \mathcal{D}_{\mathcal{E}}, \int_V \psi^2(x)\,\mathrm{d}x = s^2 - \sigma^2\}\,. \tag{1.43}$$

The unique solution ψ_m of (1.42) is of the form

$$\psi_m(x) = \frac{m}{c(V)}\mathcal{G}_V 1_V(x), \quad \text{where } c(V) = \int_V \mathcal{G}_V 1_V(x)\,\mathrm{d}x$$

(see appendix A for a definition of the integral operator $\mathcal{G}_V$). Next, let e_1 be the $L^2(V)$-normalized eigenfunction associated to the largest eigenvalue λ_1 of $\mathcal{G}_V$, cf. (A.10), and, for $s^2 > \sigma^2$, let

$$\xi_{s^2}(x) = \left(\sqrt{s^2-\sigma^2}\right) e_1(x), \quad x \in V.$$

Then ξ_{s^2} and $-\xi_{s^2}$ are the unique solutions of the variational problem (1.43).

Next, for given $\epsilon > 0$ and $x \in V$ let

$$\Lambda_N^{\epsilon,x} = \Lambda_{[\epsilon N]} + [xN]$$

be the cubic box of side $[\epsilon N]$ centered at $[xN]$ and let $R_N^{\epsilon,x}$ denote the corresponding empirical field:

$$R_N^{\epsilon,x}(\phi) = [\epsilon N]^{-d} \sum_{y \in \Lambda_N^{\epsilon,x}} \delta_{\theta_y \phi}.$$

Let also $B_\rho(\mu)$ denote the Prohorov ball of radius ρ around $\mu \in \mathcal{M}_1(\Omega)$. The next convergence result for the conditional laws follows from the above LDP for $\mathcal{Y}_N$:

Proposition 1.5. *Let $m > 0$, $s^2 > \sigma^2$ and $x \in V$ then for each $\rho > 0$*

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} P(R_N^{\epsilon,x} \notin B_\rho(\gamma_{\psi_m(x)}) \,|A_N(m)) = 0$$

and

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} P(R_N^{\epsilon,x} \notin B_\rho(\gamma_{\xi_{s^2}(x)}) \cup B_\rho(\gamma_{-\xi_{s^2}(x)}) \,|C_N(s^2)) = 0$$

In fact, by symmetry

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} P(R_N^{\epsilon,x} \in B_\rho(\gamma_{\xi_{s^2}(x)}) \,|C_N(s^2)) = \lim_{\epsilon\to 0}\lim_{N\to\infty} P(R_N^{\epsilon,x} \in B_\rho(\gamma_{-\xi_{s^2}(x)}) \,|C_N(s^2)) = \frac{1}{2}.$$

We expect a stronger result[2], that is a convergence for the measures

$$P(\theta_{[Nx]}\phi \in \cdot \,|A_N(m)) \quad \text{to} \quad \gamma_{\psi_m(x)}$$

and

$$P(\theta_{[Nx]}\phi \in \cdot \,|C_N(m)) \quad \text{to} \quad \frac{1}{2}\gamma_{\xi_{s^2}(x)} + \frac{1}{2}\gamma_{-\xi_{s^2}(x)},$$

without having to take macroscopic averages on the box $\Lambda_N^{\epsilon,x}$.

The standard way to prove such a convergence is to introduce the field with periodic boundary conditions on Λ_N, which in our case does not exist. An alternative way, would be to consider *mesoscopic scales* as used in the description of the Wulff droplet for the 3-d Ising model, cf. [3] or [8]. Thus replace the macroscopic box $\Lambda_N^{\epsilon,x}$ by the mesoscopic box $\Lambda_N^{\epsilon(N),x}$ of side $\epsilon(N)N$ with $\lim_{N\to\infty}\epsilon(N) = 0$. We believe that Proposition 1.5 holds for such scaling, but this would require a non-trivial improvement of Theorem 1.3.

[2]The result for local interactions as stated in Theorem 0.13 of [4] is wrong, one should instead consider the conditional measure of the empirical field R_N.

The rest of the paper is devided into 3 main sections. In section 2 we study the infinite volume problem, Theorem 1.1 and Theorem 1.3, adapting the proof of [4]. In section 3 we consider the case of zero boundary conditions, Theorem 1.2 and Theorem 1.4. In the Appendix we collect a few basic facts dealing with the α-stable process and α-stable random walk. In particular we show the convergence of the properly rescaled capacity of the random walk to the capacity of the stable process. Our main result here is Theorem A.3, which gives the variational formula (1.12).

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2. Infinite volume field

In this section we prove Theorem 1.3 and Theorem 1.1. Since the proofs follow very closely the ideas of [4], we shall sometimes omit details of those arguments which carry over unchanged to our setting. A more extensive treatment of the matter may be found in [6], Chapter 7.

2.1. **Proof of Theorem 1.1.** Observe first that compactness of level sets of $\mathcal{C}_\alpha$, and therefore also lower semi-continuity, simply follow from (1.12), the continuity of the map

$$L^2(V) \ni \varphi \to \gamma(\varphi) \in \mathcal{M}_1(\Omega)\,. \tag{2.1}$$

and Lemma A.2, see Appendix B.

In view of (1.12), to prove the lower bound (1.15) it is sufficient to show

$$\liminf_{N\to\infty} N^{\alpha-d} \log P\big(R_N \in G\big) \geqslant -\frac{1}{2}\,\mathcal{E}_V(\psi,\psi) \tag{2.2}$$

for any $\psi \in L^2(V)$ such that $\gamma(\psi) \in G$. Moreover, due to the continuity of the map (2.1) and the semi-continuity of $\mathcal{E}_V$, we may restrict ourselves to $\psi \in \mathcal{C}^\infty(V)$. Given a smooth profile $\psi \in \mathcal{C}^\infty(V)$, consider the Gaussian measure γ^{ψ_N} defined in (1.23). As discussed in the previous section, all one has to check is

$$\lim_{N\to\infty} \gamma^{\psi_N}\big(R_N \in G\big) = 1\,, \tag{2.3}$$

for any open set G containing $\gamma(\psi)$. Note that if ψ is a constant, then (2.3) is a straightforward consequence of the ergodic theorem. Moreover, (2.3) can be extended to piecewise constant functions. Following [4], Theorem 2.9, one proves that (2.3) holds for any $\psi \in \mathcal{C}^\infty(V)$.

We turn to the upper bound of Theorem 1.1. We shall assume the validity of Theorem 1.3, whose proof will be given in the next subsection. We start with a proof of the exponential tightness of R_N. Namely, there exist $c > 0$, $L_0 > 0$, such that, for any $L > L_0$,

$$\limsup_{N\to\infty} N^{\alpha-d} \log P\big(R_N \notin \mathcal{K}_L\big) \leqslant -cL\,, \tag{2.4}$$

where $\mathcal{K}_L$ is the compact set

$$\mathcal{K}_L = \big\{\mu \in \mathcal{M}_1^S(\Omega)\,,\ \mu(\phi_0^2) \leqslant L\big\}\,. \tag{2.5}$$

In order to prove (2.4), notice that for any $a > 0$

$$P\big(R_N \notin \mathcal{K}_L\big) \leqslant\ \leqslant \exp\left(-aL\right) P\big(\exp aN^{-d} \sum_{x\in\Lambda_N} \phi_x^2\big)\,. \tag{2.6}$$

From Lemma C.1 we obtain

$$\log P\big(R_N \notin \mathcal{K}_L\big) \leqslant -aL + aN^{-d}\,\mathrm{Tr}\,G_N + 2a^2N^{-2d}W(2aN^{-d},\lambda_1)\,\mathrm{Tr}\,G_N^2\,,$$

where $\lambda_1 = \lambda_1(N)$ is the maximal eigenvalue of G_N and we have set $\beta = 2aN^{-d}$ in Lemma C.1. Recalling (B.10) it is easy to check:

$$\lambda_1(N) \leqslant \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} G(x,y) \leqslant O(N^\alpha) \tag{2.7}$$

$$\mathrm{Tr}\,G_N^2 = \sum_{x,y\in\Lambda_N} G(x,y)^2 \leqslant o(N^{d+\alpha})\,. \tag{2.8}$$

Choosing now $a = a(N) = 2cN^{d-\alpha}$ in (2.1), for $c > 0$ sufficiently small, we see that $W(2aN^{-d},\lambda_1)$ is uniformly bounded in N and the last term in the r.h.s. is $o(N^{d-\alpha})$. Using $\mathrm{Tr}\,G_N = N^d\sigma^2$, we have (2.4) for all sufficiently large L.

To prove (1.14) we may now assume F to be a compact set. Then, the classical Heine-Borel argument shows that to establish (1.14), it is sufficient to prove, for any $\mu \in F$,

$$\lim_{\rho\searrow 0}\limsup_{N\to\infty} N^{\alpha-d}\log P\big(R_N \in B_\rho(\mu)\big) \leqslant -\mathcal{C}_\alpha(\mu)\,, \tag{2.9}$$

where $B_\rho(\mu)$ is the Prohorov ball of radius ρ around μ. In view of (1.4) and (1.6), (2.9) is always satisfied for $\mu \notin \mathcal{G}_2^S$. We may therefore assume $\mu = \gamma(\psi)$ for some $\psi \in L^2(V)$, cf. (2.1). Notice that

$$\mu\circ\pi_0^{-1} = \gamma(\psi)\circ\pi_0^{-1} = \int_V \mathcal{N}(\psi(x),\sigma^2)\,\mathrm{d}x\,. \tag{2.10}$$

Consider the level 2 empirical measure

$$L_N(\phi) = R_N(\phi)\circ\pi_0^{-1} = N^{-d}\sum_{x\in\Lambda_N}\delta_{\phi_x} \in \mathcal{M}_1(\mathbb{R})\,.$$

In particular,

$$\langle L_N(\phi), f\rangle = \langle Y_N(\phi), 1_V\otimes f\rangle\,, \quad f\in\mathcal{C}_b(\mathbb{R})\,.$$

Let B^1_ρ denote the Prohorov ball of radius ρ in $\mathcal{M}_1(\mathbb{R})$. Then if $R_N \in B_\rho(\mu)$, there exists $\rho' > 0$ such that $L_N \in B^1_{\rho'}(\mu\circ\pi_0^{-1})$, with $\rho' \to 0$, as $\rho \to 0$. Applying the contraction principle (see e.g. [10], Theorem 4.2.1) to the variables Y_N and using Theorem 1.3, we see that

$$\begin{aligned}&\lim_{\rho\searrow 0}\limsup_{N\to\infty} N^{\alpha-d}\log P\big(R_N\in B_\rho(\mu)\big)\\ &\qquad\leqslant \lim_{\rho'\searrow 0}\limsup_{N\to\infty} N^{\alpha-d}\log P\big(L_N\in B^1_{\rho'}(\mu\circ\pi_0^{-1})\big) \leqslant -\widetilde{\mathcal{I}}_\alpha(\mu)\,,\end{aligned}$$

where

$$\widetilde{\mathcal{I}}_\alpha(\mu) \equiv \inf\big\{\mathcal{I}_\alpha(\nu),\ \nu\in\mathcal{M}_1(V\times\mathbb{R}):\ \nu(1_V\otimes\cdot) = \mu\circ\pi_0^{-1}\big\}.$$

By the definition (1.28) of $\mathcal{I}_\alpha$ and (2.10), we see that $\widetilde{\mathcal{I}}_\alpha(\mu) = \mathcal{C}_\alpha(\mu)$. This concludes the proof of (2.9). □

2.2. **The Profile Measures.** Here we prove Theorem 1.3. Let us first introduce some notations. For every bounded Lipshitz continuous function f on V or on $\mathbb{R}$, we write

$$\|f\|_{BL} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}\,.$$

We define the metric d_* by

$$d_*(\mu, \nu) = \sup\big\{ |\,\langle \nu - \mu, f \otimes g\rangle\,|\,,\ f \in \mathcal{C}(V),\ \|f\|_{BL} \leqslant 1\,,\ g \in \mathcal{C}(\mathbb{R}),\ \|g\|_{BL} \leqslant 1\big\}\,, \quad (2.11)$$

where $\mu, \nu \in \mathcal{M}_1(V \times \mathbb{R})$ and we have used the notation

$$\langle \mu, f \otimes g\rangle = \int_V \int_{\mathbb{R}} f(x) g(u) \mu(\,\mathrm{d}x,\,\mathrm{d}u)\,.$$

Recall that the weak topology is compatible with the Wasserstein metric, which is obtained from (2.11) by replacing $f \otimes g$ with $h \in \mathcal{C}(V \times \mathbb{R})$ such that $\|h\|_{BL} \leqslant 1$. Since the functions $f \otimes g$ are separating in $\mathcal{M}_1(V \times \mathbb{R})$, one concludes that d_* metrizes the weak topology on $\mathcal{M}_1(V \times \mathbb{R})$.

The main step in the proof of Theorem 1.3 is a large deviation principle for the profiles Z_N. These are defined by

$$Z_N(\zeta) = N^{-d} \sum_{x \in \Lambda_N} \delta_{x/N} \otimes\ \mathcal{N}(\zeta_x, \sigma^2)\,, \quad (2.12)$$

where $\zeta_x = P(\phi_x | \mathcal{F}_{\mathcal{L}})$ is the field defined in (1.36) with $\mathcal{L} = L\mathbb{Z}^d$. The next step is to establish the exponential equivalence of Y_N and Z_N.

Proposition 2.1. *The profiles Z_N satisfy the strong $N^{d-\alpha}$-LDP with rate function $\mathcal{I}_\alpha$.*

Proposition 2.2. *For any $\beta > 0$, for any $f \in \mathcal{C}(V)$ and $g \in \mathcal{C}(\mathbb{R})$ such that $\|g\|_{BL} < \infty$, we have*

$$\limsup_{N \to \infty} N^{\alpha - d} \log P\big(|\,\langle Y_N - Z_N, f \otimes g\rangle\,| \geqslant \beta\big) = -\infty\,. \quad (2.13)$$

To prove Proposition 2.2 we may repeat the arguments of [4], Proposition 3.10. The crucial point is the weak dependence of the field $\tau = \phi - \zeta$, which in our setting is expressed by the estimates of Lemma C.2. With this new ingredient, the proof given in [4] applies. We do not provide further details.

Assuming the validity of Propositions 2.1, whose proof will be given below, we now complete the proof Theorem 1.3.

Proof of Theorem 1.3. The statements about the rate function $\mathcal{I}_\alpha$ follow from the same arguments as for the rate function $\mathcal{C}_\alpha$, and the continuity of the map $\Phi : L^2(V) \to \mathcal{M}_1(V \times \mathbb{R})$ given in (1.29).

Let us show how to prove the upper bound (1.30). To prove that the profiles Y_N are exponentially tight, we may define the compact sets

$$\mathcal{R}_L = \big\{\mu \in \mathcal{M}_1(V \times \mathbb{R}),\ \big\langle \mu, 1_V \otimes z^2\big\rangle \leqslant L\big\}\,, \quad L > 0\,, \quad (2.14)$$

and use the same argument as for (2.4). Then, by the Heine-Borel argument, it is sufficient to show that for any $\mu \in F$ we can find a sequence $\mathcal{U}_\rho(\mu)$, $\rho > 0$, of neighbourhoods of μ such that

$$\limsup_{\rho \searrow 0} \limsup_{N \to \infty} N^{\alpha - d} \log P\big(Y_N \in \mathcal{U}_\rho(\mu)\big) \leqslant -\,\mathcal{I}_\alpha(\mu)\,. \quad (2.15)$$

For any $\rho > 0$, we may choose functions $u_i \in \mathcal{C}_b(V \times \mathbb{R})$, $i = 1, \dots, n$, and $\varepsilon > 0$ such that

$$\begin{aligned} &\mathcal{U}_\rho(\mu) = \{\nu \in \mathcal{M}_1(V \times \mathbb{R}) : \ |\langle \nu - \mu, u_i \rangle| < \varepsilon\,, \ i = 1, \dots, n\} \\ \text{and} \quad &\{\nu \in \mathcal{M}_1(V \times \mathbb{R}) : \ |\langle \nu - \mu, u_i \rangle| < 2\varepsilon\,, \quad i = 1, \dots, n\} \ \subset \ B_\rho(\mu) \end{aligned}$$

where $B_\rho(\mu)$ denotes the ball of radius ρ around μ for a metric which is compatible with the weak topology (e.g. d_*). Moreover, by a density argument we may further assume the functions u_i to be (a linear combination of functions) of the form $f \otimes g$, with $f \in \mathcal{C}(V)$ and $g \in \mathcal{C}(\mathbb{R})$ with $\|g\|_{BL} < \infty$. At this point it is an easy exercise to show that Proposition 2.2 implies

$$\limsup_{N\to\infty} N^{\alpha-d} \log P\big(Y_N \in \mathcal{U}_\rho(\mu)\big) \leqslant \limsup_{N\to\infty} N^{\alpha-d} \log P\big(Z_N \in B_\rho(\mu)\big)\,. \tag{2.16}$$

From (2.16) and the upper bound for Z_N, cf. Proposition 2.1, we arrive at (2.15). The lower bound can be proven in a similar way. □

Large deviations for Z_N. We turn to the proof of Proposition 2.1. Here we follow Proposition 3.9 in [4]. We recall below the main steps needed.

Step 1. The first step consists in proving a large deviation principle for level 1 profile measures. The profiles

$$X_N(\phi) = N^{-d} \sum_{x \in \Lambda_N} \phi_x \delta_{x/N}\,.$$

are Gaussian random variables with values in $\mathcal{M}(V)$, the set of all finite signed measures on V. We equip $\mathcal{M}(V)$ with the locally convex topology generated by the functionals

$$\mu \to \int_V f \,\mathrm{d}\mu\,, \quad f \in \mathcal{C}(V)\,.$$

With this choice, the sets

$$\mathcal{M}_r(V) = \big\{\mu \in \mathcal{M}(V), \ \|\mu\|_{\mathrm{var}} \leqslant r\big\}\,, \quad r > 0\,,$$

are compact ($\|\cdot\|_{\mathrm{var}}$ denotes the total variation norm).

Proposition 2.3. *The measures X_N satisfy the strong $N^{d-\alpha}$-LDP with rate function*

$$J(\mu) = \frac{1}{2} \sup_{f \in \mathcal{C}(V)} \big\{2\mu(f) - (f, \mathcal{G}_V f)_V\big\}\,, \tag{2.17}$$

where $\mathcal{G}_V$ is the integral operator defined in (A.9).

Proof. The function J is clearly convex and lower semi-continuous. Moreover, using Lemma A.2, one checks the identity

$$J(\mu) = \begin{cases} \frac{1}{2}\,\mathcal{E}_V(\psi, \psi) & \text{if } \ \mu(\mathrm{d}x) = \psi(x)\,\mathrm{d}x\,, \ \psi \in L^2(V) \\ +\infty & \text{otherwise} \end{cases}\,. \tag{2.18}$$

Lemma A.2 also implies compactness of the level sets of J. The argument used for (2.2) then implies the lower bound

$$\liminf_{N\to\infty} N^{\alpha-d} \log P_\alpha\big(X_N \in G\big) \geqslant -\inf_{\mu \in G} J(\mu)\,, \tag{2.19}$$

for any open set $G \subset \mathcal{M}(V)$.

To prove the upper bound we use the asymptotics of the logarithmic moment generating function (1.33). The Legendre transform of (1.33) coincides, by definition with the function

J, and by the standard argument (cf. e.g. Theorem 4.5.3 of [10]) this leads to the upper bound

$$\limsup_{N\to\infty} N^{\alpha-d}\log P_\alpha\big(X_N\in F\big)\leqslant -\inf_{\mu\in F} J(\mu)\,, \tag{2.20}$$

for any compact set $F\subset\mathcal{M}(V)$. In order to remove the compactness restriction observe that by the same argument used for (2.4) we have

$$\limsup_{N\to\infty} N^{\alpha-d}\log P\big(X_N\notin\mathcal{M}_r(V)\big)\leqslant -cr^2\,,\quad r>0\,,$$

for some contant $c>0$. □

Step 2. In the second step we lift the LDP of step 1 to an LDP for smooth level 2 profiles. Let ρ be a smooth probability density with support contained in the unit ball of $\mathbb{R}^d$, and define $\rho_\varepsilon(x)=\varepsilon^{-d}\rho(x/\varepsilon)$, $\varepsilon>0$. For each $\mu\in\mathcal{M}(V)$, $\varepsilon>0$, we form the function $\mu_\varepsilon\in\mathcal{C}(V)$,

$$\mu_\varepsilon(x)=\rho_\varepsilon*\mu(x)=\int_V\rho_\varepsilon(x-y)\mu(\,\mathrm{d}y)\,.$$

We define

$$\Phi_\varepsilon:\mathcal{M}(V)\to\mathcal{M}_1(V\times\mathbb{R})\,,\quad\Phi_\varepsilon(\mu)=\Phi(\mu_\varepsilon)\,,$$

where Φ is defined by (1.29). Then Φ_ε is continuous for each $\varepsilon>0$. From Step 1 and the contraction principle we see that the bounds of Theorem 1.3 are satisfied for each $\varepsilon>0$ if we replace Y_N with $\Phi_\varepsilon(X_N)$ and the rate function $\mathcal{I}$ with

$$J_\varepsilon(\nu)=\inf\big\{J(\mu)\,,\quad\mu\in\mathcal{M}(V):\ \Phi_\varepsilon(\mu)=\nu\,\big\}\,. \tag{2.21}$$

Step 3. The next step is to prove exponential approximation of $\Phi_\varepsilon(X_N)$ and Z_N, as $\varepsilon\searrow 0$. Recall the definition of the metric d_*, see (2.11).

Lemma 2.4. *For any $\beta>0$,*

$$\lim_{\varepsilon\searrow0}\limsup_{N\to\infty}N^{\alpha-d}\log P\big(d_*(\Phi_\varepsilon(X_N),Z_N)\geqslant\beta\big)=-\infty\,. \tag{2.22}$$

Proof. Let $f\in\mathcal{C}(V)$, $g\in\mathcal{C}(\mathbb{R})$ be given, with $\|f\|_{BL}\leqslant1$, $\|g\|_{BL}\leqslant1$. By definition,

$$\langle f\otimes g,\Phi_\varepsilon(X_N)-Z_N\rangle=\int_V f(x)\overline{g}_\sigma(\phi_{N,\varepsilon}(x))\,\mathrm{d}x-N^{-d}\sum_{x\in\Lambda_N}f(x/N)\overline{g}_\sigma(\zeta_x)\,, \tag{2.23}$$

where $\overline{g}_\sigma(m)$ is the function

$$\overline{g}_\gamma(m)\equiv\int_{\mathbb{R}}g(r)\,\mathcal{N}(m,\gamma^2)(\,\mathrm{d}r)\,,$$

and we have introduced the random profile

$$\phi_{N,\varepsilon}(x)=N^{-d}\sum_{y\in\Lambda_N}\rho_\varepsilon(x-y/N)\phi_y\,,\quad x\in V\,,\ \phi\in\Omega\,. \tag{2.24}$$

We can rewrite (2.23) as

$$N^{-d}\sum_{z\in\Lambda_N}\int_V\mathrm{d}x\big[f(x/N+z/N)\overline{g}_\sigma(\phi_{N,\varepsilon}(x/N+z/N))-f(z/N)\,\overline{g}_\sigma(\zeta_z)\big]\,.$$

Using the assumptions on f, g and the smoothness of $\overline{g}_\sigma$, one easily checks the following estimates, valid uniformly on $x \in V$ and $z \in \Lambda_N$:

$$|\overline{g}_\sigma(\phi_{N,\varepsilon}(x/N+z/N))|\,|f(x/N+z/N)-f(z/N)| \leqslant N^{-1}\,,$$
$$|f(z/N)|\,|\overline{g}_\sigma(\phi_{N,\varepsilon}(x/N+z/N))-\overline{g}_\sigma(\phi_{N,\varepsilon}(z/N))| \leqslant C\varepsilon^{-d-1}N^{-d-1}\sum_{y\in\Lambda_N}|\phi_y|\,,$$
$$|f(z/N)|\,|\overline{g}_\sigma(\phi_{N,\varepsilon}(z/N))-\overline{g}_\sigma(\zeta_z)| \leqslant C|\eta_z^\varepsilon|\,,$$

where $C<\infty$ is independent of N, ε, and we have defined the field

$$\eta_z^\varepsilon = \zeta_z - \phi_{N,\varepsilon}(z/N)\,, \quad z\in\mathbb{Z}^d\,. \tag{2.25}$$

Thanks to these bounds we have

$$d_*(\Phi_\varepsilon(X_N), Z_N) \leqslant N^{-1} + CN^{-d}\sum_{z\in\Lambda_N}\big(|\eta_z^\varepsilon| + N^{-1}\varepsilon^{-d-1}|\phi_z|\big) \tag{2.26}$$

We first observe that for any $\beta>0$,

$$\limsup_{N\to\infty} N^{\alpha-d}\log P\big(N^{-d-1}\sum_{z\in\Lambda_N}|\phi_z|\geqslant\beta\big) = -\infty\,. \tag{2.27}$$

This is an obvious consequence of the argument used in the proof of (2.4). The proof of the lemma will then be completed by showing

$$\lim_{\varepsilon\searrow 0}\limsup_{N\to\infty} N^{\alpha-d}\log P\big(N^{-d}\sum_{z\in\Lambda_N}|\eta_z^\varepsilon|\geqslant\beta\big) = -\infty\,, \quad \beta>0\,. \tag{2.28}$$

Let

$$\Gamma_N^\varepsilon(y,z) = P(\eta_y^\varepsilon\eta_z^\varepsilon)\,, \quad y,z\in\Lambda_N\,,$$

denote the covariance of the centered Gaussian field η^ε on Λ_N, and let $\lambda_1(N,\varepsilon)$ be its largest eigenvalue. By Chebyshev's inequality and Lemma C.1, for any $a>0$ we have the bound

$$\begin{aligned}&\log P\big(N^{-d}\sum_{z\in\Lambda_N}|\eta_z^\varepsilon|\geqslant\beta\big)\\ &\qquad\leqslant -\,a\,\beta^2 + a\,N^{-d}\,\mathrm{Tr}\,\Gamma_N^\varepsilon + 2a^2\,N^{-2d}\,\mathrm{Tr}\,(\Gamma_N^\varepsilon)^2\,W(2aN^{-d},\lambda_1)\,,\end{aligned} \tag{2.29}$$

where W is defined by (C.2). We first claim that, for each $\varepsilon>0$,

$$\lim_{N\to\infty} N^{-d}\,\mathrm{Tr}\,\Gamma_N^\varepsilon = 0\,. \tag{2.30}$$

Indeed,

$$\mathrm{Tr}\,\Gamma_N^\varepsilon = \sum_{z\in\Lambda_N}P(\eta_z^{\varepsilon\,2}) \leqslant 2\sum_{z\in\Lambda_N}P(\zeta_z^2) + 2\sum_{z\in\Lambda_N}P(\phi_{N,\varepsilon}(z/N)^2)\,.$$

The second term above is easily seen to be at most $O(N^\alpha)$, while from Lemma C.2 with $L=\log N$, we have

$$N^{-d}\sum_{z\in\Lambda_N}P(\zeta_z^2) = N^{-d}\sum_{z\in\Lambda_N}\big[G(z,z)-G^L(z,z)\big]\to 0\,, \quad N\to\infty\,.$$

This proves (2.30). It is also not difficult, cf. (2.8), to see that

$$\mathrm{Tr}\,(\Gamma_N^\varepsilon)^2 \leqslant o(N^{d+\alpha})\,. \tag{2.31}$$

Let us define, for each $\varepsilon > 0$

$$\gamma(\varepsilon) \;=\; \limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{z\in\Lambda_N} |\Gamma^\varepsilon_N(x,z)| \,. \tag{2.32}$$

We claim that

$$\lim_{\varepsilon\searrow 0} \gamma(\varepsilon) = 0\,. \tag{2.33}$$

We postpone for a moment the proof of this last fact, and show that (2.33) is in fact sufficient to complete the proof of Lemma 2.4. Let us assume $\gamma(\varepsilon) > 0$ for each $\varepsilon > 0$ (the proof is simpler otherwise) and fix

$$a = a(N,\varepsilon) = \gamma(\varepsilon)^{-1/2} N^{d-\alpha}\,. \tag{2.34}$$

With this choice of a, we see that (2.33) implies, for large N,

$$a\, N^{-d} \lambda_1(N,\varepsilon) \;\leqslant\; \gamma(\varepsilon)^{-1/2} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{z\in\Lambda_N} |\Gamma^\varepsilon_N(x,z)| \;\leqslant\; 2\gamma(\varepsilon)^{1/2}\,,$$

which shows that $W(2aN^{-d}, \lambda_1(N,\varepsilon))$ is uniformly bounded in (2.29). Inserting (2.34) in (2.29) and using (2.30) and (2.31), we have the upper bound

$$\limsup_{N\to\infty} N^{\alpha-d} \log P\big(N^{-d} \sum_{z\in\Lambda_N} |\eta^\varepsilon_z| \geqslant \beta\big) \;\leqslant\; -\,\gamma(\varepsilon)^{-1/2}\beta^2\,,$$

which implies (2.28).

It remains to prove (2.33). Let us define the field

$$\tilde\eta^\varepsilon_z = \zeta_z - N^{-d} \sum_{y\in\Lambda_N} \rho_\varepsilon(z/N - y/N)\zeta_y = \eta^\varepsilon_z + N^{-d} \sum_{y\in\Lambda_N} \rho_\varepsilon(z/N - y/N)\tau_y$$

where $\tau = \phi - \zeta$ is the centered Gaussian field with covariance G^L. From the independence of ζ and τ we have

$$\Gamma^\varepsilon_N(x,y) = P(\tilde\eta^\varepsilon_x \tilde\eta^\varepsilon_y) + N^{-2d} \sum_{z,z'\in\Lambda_N} \rho_\varepsilon(x/N - z/N)\rho_\varepsilon(y/N - z'/N) G^L(z,z')\,.$$

Moreover, we easily see from Lemma C.2 that

$$\limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} |G^L(x,y)| = 0$$

and therefore (2.33) will follow from

$$\lim_{\varepsilon\searrow 0} \limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} |P(\tilde\eta^\varepsilon_x \tilde\eta^\varepsilon_y)| = 0\,. \tag{2.35}$$

Letting $G^\zeta = G - G^L$, and using $P(\zeta_x\zeta_y) = G^\zeta(x,y)$, we have

$$\begin{aligned} P(\tilde\eta^\varepsilon_x \tilde\eta^\varepsilon_y) &= G^\zeta(x,y) \\ &\quad - N^{-d} \sum_{z\in\Lambda_N} \big[\rho_\varepsilon(y/N - z/N) G^\zeta(x,z) + \rho_\varepsilon(x/N - z/N) G^\zeta(y,z)\big] \\ &\quad + N^{-2d} \sum_{z,z'\in\Lambda_N} \rho_\varepsilon(x/N - z/N)\rho_\varepsilon(y/N - z'/N) G^\zeta(z,z') \end{aligned} \tag{2.36}$$

Again we may neglect the contribution of G^L to this sums. It is therefore sufficient to prove (2.35) by replacing G^ζ with G in (2.36). At this point we use (B.10) to write

$$G(x,y) = g_\alpha(x-y) + R(x-y)$$

with $g_\alpha(x) = \omega_{\alpha,d}|x|^{\alpha-d}$, and

$$\lim_{|x|\to\infty} |x|^{d-\alpha}|R(x)| = 0\,.$$

In particular

$$\limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} |R(x-y)| = 0\,,$$

and we are left with the contribution of g_α alone in (2.36). By Riemann integration (see Lemma B.4),

$$\begin{aligned}\limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} |\Gamma_N^\varepsilon(x,y)| \\ \leqslant \max_{x\in V} \int_V \big|g_\alpha(x-y) - 2\rho_\varepsilon * g_\alpha(x-y) + \rho_\varepsilon * \rho_\varepsilon * g_\alpha(x-y)\big|\,\mathrm{d}y\,.\end{aligned}$$

Since the densities ρ_ε converge to the Dirac mass, a dominated convergence argument implies the claim (2.35). This completes the proof of (2.33). $\square$

Step 4. From Step 2 and Step 3 above together with the exponential approximation lemma (see e.g. Theorem 4.2.16 of [10]) we see that Z_N satisfies the weak $N^{d-\alpha}$-LDP with rate function given by

$$\tilde{J}(\nu) = \sup_{\delta>0} \liminf_{\varepsilon\searrow 0} \inf_{\nu'\in B_\delta(\nu)} J_\varepsilon(\nu')\,, \tag{2.37}$$

where $B_\delta(\nu)$ is the ball of radius δ around ν w.r.t. the d_* metric in $\mathcal{M}_1(V\times\mathbb{R})$. To extend the upper bound to all closed sets we observe that the variable Z_N are exponentially tight (choose e.g. the compact sets (2.14)). In order to conclude Proposition 2.1 we only have to check that (2.37) yields the correct rate function. The final step consists then in proving that

$$\mathcal{I}(\nu) = \tilde{J}(\nu)\,, \quad \nu \in \mathcal{M}_1(V\times\mathbb{R})\,. \tag{2.38}$$

The above identity can be obtained as in Lemma 3.13 of [4].

3. Zero boundary conditions field

3.1. **Proof of Theorem 1.2.** We proceed in analogy with the proof of Theorem 1.1. To prove the lower bound in Theorem 1.2 we have to replace (1.25) with

$$H_N(\gamma^{\psi_N}|P^0_{\alpha,N}) = \frac{1}{2}\left\langle \psi_N, \Gamma_N^{-1}\psi_N\right\rangle_{\Lambda_N}\,. \tag{3.1}$$

By Lemma B.4 and (A.28)

$$\lim_{N\to\infty} N^{\alpha-d} H_N(\gamma^{\psi_N}|P^0_{\alpha,N}) = \frac{1}{2}\,\mathcal{E}^0_V(\psi,\psi)\,. \tag{3.2}$$

As before, the lower bound now follows from standard arguments. Moreover, assuming that the profiles Y_N satisfy Theorem 1.4 the upper bound in Theorem 1.2 can be proven by the same argument used for the infinite volume field in Section 2.

3.2. **Profiles with zero boundary conditions.** Here we prove Theorem 1.4. Before starting we need a slight extension of the results of the previous section on the infinite volume field. Let $k > 0$ and define

$$V_k = [-k/2, k/2]^d\,, \quad \Lambda_{kN} = NV_k \cap \mathbb{Z}^d\,.$$

Let also

$$Y_N^{(k)}(\phi) = N^{-d} \sum_{x\in\Lambda_{kN}} \delta_{x/N} \otimes \delta_{\phi_x}\,. \tag{3.3}$$

$Y_N^{(k)}$ is a $\mathcal{F}_{\Lambda_{kN}}$-measurable random variable with values in $\mathcal{M}^{(k)}(V_k \times \mathbb{R})$, the space of nonnegative measures on the product $V_k \times \mathbb{R}$ with total variation norm bounded by k^d, equipped with the weak topology. Then the proof of Theorem 1.3 shows that for each integer $k \geqslant 1$, the profiles $Y_N^{(k)}$ satisfy the strong $N^{d-\alpha}$-LDP with rate function

$$\mathcal{I}^{(k)}(\mu) = \begin{cases} \frac{1}{2}\,\mathcal{E}_{V_k}(\varphi,\varphi) & \text{if } \mu = \mathrm{d}x \otimes \mathcal{N}(\varphi(x), \sigma^2),\ \ \varphi \in L^2(V_k) \\ +\infty & \text{otherwise} \end{cases} \tag{3.4}$$

where now μ is a measure in $\mathcal{M}^{(k)}(V_k\times\mathbb{R})$, and $\mathcal{E}_{V_k}$ stands for the Dirichlet form embedded in V_k, i.e.

$$\mathcal{E}_{V_k}(\varphi,\varphi) = \inf\left\{\mathcal{E}(\phi,\phi),\ \ \phi \in \mathcal{D}_{\mathcal{E}}\,,\ \ \phi = \varphi \text{ a.e. on } V_k\right\}\,,\ \ \varphi \in L^2(V_k)\,.$$

Soft killing measures. For each $L > 0$, $k \geqslant 1$, define the measure $P_k^{L,N}$ by

$$\frac{\mathrm{d}P_k^{N,L}}{\mathrm{d}P}(\phi) = \left(Z_k^{N,L}\right)^{-1} \exp\left(-\frac{1}{2}\,LN^{-\alpha} \sum_{x\in\Lambda_{kN}\setminus\Lambda_N} \phi_x^2\right)\,. \tag{3.5}$$

Then $P_k^{N,L}$ is the centered Gaussian field with covariance $G_k^{N,L}$, given by the Green function of the stable random walk on $\mathbb{Z}^d$, with soft killing on the set $\Lambda_{kN}\setminus\Lambda_N$ only,

$$G_k^{N,L}(x,y) = \mathbb{E}_x \sum_{n=0}^{\infty} 1_y(\xi_n) \exp\left(-\log\left(1 + LN^{-\alpha}\right) \sum_{l=0}^{n} 1_{W_{k,N}}(\xi_l)\right)\,, \tag{3.6}$$

with

$$W_{k,N} \equiv \Lambda_{kN}\setminus\Lambda_N\,.$$

The expression (3.6) can be easily obtained by explicit computations, see e.g. [7], Lemma 4.5, for similar derivations.

Proposition 3.1. *For each $L > 0$, $k > 1$, the profiles Y_N satisfy the strong $N^{d-\alpha}$-LDP under the measure $P_k^{N,L}$ with rate function*

$$\mathcal{I}_{k,L}(\mu) = \begin{cases} \frac{1}{2}\,\mathcal{J}_{k,L}(\psi,\psi) & \textit{if } \mu = \Phi(\psi),\ \ \psi \in L^2(V) \\ +\infty & \textit{otherwise} \end{cases} \tag{3.7}$$

where $\mu \in \mathcal{M}_1(V\times\mathbb{R})$ and

$$\mathcal{J}_{k,L}(\psi,\psi) \equiv \inf_{\substack{\varphi\in L^2(V_k):\\ \varphi 1_V = \psi \text{ a.e.}}} \left\{\mathcal{E}_{V_k}(\varphi,\varphi) + L\int_{V_k\setminus V} \varphi(x)^2\,\mathrm{d}x\right\}\,. \tag{3.8}$$

Proof. The proof uses the LDP for $Y_N^{(k)}$ and Varadhan's Lemma. Note in fact that

$$\frac{1}{2} L N^{-\alpha} \sum_{x \in W_{k,N}} \phi_x^2 = \frac{1}{2} L N^{d-\alpha} \left\langle Y_N^{(k)}, f \right\rangle , \tag{3.9}$$

with

$$f(x,r) = 1_{V_k \setminus V}(x) \otimes r^2 , \quad x \in V_k , r \in \mathbb{R} . \tag{3.10}$$

However, a straightforward application of Varadhan's Lemma is not allowed since the function (3.10) is not continuous. We first have to deal with the following approximation.

Let $\varepsilon > 0$ and define the sets

$$V^{\varepsilon,\pm} = [-(1 \pm \varepsilon)/2, (1 \pm \varepsilon)/2]^d , \quad \Lambda_N^{\varepsilon,\pm} = N V^{\varepsilon,\pm} \cap \mathbb{Z}^d .$$

Let $f_0^{\varepsilon,-} \in \mathcal{C}(V_k)$ be the function which is zero on V, 1 on $V_k \setminus V^{\varepsilon,+}$ and interpolates linearly in the set $V^{\varepsilon,+} \setminus V$. Let also $f_0^{\varepsilon,+} \in \mathcal{C}(V_k)$ be the function which is zero on $V^{\varepsilon,-}$, 1 on $V_k \setminus V$ and interpolates linearly in $V \setminus V^{\varepsilon,-}$. We define

$$f^{\varepsilon,\pm}(x,r) = f_0^{\varepsilon,\pm}(x) \otimes r^2 , \quad x \in V_k , r \in \mathbb{R} . \tag{3.11}$$

We also define the probability measures $P_{\varepsilon,\pm}^{N,L}$ on Ω by

$$\frac{\mathrm{d} P_{\varepsilon,\pm}^{N,L}}{\mathrm{d} P} = \left(Z_{\varepsilon,\pm}^{N,L} \right)^{-1} \exp \left(-\frac{1}{2} L N^{d-\alpha} \left\langle Y_N^{(k)}, f^{\varepsilon,\pm} \right\rangle \right) . \tag{3.12}$$

From Varadhan's Lemma (see e.g. Theorem 4.3.1 in [10]) we have[3]

$$\begin{aligned}
\lim_{N\to\infty} N^{\alpha-d} \log Z_{\varepsilon,\pm}^{N,L} &= - \inf_{\mu \in \mathcal{M}_1(V\times\mathbb{R})} \left\{ \mathcal{I}^{(k)}(\mu) + \frac{L}{2} \left\langle \mu, f_0^{\varepsilon,\pm} \right\rangle \right\} \\
&= -\frac{\sigma^2 L}{2} \int_{V_k} f_0^{\varepsilon,\pm}(x)\, \mathrm{d}x + \frac{1}{2} \inf_{\varphi \in L^2(V_k)} \left\{ \mathcal{E}_{V_k}(\varphi,\varphi) + L \int_{V_k} f_0^{\varepsilon,\pm}(x) \varphi(x)^2 \, \mathrm{d}x \right\} \\
&= -\frac{\sigma^2 L}{2} \int_{V_k} f_0^{\varepsilon,\pm}(x)\, \mathrm{d}x
\end{aligned} \tag{3.13}$$

Also, it follows that for each $L > 0$ and $\varepsilon > 0$, the profiles Y_N satisfy the strong $N^{d-\alpha}$-LDP under the measure $P_{\varepsilon,\pm}^{N,L}$ with rate

$$\mathcal{I}_{k,L}^{\varepsilon,\pm}(\mu) = \frac{1}{2} \inf_{\substack{\varphi \in L^2(V_k): \\ \mu = \Phi(\varphi 1_V)}} \left\{ \mathcal{E}_{V_k}(\varphi,\varphi) + L \int_{V_k} f_0^{\varepsilon,\pm}(x) \varphi(x)^2 \, \mathrm{d}x \right\} . \tag{3.14}$$

To complete the proof of the proposition, note that

$$\left\langle Y_N, f^{\varepsilon,-} \right\rangle \leqslant \left\langle Y_N, f \right\rangle \leqslant \left\langle Y_N, f^{\varepsilon,+} \right\rangle .$$

In order to prove the large deviations upper bound, let $F \subset \mathcal{M}_1(V \times \mathbb{R})$ be a closed set. We have

$$P_k^{N,L}\big(Y_N \in F\big) \leqslant \frac{Z_{\varepsilon,-}^{N,L}}{Z_{\varepsilon,+}^{N,L}} P_{\varepsilon,-}^{N,L}\big(Y_N \in F\big) . \tag{3.15}$$

[3]Note that the functions $f^{\varepsilon,\pm}$ are not bounded, but the exponential tightness of the sets (2.14) shows that there is no difficulty in applying Varadhan's Lemma.

Therefore,

$$\limsup_{N\to\infty} N^{\alpha-d}\log P_k^{N,L}\big(Y_N\in F\big)$$
$$\leqslant -\inf_{\mu\in F}\mathcal{I}_{k,L}^{\varepsilon,-}(\mu)+\frac{\sigma^2 L}{2}\int_{V_k}\Big(f_0^{\varepsilon,+}(x)-f_0^{\varepsilon,-}(x)\Big)\,\mathrm{d}x\,. \tag{3.16}$$

Since the last term in the r.h.s. of (3.16) vanishes as $\varepsilon\to 0$, in order to prove the upper bound we only have to show

$$\inf_{\mu\in F}\mathcal{I}_{k,L}^{\varepsilon,-}(\mu)\ \nearrow\ \inf_{\mu\in F}\mathcal{I}_{k,L}(\mu)\,,\quad \varepsilon\to 0\,. \tag{3.17}$$

Let $\varphi\in L^2(V_k)$ be the minimizer in (3.14). With $\mu=\Phi(\varphi 1_V)\in F$, we have

$$\begin{aligned}\mathcal{I}_{k,L}^{\varepsilon,-}(\mu)&=\frac{1}{2}\,\mathcal{E}_{V_k}(\varphi,\varphi)+\frac{L}{2}\int_{V_k} f_0^{\varepsilon,-}(x)\varphi(x)^2\,\mathrm{d}x\\ &\geqslant \frac{1}{2}\,\mathcal{E}_{V_k}(\varphi,\varphi)+\frac{L}{2}\int_{V_k\setminus V}\varphi(x)^2\,\mathrm{d}x-\frac{L}{2}\int_{V^{\varepsilon,+}\setminus V}\varphi(x)^2\,\mathrm{d}x\,.\end{aligned} \tag{3.18}$$

An application of the Sobolev inequality of [24], Theorem 1 in chapter 5 (see also [14], Example 1.5.2) shows that there exists a constant $K<\infty$ such that, if $p_0=2d/(d-\alpha)$,

$$\|\varphi\|_{p_0,V_k}^2=\left(\int_{V_k}\varphi(x)^{p_0}\,\mathrm{d}x\right)^{2/p_0}\leqslant K\,\mathcal{E}_{V_k}(\varphi,\varphi)\,, \tag{3.19}$$

for any $\varphi\in L^2(V_k)$. Let $v(\varepsilon)$ denote the volume of $V^{\varepsilon,+}\setminus V$. Hölder's inequality and (3.19) imply

$$\int_{V^{\varepsilon,+}\setminus V}\varphi(x)^2\,\mathrm{d}x\leqslant v(\varepsilon)^{1/q}\|\varphi\|_{p_0,V_k}^2\leqslant v(\varepsilon)^{1/q}K\mathcal{E}_{V_k}(\varphi,\varphi)\,, \tag{3.20}$$

where $q=p_0/2/(p_0/2-1)$. Letting $c(\varepsilon)=v(\varepsilon)^{1/q}KL$, by (3.18) and (3.20) we have

$$\mathcal{I}_{k,L}^{\varepsilon,-}(\mu)\geqslant(1-c(\varepsilon))\mathcal{I}_{k,L}(\mu)\,, \tag{3.21}$$

for any μ such that $\mu=\Phi(\varphi 1_V)$ for some $\varphi\in L^2(V_k)$. Taking the infimum over such $\mu\in F$, (3.17) follows from $c(\varepsilon)\to 0$ as $\varepsilon\to 0$. The lower bound can be proven by the same argument (inverting the role of the functions $f^{\varepsilon,\pm}$). ☐

Exponential approximation. Let us consider the field

$$\omega_x=\omega(\phi)_x=\phi_x-P_k^{N,L}(\phi_x|\mathcal{F}_{W_{k,N}})\,,\quad x\in\Lambda_N\,,\ k>1\,. \tag{3.22}$$

Observe that $P_k^{N,L}\circ\omega^{-1}$ is the centered Gaussian field on $\mathbb{R}^{\Lambda_N}$ with covariance $\Gamma_{k,N}$, the Green function of the random walk which is killed upon hitting $W_{k,N}$. We define the profiles of the ω-field,

$$\mathcal{W}_N(\phi)=Y_N(\omega)=N^{-d}\sum_{x\in\Lambda_N}\delta_{x/N}\otimes\delta_{\omega_x}\,. \tag{3.23}$$

Then

$$P_k^{N,L}\circ\mathcal{W}_N^{-1}=P_k^{N,\infty}\circ Y_N^{-1}\,,\quad L>0\,,\ k>1\,, \tag{3.24}$$

where $P_k^{N,\infty}$ is the centered Gaussian measure on $\mathbb{R}^{\Lambda_N}$ with covariance $\Gamma_{k,N}$. The zero boundary condition field $P_N^0=P_{\alpha,N}^0$ of Theorem 1.4 shall be recovered by letting $k\to\infty$ in the end. Recall the definition (2.11) of the metric d_*.

Proposition 3.2. *For any $k>1$, $\beta>0$*

$$\lim_{L\to\infty}\limsup_{N\to\infty} N^{\alpha-d}\log P_k^{N,L}\big(d_*(Y_N,\mathcal{W}_N)\geqslant\beta\big)=-\infty\,. \tag{3.25}$$

Proof. Let η_x be the field

$$\eta_x=\phi_x-\omega_x=P_k^{N,L}(\phi_x|\mathcal{F}_{W_{k,N}})\,,\quad x\in\Lambda_N\,. \tag{3.26}$$

As in the proof of Lemma 2.4, the claim (3.25) will follow from

$$\lim_{L\to\infty}\limsup_{N\to\infty} N^{\alpha-d}\log P_k^{N,L}\big(N^{-d}\sum_{x\in\Lambda_N}|\eta_x|\geqslant\beta\big)=-\infty\,,\quad \beta>0\,. \tag{3.27}$$

Observe that η is the centered Gaussian field with covariance $\Gamma_k^{N,L}=G_k^{N,L}-\Gamma_{k,N}$ (see (3.6)), i.e. for $x,y\in\Lambda_N$

$$\Gamma_k^{N,L}(x,y)=\mathbb{E}_x\left[\sum_{n=\tau_{k,N}}^{\infty}1_y(\xi_n)\exp\left(-\log\left(1+LN^{-\alpha}\right)\sum_{l=0}^{n}1_{W_{k,N}}(\xi_l)\right)\right]\,, \tag{3.28}$$

where

$$\tau_{k,N}=\inf\left\{n\geqslant1\,,\ \xi_n\in W_{k,N}\right\}\,. \tag{3.29}$$

Then from the proof of Lemma 2.4 we see that it will be enough to prove that, for each $k>1$,

$$\lim_{N\to\infty} N^{-d-\alpha}\,\mathrm{Tr}\left(\Gamma_k^{N,L}\right)^2=0\,,\quad L>0\,, \tag{3.30}$$

$$\lim_{L\to\infty}\limsup_{N\to\infty} N^{-d}\sum_{x\in\Lambda_N}\Gamma_k^{N,L}(x,x)=0\,, \tag{3.31}$$

$$\lim_{L\to\infty}\limsup_{N\to\infty} N^{-\alpha}\max_{x\in\Lambda_N}\sum_{y\in\Lambda_N}\Gamma_k^{N,L}(x,y)=0\,. \tag{3.32}$$

Since $\Gamma_k^{N,L}(x,y)\leqslant G(x,y)$, it is clear that (3.30) simply follows from (2.8). In order to prove (3.31) and (3.32) we may proceed as follows. For each $\delta\in(0,1/4)$, recall the boxes

$$\Lambda_N^{\delta,-}=\Lambda_{(1-2\delta)N}\,,\quad \Lambda_N^{\delta,+}=\Lambda_{(1+2\delta)N}$$

so that a site in $\Lambda_N^{\delta,-}$ has a distance at least δN from any site in Λ_N^c. We also choose $\delta>0$, such that $\Lambda_N^{\delta,+}\subset\Lambda_{kN}$. It is clear that it is sufficient to prove (3.27) with Λ_N replaced by $\Lambda_N^{\delta,-}$, for any $\delta>0$. In particular, it will be sufficient to replace (3.31) and (3.32) by

$$\lim_{L\to\infty}\limsup_{N\to\infty} N^{-d}\sum_{x\in\Lambda_N^{\delta,-}}\Gamma_k^{N,L}(x,x)=0\,, \tag{3.33}$$

$$\lim_{L\to\infty}\limsup_{N\to\infty} N^{-\alpha}\max_{x\in\Lambda_N^{\delta,-}}\sum_{y\in\Lambda_N^{\delta,-}}\Gamma_k^{N,L}(x,y)=0\,,\quad \delta>0\,. \tag{3.34}$$

Consider the random walk $\xi_0,\xi_1,\xi_2,\ldots$ starting at $\xi_0=x\in\Lambda_N^{\delta,-}$, and let $\tau_{k,N}$ be as in (3.29). Define also the function

$$\Psi_{N,L}(\xi,n)=\log\left(1+LN^{-\alpha}\right)\sum_{l=0}^{n}1_{W_{k,N}}(\xi_l)\,,\quad n\in\mathbb{Z}_+\,. \tag{3.35}$$

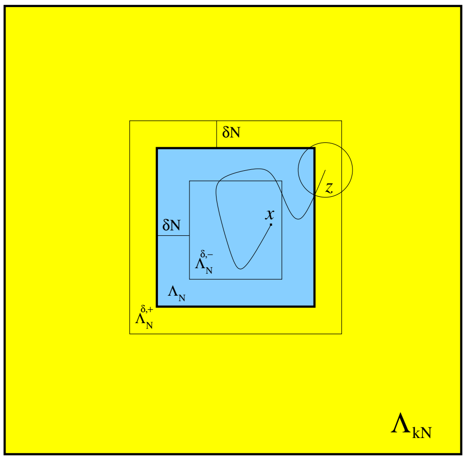

FIGURE 1. The sets $\Lambda_N^{\delta,-}$,$\Lambda_N^{\delta,+}$ and $W_{k,N} = \Lambda_{kN} \setminus \Lambda_N$.

Then by (3.6) and (3.28), using the strong Markov property we have,

$$\begin{aligned}\Gamma_k^{N,L}(x,y) &= \mathbb{E}_x \, \mathbb{E}_{\xi_{\tau_{k,N}}} \left[\sum_{n=0}^{\infty} 1_{\{y\}}(\xi_n) \exp\left(-\Psi_{N,L}(\xi,n)\right) \right] \\ &= \mathbb{E}_x \, G_k^{N,L}(\xi_{\tau_{k,N}}, y)\,, \qquad x,y \in \Lambda_N^{\delta,-}\,. \end{aligned} \tag{3.36}$$

Let us call s_N the time of first come back to $\Lambda_N^{\delta,-}$ after exiting Λ_N. By the strong Markov property, from (3.36) we have

$$\Gamma_k^{N,L}(x,y) = \mathbb{E}_x \, \mathbb{E}_{\xi_{\tau_{k,N}}} \left[\exp\left(-\Psi_{N,L}(\xi, s_N - 1)\right) G_k^{N,L}(\xi_{s_N}, y) \right] \tag{3.37}$$

Using $G_k^{N,L}(z,y) \leqslant \sigma^2$, this shows that

$$\Gamma_k^{N,L}(x,x) \leqslant \sigma^2 \, \mathbb{E}_x \, \mathbb{E}_{\xi_{\tau_{k,N}}} \exp\left(-\Psi_{N,L}(\xi, s_N - 1)\right), \quad x \in \Lambda_N^{\delta,-}\,. \tag{3.38}$$

On the other hand, letting

$$\Gamma_k^{N,L}(x, \Lambda_N^{\delta,-}) \equiv \sum_{y \in \Lambda_N^{\delta,-}} \Gamma_k^{N,L}(x,y)\,, \quad G_k^{N,L}(z, \Lambda_N^{\delta,-}) \equiv \sum_{y \in \Lambda_N^{\delta,-}} G_k^{N,L}(z,y)$$

we see from (3.37) that

$$\Gamma_k^{N,L}(x, \Lambda_N^{\delta,-}) = \mathbb{E}_x \, \mathbb{E}_{\xi_{\tau_{k,N}}} \left[\exp\left(-\Psi_{N,L}(\xi, s_N - 1)\right) G_k^{N,L}(\xi_{s_N}, \Lambda_N^{\delta,-}) \right] .$$

Since (cf. (B.10))

$$\sup_N \max_{x \in \Lambda_N^{\delta,-}} N^{-\alpha} G(x, \Lambda_N^{\delta,-}) < \infty\,,$$

using $G_k^{N,L} \leqslant G$, there exists a constant $C < \infty$ such that for any $N \in \mathbb{Z}_+$ we have

$$N^{-\alpha} \Gamma_k^{N,L}(x, \Lambda_N^{\delta,-}) \leqslant C \, \mathbb{E}_x \, \mathbb{E}_{\xi_{\tau_{k,N}}} \exp\left(-\Psi_{N,L}(\xi, s_N - 1)\right), \quad x \in \Lambda_N^{\delta,-}\,. \tag{3.39}$$

We may restrict to the case $\xi_{\tau_{k,N}} \in \Lambda_N^{\delta,+} \cap W_{k,N}$ in (3.39), i.e. we may exclude that the walk enters $W_{k,N}$ by hitting $W_{k,N} \setminus \Lambda_N^{\delta,+}$. Indeed, for any $k > \delta$, the probability of a single jump from inside Λ_N to $W_{k,N} \setminus \Lambda_N^{\delta,+}$ is bounded by $(k-\delta)^{-\alpha} N^{-\alpha}$, therefore

$$\max_{x \in \Lambda_N^{\delta,-}} \mathbb{P}_x \left(\xi_{\tau_{k,N}} \in W_{k,N} \setminus \Lambda_N^{\delta,+} \right) \to 0\,, \quad N \to \infty\,.$$

In particular, (3.38) and (3.39) show that both claims (3.33) and (3.34) will follow from the following lemma.

Lemma 3.3. *For any $\delta > 0$,*

$$\lim_{L\to\infty} \limsup_{N\to\infty} \max_{z \in \Lambda_N^{\delta,+} \cap W_{k,N}} \mathbb{E}_z \exp\left(-\Psi_{N,L}(\xi, s_N - 1)\right) = 0\,. \tag{3.40}$$

Proof. We are going to exploit the fact that $z \in W_{N,L}$ is at least at distance δN from any site $x \in \Lambda_N^{\delta,-}$. Let us define the first exit time for the euclidean ball $B_{\delta N}(\xi_0)$ of radius δN around ξ_0

$$s_N^\delta = \inf\left\{ n \geqslant 1 : \ |\xi_n - \xi_0| \geqslant \delta N \right\} \tag{3.41}$$

Note that by construction $B_{\delta N}(z) \cap \Lambda_N^{\delta,-} = \emptyset$ if $\xi_0 = z \in \Lambda_N^{\delta,+} \cap W_{k,N}$, therefore in this case $s_N \geqslant s_N^\delta$. We claim that

$$\limsup_{N\to\infty} \mathbb{P}_0 \left(s_N^\delta < \varepsilon N^\alpha \right) \to 0\,, \quad \varepsilon \to 0\,. \tag{3.42}$$

Assuming the validity of (3.42), we may (by the arbitrariness of ε) rule out the possibility that ξ_n visits $\Lambda_N^{\delta,-} \cup \Lambda_{kN}^c$ up to time $[\varepsilon N^\alpha]$. In particular the lemma will follow from

$$\lim_{L\to\infty} \limsup_{N\to\infty} \sup_{z \in \Lambda_N^c} \mathbb{E}_z \left[\exp\Big(-\frac{1}{2} L N^{-\alpha} \sum_{n=0}^{[\varepsilon N^\alpha]-1} 1_{\Lambda_N^c}(\xi_n) \Big) \right] = 0 \tag{3.43}$$

for all $\varepsilon > 0$, where we use the fact that for $LN^{-\alpha} \leqslant 1$, $\log\left(1 + LN^{-\alpha}\right) \geqslant \frac{1}{2} LN^{-\alpha}$. To prove (3.43), it is sufficient to show that for all $\varepsilon > 0$,

$$\limsup_{N\to\infty} \sup_{z \in \Lambda_N^c} \mathbb{P}_z \Big(N^{-\alpha} \sum_{n=0}^{[\varepsilon N^\alpha]-1} 1_{\Lambda_N^c}(\xi_n) \leqslant \gamma \Big) \to 0\,, \quad \gamma \to 0\,. \tag{3.44}$$

We are left with the proof of (3.42) and (3.44).

Proof of (3.42). By a well known inequality of Levy (cf. e.g. [25], Theorem 1.4.15), we have

$$\begin{aligned} \mathbb{P}_0 \left(s_N^\delta < \varepsilon N^\alpha \right) &= \mathbb{P}_0 \Big(\max_{1 \leqslant n \leqslant [\varepsilon N^\alpha]} |\xi_n| \geqslant \delta N \Big) \\ &\leqslant 2d\, \mathbb{P}_0 \left(|\xi_{[\varepsilon N^\alpha]}| \geqslant \delta N / d \right) \end{aligned} \tag{3.45}$$

Let $\bar\xi(n) = n^{-1} \xi_{[n^\alpha]}$. Then (3.45) implies

$$\mathbb{P}_0 \left(s_N^\delta < \varepsilon N^\alpha \right) \leqslant 2d \mathbb{P}_0 \left(|\bar\xi(\varepsilon^{1/\alpha} N)| > \delta \varepsilon^{-1/\alpha}/d \right)\,. \tag{3.46}$$

We now use the invariance principle $\bar\xi(n) \to^{(d)} U_\alpha$, cf. (B.8), to write

$$\limsup_{N\to\infty} \mathbb{P}_0 \left(s_N^\delta < \varepsilon N^\alpha \right) \leqslant 2d\, \mathcal{P} \left(|U_\alpha| > \delta\, \varepsilon^{-1/\alpha}/d \right) \leqslant \varepsilon\, c(d,\delta)\,, \tag{3.47}$$

for some finite constant $c(d,\delta)$, where $\mathcal{P} \in \mathcal{M}_1(\mathbb{R}^d)$ is the distribution of the symmetric α-stable variable U_α, with density $q_\alpha(1,x)$, cf. (A.1). This proves (3.42).

Proof of (3.44). We first observe that the probability appearing in (3.44) is maximized when $z \in \partial^+\Lambda_N$, with $\partial^+\Lambda_N$ denoting the set of sites $x \notin \Lambda_N$ at euclidean distance $d(x, \Lambda_N) = 1$ from Λ_N. By symmetry, the situation may be further simplified as follows. Let $(\xi_n) = (\xi_n^1, \ldots, \xi_n^d)$ be the $d-$dimensional random walk vector and let J_T, $T \in \mathbb{Z}_+$, be the time spent up to time T on the half space corresponding to nonnegative values of the first coordinate, i.e. $J_T = \sum_{n=0}^T 1_{\mathbb{Z}_+}(\xi_n^1)$. Then, for any $z \in \Lambda_N^c$, letting $T = [\varepsilon N^\alpha] - 1$

$$\mathbb{P}_z\Big(N^{-\alpha} \sum_{n=0}^{[\varepsilon N^\alpha]-1} 1_{\Lambda_N^c}(\xi_n) \leqslant \gamma\Big) \leqslant \mathbb{P}_0\left(J_T/T \leqslant \gamma/\varepsilon\right) . \tag{3.48}$$

We now apply to ξ_n^1 the arcsine law for an arbitrary one-dimensional, symmetric random walk (cf. [13], chapter XII). Namely, the distribution of J_T/T converges, as $T \to \infty$, to the arcsine function

$$\mathbb{P}_0\left(J_T/T \leqslant \gamma/\varepsilon\right) \to \frac{2}{\pi} \arcsin \sqrt{\gamma/\varepsilon}\,, \quad T \to \infty\,. \tag{3.49}$$

From this and (3.48) the claim (3.44) follows immediately. $\square$

Proposition 3.4. *For each $k \geqslant 1$, the profiles Y_N satisfy the strong $N^{d-\alpha}$-LDP under the measure $P_k^{N,\infty}$ with rate function*

$$\mathcal{I}_k(\mu) = \begin{cases} \frac{1}{2}\,\mathcal{J}_{k,\infty}(\psi,\psi) & \textit{if } \mu = \Phi(\psi), \;\; \psi \in L^2(V) \\ +\infty & \textit{otherwise} \end{cases} \tag{3.50}$$

where $\mu \in \mathcal{M}_1(V \times \mathbb{R})$ and

$$\mathcal{J}_{k,\infty}(\psi,\psi) \equiv \mathcal{E}_{V_k}(\tilde\psi,\tilde\psi)\,, \quad \tilde\psi \in L^2(V_k)\,, \quad \tilde\psi = \begin{cases} \psi & \textit{on } V \\ 0 & \textit{on } V_k \setminus V \end{cases} \quad \text{a.e.} \tag{3.51}$$

Proof. We see from (3.24) and Propositions 3.1 and 3.2, that as $L \to \infty$, the measures $P_k^{N,L}$ are an exponentially good approximation of the measure $P_k^{N,\infty}$. Therefore, using the classical exponential approximation lemma (cf. Theorem 4.2.16 of [10]), we have that Y_N satisfies the weak $N^{d-\alpha}$-LDP under the measure $P_k^{N,\infty}$, with rate given by

$$\tilde{\mathcal{I}}_k(\mu) = \sup_{\delta>0} \liminf_{L\to\infty} \inf_{\nu \in B_\delta(\mu)} \mathcal{I}_{k,L}(\nu)\,, \tag{3.52}$$

where $\mathcal{I}_{k,L}$ is the rate function defined in (3.7). As in the previous section, there is no difficulty in turning the weak LDP into a strong LDP. The proposition then follows if we prove

$$\mathcal{I}_k(\mu) = \tilde{\mathcal{I}}_k(\mu)\,, \quad \mu \in \mathcal{M}_1(V \times \mathbb{R})\,. \tag{3.53}$$

From (3.7) and (3.8) it is easily seen that $\mathcal{I}_{k,L}(\mu) \leqslant \mathcal{I}_k(\mu)$ and $\mathcal{I}_{k,L}(\mu)$ is increasing in L for fixed μ. It follows $\mathcal{I}_k(\mu) \geqslant \tilde{\mathcal{I}}_k(\mu)$, $\mu \in \mathcal{M}_1(V \times \mathbb{R})$. To prove the converse, assume $\mu \in \mathcal{M}_1(V \times \mathbb{R})$ and $\tilde{\mathcal{I}}_k(\mu) < \infty$. By lower semi-continuity, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{I}_{k,L}(\nu) \geqslant \mathcal{I}_{k,L}(\mu) - \varepsilon$, for any $\nu \in B_\delta(\mu)$. In particular, there exists $\psi \in L^2(V)$ such that $\mu = \Phi(\psi)$ and by (3.8)

$$\tilde{\mathcal{I}}_k(\mu) \geqslant \frac{1}{2} \liminf_{L\to\infty} \mathcal{J}_{k,L}(\psi,\psi) - \varepsilon\,. \tag{3.54}$$

By compactness we may now find a sequence $\varphi_{k,L} \in L^2(V_k)$, $L > 0$, such that $\varphi_{k,L} 1_V = \psi$ a.e. and

$$\mathcal{J}_{k,L}(\psi, \psi) = \mathcal{E}_{V_k}(\varphi_{k,L}, \varphi_{k,L}) + L \int_{V_k \setminus V} \varphi_{k,L}(x)^2 \, \mathrm{d}x$$

This implies $\varphi_{k,L} 1_{V_k \setminus V} \to 0$ as $L \to \infty$, in $L^2(V_k)$, and therefore $\varphi_{k,L} \to \varphi_{k,\infty}$ in $L^2(V_k)$ where $\varphi_{k,\infty} 1_V = \psi$ a.e. and $\varphi_{k,\infty} 1_{V_k \setminus V} = 0$ a.e. By lower semi-continuity we then have

$$\begin{aligned}\liminf_{L\to\infty} \mathcal{J}_{k,L}(\psi, \psi) &\geqslant \liminf_{L\to\infty} \mathcal{E}_{V_k}(\varphi_{k,L}, \varphi_{k,L}) \\ &\geqslant \mathcal{E}_{V_k}(\varphi_{k,\infty}, \varphi_{k,\infty}) = \mathcal{J}_{k,\infty}(\psi, \psi) \,.\end{aligned}$$

From (3.54) and the arbitrariness of ε, this implies $\mathcal{I}_k(\mu) \leqslant \tilde{\mathcal{I}}_k(\mu)$. □

Proof of Theorem 1.4. We are now in the position to conclude the proof of Theorem 1.4. We first prove that $P_k^{N,\infty}$ is an exponentially good approximation of P_N^0 as $k \to \infty$. In order to do so, observe that (cf. Proposition 3.2) it is sufficient to prove

$$\lim_{k\to\infty} \limsup_{N\to\infty} N^{-d} \sum_{x\in\Lambda_N} \tilde{\Gamma}_{k,N}(x, x) = 0 \,, \tag{3.55}$$

$$\lim_{k\to\infty} \limsup_{N\to\infty} N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} \tilde{\Gamma}_{k,N}(x, y) = 0 \,, \tag{3.56}$$

where we have defined

$$\tilde{\Gamma}_{k,N}(x, y) \equiv \Gamma_{k,N}(x, y) - \Gamma_N(x, y) \,. \tag{3.57}$$

Recall that $\Gamma_{k,N}$ is the Green function of the random walk with killing on $W_{k,N} = \Lambda_{kN} \setminus \Lambda_N$, while Γ_N is the Green function of the random walk with killing on Λ_N^c. We then have

$$\tilde{\Gamma}_{k,N}(x, y) = \mathbb{E}_x \left[\sum_{n=\tau_{k,N}}^{\infty} 1_y(\xi_n) \,;\ \tau_{k,N} > \sigma_{k,N} \right] \,, \tag{3.58}$$

where

$$\begin{aligned}\sigma_{k,N} &= \inf \{ n \geqslant 1 \,,\ \xi_n \in W_{k,N} \} \,, \\ \tau_{k,N} &= \inf \{ n \geqslant 1 \,,\ \xi_n \notin \Lambda_{kN} \} \,.\end{aligned}$$

Using the strong Markov property we can write

$$\tilde{\Gamma}_{k,N}(x, y) \leqslant \mathbb{E}_x \mathbb{E}_{\xi_{\tau_{k,N}}} G(\xi_{\tau_{k,N}}, y) \leqslant \sup_{z\notin\Lambda_{kN}} G(z, y) \,, \quad x, y \in \Lambda_N \,.$$

Then using assumption H2, cf. (B.10), we have

$$N^{-d} \sum_{x\in\Lambda_N} \tilde{\Gamma}_{k,N}(x, x) \leqslant \sup_{x\in\Lambda_N} \sup_{z\notin\Lambda_{kN}} G(x, z) \leqslant C\, [(k-1)N]^{\alpha-d} \,,$$

for some $C < \infty$ as $N \to \infty$. This obviously implies (3.55). On the other hand, by the same argument,

$$N^{-\alpha} \max_{x\in\Lambda_N} \sum_{y\in\Lambda_N} \tilde{\Gamma}_{k,N}(x, y) \leqslant N^{d-\alpha} C\, [(k-1)N]^{\alpha-d} = C\, (k-1)^{\alpha-d} \,,$$

for all sufficiently large N, and (3.56) follows by letting $k \to \infty$.

From the preceding observations we know (cf. the proof of Proposition 3.4) that Y_N satisfies the strong $N^{d-\alpha}$-LDP under the measure P_N^0, with rate function given by

$$\tilde{\mathcal{I}}_\infty(\mu) = \sup_{\delta>0} \liminf_{k\to\infty} \inf_{\nu\in B_\delta(\mu)} \mathcal{I}_k(\nu)\,, \tag{3.59}$$

where $\mathcal{I}_k$ has been defined in (3.50). We are left with the proof of

$$\tilde{\mathcal{I}}_\infty(\mu) = \mathcal{I}_\alpha^0(\mu) \quad \mu \in \mathcal{M}_1(V\times\mathbb{R})\,, \tag{3.60}$$

where $\mathcal{I}_\alpha^0$ is the rate function defined in (1.39). Observing that $\mathcal{J}_{k,\infty}(\psi,\psi) \leqslant \mathcal{E}_V^0(\psi,\psi)$, for any $\psi \in L^2(V)$ (cf. (3.51) and (A.28)), it is clear that $\mathcal{I}_\alpha^0(\mu) \geqslant \tilde{\mathcal{I}}_\infty(\mu)$, $\mu \in \mathcal{M}_1(V\times\mathbb{R})$. To prove the converse, as in Proposition 3.4, it is sufficient to show that if $\psi\in L^2(V)$ is such that $\mathcal{J}_{k,\infty}(\psi,\psi)<\infty$, then (cf. (3.51))

$$\lim_{k\to\infty} \mathcal{J}_{k,\infty}(\psi,\psi) = \lim_{k\to\infty} \mathcal{E}_{V_k}(\tilde\psi,\tilde\psi) = \mathcal{E}_V^0(\psi,\psi)\,. \tag{3.61}$$

Moreover, by a density argument, we may restrict to $\psi \in \mathcal{C}^1(V)$. In this case, using Lemma B.4 and Lemma B.5, we see that (3.61) follows if we can show

$$\lim_{k\to\infty}\lim_{N\to\infty} N^{\alpha-d} \sum_{x,y\in\Lambda_N} \psi(x/N)\psi(y/N)\left|G_{kN}^{-1}(x,y) - G^{-1}(x,y)\right| = 0\,, \tag{3.62}$$

where $G^{-1}(x,y) = \delta(x,y) - p(x,y)$ and G_{kN} stands for the matrix $\{G(x,y)\}_{x,y\in\Lambda_{kN}}$. We recall that (cf. [7], Lemma A.1 and Proposition 3.1)

$$\sum_{y\in\Lambda_N} \left|G_{kN}^{-1}(x,y) - G^{-1}(x,y)\right| \leqslant \mathbb{P}_x\left(\xi_1 \notin \Lambda_{kN}\right)\,.$$

The r.h.s. above is the probability that, starting in $x\in\Lambda_N$, the first jump brings the particle outside Λ_{kN}, and is easily seen to be bounded by $C[(k-1)N]^{-\alpha}$, for sufficiently large N, with $C<\infty$ independent of k and N. Going back to (3.62), we then see that

$$\begin{aligned} N^{\alpha-d} &\sum_{x,y\in\Lambda_N} \psi(x/N)\psi(y/N)\left|G_{kN}^{-1}(x,y) - G^{-1}(x,y)\right| \\ &\leqslant \|\psi\|_\infty^2\, N^\alpha \sup_{x\in\Lambda_N} \mathbb{P}_x\left(\xi_1\notin\Lambda_{kN}\right) \leqslant C\,\|\psi\|_\infty^2\,(k-1)^{-\alpha}\,, \end{aligned}$$

for all sufficiently large N. This proves (3.62) and completes the proof of Theorem 1.4.

Appendix A. The symmetric α-stable process

The Appendix is divided in three parts: Appendix A deals with the symmetric α-stable process, Appendix B is devoted to the stable random walk model and Appendix C contains a few useful estimates for Gaussian expectations.

A.1. **Free process.** Let $\alpha\in(0,2)$. The symmetric α-stable process is defined as the Markov process on $\mathbb{R}^d$ whose transition kernel has density

$$q_\alpha(t,x-y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{-i(x-y)\cdot\xi}\,\mathrm{e}^{-t|\xi|^\alpha}\,\mathrm{d}\xi\,. \tag{A.1}$$

The case $\alpha=2$ is singular, it gives brownian motion in $\mathbb{R}^d$ with $q_2(t,x)$ the centered Gaussian density with variance $2t$. On the other hand, for $\alpha<2$ one shows that

$$\lim_{|x|\to\infty} |x|^{d+\alpha} q_\alpha(1,x) = c_{\alpha,d} \equiv \Big(\int_{\mathbb{R}^d} \frac{1-\cos\hat\xi\cdot x}{|x|^{d+\alpha}}\,\mathrm{d}x\Big)^{-1}\,, \tag{A.2}$$

with $\hat{\xi}$ a unit vector in $\mathbb{R}^d$, see e.g. [2]. We shall assume $\alpha < d$ in order to have a transient process. In particular, we may define the resolvent kernel

$$g_\alpha(x) = \int_0^\infty q_\alpha(t,x)\,\mathrm{d}t = \omega_{\alpha,d}\,|x|^{\alpha-d}\,, \quad \omega_{\alpha,d} = \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha \pi^{d/2}\Gamma(\frac{\alpha}{2})}\,. \tag{A.3}$$

A detailed account of the potential theory associated to the kernel g_α, also known as the M. Riesz kernel, can be found in the treatise [15].

The *Dirichlet Form* of the symmetric α-stable process is given by

$$\mathcal{E}(\psi,\psi) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_{-\alpha}(x-y)\big(\psi(x)-\psi(y)\big)^2\,\mathrm{d}x\;\mathrm{d}y\,, \tag{A.4}$$

where we have introduced the kernel $g_{-\alpha}(x) \equiv c_{\alpha,d}|x|^{-d-\alpha}$, with $c_{\alpha,d}$ the constant given in (A.2), cf. chapter 1 of [15]. Its domain is given by

$$\tilde{\mathcal{D}}_{\mathcal{E}} = \left\{\psi \in L^2(\mathbb{R}^d):\ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_{-\alpha}(x-y)\big(\psi(x)-\psi(y)\big)^2\,\mathrm{d}x\;\mathrm{d}y < \infty\right\}. \tag{A.5}$$

In terms of Fourier transforms we have

$$\mathcal{E}(\psi,\psi) = \int_{\mathbb{R}^d} |\hat{\psi}(\xi)|^2\,|\xi|^\alpha\,\mathrm{d}\xi \tag{A.6}$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. Following Example 1.5.2 of [14] we can describe the extended domain of $\mathcal{E}$ as the range of the integral operator given by $g_{\alpha/2}$,

$$\mathcal{D}_{\mathcal{E}} = \big\{\psi = g_{\alpha/2} * \varphi\,,\ \ \varphi \in L^2(\mathbb{R}^d)\big\}$$

and, for $\psi = g_{\alpha/2} * \varphi$ we have

$$\mathcal{E}(\psi,\psi) = (\varphi,\varphi)\,, \tag{A.7}$$

where $(\cdot,\cdot)$ denotes scalar product in $L^2(\mathbb{R}^d)$. Notice that $\tilde{\mathcal{D}}_{\mathcal{E}} \subset \mathcal{D}_{\mathcal{E}}$, but $\psi \in \mathcal{D}_{\mathcal{E}}$ is not necessarily in $L^2(\mathbb{R}^d)$, because g_α is only locally in $L^1(\mathbb{R}^d)$. In fact, $\mathcal{D}_{\mathcal{E}}$ is the space of $\psi \in L^1_{loc}(\mathbb{R}^d)$ such that ψ is a tempered distributions and $\hat{\psi}(\xi)$ is square integrable w.r.t. $|\xi|^\alpha$. Since the process is transient, the Dirichlet space $\mathcal{D}_{\mathcal{E}}$ with inner product $\mathcal{E}(\cdot,\cdot)$ is a Hilbert space, [14].

The *capacity* of a compact set $\Gamma \subset \mathbb{R}^d$ is defined by the variational formula

$$\mathrm{cap}_\alpha(\Gamma) = \inf\Big\{\mathcal{E}(\psi,\psi),\quad \psi \in \mathcal{D}_{\mathcal{E}},\quad \psi = 1 \ \text{ a.e. on } \Gamma\Big\}. \tag{A.8}$$

A.2. **Embedded process.** Let us consider the $\alpha-$stable process *embedded* in the unit cube $V \equiv [0,1]^d$. We first introduce the integral operator on $L^2(V)$ given by

$$\mathcal{G}_V\phi(x) = \int_V g_\alpha(x-y)\phi(y)\,\mathrm{d}y\,, \quad x \in V\,. \tag{A.9}$$

By approximating the singular kernel g_α by a sequence of non-singular kernels one proves:

Lemma A.1. *The operator $\mathcal{G}_V : L^2(V) \to L^2(V)$ is compact and nonnegative.*

By the above lemma, $\mathcal{G}_V$ has eigenvalues $\{\lambda_1 \geqslant \lambda_2 \geqslant \ldots\} \subset \mathbb{R}^+$. We call $\{e_n\}$ the corresponding $L^2(V)$-normalized eigenfunctions, and denote $(\cdot,\cdot)_V$ the scalar product in $L^2(V)$. Let also $\mathcal{G}_V^{1/2}$ denote the square root of the operator $\mathcal{G}_V$, i.e.

$$\mathcal{G}_V^{1/2}\phi = \sum_n \lambda_n^{1/2}\,(\phi,e_n)_V\,e_n\,, \quad \phi \in L^2(V)\,. \tag{A.10}$$

Note that $\mathcal{G}_V^{1/2}$ is also nonnegative and compact.

The *embedded Dirichlet form* $\mathcal{E}_V$ is defined by

$$\mathcal{E}_V(\psi,\psi) = \sum_n \frac{1}{\lambda_n}(\psi, e_n)_V^2, \quad \psi \in L^2(V) \tag{A.11}$$

with domain

$$\mathcal{D}_V = \left\{\psi \in L^2(V) : \mathcal{E}_V(\psi,\psi) < \infty\right\}. \tag{A.12}$$

Lemma A.2. *We have*

$$\begin{aligned}\mathcal{D}_V &= \left\{\psi = \mathcal{G}_V^{1/2}\phi, \ \phi \in L^2(V)\right\}, \\ \mathcal{E}_V(\psi,\psi) &= (\phi,\phi)_V, \quad \phi = \mathcal{G}_V^{-1/2}\psi.\end{aligned} \tag{A.13}$$

Moreover, for any $\psi \in \mathcal{D}_V$ and any dense subset $\mathcal{S}_V$ of $L^2(V)$,

$$\begin{aligned}\mathcal{E}_V(\psi,\psi) &= \sup_{f\in\mathcal{S}_V}\left\{2(f,\psi)_V - (f,\mathcal{G}_V f)_V\right\} \\ &= \sup_{f\in\mathcal{S}_V}\left\{\frac{(f,\psi)_V^2}{(f,\mathcal{G}_V f)_V}\right\}.\end{aligned} \tag{A.14}$$

Finally, the function $\phi \to \mathcal{E}_V(\phi,\phi)$ is lower semi-continuous and has compact level sets in $L^2(V)$.

Proof. First observe that (A.13) is an easy consequence of the spectral representations (A.11) and (A.10). Using again the spectral theorem, the first line in (A.14) follows from non-negativity of $\mathcal{G}_V$ and the fact that the function $x(2a-\lambda x)$, for fixed $a \in \mathbb{R}$ and $\lambda > 0$, is maximized at $x = a/\lambda$. Moreover, by the continuity of $\mathcal{G}_V$, the choice of the dense set $\mathcal{S}_V$ is arbitrary. The second variational formula in (A.14) follows by changing f into βf in the first line of (A.14) and optimizing over $\beta \in \mathbb{R}$. The lower semi-continuity of $\mathcal{E}_V$ is a straightforward consequence of (A.14). In particular, the level sets

$$\Phi_L = \left\{\mathcal{G}_V^{1/2}\varphi, \ \varphi \in L^2(V), \ (\varphi,\varphi)_V^2 \leqslant L\right\},$$

are closed. But since $\mathcal{G}_V^{1/2}$ is compact, Φ_L is also relatively compact in $L^2(V)$, and thus compact. $\square$

The following theorem shows the deep connection between the problem of *balayage* (or sweeping out) on V and the embedded Dirichlet form $\mathcal{E}_V$. In particular, it shows that $\mathcal{E}_V$ coincides with the so-called *balayaged Dirichlet form* introduced in [19]. Also, it will turn out to be essential to prove convergence of capacities as discussed in the next section. We introduce the hitting distribution kernel

$$H_V(x,E) = \tilde{\mathbb{P}}_x\left(\omega(\sigma_V) \in E\right), \tag{A.15}$$

where $\tilde{\mathbb{P}}_x$ denotes the distribution of the symmetric α-stable process on the space of right continuous paths with left limits $\omega : [0,\infty) \to \mathbb{R}^d$, such that $\omega(0) = x$, E is any measurable set and σ_V denotes the hitting time of V

$$\sigma_V = \inf\left\{t \geqslant 0, \ \omega(t) \in V\right\}. \tag{A.16}$$

Then H_V defines a probability kernel on $\mathbb{R}^d$ and, for a measurable f we write

$$H_V f(x) = \int_{\mathbb{R}^d} f(y) H_V(x, \mathrm{d}y). \tag{A.17}$$

Then, for any $f \in \mathcal{D}_{\mathcal{E}}$, $H_V f$ is a function which agrees with f on V and is $\alpha-$harmonic outside of V, in the sense that (see Theorem 4.3.2 in [14])

$$\mathcal{E}(H_\Gamma f, g) = 0\,, \quad f, g \in \mathcal{D}_{\mathcal{E}}\,,\ g = 0 \text{ q.e. on } \Gamma\,,$$

where *q.e.* stands for "quasi everywhere" (i.e. everywhere apart from sets of zero capacity).

We shall use the following notation. Let $\mathcal{M}$ denote the set of (signed) Radon measures on $\mathbb{R}^d$, and U_α^μ the *potential* of $\mu \in \mathcal{M}$, i.e.

$$U_\alpha^\mu(x) = \int_{\mathbb{R}^d} g_\alpha(x-y)\mu(\mathrm{d}y)\,.$$

To $\mu \in \mathcal{M}$ we associate the quadratic functional (the energy of μ)

$$E_\alpha(\mu,\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g_\alpha(x-y)\mu(\mathrm{d}x)\mu(\mathrm{d}y)\,. \tag{A.18}$$

Measures of finite energy are denoted

$$\mathcal{M}_0 = \{\mu \in \mathcal{M}\ :\quad E_\alpha(|\mu|,|\mu|) < \infty\}\ ,$$

where $|\mu| = \mu^+ + \mu^-$ is the total variation of $\mu = \mu^+ - \mu^-$. For $\mu,\nu \in \mathcal{M}_0$, we use (A.18) to define $E_\alpha(\mu,\nu)$, by polarization. If $\mu \in \mathcal{M}_0$, then by [14], Theorem 2.2.2, $U_\alpha^\mu \in \mathcal{D}_{\mathcal{E}}$ and

$$\mathcal{E}(U_\alpha^\mu, U_\alpha^\mu) = E_\alpha(\mu,\mu)\,. \tag{A.19}$$

Theorem A.3. *For any $f \in \mathcal{D}_V$, we have*

$$\begin{aligned}\mathcal{E}_V(f,f) &= \mathcal{E}(H_V f, H_V f)\\ &= \inf\{\mathcal{E}(\psi,\psi),\ \ \psi \in \mathcal{D}_{\mathcal{E}}\,,\ \ \psi = f \text{ a.e. on } V\}\end{aligned} \tag{A.20}$$

Proof. Let us define

$$\overline{\mathcal{E}}_V(f) = \inf\{\mathcal{E}(\psi,\psi),\ \ \psi \in \mathcal{D}_{\mathcal{E}}\,,\ \ \psi = f \text{ a.e. on } V\}\ .$$

Let $\psi \in \mathcal{D}_{\mathcal{E}}$ such that $\psi = g_{\alpha/2} * \varphi$, with $\varphi \in L^2(\mathbb{R}^d)$ as in (A.7). Then, for any $h \in L^2(\mathbb{R}^d)$

$$\mathcal{E}(\psi,\psi) = (\varphi,\varphi) \geqslant 2(\varphi,h) - (h,h)\,.$$

By choosing $h = g_{\alpha/2} * \phi$, $\phi \in L^2(\mathbb{R}^d)$, since $g_{\alpha/2} * g_{\alpha/2} = g_\alpha$, we obtain

$$\mathcal{E}(\psi,\psi) \geqslant \sup_\phi\{2(\psi,\phi) - (\phi, g_\alpha * \phi)\}\ , \tag{A.21}$$

where the supremum is taken over $\phi \in L^2(\mathbb{R}^d)$ such that $g_{\alpha/2} * \phi$ is in $L^2(\mathbb{R}^d)$. Restricting (A.21) to ϕ's vanishing out of V and using (A.14), we see that

$$\mathcal{E}(\psi,\psi) \geqslant \mathcal{E}_V(f,f)\,,\quad \psi = f \text{ a.e. on } V, \tag{A.22}$$

and therefore $\overline{\mathcal{E}}_V(f) \geqslant \mathcal{E}_V(f,f)$.

Since $H_V f = f$ q.e. (and therefore a.e.), we always have

$$\overline{\mathcal{E}}_V(f) \leqslant \mathcal{E}(H_V f, H_V f)\,.$$

To finish the proof of the theorem it is then sufficient to show

$$\mathcal{E}(H_V f, H_V f) = \mathcal{E}_V(f,f)\,. \tag{A.23}$$

Let us first prove (A.23) when f is a potential, i.e. assume there exists $\mu \in \mathcal{M}_0$ such that $f = U_\alpha^\mu$ a.e. on V (this is the case if e.g. $f \in \mathcal{C}^{d+2}(V)$ by Lemma 1.1 in [15]). In this case, by Theorem 4.3.2 in [14], $H_V f$ is known to be the *reduced function* of f on V, i.e. the potential $H_V f = U_\alpha^\nu$, for some $\nu \in \mathcal{M}_0$ with $\mathrm{Supp}[\nu] \subset V$[4]. Then, by (A.19)

$$\mathcal{E}(H_V f, H_V f) = \mathcal{E}(U_\alpha^\nu, U_\alpha^\nu) = E_\alpha(\nu,\nu)\,. \tag{A.24}$$

Observe now that for any $\lambda \in \mathcal{M}_0$, $E_\alpha(\nu-\lambda,\nu-\lambda) \geqslant 0$, and therefore

$$E_\alpha(\nu,\nu) = \sup\{2E_\alpha(\nu,\lambda) - E_\alpha(\lambda,\lambda)\,,\ \ \lambda \in \mathcal{M}_0,\ \mathrm{Supp}[\lambda] \subset V\}\,.$$

Any $\lambda \in \mathcal{M}_0$ supported on V can be approximated by absolutely continuous measures $\lambda_n(\mathrm{d}x) = \phi_n(x)\,\mathrm{d}x$, $\phi_n \in \mathcal{C}(V)$, (cf. [15], Lemma 1.2), so that

$$\begin{aligned} 2E_\alpha(\nu,\lambda_n) - E_\alpha(\lambda_n,\lambda_n) &= 2(U_\alpha^\nu,\phi_n)_V - (\phi_n,\mathcal{G}_V\phi_n)_V \\ &\to 2E_\alpha(\nu,\lambda) - E_\alpha(\lambda,\lambda)\,,\quad n\to\infty\,. \end{aligned}$$

Thus, using $U_\alpha^\nu = f$ a.e. on V and the variational principle (A.14), we see that

$$E_\alpha(\nu,\nu) = \sup_{\phi\in\mathcal{C}(V)} \{2(f,\phi)_V - (\phi,\mathcal{G}_V\phi)_V\} = \mathcal{E}_V(f,f)\,. \tag{A.25}$$

This, together with (A.24), proves the claim when f is a potential. In order to treat the general case, we use the results on spectral synthesis described in [14]. Namely, by Theorem 2.3.2 of [14], $H_V f$ can be approximated by a sequence of potentials $U_\alpha^{\nu_n}$ such that $\mathrm{Supp}[\nu_n] \subset V$ and $U_\alpha^{\nu_n}$ converges to $H_V f$ in the Hilbert space $(\mathcal{D}_\mathcal{E}, \mathcal{E}(\cdot,\cdot))$. Then, by (A.22) we see that the functions $f_n = U_\alpha^{\nu_n} 1_V$ are $\mathcal{E}_V$-converging to f and by (A.25) we have

$$\begin{aligned} \mathcal{E}(H_V f, H_V f) &= \lim_{n\to\infty} \mathcal{E}(U_\alpha^{\nu_n}, U_\alpha^{\nu_n}) \\ &= \lim_{n\to\infty} E_\alpha(\nu_n,\nu_n) = \lim_{n\to\infty} \mathcal{E}_V(f_n,f_n) = \mathcal{E}_V(f,f)\,. \end{aligned} \tag{A.26}$$

This concludes the proof of (A.23). □

A.3. **Killed process.** Let $\mathcal{D}([0,\infty),\mathbb{R}^d)$ be the space of right continuous paths with left limits, equipped with the Skorokhod topology and the corresponding Borel σ-algebra of measurable sets. If $\omega(\cdot) \in \mathcal{D}([0,\infty),\mathbb{R}^d)$ denotes the coordinate process generated by the symmetric α-stable density (A.1), we define the killed process by

$$\omega^V(t) = \begin{cases} \omega(t) & \text{if } t < \tau_V \\ \partial & \text{if } t \geqslant \tau_V \end{cases}, \tag{A.27}$$

where ∂ is an extra point (the cemetery) and τ_V is the exit time

$$\tau_V = \inf\{t \geqslant 0,\ \omega(t) \notin V\}\,.$$

The process ω^V is called the symmetric α-stable process killed upon exiting V. Its Dirichlet form is given by (cf. e.g. [20])

$$\mathcal{E}_V^0(\psi,\psi) = \mathcal{E}(\tilde\psi,\tilde\psi)\,,\quad \psi \in L^2(V)\,, \tag{A.28}$$

where $\tilde\psi \in \mathcal{D}_\mathcal{E}$ agrees with ψ on V and vanishes on V^c, with domain

$$\mathcal{D}_V^0 = \{\psi \in \mathcal{D}_\mathcal{E} :\ \ \psi = 0 \text{ a.e. on } V^c\}\,. \tag{A.29}$$

The corresponding semigroup is defined by

$$P_t^V\psi(x) = \tilde{\mathbb{E}}_x\left[\psi(\omega(t))\,;\ t < \tau_V\right]\,,\quad \psi \in \mathcal{C}_b(V)\,,\ t \geqslant 0\,, \tag{A.30}$$

[4]The measure ν is called the *balayage* of μ on V. We refer to chapter 4 of [15] for an alternative construction.

where $\tilde{\mathbb{E}}_x$ denotes expectation w.r.t. the symmetric α-stable process in $\mathbb{R}^d$ started in $x \in V$. P_t^V can be extended to $L^2(V)$ and, following Theorem 2.5 in [9], one shows that it is a compact operator with eigenvalues $e^{-\mu_k t}$, $k = 1, 2, \ldots$ and $0 < \mu_1 < \mu_2 < \ldots$. In particular, the Green operator

$$\mathcal{G}_V^0 = \int_0^\infty P_t^V \,\mathrm{d}t \tag{A.31}$$

is a bounded, compact operator in $L^2(V)$, with eigenvalues $\mu_1^{-1} > \mu_2^{-1} > \ldots \geqslant 0$. We may then repeat Lemma A.2 by replacing $\mathcal{G}_V$ with $\mathcal{G}_V^0$. In particular, the Dirichlet form $\mathcal{E}_V^0$ has compact level sets in $L^2(V)$.

We close this section with a computation of $\mathcal{E}(f,f)$ when f is the indicator function of a euclidean ball.

Lemma A.4. *Let $f = 1_{B_\delta}$, $\delta > 0$, $B_\delta = \{x \in \mathbb{R}^d : \ |x| < \delta\}$. Then,*

$$\mathcal{E}(f,f) < \infty\,, \quad \text{if } \alpha \in (0,1)\,,\ d \geqslant 1\,, \tag{A.32}$$

while

$$\mathcal{E}(f,f) = \infty\,, \quad \text{if } \alpha \in [1,2)\,,\ d \geqslant 2\,,\ \text{or } \alpha = 2\,,\ d \geqslant 3\,. \tag{A.33}$$

Proof. We first consider the case $\alpha \in (0,1)$, $d = 1$. Then the Fourier transform of f is given by

$$\hat{f}(\xi) = \int_{-\delta}^{\delta} \cos x\xi \,\mathrm{d}x = 2\,\frac{\sin \delta\xi}{\xi}\,, \quad \xi \in \mathbb{R}\,.$$

From (A.6) we see that $\mathcal{E}(f,f) < \infty$, and therefore (A.32) follows for $d = 1$. Let now $d \geqslant 2$ and $\alpha \in (0,2)$. Then, for any $\xi \in \mathbb{R}^d$, using the Bessel function (B.21), we have

$$\begin{aligned}
\hat{f}(\xi) &= \int_{B_\delta} \mathrm{e}^{i\xi\cdot x}\,\mathrm{d}x \\
&= \int_0^\delta \rho^{d-1} \frac{1}{(\rho|\xi|)^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(\rho|\xi|)\,\mathrm{d}\rho \\
&= |\xi|^{-d} \int_0^{\delta|\xi|} s^{d/2} J_{\frac{d-2}{2}}(s)\,\mathrm{d}s\,.
\end{aligned}$$

From the asymptotic expansion for Bessel functions (cf. e.g. [12], chapter 15) one has, for large t

$$\int_0^t s^{d/2} J_{\frac{d-2}{2}}(s)\,\mathrm{d}s = \sqrt{\frac{2}{\pi}}\, t^{\frac{d-1}{2}} \cos\left(t - (d+1)\pi/4\right) + O(t^{\max(\frac{d-3}{2},0)})\,. \tag{A.34}$$

For large $|\xi|$ we then have

$$|\xi|^{d+1} |\hat{f}(\xi)|^2 = \frac{2\delta^{d-1}}{\pi} \cos^2\left(\delta|\xi| - (d+1)\pi/4\right) \,+\, o(1)\,. \tag{A.35}$$

Then both claims follow from (A.35) and (A.6). $\square$

Appendix B. The stable RW

For each $\alpha \in (0, 2 \wedge d)$, we consider the jump process on $\mathbb{Z}^d$ defined by a symmetric, homogeneous stochastic matrix p_α,

$$p_\alpha(x,y) = p_\alpha(0, y-x) = p_\alpha(y,x)\,, \quad x,y \in \mathbb{Z}^d\,,$$
$$\sum_{x\in\mathbb{Z}^d} p_\alpha(0,x) = 1\,, \tag{B.1}$$

satisfying the following assumptions.

Assumption H1.

- there exists a function $\rho_\alpha : [0,\infty) \to [0,1]$ such that
$$p_\alpha(0,x) = \rho_\alpha(|x|) \quad x \in \mathbb{Z}^d\,. \tag{B.2}$$
- there exists a constant $v_{\alpha,d} \in (0,\infty)$ such that
$$\lim_{t\to\infty} t^{d+\alpha}\rho_\alpha(t) = v_{\alpha,d}\,. \tag{B.3}$$

Note that if p_α satisfies H1 then it is automatically irreducible. Let $\hat p_\alpha$ be the characteristic function of p_α, i.e.

$$\hat p_\alpha(\theta) = \sum_{x\in\mathbb{Z}^d} p_\alpha(0,x)\,\mathrm{e}^{i\,x\cdot\theta}\,, \quad \theta \in \mathbb{T}^d\,, \tag{B.4}$$

where $\mathbb{T}^d$ stands for $(-\pi,\pi]^d$. The Green function associated to p_α is given by

$$G(x,y) = \sum_{n=0}^\infty p_\alpha^n(x,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \frac{\mathrm{e}^{i\,(y-x)\cdot\theta}}{1-\hat p_\alpha(\theta)}\,\mathrm{d}\theta\,. \tag{B.5}$$

Lemma B.1. *Let $d \geqslant 1$, $\alpha \in (0, 2\wedge d)$. Assume p_α satisfies H1. Then the random walk p_α is transient and (B.5) is well defined. Moreover, letting $\gamma_{\alpha,d} = v_{\alpha,d}/c_{\alpha,d}$, cf. (A.2) and (B.3), we have*

$$\lim_{|\theta|\to 0} |\theta|^{-\alpha}(1-\hat p_\alpha(\theta)) = \gamma_{\alpha,d}\,. \tag{B.6}$$

Proof. The transience follows from integrability of $(1-\hat p_\alpha)^{-1}$, cf. [23], which in turn is a consequence of (B.6) since $\alpha < d$. To prove (B.6) we follow Example 2 in section 8 of [23], in which the claim was proven for $d = 1$. Let $\varepsilon > 0$ and set $\theta = \varepsilon\hat\theta$, with $|\hat\theta| = 1$. We write

$$|\theta|^{-\alpha}(1-\hat p_\alpha(\theta)) = \varepsilon^d \sum_{x\in\mathbb{Z}^d} |x|^{d+\alpha} p_\alpha(0,x)\frac{1-\cos\varepsilon x\cdot\hat\theta}{|\varepsilon x|^{d+\alpha}}\,.$$

From Riemann integration, (B.3) and (A.2), we see that the above expression converges, as $\varepsilon \to 0$, to

$$v_{\alpha,d}\int_{\mathbb{R}^d}(1-\cos\hat\theta\cdot x)\,|x|^{-d-\alpha}\,\mathrm{d}x = v_{\alpha,d}/c_{\alpha,d}\,. \quad \square$$

Remark. Let $\xi_1, \xi_2, \dots$ be the random walk generated by p_α, and denote $\phi_{\alpha,n}$ the characteristic function of the $\mathbb{R}^d$-valued random variable $\bar\xi_n = n^{-1/\alpha}\xi_n$. By (B.6) we have

$$\phi_{\alpha,n}(\theta) = \hat p_\alpha(n^{-1/\alpha}\theta)^n \to \exp\left(-\gamma_{\alpha,d}|\theta|^\alpha\right), \quad n\to\infty\,, \tag{B.7}$$

which implies the invariance principle

$$\bar\xi_n \to^{(d)} U_\alpha\,, \quad n\to\infty\,, \tag{B.8}$$

where (d) stands for convergence in distribution and U_α is an $\mathbb{R}^d$-valued random variable distributed according to the symmetric α-stable density $q_\alpha(t = \gamma_{\alpha,d}, x)$, cf. (A.1). In this sense we say that the random walk p_α is in the *domain of attraction* of the symmetric α-stable law[5].

Convention. From now on we shall also assume

$$\gamma_{\alpha,d} = 1\,. \tag{B.9}$$

This is done merely for notational convenience. Indeed (B.9) amounts to fix $v_{\alpha,d} = c_{\alpha,d}$ in H1 or, equivalently, it corresponds to a different tuning of the time scale for the limiting stable process. It is immediate to see that the constant $\gamma_{\alpha,d}$ enters in the rate functions $\mathcal{C}_\alpha, \mathcal{C}_\alpha^0$ and $\mathcal{I}_\alpha, \mathcal{I}_\alpha^0$ only as a prefactor in front of the Dirichlet form $\mathcal{E}$. We also need:

Assumption H2.

- Let $g_\alpha(z) = \omega_{\alpha,d}|z|^{\alpha-d}$, $z \in \mathbb{R}^d$, be the Riesz kernel defined in (A.3). Then, for any $\eta \in \mathbb{R}^d$ with $|\eta| = 1$, we have

$$\lim_{|x|\to\infty} |x|^{d-\alpha} G(0,x) = g_\alpha(\eta)\,. \tag{B.10}$$

Remark. It might appear at first sight that the result (B.10) simply follows from (B.6), since a change of variables in (B.5) yields

$$G(0,x) = |x|^{\alpha-d}\,\frac{1}{(2\pi)^d}\int_{|x|\mathbb{T}^d} \frac{\mathrm{e}^{i\,\xi\cdot x/|x|}}{(1-\hat{p}_\alpha(\xi/|x|))|x|^\alpha}\,\mathrm{d}\xi\,. \tag{B.11}$$

But the passage to the limit under the integral is not obvious. For symmetric irreducible random walks in dimension $d = 3$ (B.10) is known to hold (with $\alpha = 2$) as soon as the variance is finite, see Proposition 26.1 in [23]. In our setting we shall rely on the analogous results of [26] for stable random walks. The interested reader is also referred to section 5 of [18] where (B.10) is proven for a general class of stable random walks in the regime $\alpha > (d-1)/2$. In the next subsection we discuss several examples for which (B.10) can be shown to hold.

The case $\alpha = 2$. The case $\alpha = 2$, $d \geqslant 3$, is often not treated explicitly in this work, but the techniques we use certainly apply to it. Namely, one defines the stable random walk p_2 assuming that (B.2) is satisfied and replacing (B.3) by the finite variance condition

$$\sum_{x\in\mathbb{Z}^d} |x|^2 p_2(0,x) < \infty\,. \tag{B.12}$$

Provided H2 is satisfied[6], all results contained in this paper hold. The finite range case (a special case of (B.12)) was studied in [4].

[5] One can also prove the convergence of the random walk to the symmetric α-stable process on the space of right continuous paths with left limits, when the latter is given the Skorokhod topology, see [21]. We shall not need this stronger result in the following.

[6] As we mentioned above, (B.12) implies H2 (with $\alpha = 2$ and g_2 the Newtonian potential) when $d = 3$. It has been shown that for $d = 4$, $\sum_{x\in\mathbb{Z}^d} |x|^2 \log|x|\, p_2(0,x) < \infty$ is sufficient to prove H2, see [16]. On the other hand for $d \geqslant 5$ H2 is known to follow from $\sum_{x\in\mathbb{Z}^d} |x|^{d-1} p_2(0,x) < \infty$, see [17].

B.1. **Examples.** We shall be working under assumptions H1 and H2 for the rest of the paper. In order to check assumption H2 in specific cases, we recall a basic result of [26]. Given a function $f : \mathbb{Z}^d \to \mathbb{R}$, denote $\nabla_i f$, $i = 1, \dots, d$, its discrete gradient,

$$\nabla_i f(x) = f(x + e_i) - f(x)\,,$$

where e_i is the unit vector in the positive i-th direction. From Corollary 3-A and 3-B of [26] we obtain:

Lemma B.2. *Let $d \geqslant 1$, $\alpha \in (0, 2 \wedge d)$. Assume p_α satisfies H1 and assume there exists a constant $c \in (0,\infty)$ such that for any $x \in \mathbb{Z}^d_+$ with $|x| > c$, we have*

$$(-1)^d \nabla_1 \cdots \nabla_d\, p_\alpha(0, x) \geqslant 0\,. \tag{B.13}$$

Then (B.10) holds.

Note that in dimension $d = 1$ the condition (B.13) simply means that $p_\alpha(0, x)$ is monotone decaying after a certain distance. Let us discuss some more examples:

1. Let $d \geqslant 1$, $\alpha \in (0, 2 \wedge d)$, and define

$$p_\alpha(0, x) = b(a + |x|^2)^{-(d+\alpha)/2}\,, \quad x \in \mathbb{Z}^d\,, \tag{B.14}$$

with positive constants a, b compatible with (B.1). Assumption H1 is obviously satisfied. In order to check condition (B.13) let us consider $p_\alpha(0, x)$ in (B.14) as a function defined for any $x \in \mathbb{R}^d$. Let $r = |x|$ and $p_\alpha(r) = p_\alpha(0, x)$. Then

$$\frac{\partial^d}{\partial x_1 \cdots \partial x_d}\, p_\alpha(0, x) = x_1 \cdots x_d\, D_r^d\, p_\alpha(r) \tag{B.15}$$

where we have introduced the differential operator $D_r = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}$. It is easily seen that

$$D_r^d\, p_\alpha(r) = (-1)^d 2^{-d}\, b(a + r^2)^{-(3d+\alpha)/2} \prod_{k=0}^{d-1} (d + \alpha + 2k)\,.$$

Restricting to $\mathbb{R}^d_+$ and integrating along each coordinate x_i, $i = 1, \dots, d$ on an interval of length 1 we obtain (B.13). By lemma B.2 this implies p_α satisfies H2.

2. Example 1 can be generalized by letting b depend (smoothly) on $r = |x|$ in (B.14). Let $b(r) \geqslant 0$, $b(r) \to b > 0$ as $r \to \infty$ and assume

$$\frac{\mathrm{d}^k}{\mathrm{d}r^k}\, b(r) \to 0\,, \quad r \to \infty\,, \quad k = 1, \dots, d\,. \tag{B.16}$$

To see that (B.16) is sufficient for p_α to satisfy H2, we observe that

$$D_r^d\, p_\alpha(r) = \sum_{k=0}^{d} \binom{d}{k} \left(D_r^{d-k} (a + r^2)^{-(d+\alpha)/2} \right) \left(D_r^k\, b(r) \right)\,. \tag{B.17}$$

Now from (B.16) we see that $r^k D_r^k\, b(r) \to 0$ when $r \to \infty$ for any $k = 1, \dots, d$. Therefore the first term ($k = 0$) in (B.17) dominates when r is large. As before, this implies that for sufficiently large $|x|$ (B.13) is satisfied, and H2 follows from Lemma B.2.

3. Let $d \geqslant 1$, $\alpha \in (0, 2 \wedge d)$, and consider the discretized α-stable process

$$p_\alpha(0, x) = Z^{-1} q_\alpha(1, x)\,. \tag{B.18}$$

Here $q_\alpha(1,x)$ is the symmetric α-stable density at time $t=1$, cf. (A.1), and

$$Z = \sum_{x\in\mathbb{Z}^d} q_\alpha(1,x)\,, \quad \hat{p}_\alpha(\theta) = Z^{-1}\sum_{y\in\mathbb{Z}^d} \exp\left(-|\theta+2\pi y|^\alpha\right).$$

In this case H1 follows from (A.2) with $\gamma_{\alpha,d} = Z^{-1}$. In the case $d=1$, H2 follows from Lemma B.2 and the monotonicity of q_α, or directly from the method of [18]. For $d\geqslant 2$, H2 holds as a consequence of Lemma B.2 and the following identity for stable densities.

Lemma B.3. *Let $d\geqslant 2$ and $\alpha\in(0,2)$. Let $q_\alpha^{(d)}$ be the isotropic stable density (A.1) in dimension d. Then, for any $x\in\mathbb{R}^d$,*

$$\frac{\partial^d}{\partial x_1\cdots\partial x_d}\, q_\alpha^{(d)}(1,x) = (-1)^d\, x_1\cdots x_d\,(2\pi)^d\, q_\alpha^{(3d)}(1,x)\,. \tag{B.19}$$

Proof. Let $r=|x|$ and set $q_\alpha^{(d)}(r) = q_\alpha^{(d)}(1,x)$. From (B.15) it is clear that the claim (B.19) will follow from the identity

$$D_r\, q_\alpha^{(d)}(r) = -(2\pi) q_\alpha^{(d+2)}(r)\,. \tag{B.20}$$

To prove (B.20) we write

$$q_\alpha^{(d)}(r) = (2\pi)^{-d/2}\int_0^\infty \rho^{d-1}\mathrm{e}^{-\rho^\alpha}\,\frac{1}{(r\rho)^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(r\rho)\,\mathrm{d}\rho$$

where we have introduced the usual Bessel function of first kind

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}}\mathrm{e}^{itz}\,\mathrm{d}t\,. \tag{B.21}$$

Using the known identity (see e.g. [1], 9.1.30)

$$D_z z^{-\nu} J_\nu(z) = -z^{-(\nu+1)} J_{\nu+1}(z)\,,$$

we conclude

$$\begin{aligned} D_r\, q_\alpha^{(d)}(r) &= (2\pi)^{-d/2}\int_0^\infty \rho^{d+1}\mathrm{e}^{-\rho^\alpha} D_{r\rho}\,\frac{1}{(r\rho)^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(r\rho)\;\mathrm{d}\rho \\ &= -2\pi\, q_\alpha^{(d+2)}(r)\,. \end{aligned}$$

□

Remark. In the above lemma the function $\mathrm{e}^{-\rho^\alpha}$ plays no essential role. In particular, this means that any probability density q which is given by the Fourier transform of a (sufficiently rapidly decaying) radial function will satisfy (B.19). Now, if this density also satisfies $q(x)\sim |x|^{-d-\alpha}$ then we may set $p_\alpha(0,x) = \text{const.}\times q(x)$, and both H1 and H2 are satisfied.

B.2. **Limit theorems.** In the rest of this section we exploit the convergence (B.10) to prove that certain quadratic forms associated to the lattice random walk converge, after rescaling, to the forms of the corresponding stable process in $\mathbb{R}^d$ (see previous section). The following convergence results were first obtained in [4], for the local case. The generalization of Lemma B.4 to our setting is straightforward. On the other hand, the capacity-order result of Lemma B.5 appears to be non trivial, since Theorem A.3 is essential.

We denote G_N the matrix

$$G_N = \{G(x,y)\}_{x,y\in\Lambda_N} \tag{B.22}$$

We also use the notation $G^{-1}(x,y)=\delta(x,y)-p_\alpha(x,y)$. Recall that $\Gamma_N^{-1}=G^{-1}1\!\!1_N$, i.e. the Green function of the random walk killed upon exiting Λ_N coincides with the restriction of G^{-1} to Λ_N. For any subset Λ of $\mathbb{Z}^d$ we let $\langle\cdot,\cdot\rangle_\Lambda$ denote the scalar product in $l^2(\Lambda)$. We refer to (A.6) and (A.9) for the definition of the Dirichlet form $\mathcal{E}(\cdot,\cdot)$ and the integral operator $\mathcal{G}_V$ respectively.

Lemma B.4. *Let f be a Riemann integrable function on V and let $h\in\mathcal{C}^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. Define $f_N(x)=f(x/N)$, for $x\in\Lambda_N$, and $h_N(x)=h(x/N)$, for $x\in\mathbb{Z}^d$, then*

$$\lim_{N\to\infty}N^{-d-\alpha}\langle f_N,G_Nf_N\rangle_{\Lambda_N}=(f,\mathcal{G}_Vf)_V\,,$$
$$\lim_{N\to\infty}N^{-d+\alpha}\big\langle h_N,G^{-1}h_N\big\rangle_{\mathbb{Z}^d}=\mathcal{E}(h,h)\,.$$

In particular, for any $h\in L^2(\mathbb{R}^d)$, such that $h1_{V^c}=0$ a.e., and $h1_V\in\mathcal{C}^1(V)$, we have

$$\lim_{N\to\infty}N^{-d+\alpha}\big\langle h_N,\Gamma_N^{-1}h_N\big\rangle_{\Lambda_N}=\mathcal{E}(h,h)\,.$$

Proof. From Riemann integration and (B.10)

$$\begin{aligned}
&\lim_{N\to\infty}N^{-d-\alpha}\langle f_N,G_Nf_N\rangle_{\Lambda_N}\\
&\quad=\lim_{M\to\infty}\lim_{N\to\infty}N^{-2d}\sum_{\substack{x,y\in\Lambda_N:\\|x-y|>M}}f(x/N)N^{d-\alpha}G(x-y)f(y/N)\\
&\quad=\lim_{N\to\infty}N^{-2d}\sum_{x,y\in\Lambda_N,x\neq y}f(x/N)g_\alpha(x/N-y/N)f(y/N)\\
&\quad=(f,\mathcal{G}_Vf)_V\,.
\end{aligned}$$

To prove the second identity we write

$$\begin{aligned}
&\lim_{N\to\infty}N^{-d+\alpha}\big\langle h_N,G^{-1}h_N\big\rangle_{\mathbb{Z}^d}\\
&\quad=\frac12\lim_{M\to\infty}\lim_{N\to\infty}N^{-2d}\sum_{\substack{x,y\in\mathbb{Z}^d:\\|x-y|>M}}N^{d+\alpha}p_\alpha(0,x-y)\big(h(x/N)-h(y/N)\big)^2\\
&\quad=\frac12\,c_{\alpha,d}\lim_{N\to\infty}N^{-2d}\sum_{x,y\in\mathbb{Z}^d,x\neq y}|x-y|^{-d-\alpha}\big(h(x/N)-h(y/N)\big)^2\\
&\quad=\mathcal{E}(h,h)\,,
\end{aligned}$$

where we have used (B.3) with $v_{\alpha,d}=c_{\alpha,d}$, cf. (B.9), and the expression (A.4) for $\mathcal{E}(\cdot,\cdot)$.

Finally, the last assertion is an immediate consequence of the previous one, since if $h1_{V^c}=0$ a.e. we have

$$\big\langle h_N,\Gamma_N^{-1}h_N\big\rangle_{\Lambda_N}=\big\langle h_N,G^{-1}h_N\big\rangle_{\mathbb{Z}^d}$$

□

Lemma B.5. *Let $f\in\mathcal{C}^1(V)$ and define $f_N(\cdot)=f(\cdot/N)$ on Λ_N. Then*

$$\lim_{N\to\infty}N^{-d+\alpha}\big\langle f_N,G_N^{-1}f_N\big\rangle_{\Lambda_N}=\mathcal{E}_V(f,f)\tag{B.23}$$

Proof. Recall that

$$\big\langle f_N,G_N^{-1}f_N\big\rangle_{\Lambda_N}=\sup\left\{2\langle f_N,\phi_N\rangle_{\Lambda_N}-\langle\phi_N,G_N\phi_N\rangle_{\Lambda_N}\,,\phi_N\in\mathbb{R}^{\Lambda_N}\right\}$$

Therefore

$$N^{-d+\alpha}\left\langle f_N, G_N^{-1} f_N\right\rangle_{\Lambda_N} \geqslant 2N^{-d}\left\langle f_N, \phi_N\right\rangle_{\Lambda_N} - N^{-d-\alpha}\left\langle \phi_N, G_N\phi_N\right\rangle_{\Lambda_N},$$

for any $\phi \in \mathcal{C}(V)$, with $\phi_N(\cdot) = \phi(\cdot/N)$. By Lemma B.4 and (A.14), we see that

$$\liminf_{N\to\infty} N^{-d+\alpha}\left\langle f_N, G_N^{-1} f_N\right\rangle_{\Lambda_N} \geqslant \mathcal{E}_V(f,f).$$

On the other hand, for each $h \in \mathcal{C}^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, with $h = f$ on V

$$\begin{aligned}\left\langle f_N, G_N^{-1} f_N\right\rangle_{\Lambda_N} &\leqslant \sup_{\phi\in\ell^2(\mathbb{Z}^d)} \left\{2\left\langle h_N, \phi\right\rangle_{\mathbb{Z}^d} - \left\langle \phi, G\phi\right\rangle_{\mathbb{Z}^d}\right\} \\ &= \left\langle h_N, G^{-1} h_N\right\rangle_{\mathbb{Z}^d}.\end{aligned}$$

Using Lemma B.4 and Theorem A.3, we see that

$$\begin{aligned}&\limsup_{N\to\infty} N^{-d+\alpha}\left\langle f_N, G_N^{-1} f_N\right\rangle_{\Lambda_N} \\ &\qquad\leqslant \inf\left\{\mathcal{E}(h,h),\quad h\in\mathcal{C}^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d),\quad h = f \text{ a.e. on } V\right\} \\ &\qquad= \inf\left\{\mathcal{E}(h,h),\quad h\in\mathcal{D}_{\mathcal{E}},\ h = f \text{ a.e. on } V\right\} \\ &\qquad= \mathcal{E}_V(f,f).\end{aligned}$$

□

Remark. As a consequence of Lemma B.5 we obtain the convergence of *capacity*. Choosing $f = 1_V$, we have

$$\lim_{N\to\infty} N^{-d+\alpha}\mathrm{cap}_N(\Lambda_N) = \mathrm{cap}_\alpha(V) \tag{B.24}$$

where

$$\mathrm{cap}_N(\Lambda_N) = \left\langle 1_N, G_N^{-1} 1_N\right\rangle_{\Lambda_N}$$

is the capacity of the box Λ_N for the random walk (cf. e.g. chapter 25 of [23]), while

$$\mathrm{cap}_\alpha(V) = \mathcal{E}_V(1_V, 1_V)$$

is the capacity of V for the α-stable process, cf. (A.8) and Theorem A.3.

Appendix C. Some Gaussian tools

Let μ be the centered Gaussian law on $\mathbb{R}^n$, $n\in\mathbb{Z}_+$, with positive definite covariance matrix

$$\Gamma = \Gamma\,(i,j)_{i,j=1,\ldots,n},$$

and denote by $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n > 0$ the eigenvalues of Γ.

Lemma C.1. *For any $\beta > 0$ such that $\beta\lambda_1 < 1$, we have*

$$\log\mu\Big(\exp\frac{1}{2}\beta\sum_{i=1}^n \phi_i^2\Big) \leqslant \frac{1}{2}\beta\,\mathrm{Tr}\,\Gamma + \frac{1}{2}\beta^2 W(\beta,\lambda_1)\,\mathrm{Tr}\,\Gamma^2 \tag{C.1}$$

where

$$W(\beta,\lambda_1) = \frac{1}{2} - \log\left(1-\beta\lambda_1\right). \tag{C.2}$$

Proof. Direct computation yields

$$\mu\big(\exp\frac{1}{2}\beta\sum_{i=1}^{n}\phi_i^2\big) = \det\big(1-\beta\Gamma\big)^{-\frac{1}{2}} = \prod_{j=1}^{n}\big(1-\beta\lambda_j\big)^{-\frac{1}{2}}\,.$$

Using

$$\sum_{j=1}^{n}\lambda_j = \operatorname{Tr}\Gamma\,,\qquad \sum_{j=1}^{n}\lambda_j^2 = \operatorname{Tr}\Gamma^2\,,$$

and the definition (C.2) of W, we have

$$\begin{aligned}\log\,\mu\big(\exp\frac{1}{2}\beta\sum_{i=1}^{n}\phi_i^2\big) &= -\frac{1}{2}\sum_{j=1}^{n}\log\big(1-\beta\lambda_j\big) = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{(\beta\lambda_j)^k}{k}\\ &= \frac{1}{2}\beta\operatorname{Tr}\Gamma + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=2}^{\infty}\frac{(\beta\lambda_j)^k}{k}\\ &\leqslant \frac{1}{2}\beta\operatorname{Tr}\Gamma + \frac{1}{2}\beta^2\sum_{k=0}^{\infty}\frac{(\beta\lambda_1)^k}{k+2}\operatorname{Tr}\Gamma^2\\ &\leqslant \frac{1}{2}\beta\operatorname{Tr}\Gamma + \frac{1}{2}\beta^2 W(\beta,\lambda_1)\operatorname{Tr}\Gamma^2\,.\end{aligned}$$

□

C.1. **Conditioning on a sub-grid.** Let p_α be the transition of the stable random walk defined in section 2. Denote $\xi_0 = x, \xi_1, \xi_2, \dots$ the paths with starting point $x\in\mathbb{Z}^d$ and let $\mathbb{P}_x$ and $\mathbb{E}_x$ denote the corresponding probability and expectation respectively. Let $L>0$ and define

$$\tau_L = \inf\{n\geqslant 0\,,\;\; \xi_n\in L\mathbb{Z}^d\} \tag{C.3}$$

Define the transition

$$p_\alpha^L(x,z) = \mathbb{P}_x(\xi_{\tau_L}=z)\,,\quad x\in\mathbb{Z}^d,\; z\in L\mathbb{Z}^d. \tag{C.4}$$

The Green function of the random walk which is killed upon hitting the sub-grid $L\mathbb{Z}^d$ is given by

$$G^L(x,y) = \mathbb{E}_x\left[\sum_{n=0}^{\tau_L-1}1_{\xi_n=y}\right]\,,\quad x,y\in\mathbb{Z}^d. \tag{C.5}$$

We define the ζ-field

$$\zeta_x = P_\alpha(\phi_x|\mathcal{F}_{L\mathbb{Z}^d})\,,\quad x\in\mathbb{Z}^d, \tag{C.6}$$

where $\mathcal{F}_{L\mathbb{Z}^d}$ denotes the σ-algebra generated by the variables ϕ_x, $x\in L\mathbb{Z}^d$. We have the following random walk representation. If $\tau_x = \phi_x - \zeta_x$, for any $L>0$, $x,y\in\mathbb{Z}^d$,

$$\zeta_x = \sum_{z\in L\mathbb{Z}^d} p_\alpha^L(x,z)\phi_z\,,\quad P_\alpha(\tau_x\tau_y) = G^L(x,y)\,. \tag{C.7}$$

Moreover, the fields ζ and τ are independent.

The following has been proven in [5], Proposition A.12.

Lemma C.2. *Assume the random walk p_α satisfies H1. Then*

$$\lim_{L\to\infty} G^L(x,x) = G(x,x) = \sigma^2\,, \quad x \in \mathbb{Z}^d\,. \tag{C.8}$$

Moreover, there exist constants $a, b > 0$ such that for any $L > 0$, any $x, y \in \mathbb{Z}^d$ with $x \neq y$,

$$G^L(x,y) \leqslant aL^b(\log|x-y|)^{d+2+\alpha}|x-y|^{-d-\alpha}\,. \tag{C.9}$$

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Fachbereich Mathematik, Sekr. MA 7-4, TU-Berlin, Strasse des 17. Juni 136, D-10623 Berlin, Germany

E-mail address: caputo@math.tu-berlin.de

Fachbereich Mathematik, Sekr. MA 7-4, TU-Berlin, Strasse des 17. Juni 136, D-10623 Berlin, Germany

E-mail address: deuschel@math.tu-berlin.de