Resummation of Feynman Diagrams and the Inversion of Matrices

by

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Abstract: In many field theoretical models one has to resum two- and four-legged subdiagrams in order to determine their behaviour. In this article we present a novel formalism which does this in a nice way. It is based on the central limit theorem of probability and an inversion formula for matrices which is obtained by repeated application of the Feshbach projection method. We discuss applications to the Anderson model, to the many-electron system and to the φ^4-model. In particular, for the many-electron system with attractive delta-interaction, we find that the expectation value of the Hubbard-Stratonovich field for small momentum q has a delta-function singularity instead of the commonly expected $1/q^2$-type singularity.

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1 Introduction

The computation of correlation functions in field theoretical models is a difficult problem. In this article we present a novel approach which applies to models where a two point function can be written as

$$S(x,y) = \int [P+Q]^{-1}_{x,y} \, d\mu(Q) \,. \tag{1.1}$$

Here P is some operator diagonal in momentum space, typically determined by the unperturbed Hamiltonian, and Q is diagonal in coordinate space. The functional integral is taken with respect to some probability measure $d\mu(Q)$ and goes over the matrix elements of Q. $[\,\cdot\,]^{-1}_{x,y}$ denotes the x,y-entry of the matrix $[P+Q]^{-1}$. Our starting point is always a model in finite volume and with positive lattice spacing in which case the operator $P+Q$ and the functional integral in (1.1) becomes huge- but finite-dimensional. In the end we take the infinite volume limit and, if wanted, the continuum limit.

Our treatment is based on the following identity which is obtained by repeated application of the Feshbach formula (Lemma 3.2 below). It is proven in Theorem 3.3. Let $B = (B_{kp})_{k,p\in\mathcal{M}} \in \mathbb{C}^{N\times N}$, $\mathcal{M}$ some index set, $|\mathcal{M}| = N$ and let

$$G(k) := \left[B^{-1}\right]_{kk} \tag{1.2}$$

Then one has

$$G(k) = \frac{1}{B_{kk} + \sum\limits_{r=2}^{N} (-1)^{r+1} \sum\limits_{\substack{p_2\cdots p_r\in\mathcal{M}\setminus\{k\}\\ p_i\neq p_j}} B_{kp_2} G_k(p_2) B_{p_2p_3} \cdots B_{p_{r-1}p_r} G_{kp_2\cdots p_{r-1}}(p_r) B_{p_rk}} \tag{1.3}$$

where $G_{k_1\cdots k_j}(p) = \left[(B_{st})_{s,t\in\mathcal{M}\setminus\{k_1\cdots k_j\}}\right]^{-1}_{pp}$ is the p,p entry of the inverse of the matrix which is obtained from B by deleting the rows and columns labelled by $k_1,\cdots,k_j$. In Section 2 we apply this formula to a matrix of the form $B =$ self adjoint $+\, i\varepsilon\, Id$, which, for $\varepsilon \neq 0$, has the property that all submatrices $(B_{st})_{s,t\in\mathcal{M}\setminus\{k_1\cdots k_j\}}$ are invertible.

There is also a formula for the off-diagonal inverse matrix elements. It reads

$$\left[B^{-1}\right]_{kp} = -G(k)B_{kp}G_k(p) + \sum_{r=3}^{N} (-1)^{r+1} \sum_{\substack{t_3\cdots t_r\in\mathcal{M}\setminus\{k,p\}\\ t_i\neq t_j}} G(k) B_{kt_3} G_k(t_3) B_{t_3t_4} \cdots B_{t_rp} G_{kt_3\cdots t_r}(p) \tag{1.4}$$

These formulae also hold in the case where the matrix B has a block structure $B_{kp} = (B_{k\sigma,p\tau})$ where, say, $\sigma,\tau \in \{\uparrow,\downarrow\}$ are some spin variables. In that case the B_{kp} are small matrices, the $G_{k_1\cdots k_j}(p)$ are matrices of the same size and the $1/\cdot$ in (1.3) means inversion of matrices, see Theorem 3.3 below.

The paper is organized as follows. In the next section we demonstrate the method by applying it to the averaged Green function of the Anderson model. The Schwinger-Dyson equation for that model reads $G^{-1} = G_0^{-1} + \Sigma(G_0)$ where $\Sigma(G_0)$ is the sum of all two-legged one-particle irreducible diagrams. Application of (1.3) leads to an integral equation $G^{-1} = G_0^{-1} + I(G)$ where $I(G)$ is the sum of all two-legged graphs without two-legged subgraphs. The latter equation has two advantages. First, Σ is the sum of one-particle irreducible diagrams, but these diagrams may very well have two-legged subdiagrams and usually these are the diagrams which produce anomalously large contributions. And second, the propagator for $I(G)$ is the interacting two point function G, which, for the Anderson model, is more regular than the free two point function G_0 which is the propagator for the diagrams contributing to $\Sigma(G_0)$. More precisely, the series for $I(G)$ can be expected to be asymptotic, that is, its lowest order contributions are a good approximation if the coupling is small, but, usually, the series for $\Sigma(G_0)$ is not asymptotic.

For the many-electron system and for the φ^4 model repeated application of (1.3,4) amounts to a resummation of two- and four-legged subgraphs. This is discussed in Section 4. In particular, for the many-electron system with attractive delta-interaction we find, as one expects, the formation of a BCS gap, but we are also able to obtain information on the expectation value of the Hubbard-Stratonovich field $\langle |\phi_q|^2 \rangle$ which is related to a certain four point function (see (4.11,12) and (4.3,4)). This function has been investigated by several authors [FMRT,CFS,B]. One expects a singularity at $q = 0$. The usual argument, based on a Taylor expansion of the effective potential around its minimum predicts a $1/q^2$ behaviour. Our analysis, given in Section 4.1, results in $\langle |\phi_q|^2 \rangle = \frac{|\Delta|^2}{\lambda}\,\delta(q) + regular$ where Δ is the BCS gap and the regular term is bounded for small q. In particular, we find that the existence of a gap and a macroscopic $\langle |\phi_0|^2 \rangle = \beta L^d \frac{|\Delta|^2}{\lambda}$ enforce each other (L^d volume, $1/\beta = T$ temperature). The proof of the inversion formula is given in Section 3.

2 Application to the Anderson Model

Let coordinate space be a lattice of finite volume with periodic boundary conditions, lattice spacing $1/M$ and volume $[0, L]^d$:

$$\Gamma = \left\{ x = \tfrac{1}{M}(n_1, \cdots, n_d) \mid 0 \le n_i \le ML - 1 \right\} = \left(\tfrac{1}{M}\mathbb{Z}\right)^d / (L\mathbb{Z})^d \tag{2.1}$$

Momentum space is given by

$$\mathcal{M} := \Gamma^\sharp = \left\{ k = \tfrac{2\pi}{L}(m_1, \cdots, m_d) \mid 0 \le m_i \le ML - 1 \right\} = \left(\tfrac{2\pi}{L}\mathbb{Z}\right)^d / (2\pi M\mathbb{Z})^d \tag{2.2}$$

We consider the averaged Green function of the Anderson model given by

$$\langle G \rangle(x, x') := \int [-\Delta - z + \lambda V]^{-1}_{x,x'}\, dP(V) \tag{2.3}$$

where the random potential is Gaussian,

$$dP(V) = \prod_{x\in\Gamma} e^{-\frac{V_x^2}{2}} \frac{dV_x}{\sqrt{2\pi}}. \tag{2.4}$$

Here $z = E + i\varepsilon$ and Δ is the discrete Laplacian,

$$[-\Delta - z + \lambda V]_{x,x'} = -M^2 \sum_{i=1}^{d} \left(\delta_{x',x+e_i/M} + \delta_{x',x-e_i/M} - 2\delta_{x',x}\right) - z\,\delta_{x,x'} + \lambda V_x\,\delta_{x,x'} \tag{2.5}$$

By taking the Fourier transform, one has

$$\langle G\rangle(x,x') = \tfrac{1}{M^dL^d} \sum_{k\in\mathcal{M}} e^{ik(x'-x)} \langle G\rangle(k) \tag{2.6}$$

$$\langle G\rangle(k) = \int_{\mathbb{R}^{N^d}} \left[a_k\delta_{k,p} + \tfrac{\lambda}{\sqrt{N^d}} v_{k-p}\right]^{-1}_{k,k} dP(v) \tag{2.7}$$

where $N^d = (ML)^d = |\Gamma| = |\mathcal{M}|$, $dP(v)$ is given by (2.10) or (2.11) below, depending on whether N^d is even or odd, and

$$a_k = 4M^2 \sum_{i=1}^{d} \sin^2\left[\tfrac{k_i}{2M}\right] - E - i\varepsilon \tag{2.8}$$

The rigorous control of $\langle G\rangle(k)$ for small disorder λ and energies inside the spectrum of the unperturbed Laplacian, $E \in [0, 4M^2]$, in which case a_k has a root if $\varepsilon \to 0$, is still an open problem [AG,K,MPR,P,W]. It is expected that $\lim_{\varepsilon\searrow 0} \lim_{L\to\infty} \langle G\rangle(k) = 1/(a_k - \sigma_k)$ where $\mathrm{Im}\sigma = O(\lambda^2)$.

The integration variables v_q in (2.7) are given by the discrete Fourier transform of the V_x. In particular, observe that, if F denotes the unitary matrix of discrete Fourier transform, the variables

$$v_q \equiv (FV)_q = \tfrac{1}{\sqrt{N^d}} \sum_{x\in\Gamma} e^{-iqx} V_x = \left(\tfrac{M}{L}\right)^{\frac{d}{2}} \tfrac{1}{M^d} \sum_{x\in\Gamma} e^{-iqx} V_x \equiv \left(\tfrac{M}{L}\right)^{\frac{d}{2}} \hat{V}_q \tag{2.9}$$

would not have a limit if V_x would be deterministic and cutoffs are removed, since the $\hat{V}_q$ are the quantities which have a limit in that case. But since the V_x are integration variables, we choose a unitary transform to keep the integration measure invariant. Observe also that v_q is complex, $v_q = u_q + iw_q$. Since V_x is real, $u_{-q} = u_q$ and $w_{-q} = -w_q$. In order to transform $dP(V)$ to momentum space, we have to choose a set $\mathcal{M}^+ \subset \mathcal{M}$ such that either $q \in \mathcal{M}^+$ or $-q \in \mathcal{M}^+$. If N is odd, the only momentum with $q = -q$ or $w_q = 0$ is $q = 0$. In that case $dP(V)$ becomes

$$dP(v) = e^{-\frac{u_0^2}{2}} \frac{du_0}{\sqrt{2\pi}} \prod_{q\in\mathcal{M}^+} e^{-(u_q^2+w_q^2)} \frac{du_q\,dw_q}{\pi} \tag{2.10}$$

For even N we get

$$dP(v) = e^{-\frac{1}{2}(u_0^2+u_{q_0}^2)}\frac{du_0du_{q_0}}{2\pi}\prod_{q\in\mathcal{M}^+} e^{-(u_q^2+w_q^2)}\frac{du_q\,dw_q}{\pi} \tag{2.11}$$

where $q_0 = \frac{2\pi m}{L}$ is the unique nonzero momentum for which $\frac{2\pi}{L}m = 2\pi M(1,\cdots,1) - \frac{2\pi}{L}m$.

Now we apply the inversion formula (1.3) to the inverse matrix element in

$$\langle G\rangle(k) = \int_{\mathbb{R}^{N^d}} \left[a_k\delta_{k,p} + \tfrac{\lambda}{\sqrt{N^d}}v_{k-p}\right]^{-1}_{k,k} dP(v) \tag{2.7}$$

We start with the 'two loop approximation', which we define by retaining only the $r=2$ term in the denominator of the right hand side of (1.3),

$$G(k) \approx \frac{1}{B_{kk} - \sum_{p\in\mathcal{M}\setminus\{k\}} B_{kp}G_k(p)B_{pk}} \tag{2.12}$$

Thus, let

$$G(k) := \left[a_k\delta_{k,p} + \tfrac{\lambda}{\sqrt{N^d}}v_{k-p}\right]^{-1}_{k,k} = G(k;v,z) \tag{2.13}$$

In the infinite volume limit the spacing $2\pi/L$ of the momenta becomes arbitrary small. Hence, in computing an inverse matrix element, it should not matter whether a single column and row labelled by some momentum t is absent or not. In other words, in the infinite volume limit one should have

$$G_t(p) = G(p) \quad \text{for} \quad L\to\infty \tag{2.14}$$

and similarly $G_{t_1\cdots t_j}(p) = G(p)$ as long as j is independent of the volume. We remark however that if the matrix has a block structure, say $B = (B_{k\sigma,p\tau})$ with $\sigma,\tau\in\{\uparrow,\downarrow\}$ some spin variables, this structure has to be respected. That is, for a given momentum k all rows and columns labelled by $k\uparrow$, $k\downarrow$ have to be eliminated, since otherwise (2.14) may not be true.

Thus the two loop approximation gives

$$G(k) = \frac{1}{a_k + \frac{\lambda}{\sqrt{N^d}}v_0 - \frac{\lambda^2}{N^d}\sum_{p\neq k} v_{k-p}\,G(p)\,v_{p-k}} \tag{2.15}$$

For large L, we can disregard the $\frac{\lambda}{\sqrt{N^d}}v_0$ term. Introducing $\sigma_k = \sigma_k(v,z)$ according to

$$G(k) =: \frac{1}{a_k - \sigma_k}\ , \tag{2.16}$$

we get

$$\sigma_k = \tfrac{\lambda^2}{N^d}\sum_{p\neq k}\frac{|v_{k-p}|^2}{a_p-\sigma_p} \approx \tfrac{\lambda^2}{N^d}\sum_{p}\frac{|v_{k-p}|^2}{a_p-\sigma_p} \tag{2.17}$$

and arrive at

$$\langle G\rangle(k) = \int \frac{1}{a_k - \frac{\lambda^2}{N^d}\sum_p \frac{|v_{k-p}|^2}{a_p - \frac{\lambda^2}{N^d}\sum_t \frac{|v_{p-t}|^2}{a_t - \frac{\lambda^2}{N^d}\Sigma\cdots}}}\, dP(v) \tag{2.18}$$

Now consider the infinite volume limit $L \to \infty$ or $N = ML \to \infty$. By the central limit theorem of probability $\frac{1}{\sqrt{N^d}}\sum_q (|v_q|^2 - \langle |v_q|^2\rangle)$ is, as a sum of independent random variables, normal distributed. Note that only a prefactor of $1/\sqrt{N^d}$ is required for that property. In particular, if F is some bounded function independent of N, sums which come with a prefactor of $1/N^d$ like $\frac{1}{N^d}\sum_q c_q |v_q|^2$ can be substituted by their expectation value,

$$\lim_{N\to\infty} \int F\Big(\tfrac{1}{N^d}\sum_k c_k |v_k|^2\Big)\, dP(v) = F\Big(\lim_{N\to\infty} \tfrac{1}{N^d}\sum_k c_k \langle |v_k|^2\rangle\Big) \tag{2.19}$$

Therefore, in the two loop approximation, one obtains in the infinite volume limit

$$\langle G\rangle(k) = \frac{1}{a_k - \frac{\lambda^2}{N^d}\sum_p \frac{\langle |v_{k-p}|^2\rangle}{a_p - \frac{\lambda^2}{N^d}\sum_t \frac{\langle |v_{p-t}|^2\rangle}{a_t - \frac{\lambda^2}{N^d}\Sigma\cdots}}} =: \frac{1}{a_k - \langle \sigma_k\rangle} \tag{2.20}$$

where the quantity $\langle \sigma_k\rangle$ satisfies the integral equation

$$\langle \sigma_k\rangle = \tfrac{\lambda^2}{N^d}\sum_p \frac{\langle |v_{k-p}|^2\rangle}{a_p - \langle \sigma_p\rangle} \overset{L\to\infty}{\to} \tfrac{\lambda^2}{M^d}\int_{[0,2\pi M]^d} \frac{d^d p}{(2\pi)^d}\, \frac{\langle |v_{k-p}|^2\rangle}{a_p - \langle \sigma_p\rangle} \tag{2.21}$$

For a Gaussian distribution $\langle |v_q|^2\rangle = 1$ for all q such that $\langle \sigma_k\rangle = \langle \sigma\rangle$ becomes independent of k. Thus we end up with

$$\langle G\rangle(k) = \frac{1}{4M^2 \sum_{i=1}^d \sin^2\left[\frac{k_i}{2M}\right] - E - i\varepsilon - \langle \sigma\rangle} \tag{2.22}$$

where $\langle \sigma\rangle$ is a solution of

$$\begin{aligned} \langle \sigma\rangle &= \tfrac{\lambda^2}{M^d}\int_{[0,2\pi M]^d} \frac{d^d p}{(2\pi)^d}\, \frac{1}{4M^2 \sum_{i=1}^d \sin^2\left[\frac{p_i}{2M}\right] - z - \langle \sigma\rangle} \\ &= \lambda^2 \int_{[0,2\pi]^d} \frac{d^d p}{(2\pi)^d}\, \frac{1}{4M^2 \sum_{i=1}^d \sin^2\left[\frac{p_i}{2}\right] - z - \langle \sigma\rangle}\,. \end{aligned} \tag{2.23}$$

This equation is of course well known and one deduces from it that it generates a small imaginary part $\operatorname{Im}\sigma = O(\lambda^2)$ if the energy E is within the spectrum of $-\Delta$.

We now add the higher loop terms (the terms for $r > 2$ in the denominator of (1.3)) to our discussion and give an interpretation in terms of Feynman diagrams. To make the volume factors more explicit, asume that the lattice spacing in coordinate space is $1/M = 1$ such that $N = L$.

For the Anderson model, Feynman graphs may be obtained by brutally expanding

$$\int \left[a_k \delta_{k,p} + \tfrac{\lambda}{\sqrt{L}^d} v_{k-p}\right]^{-1}_{k,k} dP = \sum_{r=0}^{\infty} \int (C[VC]^r)_{kk}\, dP \tag{2.24}$$

$$= \sum_{r=0}^{\infty} \tfrac{(-\lambda)^r}{\sqrt{L^d}^r} \sum_{p_2 \cdots p_r} \tfrac{1}{a_k a_{p_2} \cdots a_{p_r} a_k} \int v_{k-p_2} v_{p_2-p_3} \cdots v_{p_r-k}\, dP$$

For a given r, we may represent this by ($c_k := 1/a_k$)

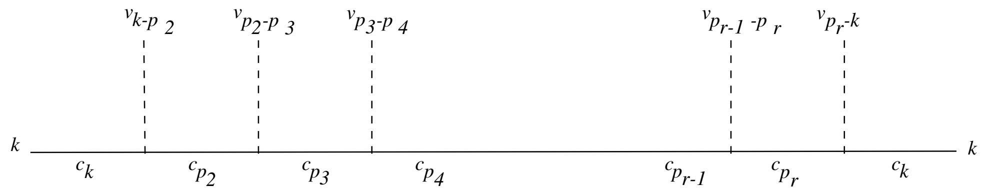

Figure 1: A string of particle lines with unpaired squiggles (dashed lines)

The integral over the v gives a sum of $(r-1)!!$ terms where each term is a product of $r/2$ Kroenecker-delta's, the terms for odd r vanish. If this is substituted in (2.24), the number of independent momentum sums is cut down to $r/2$ and each of the $(r-1)!!$ terms may be represented by a diagram

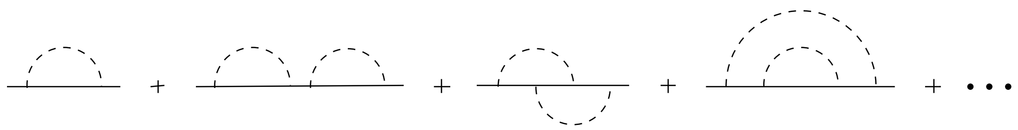

Figure 2: Lowest order diagrams contributing to (2.24) for $r = 2$ and $r = 4$

where, say, the value of the third diagram is given by $\frac{\lambda^4}{L^{2d}} \sum_{p_1,p_2} \frac{1}{a_k a_{k+p_1} a_{k+p_1+p_2} a_{k+p_2} a_k}$. For short:

$$\langle G \rangle(k) = \text{sum of all two legged diagrams}\,. \tag{2.25}$$

Since the value of a diagram depends on its subgraph structure, one distinguishes, in the easiest case, two types of diagrams. Diagrams with or without two-legged subdiagrams. Those diagrams with two-legged subgraphs usually produce anomalously large contributions. They are devided further into the one-particle irreducible ones and the reducible ones. Thereby a diagram is called one-particle reducible if it can be made disconnected by cutting one solid or 'particle' line (no squiggle or dashed line), see also figure 3.

The reason for introducing reducible and irreducible diagrams is that the reducible ones can be easily resummed by writing down the Schwinger-Dyson equation which states that

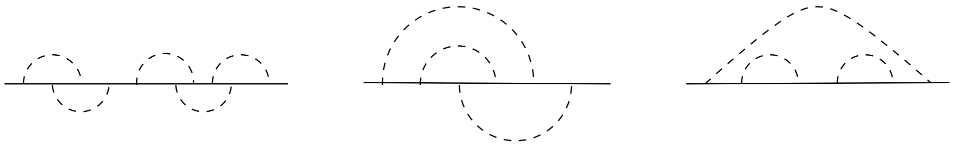

Figure 3: Reducible and irreducible diagrams

if the self energy Σ_k is defined through

$$\langle G\rangle(k) = \frac{1}{a_k - \Sigma_k(G_0)} \tag{2.26}$$

then $\Sigma_k(G_0)$ is the sum of all amputated (no $1/a_k$'s at the ends) one particle irreducible diagrams. Here we wrote $\Sigma_k(G_0)$ to indicate that the factors ('propagators') assigned to the solid lines of the diagrams contributing to Σ_k are given by the free two point function $G_0(p) = \frac{1}{a_p}$. However, the diagrams contributing to $\Sigma_k(G_0)$ still contain anomalously large contributions, namely irreducible graphs which contain two-legged subgraphs like diagram (c) in figure 3.

In the following we show, using the inversion formula (1.3) including all higher loop terms, that all graphs with two-legged subgraphs can be eliminated or resummed by writing down the following integral equation for $\langle G\rangle$:

$$\langle G\rangle(k) = \frac{1}{a_k - \sigma_k(\langle G\rangle)} \tag{2.27}$$

where

> $\sigma_k(\langle G\rangle)$ is the sum of all amputated two-legged diagrams which do not contain any two-legged subdiagrams, but now with propagators $\langle G\rangle(k) = \frac{1}{a_k - \sigma_k}$ instead of $G_0 = \frac{1}{a_k}$

which may be formalized as in (2.35) below. The advantage of this formula is that the series for $\sigma_k(\langle G\rangle)$ can be expected to be asymptotic, that is, its lowest order contributions are a good approximation if the coupling is small, but, usually, the series for $\Sigma_k(G_0)$ is not asymptotic. Thus, in order to rigorously controll $\langle G\rangle(k)$, one has to define a suitable space of propagators, to estimate the sum of all amputated two-legged graphs without two-legged subgraphs on that space and then finally to show that the equation (2.27) has a solution on this space. We intend to address this problem in another paper.

We now show (2.27) for the Anderson model. For fixed v one has

$$G(k, v) = \frac{1}{a_k - \sigma_k(v)} \tag{2.28}$$

where

$$\sigma_k(v) = \sum_{r=2}^{L^d} (-1)^r \sum_{\substack{p_2\cdots p_r \\ p_i\neq p_j,\, p_i\neq k}} \frac{\lambda^r}{\sqrt{L^d}^r} G_k(p_2)\cdots G_{kp_2\cdots p_{r-1}}(p_r)\, v_{k-p_2}\cdots v_{p_r-k} \tag{2.29}$$

We cutoff the r-sum in (2.29) at some arbitrary but fixed order $\ell < L^d$ where ℓ is choosen to be independent of the volume. Furthermore we substitute $G_{kp_2\cdots p_j}(p)$ by $G(p)$. Thus

$$\langle G\rangle(k) = \left\langle \frac{1}{a_k - \sum_{r=2}^{\ell} \sigma_k^r(v)} \right\rangle \tag{2.30}$$

where

$$\sigma_k^r(v) = (-1)^r \frac{\lambda^r}{\sqrt{L^d}^r} \sum_{\substack{p_2\cdots p_r \\ p_i\neq p_j,\, p_i\neq k}} G(p_2)\cdots G(p_r)\, v_{k-p_2}\cdots v_{p_r-k} \tag{2.31}$$

Consider first two strings $s_k^{r_1}$, $s_k^{r_2}$ where

$$s_k^r(v) = \frac{\lambda^r}{\sqrt{L^d}^r} \sum_{\substack{p_2\cdots p_r \\ p_i\neq p_j,\, p_i\neq k}} c^r_{kp_2\cdots p_r}\, v_{k-p_2}\cdots v_{p_r-k} \tag{2.32}$$

and the $c^r_{kp_2\cdots p_r}$ are some numbers. Then in the infinite volume limit

$$\langle s_k^{r_1} s_k^{r_2}\rangle = \langle s_k^{r_1}\rangle\, \langle s_k^{r_2}\rangle \tag{2.33}$$

because all pairings which connect the two strings have an extra volume factor $1/L^d$. Namely, if the two strings are disconnected, there are $(r_1+r_2)/2$ loops and a volume factor of $1/\sqrt{L^d}^{(r_1+r_2)}$ giving $(r_1+r_2)/2$ Riemannian sums. If the two strings are connected, there are only $(r_1+r_2-2)/2$ loops leaving an extra factor of $1/L^d$. By the same argument one has in the infinite volume limit

$$\langle (s_k^{r_1})^{n_1}\cdots (s_k^{r_m})^{n_m}\rangle = \langle s_k^{r_1}\rangle^{n_1}\cdots \langle s_k^{r_m}\rangle^{n_m} \tag{2.34}$$

which results in

$$\langle G\rangle(k) = \frac{1}{a_k - \sum_{r=2}^{\ell} \frac{(-\lambda)^r}{\sqrt{L^d}^r} \sum_{\substack{p_2\cdots p_r \\ p_i\neq p_j,\, p_i\neq k}} \langle G\rangle(p_2)\cdots \langle G\rangle(p_r)\, \langle v_{k-p_2}\cdots v_{p_r-k}\rangle} \tag{2.35}$$

The condition $p_2,\cdots,p_r \neq k$ and $p_i \neq p_j$ means exactly that two-legged subgraphs are forbidden. Namely, for a two-legged subdiagram as in (c) in figure 3, the incomming and outgoing momenta p, p' (to which are assigned propagators $\langle G\rangle(p)$, $\langle G\rangle(p')$) must be equal which is forbidden by the condidtion $p_i \neq p_j$ in (2.35).

However, we cannot take the limit $\ell \to \infty$ in (2.35) since the series in the denominator of (2.35) is only an asymptotic one. To see this a bit more clearly suppose for the moment that there were no restrictions on the momentum sums. Then, if $V = (\frac{\lambda}{\sqrt{L^d}} v_{k-p})_{k,p}$ and $\langle G\rangle = (\langle G\rangle(k)\,\delta_{k,p})_{k,p}$,

$$\frac{\lambda^r}{\sqrt{L^d}^r} \sum_{p_2\cdots p_r} \langle G\rangle(p_2)\cdots\langle G\rangle(p_r)\,\langle v_{k-p_2}\cdots v_{p_r-k}\rangle = \left\langle (V[\langle G\rangle V]^{r-1})_{kk}\right\rangle \tag{2.36}$$

and for $\ell\to\infty$ we would get

$$\langle G\rangle(k) = \frac{1}{a_k - \langle (V\frac{\langle G\rangle V}{Id+\langle G\rangle V})_{kk}\rangle} = \frac{1}{a_k - \langle (V\frac{1}{\langle G\rangle^{-1}+V}V)_{kk}\rangle} \tag{2.37}$$

That is, the factorials produced by the number of diagrams in the denominator of (2.35) are basically the same as those in the expansion

$$\int_{\mathbb{R}} \tfrac{x^2}{z+\lambda x}\, e^{-\frac{x^2}{2}}\, \tfrac{dx}{\sqrt{2\pi}} \;=\; \sum_{r=0}^{\ell} \tfrac{\lambda^{2r}}{z^{2r+1}}\,(2r+1)!! \;+ R_{\ell+1}(\lambda) \tag{2.38}$$

where the remainder satisfies the bound $|R_{\ell+1}(\lambda)| \le \ell!\, const_z^{\ell}\,\lambda^{\ell}$.

We close this section with two further remarks. So far the computations were done in momentum space. One may wonder what one gets if the inversion formula (1.3) is applied to $[-\Delta+z+\lambda V]^{-1}$ in coordinate space. Whereas a geometric series expansion of $[-\Delta+z+\lambda V]^{-1}$ gives a representation in terms of the simple random walk, application of (1.3) results in a representation in terms of the self avoiding walk:

$$[-\Delta+z+\lambda V]^{-1}_{0,x} = \sum_{\substack{\gamma:0\to x\\ \gamma \text{ self avoiding}}} \frac{\det\left[(-\Delta+z+\lambda V)_{y,y'\in\Gamma\setminus\gamma}\right]}{\det\left[(-\Delta+z+\lambda V)_{y,y'\in\Gamma}\right]} \tag{2.39}$$

where Γ is the lattice in coordinate space. Namely, if $|x|>1$, the inversion formula (1.4) for the off-diagonal elements gives

$$\begin{aligned}
[-\Delta+\lambda V]^{-1}_{0,x} &= \sum_{r=3}^{L^d}(-1)^{r+1}\sum_{\substack{x_3\cdots x_r\in\Gamma\setminus\{0,x\}\\ x_i\neq x_j}} G(0)G_0(x_3)\cdots G_{0x_3\cdots x_r}(x)\,(-\Delta)_{0x_3}\cdots(-\Delta)_{x_r x}\\
&= \sum_{r=3}^{L^d}\sum_{\substack{x_2=0,x_3,\cdots,x_r,x_{r+1}=x\in\Gamma\\ |x_i-x_{i+1}|=1\ \forall i=2\cdots r}} \frac{\det\left[(-\Delta+\lambda V)_{y,y'\in\Gamma\setminus\{0\}}\right]}{\det\left[(-\Delta+\lambda V)_{y,y'\in\Gamma}\right]}\cdots\frac{\det\left[(-\Delta+\lambda V)_{y,y'\in\Gamma\setminus\{0,x_3\cdots x_r,x\}}\right]}{\det\left[(-\Delta+\lambda V)_{y,y'\in\Gamma\setminus\{0,x_3\cdots x_r\}}\right]}
\end{aligned}$$

which coincides with (2.39).

Finally we remark that, while the argument following (2.32) leads to a factorization property for on-diagonal elements in momentum space, $\langle G(k)\,G(p)\rangle = \langle G(k)\rangle\,\langle G(p)\rangle$, there is no such property for products of off-diagonal elements which appear in a quantity like

$$\Lambda(q) = \tfrac{1}{L^d}\sum_{k,p}\left\langle \left[a_k\delta_{k,p}+\tfrac{\lambda}{\sqrt{L}^d}v_{k-p}\right]^{-1}_{k,p}\left[\bar a_k\delta_{k,p}+\tfrac{\lambda}{\sqrt{L}^d}\bar v_{k-p}\right]^{-1}_{k-q,p-q}\right\rangle \tag{2.40}$$

which is the Fourier transform of $\langle |[-\Delta + z + \lambda V]^{-1}_{x,y}|^2 \rangle$. (Each off-diagonal inverse matrix element is proportional to $1/\sqrt{L^d}$, therefore the prefactor of $1/L^d$ in (2.40) is correct.)

3 Proof of the Inversion Formula

Lemma 3.1: Let $B \in \mathbb{C}^{k\times n}$, $C \in \mathbb{C}^{n\times k}$ and let Id_k denote the identity in $\mathbb{C}^{k\times k}$. Then:

(i) $Id_k - BC$ invertible $\quad\Leftrightarrow\quad$ $Id_n - CB$ invertible.

(ii) If the left or the right hand side of (i) fullfilled, then $C\,\frac{1}{Id_k - BC} = \frac{1}{Id_n - CB}\,C$.

Proof: Let

$$B = \begin{pmatrix} - & \vec{b}_1 & - \\ & \vdots & \\ - & \vec{b}_k & - \end{pmatrix}, \qquad C = \begin{pmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_k \\ | & & | \end{pmatrix}$$

where the $\vec{b}_j$ are n-component row vectors and the $\vec{c}_j$ are n-component column vectors. Let $\vec{x} \in \mathrm{Kern}(Id - CB)$. Then $\vec{x} = CB\vec{x} = \sum_j \lambda_j \vec{c}_j$ if we define $\lambda_j := (\vec{b}_j, \vec{x})$. Let $\vec{\lambda} = (\lambda_j)_{1\le j\le k}$. Then $[(Id - BC)\vec{\lambda}]_i = \lambda_i - \sum_j (\vec{b}_i, \vec{c}_j)\lambda_j = (\vec{b}_i, \vec{x}) - \sum_j (\vec{b}_i, \vec{c}_j)\lambda_j = 0$ since $\vec{x} = \sum_j \lambda_j \vec{c}_j$, thus $\vec{\lambda} \in \mathrm{Kern}(Id - BC)$. On the other hand, if some $\vec{\lambda} \in \mathrm{Kern}(Id - BC)$, then $\vec{x} := \sum_j \lambda_j \vec{c}_j \in \mathrm{Kern}(Id - CB)$ which proves (i). Part (ii) then follows from $C = \frac{1}{Id_n - CB}(Id_n - CB)C = \frac{1}{Id_n - CB}C(Id_k - BC)$ ■

The inversion formula (1.3,4) is obtained by iterative application of the next lemma, which states the Feshbach formula for finite dimensional matrices. For a more general version one may look in [BFS], Theorem 2.1.

Lemma 3.2: Let $h = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{C}^{n\times n}$ where $A \in \mathbb{C}^{k\times k}$, $D \in \mathbb{C}^{(n-k)\times(n-k)}$ are invertible and $B \in \mathbb{C}^{k\times(n-k)}$, $C \in \mathbb{C}^{(n-k)\times k}$. Then

$$h \text{ invertible} \quad\Leftrightarrow\quad A - BD^{-1}C \text{ invertible} \quad\Leftrightarrow\quad D - CA^{-1}B \text{ invertible} \tag{3.1}$$

and if one of the conditions in (3.1) is fullfilled, one has $h^{-1} = \left(\begin{smallmatrix} E & F \\ G & H \end{smallmatrix}\right)$ where

$$E = \frac{1}{A - BD^{-1}C}\,, \qquad H = \frac{1}{D - CA^{-1}B}\,, \tag{3.2}$$

$$F = -EBD^{-1} = -A^{-1}BH\,, \qquad G = -HCA^{-1} = -D^{-1}CE\,. \tag{3.3}$$

Proof: We have, using Lemma 3.1 in the second line,

$$\begin{aligned} A - BD^{-1}C \ \text{ inv.} \ &\Leftrightarrow \ Id_k - A^{-1}BD^{-1}C \ \text{ inv.} \\ &\Leftrightarrow \ Id_{n-k} - D^{-1}CA^{-1}B \ \text{ inv.} \\ &\Leftrightarrow \ D - CA^{-1}B \ \text{ inv.} \end{aligned}$$

Furthermore, again by Lemma 3.1,

$$D^{-1}C \, \tfrac{1}{Id - A^{-1}BD^{-1}C} = \tfrac{1}{Id - D^{-1}CA^{-1}B} \, D^{-1}C = \tfrac{1}{D - CA^{-1}B} \, C = HC$$

and

$$A^{-1}B \, \tfrac{1}{Id - D^{-1}CA^{-1}B} = \tfrac{1}{Id - A^{-1}BD^{-1}C} \, A^{-1}B = \tfrac{1}{A - BD^{-1}C} \, B = EB$$

which proves the last equalities in (3.3), $HCA^{-1} = D^{-1}CE$ and $EBD^{-1} = A^{-1}BH$. Using these equations and the definition of E, F, G and H one computes

$$\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \left(\begin{smallmatrix} E & F \\ G & H \end{smallmatrix}\right) = \left(\begin{smallmatrix} E & F \\ G & H \end{smallmatrix}\right) \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} Id & 0 \\ 0 & Id \end{smallmatrix}\right)$$

It remains to show that the invertibility of h implies the invertibility of $A - BD^{-1}C$. To this end let $P = \left(\begin{smallmatrix} Id & \\ & 0 \end{smallmatrix}\right)$, $\bar{P} = \left(\begin{smallmatrix} 0 & \\ & Id \end{smallmatrix}\right)$ such that $A - BD^{-1}C = PhP - Ph\bar{P}(\bar{P}h\bar{P})^{-1}\bar{P}hP$. Then

$$\begin{aligned} (A - BD^{-1}C)Ph^{-1}P &= PhPh^{-1}P - Ph\bar{P}(\bar{P}h\bar{P})^{-1}\bar{P}hPh^{-1}P \\ &= Ph(1 - \bar{P})h^{-1}P - Ph\bar{P}(\bar{P}h\bar{P})^{-1}\bar{P}h(1 - \bar{P})h^{-1}P \\ &= P - Ph\bar{P}h^{-1}P + Ph\bar{P}h^{-1}P = P \end{aligned}$$

and similarly $Ph^{-1}P(A - BD^{-1}C) = P$ which proves the invertibility of $A - BD^{-1}C$ ■

Theorem 3.3: Let $B \in \mathbb{C}^{nN \times nN}$ be given by $B = (B_{kp})_{k,p \in \mathcal{M}}$, $\mathcal{M}$ some index set, $|\mathcal{M}| = N$, and $B_{kp} = (B_{k\sigma, p\tau})_{\sigma,\tau \in I} \in \mathbb{C}^{n \times n}$ where I is another index set, $|I| = n$. Suppose that B and, for any $\mathcal{N} \subset \mathcal{M}$, the submatrix $(B_{kp})_{k,p \in \mathcal{N}}$ is invertible. For $k \in \mathcal{M}$ let

$$G(k) := \left[B^{-1}\right]_{kk} \in \mathbb{C}^{n \times n} \tag{3.4}$$

and, if $\mathcal{N} \subset \mathcal{M}$, $k \notin \mathcal{N}$,

$$G_{\mathcal{N}}(k) := \left[\{(B_{st})_{s,t \in \mathcal{M} \setminus \mathcal{N}}\}^{-1}\right]_{kk} \in \mathbb{C}^{n \times n} \tag{3.5}$$

Then one has

(i) The on-diagonal block matrices of B^{-1} are given by

$$G(k) = \frac{1}{B_{kk} - \sum\limits_{r=2}^{N} (-1)^r \sum\limits_{\substack{p_2 \cdots p_r \in \mathcal{M} \setminus \{k\} \\ p_i \neq p_j}} B_{kp_2} G_k(p_2) B_{p_2 p_3} \cdots B_{p_{r-1} p_r} G_{kp_2 \cdots p_{r-1}}(p_r) B_{p_r k}} \tag{3.6}$$

where $1/\cdot$ is inversion of $n \times n$ matrices.

(ii) Let $k, p \in \mathcal{M}$, $k \neq p$. Then the off-diagonal block matrices of B^{-1} can be expressed in terms of the $G_{\mathcal{N}}(s)$ and the B_{st},

$$[B^{-1}]_{kp} = -G(k)B_{kp}G_k(p) - \sum_{r=3}^{N}(-1)^r \sum_{\substack{t_3\cdots t_r \in \mathcal{M}\setminus\{k,p\} \\ t_i \neq t_j}} G(k)B_{kt_3}G_k(t_3)B_{t_3t_4}\cdots B_{t_rp}G_{kt_3\cdots t_r}(p) \tag{3.7}$$

Proof: Let k be fixed and let $p, p' \in \mathcal{M}\setminus\{k\}$ below label columns and rows. By Lemma 3.2 we have

$$\begin{pmatrix} G(k) & \\ & * \end{pmatrix} = \begin{pmatrix} B_{kk} & - \; B_{kp} \; - \\ | & \\ B_{p'k} & B_{p'p} \\ | & \end{pmatrix}^{-1} = \begin{pmatrix} E & - \; F \; - \\ | & \\ G & H \\ | & \end{pmatrix}$$

where

$$\begin{aligned} G(k) = E &= \frac{1}{B_{kk} - \sum\limits_{p,p'\neq k} B_{kp}\left[\{(B_{p'p})_{p',p\in\mathcal{M}\setminus\{k\}}\}^{-1}\right]_{pp'} B_{p'k}} \qquad (3.8) \\ &= \frac{1}{B_{kk} - \sum\limits_{p\neq k} B_{kp}G_k(p)B_{pk} - \sum\limits_{\substack{p,p'\neq k\\ p\neq p'}} B_{kp}\left[\{(B_{p'p})_{p',p\in\mathcal{M}\setminus\{k\}}\}^{-1}\right]_{pp'} B_{p'k}} \end{aligned}$$

and

$$\begin{aligned} F_{kp} = \left[B^{-1}\right]_{kp} &= -G(k)\sum_{t\neq k} B_{kt}\left[\{(B_{p'p})_{p',p\in\mathcal{M}\setminus\{k\}}\}^{-1}\right]_{tp} \qquad (3.9) \\ &= -G(k)B_{kp}G_k(p) - G(k)\sum_{t\neq k,p} B_{kt}\left[\{(B_{p'p})_{p',p\in\mathcal{M}\setminus\{k\}}\}^{-1}\right]_{tp} \end{aligned}$$

Apply Lemma 3.2 now to the matrix $\{(B_{p'p})_{p',p\in\mathcal{M}\setminus\{k\}}\}^{-1}$ and proceed by induction to obtain after ℓ steps

$$G(k) = \frac{1}{B_{kk} - \sum\limits_{r=2}^{\ell}(-1)^r \sum\limits_{\substack{p_2\cdots p_r\in\mathcal{M}\setminus\{k\}\\ p_i\neq p_j}} B_{kp_2}G_k(p_2)B_{p_2p_3}\cdots B_{p_{r-1}p_r}G_{kp_2\cdots p_{r-1}}(p_r)B_{p_rk} - R_{\ell+1}} \tag{3.10}$$

$$F_{kp} = -G(k)B_{kp}G_k(p) - \sum_{r=3}^{\ell}(-1)^r \sum_{\substack{t_3\cdots t_r\in\mathcal{M}\setminus\{k,p\}\\ t_i\neq t_j}} G(k)B_{kt_3}G_k(t_3)B_{t_3t_4}\cdots B_{t_rp}G_{kt_3\cdots t_r}(p) - \tilde{R}_{\ell+1} \tag{3.11}$$

where

$$R_{\ell+1} = (-1)^\ell \sum_{\substack{p_2\cdots p_{\ell+1}\in\mathcal{M}\setminus\{k\}\\ p_i\neq p_j}} B_{kp_2}G_k(p_2)\cdots B_{p_{\ell-1}p_\ell}\left[\left\{(B_{p'p})_{p',p\in\mathcal{M}\setminus\{kp_2\cdots p_\ell\}}\right\}^{-1}\right]_{p_\ell p_{\ell+1}} B_{p_{\ell+1}k} \tag{3.12}$$

$$\tilde{R}_{\ell+1} = (-1)^\ell \sum_{\substack{t_3\cdots t_{\ell+1}\in\mathcal{M}\setminus\{k,p\}\\ t_i\neq t_j}} G(k)B_{kt_3}\cdots G_{kt_3\cdots t_{\ell-1}}(t_\ell)B_{t_\ell t_{\ell+1}}\left[\left\{(B_{p'p})_{p',p\in\mathcal{M}\setminus\{kt_3\cdots t_\ell\}}\right\}^{-1}\right]_{t_{\ell+1}p} \tag{3.13}$$

Since $R_{N+1} = \tilde{R}_{N+1} = 0$ the theorem follows ■

4 Application to the Many-Electron System and to the φ^4-Model

4.1 The Many-Electron System

We consider the many-electron system in the grand canonical ensemble in finite volume $[0,L]^d$ and at some small but positive temperature $T = 1/\beta > 0$ with attractive delta-interaction given by the Hamiltonian

$$H = H_0 - \lambda H_{\rm int} = \tfrac{1}{L^d}\sum_{\mathbf{k}\sigma}(\tfrac{\mathbf{k}^2}{2m}-\mu)a^+_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma} - \tfrac{\lambda}{L^{3d}}\sum_{\mathbf{kpq}} a^+_{\mathbf{k}\uparrow}a^+_{\mathbf{q}-\mathbf{k}\downarrow}a_{\mathbf{q}-\mathbf{p}\downarrow}a_{\mathbf{p}\uparrow} \tag{4.1}$$

Our normalization conventions concerning the volume factors are such that the canonical anticommutation relations read $\{a_{\mathbf{k}\sigma}, a^+_{\mathbf{p}\tau}\} = L^d\,\delta_{\mathbf{k},\mathbf{p}}\delta_{\sigma,\tau}$. The momentum sums range over some subset of $\left(\frac{2\pi}{L}\mathbb{Z}\right)^d$, say $\mathcal{M} = \left\{\mathbf{k}\in\left(\frac{2\pi}{L}\mathbb{Z}\right)^d \,\middle|\, |e_\mathbf{k}| \le 1\right\}$, $e_\mathbf{k} = \mathbf{k}^2/2m - \mu$, and $\mathbf{q}\in\{\mathbf{k}-\mathbf{p}\,|\mathbf{k},\mathbf{p}\in\mathcal{M}\}$.

We are interested in the momentum distribution

$$\langle a^+_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}\rangle = Tr[e^{-\beta H}a^+_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}]/Tr\,e^{-\beta H} \tag{4.2}$$

and in the expectation value of the energy

$$\langle H_{\rm int}\rangle = \sum_{\mathbf{q}} \Lambda(\mathbf{q}) \tag{4.3}$$

where

$$\Lambda(\mathbf{q}) = \tfrac{\lambda}{L^{3d}} \sum_{\mathbf{k},\mathbf{p}} Tr[e^{-\beta H} a^+_{\mathbf{k}\uparrow} a^+_{\mathbf{q}-\mathbf{k}\downarrow} a_{\mathbf{q}-\mathbf{p}\downarrow} a_{\mathbf{p}\uparrow}]/Tr\, e^{-\beta H} \tag{4.4}$$

By writing down the perturbation series for the partition function, rewriting it as a Grassmann integral

$$\begin{aligned} \tfrac{Tr\, e^{-\beta(H_0-\lambda H_{\rm int})}}{Tr\, e^{-\beta H_0}} &= \int e^{\frac{\lambda}{(\beta L^d)^3}\sum_{kpq}\bar\psi_{k\uparrow}\bar\psi_{q-k\downarrow}\psi_{q-p\downarrow}\psi_{p\uparrow}} d\mu_C(\psi,\bar\psi) \qquad (4.5)\\ d\mu_C &= \prod_{k\sigma} \tfrac{\beta L^d}{ik_0-e_{\mathbf{k}}}\, e^{-\frac{1}{\beta L^d}\sum_{k\sigma}(ik_0-e_{\mathbf{k}})\bar\psi_{k\sigma}\psi_{k\sigma}} \prod_{k\sigma} d\psi_{k\sigma} d\bar\psi_{k\sigma}\,, \end{aligned}$$

performing a Hubbard-Stratonovich transformation ($\phi_q = u_q + iv_q$, $d\phi_q d\bar\phi_q := du_q dv_q$)

$$e^{-\sum_q a_q b_q} = \int e^{i\sum_q (a_q\phi_q + b_q\bar\phi_q)} e^{-\sum_q |\phi_q|^2} \prod_q \tfrac{d\phi_q d\bar\phi_q}{\pi} \tag{4.6}$$

with

$$a_q = \tfrac{\lambda^{\frac12}}{(\beta L^d)^{\frac32}} \sum_k \bar\psi_{k\uparrow}\bar\psi_{q-k\downarrow}, \quad b_q = \tfrac{\lambda^{\frac12}}{(\beta L^d)^{\frac32}} \sum_p \psi_{p\uparrow}\psi_{q-p\downarrow} \tag{4.7}$$

and then integrating out the $\psi,\bar\psi$ variables, one arrives at the following representation which is the starting point for our analysis (for more details, see [FKT] or [L1]):

$$\tfrac{1}{L^d}\langle a^+_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}\rangle = \tfrac{1}{\beta L^d}\,\tfrac{1}{\beta} \sum_{k_0\in\frac{\pi}{\beta}(2\mathbb{Z}+1)} \langle \psi^+_{\mathbf{k}k_0\sigma}\psi_{\mathbf{k}k_0\sigma}\rangle \tag{4.8}$$

where, abbreviating $k=(\mathbf{k},k_0)$, $\kappa=\beta L^d$, $a_k = ik_0 - e_{\mathbf{k}}$, $g=\lambda^{\frac12}$,

$$\tfrac{1}{\kappa}\langle\bar\psi_{t\sigma}\psi_{t\sigma}\rangle = \int \begin{bmatrix} a_k\delta_{k,p} & \frac{ig}{\sqrt\kappa}\,\bar\phi_{p-k} \\ \frac{ig}{\sqrt\kappa}\,\phi_{k-p} & a_{-k}\delta_{k,p}\end{bmatrix}^{-1}_{t\sigma,t\sigma} dP(\phi) \tag{4.9}$$

and $dP(\phi)$ is the probability measure

$$dP(\phi) = \tfrac{1}{Z}\,\det \begin{bmatrix} a_k\delta_{k,p} & \frac{ig}{\sqrt\kappa}\,\bar\phi_{p-k} \\ \frac{ig}{\sqrt\kappa}\,\phi_{k-p} & a_{-k}\delta_{k,p}\end{bmatrix} e^{-\sum_q|\phi_q|^2} \prod_q d\phi_q d\bar\phi_q \tag{4.10}$$

Furthermore

$$\Lambda(\mathbf{q}) = \tfrac{1}{\beta}\sum_{q_0\in\frac{2\pi}{\beta}\mathbb{Z}} \Lambda(\mathbf{q},q_0) \tag{4.11}$$

where

$$\begin{aligned} \Lambda(q) &= \tfrac{\lambda}{(\beta L^d)^3}\sum_{k,p}\langle\bar\psi_{k\uparrow}\bar\psi_{q-k\downarrow}\psi_{q-p\downarrow}\psi_{p\uparrow}\rangle \\ &= \langle|\phi_q|^2\rangle - 1 \qquad (4.12)\end{aligned}$$

and the expectation in the last line is integration with respect to $dP(\phi)$. The expectation on the ψ variables $\langle \bar{\psi}_{k\sigma}\psi_{k\sigma}\rangle = \frac{1}{Z}\int \bar{\psi}_{k\sigma}\psi_{k\sigma}\, e^{\frac{\lambda}{\kappa^3}\sum_{k,p,q}\bar{\psi}_{k\uparrow}\bar{\psi}_{q-k\downarrow}\psi_{q-p\downarrow}\psi_{p\uparrow}} d\mu_C$ is Grassmann integration, but these representations are not used in the following. The matrix and the integral in (4.9) become finite dimensional if we choose some cutoff on the k_0 variables which is removed in the end. The set $\mathcal{M}$ for the spatial momenta is already finite since we have chosen a fixed UV-cuttoff $|e_{\mathbf{k}}| = |\mathbf{k}^2/2m - \mu| \le 1$ which will not be removed in the end since we are interested in the infrared properties at $\mathbf{k}^2/2m = \mu$.

Our goal is to apply the inversion formula to the inverse matrix element in (4.9). Instead of writing the matrix in terms of four $N\times N$ blocks $(a_k\delta_{k,p})_{k,p}$, $(\bar{\phi}_{p-k})_{k,p}$, $(\phi_{k-p})_{k,p}$ and $(a_{-k}\delta_{k,p})_{k,p}$ where N is the number of the $d+1$-dimensional momenta k,p, we interchange rows and columns to rewrite it in terms of N blocks of size 2×2 (the matrix U in the next line interchanges the rows and columns):

$$U\begin{bmatrix} a_k\delta_{k,p} & \frac{ig}{\sqrt{\kappa}}\,\bar{\phi}_{p-k} \\ \frac{ig}{\sqrt{\kappa}}\,\phi_{k-p} & a_{-k}\delta_{k,p}\end{bmatrix}U^{-1} = B = (B_{kp})_{k,p}$$

where the 2×2 blocks B_{kp} are given by

$$B_{kk} = \begin{pmatrix} a_k & \frac{ig}{\sqrt{\kappa}}\,\bar{\phi}_0 \\ \frac{ig}{\sqrt{\kappa}}\,\phi_0 & a_{-k}\end{pmatrix}, \quad B_{kp} = \tfrac{ig}{\sqrt{\kappa}}\begin{pmatrix} 0 & \bar{\phi}_{p-k} \\ \phi_{k-p} & 0\end{pmatrix} \text{ if } k\ne p\,. \tag{4.13}$$

We want to compute the 2×2 matrix

$$\langle G\rangle(k) = \int G(k)\,dP(\phi) \tag{4.14}$$

where

$$G(k) = [B^{-1}]_{kk} \tag{4.15}$$

We start again with the two loop approximation which retains only the $r=2$ term in the denominator of (1.3). The result will be equation (4.20) below where the quantities $\langle\sigma_k\rangle$ and $\langle|\phi_0|^2\rangle$ appearing in (4.20) have to satisfy the equations (4.21) and (4.24) which have to be solved in conjunction with (4.26). The solution to these equations is discussed below (4.27) and leads to (4.32).

We first derive (4.20). In the two loop approximation,

$$\begin{aligned} G(k) &\approx \Big[B_{kk} - \sum_{p\ne k} B_{kp}\,G_k(p)\,B_{pk}\Big]^{-1} \\ &= \Big[\begin{pmatrix} a_k & \frac{ig}{\sqrt{\kappa}}\,\bar{\phi}_0 \\ \frac{ig}{\sqrt{\kappa}}\,\phi_0 & a_{-k}\end{pmatrix} + \tfrac{\lambda}{\kappa}\sum_{p\ne k}\begin{pmatrix} & \bar{\phi}_{p-k} \\ \phi_{k-p} & \end{pmatrix} G_k(p)\begin{pmatrix} & \bar{\phi}_{k-p} \\ \phi_{p-k} & \end{pmatrix}\Big]^{-1} \\ &=: \Big[\begin{pmatrix} a_k & \frac{ig}{\sqrt{\kappa}}\,\bar{\phi}_0 \\ \frac{ig}{\sqrt{\kappa}}\,\phi_0 & \bar{a}_k\end{pmatrix} + \Sigma(k)\Big]^{-1} \end{aligned} \tag{4.16}$$

where, substituting again $G_k(p)$ by $G(p)$ in the infinite volume limit,

$$\Sigma(k) = \tfrac{\lambda}{\kappa} \sum_{p\neq k} \begin{pmatrix} & \bar\phi_{p-k} \\ \phi_{k-p} & \end{pmatrix} \left[\begin{pmatrix} a_p & \tfrac{ig}{\sqrt{\kappa}}\,\bar\phi_0 \\ \tfrac{ig}{\sqrt{\kappa}}\,\phi_0 & \bar a_p \end{pmatrix} + \Sigma(p) \right]^{-1} \begin{pmatrix} & \bar\phi_{k-p} \\ \phi_{p-k} & \end{pmatrix} \tag{4.17}$$

Anticipating the fact that the off-diagonal elements of $\langle\Sigma\rangle(k)$ will be zero (for 'zero external field'), we make the Ansatz

$$\Sigma(k) = \begin{pmatrix} \sigma_k & \\ & \bar\sigma_k \end{pmatrix} \tag{4.18}$$

and obtain

$$\begin{pmatrix} \sigma_k & \\ & \bar\sigma_k \end{pmatrix} = \tfrac{\lambda}{\kappa} \sum_{p\neq k} \tfrac{1}{(a_p+\sigma_p)(\bar a_p+\bar\sigma_p)+\frac{\lambda}{\kappa}|\phi_0|^2} \begin{pmatrix} (a_k+\sigma_k)|\phi_{p-k}|^2 & -\tfrac{ig}{\sqrt{\kappa}}\,\phi_0\bar\phi_{k-p}\bar\phi_{p-k} \\ -\tfrac{ig}{\sqrt{\kappa}}\,\bar\phi_0\phi_{k-p}\phi_{p-k} & (\bar a_k+\bar\sigma_k)|\phi_{k-p}|^2 \end{pmatrix} \tag{4.19}$$

As for the Anderson model, we perform the functional integral by substituting the quantities $|\phi_q|^2$ by their expectation values $\langle|\phi_q|^2\rangle$. Apparently this is less obvious in this case since $dP(\phi)$ is no longer Gaussian and the $|\phi_q|^2$ are no longer identically, independently distributed. We will comment on this after (4.37) below and at the end of the next section by reinterpreting this procedure as a resummation of diagrams. For now, we simply continue in this way. Then

$$\langle G\rangle(k) = \tfrac{1}{|a_k+\langle\sigma_k\rangle|^2+\frac{\lambda}{\kappa}\langle|\phi_0|^2\rangle} \begin{pmatrix} \bar a_k+\langle\bar\sigma_k\rangle & -\tfrac{ig}{\sqrt{\kappa}}\,\langle\bar\phi_0\rangle \\ -\tfrac{ig}{\sqrt{\kappa}}\,\langle\phi_0\rangle & a_k+\langle\sigma_k\rangle \end{pmatrix} \tag{4.20}$$

where the quantity $\langle\sigma_k\rangle$ has to satisfy the equation

$$\langle\sigma_k\rangle = \tfrac{\lambda}{\kappa} \sum_{p\neq k} \tfrac{\bar a_p+\langle\bar\sigma_p\rangle}{|a_p+\langle\sigma_p\rangle|^2+\frac{\lambda}{\kappa}\langle|\phi_0|^2\rangle}\,\langle|\phi_{p-k}|^2\rangle \tag{4.21}$$

Since $dP(\phi)$ is not Gaussian, we do not know the expectations $\langle|\phi_q|^2\rangle$. However, by partial integration, we obtain

$$\langle|\phi_q|^2\rangle = 1 + \tfrac{ig}{\sqrt{\kappa}} \sum_p \int \phi_q\,[B^{-1}(\phi)]_{p\uparrow,p+q\downarrow}\,dP(\phi) \tag{4.22}$$

Namely,

$$\begin{aligned}
\langle|\phi_q|^2\rangle &= \tfrac{1}{Z}\int \phi_q\bar\phi_q\,\det\left[\{B_{kp}(\phi)\}_{k,p}\right]\,e^{-\sum_q|\phi_q|^2} d\phi_q d\bar\phi_q \\
&= 1 + \tfrac{1}{Z}\int \phi_q\left(\tfrac{\partial}{\partial\phi_q}\det\left[\{B_{kp}(\phi)\}_{k,p}\right]\right)e^{-\sum_q|\phi_q|^2} d\phi_q d\bar\phi_q \\
&= 1 + \tfrac{1}{Z}\int \phi_q \sum_{p,\tau} \det\begin{bmatrix} | & | & | \\ B_{k\sigma,p'\tau'} & \frac{\partial B_{k\sigma,p\tau}}{\partial\phi_q} & B_{k\sigma,p''\tau''} \\ | & | & | \end{bmatrix} e^{-\sum_q|\phi_q|^2} d\phi_q d\bar\phi_q
\end{aligned}$$

Since

$$\frac{\partial}{\partial\phi_q} B_{kp} = \frac{ig}{\sqrt{\kappa}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \delta_{k-p,q}$$

we have

$$\det\left[\begin{array}{ccc} | & | & | \\ B_{k\sigma,p'\tau'} & \frac{\partial B_{k\sigma,p\tau}}{\partial\phi_q} & B_{k\sigma,p''\tau''} \\ | & | & | \end{array}\right] \Big/ \det\left[\{B_{kp}\}_{k,p}\right] = \begin{cases} 0 & \text{if } \tau = \downarrow \\ \frac{ig}{\sqrt{\kappa}}\, [B^{-1}]_{p\uparrow,p+q\downarrow} & \text{if } \tau = \uparrow \end{cases}$$

which results in (4.22).

The inverse matrix element in (4.22) we compute again with (1.3,4) in the two loop approximation. Consider first the case $q = 0$. Then one gets

$$\begin{aligned} \langle|\phi_0|^2\rangle &= 1 + \frac{ig}{\sqrt{\kappa}} \sum_p \int \phi_0\, G(p)_{\uparrow\downarrow}\, dP(\phi) \\ &= 1 + \frac{ig}{\sqrt{\kappa}} \sum_p \int \phi_0\, \frac{1}{|a_p+\sigma_p|^2 + \frac{\lambda}{\kappa}|\phi_0|^2} \begin{pmatrix} \bar a_p + \bar\sigma_p & -\frac{ig}{\sqrt{\kappa}}\,\bar\phi_0 \\ -\frac{ig}{\sqrt{\kappa}}\,\phi_0 & a_p + \sigma_p \end{pmatrix}_{\uparrow\downarrow} dP(\phi) \\ &= 1 + \frac{\lambda}{\kappa} \sum_p \int \phi_0\, \frac{\bar\phi_0}{|a_p+\sigma_p|^2 + \frac{\lambda}{\kappa}|\phi_0|^2}\, dP(\phi) \end{aligned} \tag{4.23}$$

Performing the functional integral by substitution of expectation values gives

$$\langle|\phi_0|^2\rangle = 1 + \frac{\lambda}{\kappa} \sum_p \langle|\phi_0|^2\rangle\, \frac{1}{|a_p + \langle\sigma_p\rangle|^2 + \frac{\lambda}{\kappa}\langle|\phi_0|^2\rangle}$$

or

$$\langle|\phi_0|^2\rangle = \frac{1}{1 - \frac{\lambda}{\kappa} \sum_p \frac{1}{|a_p + \langle\sigma_p\rangle|^2 + \frac{\lambda}{\kappa}\langle|\phi_0|^2\rangle}} \tag{4.24}$$

Before we discuss (4.24), we write down the equation for $q \neq 0$. In that case we use (1.4) to compute $[B^{-1}(\phi)]_{p\uparrow,p+q\downarrow}$ in the two loop approximation. We get

$$\begin{aligned} &[B^{-1}(\phi)]_{p\uparrow,p+q\downarrow} \approx -\left[G(p) B_{p,p+q} G(p+q)\right]_{\uparrow\downarrow} \\ &= -\frac{1}{|a_p+\sigma_p|^2 + \frac{\lambda}{\kappa}|\phi_0|^2}\, \frac{1}{|a_{p+q}+\sigma_{p+q}|^2 + \frac{\lambda}{\kappa}|\phi_0|^2}\, \frac{ig}{\sqrt{\kappa}} \times \\ &\quad \begin{pmatrix} -\frac{ig}{\sqrt{\kappa}}[(\bar a+\bar\sigma)_{p+q}\bar\phi_0\phi_{-q} + (\bar a+\bar\sigma)_p\phi_0\bar\phi_q] & (\bar a+\bar\sigma)_p(a+\sigma)_{p+q}\bar\phi_q - \frac{\lambda}{\kappa}\bar\phi_0^2\phi_{-q} \\ (a+\sigma)_p(\bar a+\bar\sigma)_{p+q}\phi_{-q} - \frac{\lambda}{\kappa}\phi_0^2\bar\phi_q & -\frac{ig}{\sqrt{\kappa}}[(a+\sigma)_{p+q}\phi_0\bar\phi_q + (a+\sigma)_p\bar\phi_0\phi_{-q}] \end{pmatrix}_{\uparrow\downarrow} \\ &= -\frac{ig}{\sqrt{\kappa}}\, \frac{(\bar a+\bar\sigma)_p(a+\sigma)_{p+q}\bar\phi_q - \frac{\lambda}{\kappa}\bar\phi_0^2\phi_{-q}}{\left(|a_p+\sigma_p|^2 + \frac{\lambda}{\kappa}|\phi_0|^2\right)\left(|a_{p+q}+\sigma_{p+q}|^2 + \frac{\lambda}{\kappa}|\phi_0|^2\right)} \end{aligned} \tag{4.25}$$

which gives

$$\begin{aligned}\langle|\phi_q|^2\rangle &= 1+\tfrac{\lambda}{\kappa}\sum_p\int \phi_q\,\tfrac{(\bar a+\bar\sigma)_p(a+\sigma)_{p+q}\bar\phi_q-\frac{\lambda}{\kappa}\bar\phi_0^2\phi_{-q}}{\left(|a_p+\sigma_p|^2+\frac{\lambda}{\kappa}|\phi_0|^2\right)\left(|a_{p+q}+\sigma_{p+q}|^2+\frac{\lambda}{\kappa}|\phi_0|^2\right)}\,dP(\phi)\\ &= 1+\langle|\phi_q|^2\rangle\tfrac{\lambda}{\kappa}\sum_p\tfrac{(\bar a_p+\langle\bar\sigma_p\rangle)(a_{p+q}+\langle\sigma_{p+q}\rangle)}{\left(|a_p+\langle\sigma_p\rangle|^2+\frac{\lambda}{\kappa}\langle|\phi_0|^2\rangle\right)\left(|a_{p+q}+\langle\sigma_{p+q}\rangle|^2+\frac{\lambda}{\kappa}\langle|\phi_0|^2\rangle\right)}\end{aligned}$$

or

$$\langle|\phi_q|^2\rangle=\frac{1}{1-\frac{\lambda}{\kappa}\sum_p\frac{(\bar a_p+\langle\bar\sigma_p\rangle)(a_{p+q}+\langle\sigma_{p+q}\rangle)}{\left(|a_p+\langle\sigma_p\rangle|^2+\frac{\lambda}{\kappa}\langle|\phi_0|^2\rangle\right)\left(|a_{p+q}+\langle\sigma_{p+q}\rangle|^2+\frac{\lambda}{\kappa}\langle|\phi_0|^2\rangle\right)}}\,,\quad q\neq 0 \tag{4.26}$$

We now discuss the solutions to (4.24) and (4.26). We assume that the solution $\langle\sigma_k\rangle$ of (4.21) is sufficiently small such that the BCS equation

$$\tfrac{\lambda}{\kappa}\sum_p\tfrac{1}{|a_p+\langle\sigma_p\rangle|^2+|\Delta|^2}=1 \tag{4.27}$$

has a nonzero solution $\Delta\neq 0$ (in particular this excludes Luttinger liquid behaviour, for $d=1$ one should make a seperate analysis), and make the Ansatz

$$\lambda\langle|\phi_0|^2\rangle=\beta L^d\,|\Delta|^2+\eta \tag{4.28}$$

where η is independent of the volume. Then

$$\begin{aligned}\tfrac{\lambda}{\kappa}\sum_p\tfrac{1}{|a_p+\langle\sigma_p\rangle|^2+\frac{\lambda}{\kappa}\langle|\phi_0|^2\rangle} &= \tfrac{\lambda}{\kappa}\sum_p\tfrac{1}{|a_p+\langle\sigma_p\rangle|^2+|\Delta|^2+\frac{\eta}{\kappa}}\\ &= \tfrac{\lambda}{\kappa}\sum_p\tfrac{1}{|a_p+\langle\sigma_p\rangle|^2+|\Delta|^2}-\tfrac{\lambda}{\kappa}\sum_p\tfrac{\eta/\kappa}{(|a_p+\langle\sigma_p\rangle|^2+|\Delta|^2)^2}+O\left((\tfrac{\eta}{\kappa})^2\right)\\ &= 1-c_\Delta\,\tfrac{\eta}{\kappa}+O\left((\tfrac{\eta}{\kappa})^2\right)\end{aligned} \tag{4.29}$$

where we put $c_\Delta=\frac{\lambda}{\kappa}\sum_p\frac{1}{(|a_p+\langle\sigma_p\rangle|^2+|\Delta|^2)^2}$ and used the BCS equation (4.27) in the last line. Equation (4.24) becomes

$$\kappa\,|\Delta|^2+\eta \;=\; \frac{\lambda}{c_\Delta\,\frac{\eta}{\kappa}+O\left((\frac{\eta}{\kappa})^2\right)}=\kappa\,\frac{\lambda}{c_\Delta\,\eta}+O(1) \tag{4.30}$$

and has a solution $\eta=\lambda/(c_\Delta|\Delta|^2)$. Now consider $\langle|\phi_q|^2\rangle$ for small but nonzero q. Substituting (4.28) and Taylor expanding around $q=0$ we get

$$\begin{aligned}\langle|\phi_q|^2\rangle &= \frac{1}{1-\frac{\lambda}{\kappa}\sum_p\frac{(\bar a_p+\langle\bar\sigma_p\rangle)(a_p+\langle\sigma_p\rangle)}{(|a_p+\langle\sigma_p\rangle|^2+|\Delta|^2)(|a_p+\langle\sigma_p\rangle|^2+|\Delta|^2)}+O(q)}\\ &= \frac{1}{\frac{\lambda}{\kappa}\sum_p\frac{|\Delta|^2}{(|a_p+\langle\sigma_p\rangle|^2+|\Delta|^2)^2}+O(q)}\end{aligned} \tag{4.31}$$

which is bounded for $q \to 0$. Thus, in the infinite volume limit and at zero temperature we find

$$\lambda\langle|\phi_q|^2\rangle = |\Delta|^2\,\delta(q) + \text{regular} \tag{4.32}$$

The function $\Lambda(q) = \frac{\lambda}{(\beta L^d)^3}\sum_{k,p}\langle\bar\psi_{k\uparrow}\bar\psi_{q-k\downarrow}\psi_{q-p\downarrow}\psi_{p\uparrow}\rangle = \langle|\phi_q|^2\rangle - 1$ or some closely related four-point functions have been discussed by some authors [FMRT,CFS,B]. The conclusion is $\Lambda(q) \sim 1/q^2$ for small q. Basically this is infered from the second order Taylor expansion of the effective potential

$$V_{\rm eff}(\{\phi_q\}) = \sum_q |\phi_q|^2 - \log\det\begin{bmatrix} \delta_{k,p} & \frac{ig}{\sqrt{\kappa}}\,\frac{\bar\phi_{p-k}}{a_k} \\ \frac{ig}{\sqrt{\kappa}}\,\frac{\phi_{k-p}}{a_{-k}} & \delta_{k,p} \end{bmatrix} \tag{4.33}$$

around its global minimum [L2]

$$\phi_q^{\rm min} = \sqrt{\beta L^d}\,\tfrac{|\Delta|}{\sqrt{\lambda}}\,\delta_{q,0}\,e^{i\theta_0} \tag{4.34}$$

where the phase θ_0 of ϕ_0 is arbitrary. If one expands $V_{\rm eff}$ up to second order in

$$\xi_q = \phi_q - \delta_{q,0}\sqrt{\beta L^d}\,\tfrac{|\Delta|}{\sqrt{\lambda}}\,e^{i\theta_0} = \begin{cases} \left(\rho_0 - \sqrt{\beta L^d}\,\tfrac{|\Delta|}{\sqrt{\lambda}}\right)e^{i\theta_0} & \text{for } q = 0 \\ \rho_q\,e^{i\theta_q} & \text{for } q \neq 0 \end{cases} \tag{4.35}$$

one obtains [L2]

$$\begin{aligned} V_{\rm eff}(\{\phi_q\}) &= V_{\rm min} + 2\beta_0\,(\rho_0 - \sqrt{\beta L^d}\,\tfrac{|\Delta|}{\sqrt{\lambda}})^2 + \sum_{q\neq 0}(\alpha_q + i\gamma_q)\,\rho_q^2 \\ &\quad + \tfrac{1}{2}\sum_{q\neq 0}\beta_q\,|e^{-i\theta_0}\phi_q + e^{i\theta_0}\bar\phi_{-q}|^2 + O(\xi^3) \end{aligned} \tag{4.36}$$

where for small q one has $\alpha_q, \gamma_q \sim q^2$. Hence, if $V_{\rm eff}$ is substituted by the right hand side of (4.36) one obtains $\langle|\phi_q|^2\rangle \sim 1/q^2$. Thus, in view of the above computations, it is questionable to what extent one can draw any conclusions from the quadratic approximation. We believe that the probability of the configurations $\{\phi_q\}$ for which (4.36) is a good approximation , the small field region, is too small to give a significant contribution to the functional integral.

Finally we argue why it is reasonable to substitute $|\phi_0|^2$ by its expectation value while performing the functional integral. We may write the effective potential (4.33) as

$$V_{\rm eff}(\{\phi_q\}) = V_1(\phi_0) + V_2(\{\phi_q\}) \tag{4.37}$$

where

$$\begin{aligned} V_1(\phi_0) &= |\phi_0|^2 - \sum_k \log\left[1 + \tfrac{\lambda}{\kappa}\tfrac{|\phi_0|^2}{k_0^2+e_{\mathbf{k}}^2}\right] \\ &= \kappa\Big(\tfrac{|\phi_0|^2}{\kappa} - \tfrac{1}{\kappa}\sum_k \log\left[1 + \tfrac{\lambda\frac{|\phi_0|^2}{\kappa}}{k_0^2+e_{\mathbf{k}}^2}\right]\Big) \equiv \kappa\,V_{\rm BCS}\left(\tfrac{|\phi_0|}{\sqrt{\kappa}}\right) \end{aligned} \tag{4.38}$$

and

$$V_2(\{\phi_q\}) = \sum_{q\neq 0} |\phi_q|^2 - \log\det\left[\begin{pmatrix} \delta_{k,p} & \frac{ig}{\sqrt{\kappa}}\frac{\bar{\phi}_0}{a_k}\delta_{k,p} \\ \frac{ig}{\sqrt{\kappa}}\frac{\phi_0}{a_{-k}}\delta_{k,p} & \delta_{k,p} \end{pmatrix}^{-1} \begin{pmatrix} \delta_{k,p} & \frac{ig}{\sqrt{\kappa}}\frac{\bar{\phi}_{p-k}}{a_k} \\ \frac{ig}{\sqrt{\kappa}}\frac{\phi_{k-p}}{a_{-k}} & \delta_{k,p} \end{pmatrix}\right] \tag{4.39}$$

If we ignore the ϕ_0-dependence of V_2, then the ϕ_0-integral

$$\frac{\int F\left(\frac{1}{\kappa}|\phi_0|^2\right)\, e^{-V_1(\phi_0)} d\phi_0 d\bar{\phi}_0}{\int e^{-V_1(\phi_0)} d\phi_0 d\bar{\phi}_0} \quad = \quad \frac{\int F\left(\rho^2\right)\, e^{-\kappa V_{\rm BCS}(\rho)} \rho\, d\rho}{\int e^{-\kappa V_{\rm BCS}(\rho)} \rho\, d\rho} \stackrel{\kappa\to\infty}{\to} F(\rho^2_{\rm min}) = F\left(\tfrac{1}{\kappa}\langle|\phi_0|^2\rangle\right) \tag{4.40}$$

simply puts $|\phi_0|^2$ at the global minimum of the (BCS) effective potential.

4.2 The φ^4-Model

In this section we choose the φ^4-model as a typical bosonic model to demonstrate our method. As in section 2, we start in finite volume $[0,L]^d$ on a lattice with lattice spacing $1/M$. The two point function is given by

$$S(x,y) = \langle\varphi_x\varphi_y\rangle := \frac{\int_{\mathbb{R}^{N^d}} \varphi_x\varphi_y\, e^{-\frac{g^2}{2}\frac{1}{M^d}\sum_x \varphi_x^4}\, e^{-\frac{1}{M^{2d}}\sum_{x,y}(-\Delta+m^2)_{x,y}\varphi_x\varphi_y}\, \prod_x d\varphi_x}{\int_{\mathbb{R}^{N^d}} e^{-\frac{g^2}{2}\frac{1}{M^d}\sum_x \varphi_x^4}\, e^{-\frac{1}{M^{2d}}\sum_{x,y}(-\Delta+m^2)_{x,y}\varphi_x\varphi_y}\, \prod_x d\varphi_x} \tag{4.41}$$

where

$$(-\Delta+m^2)_{x,y} = M^d\Big[-M^2\sum_{i=1}^{d}(\delta_{x,y-e_i/M}+\delta_{x,y+e_i/M}-2\delta_{x,y}) + m^2\delta_{x,y}\Big] \tag{4.42}$$

First we have to bring this into the form $\int [P+Q]^{-1}_{x,y} d\mu$, P diagonal in momentum space, Q diagonal in coordinate space. This is done again by making a Hubbard Stratonovich transformation which in this case reads

$$e^{-\frac{1}{2}\sum_x a_x^2} = \int e^{i\sum_x a_x u_x} e^{-\frac{1}{2}\sum_x u_x^2} \prod_x \tfrac{du_x}{\sqrt{2\pi}} \tag{4.43}$$

with

$$a_x = \tfrac{g}{\sqrt{M^d}}\varphi_x^2 \tag{4.44}$$

The result is Gaussian in the φ_x-variables and the integral over these variables gives

$$S(x,y) = \int_{\mathbb{R}^{N^d}} \left[\tfrac{1}{M^{2d}}(-\Delta+m^2)_{x,y} - \tfrac{ig}{\sqrt{M^d}} u_x\delta_{x,y}\right]^{-1}_{x,y} dP(u) \tag{4.45}$$

where

$$dP(u) = \tfrac{1}{Z}\det\left[\tfrac{1}{M^{2d}}(-\Delta+m^2)_{x,y} - \tfrac{ig}{\sqrt{M^d}}u_x\delta_{x,y}\right]^{-\frac{1}{2}} e^{-\frac{1}{2}\sum_x u_x^2}\prod_x du_x \tag{4.46}$$

Since we have bosons, the determinant comes with a power of $-1/2$ which is the only difference compared to a fermionic system. In momentum space this reads (compare equations (2.7-11))

$$S(x-y) = \tfrac{1}{L^d}\sum_k e^{ik(x-y)}\langle G\rangle(k) \tag{4.47}$$

where ($\gamma_q = v_q + iw_q$, $\gamma_{-q} = \bar\gamma_q$, $d\gamma_q d\bar\gamma_q := dv_q dw_q$)

$$\langle G\rangle(k) = \int_{\mathbb{R}^{Nd}} \left[a_k\delta_{k,p} - \tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-1}_{kk} dP(\gamma) \tag{4.48}$$

and

$$dP(\gamma) = \tfrac{1}{Z}\det\left[a_k\delta_{k,p} - \tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-\frac{1}{2}} e^{-\frac{1}{2}v_0^2}dv_0 \prod_{q\in\mathcal{M}^+} e^{-|\gamma_q|^2}d\gamma_q d\bar\gamma_q \tag{4.49}$$

and $\mathcal{M}^+$ again is a set such that either $q\in\mathcal{M}^+$ or $-q\in\mathcal{M}^+$. Furthermore

$$a_k = 4M^2\sum_{i=1}^d \sin^2\left[\tfrac{k_i}{2M}\right] \;+\; m^2 \tag{4.50}$$

Equation (4.48) is our starting point. We apply (1.3) to the inverse matrix element in (4.48). In the two loop approximation one obtains ($\gamma_0 = v_0 \in \mathbb{R}$)

$$\left[a_k\delta_{k,p} - \tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-1}_{kk} \approx \frac{1}{a_k - \frac{igv_0}{\sqrt{L^d}} + \frac{g^2}{L^d}\sum_{p\neq k} G_k(p)|\gamma_{k-p}|^2} =: \frac{1}{a_k+\sigma_k} \tag{4.51}$$

where

$$\sigma_k = -\tfrac{ig}{\sqrt{L^d}}v_0 + \tfrac{g^2}{L^d}\sum_{p\neq k}\frac{|\gamma_{k-p}|^2}{a_p - \frac{igv_0}{\sqrt{L^d}} + \sigma_p} \tag{4.52}$$

which results in

$$\langle G\rangle(k) = \frac{1}{a_k + \langle\sigma_k\rangle} \tag{4.53}$$

where $\langle\sigma_k\rangle$ has to satisfy the equation

$$\begin{aligned}\langle\sigma_k\rangle \;&=\; -\tfrac{ig}{\sqrt{L^d}}\langle v_0\rangle + \tfrac{g^2}{L^d}\sum_{p\neq k}\frac{\langle|\gamma_{k-p}|^2\rangle}{a_p+\langle\sigma_p\rangle}\\ &=\; \tfrac{g^2}{2L^d}\sum_p\langle G\rangle(p) + \tfrac{g^2}{L^d}\sum_{p\neq k}\frac{\langle|\gamma_{k-p}|^2\rangle}{a_p+\langle\sigma_p\rangle} = \tfrac{g^2}{L^d}\sum_{p\neq k}\frac{\langle|\gamma_{k-p}|^2\rangle+\frac{1}{2}}{a_p+\langle\sigma_p\rangle}\end{aligned} \tag{4.54}$$

where the last line is due to

$$\begin{aligned}
\langle v_0\rangle &= \tfrac{1}{Z}\int v_0 \det\left[a_k\delta_{k,p}-\tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-\frac{1}{2}} e^{-\frac{1}{2}v_0^2}dv_0 \prod_{q\in\mathcal{M}^+} e^{-|\gamma_q|^2}d\gamma_q d\bar\gamma_q \\
&= \tfrac{1}{Z}\int \left\{\tfrac{\partial}{\partial v_0}\det\left[a_k\delta_{k,p}-\tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-\frac{1}{2}}\right\} e^{-\frac{1}{2}v_0^2}dv_0 \prod_{q\in\mathcal{M}^+} e^{-|\gamma_q|^2}d\gamma_q d\bar\gamma_q \\
&= -\tfrac{1}{2}\sum_p \left(-\tfrac{ig}{\sqrt{L^d}}\right)\int \left[a_k\delta_{k,p}-\tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-1}_{pp} dP(\gamma) \qquad (4.55)
\end{aligned}$$

As for the Many-Electron system, we can derive an equation for $\langle|\gamma_q|^2\rangle$ by partial integration:

$$\begin{aligned}
\langle|\gamma_q|^2\rangle &= \tfrac{1}{Z}\int \gamma_q\bar\gamma_q \det\left[a_k\delta_{k,p}-\tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-\frac{1}{2}} e^{-\frac{v_0^2}{2}}dv_0 \prod_q e^{-|\gamma_q|^2}d\gamma_q d\bar\gamma_q \\
&= 1+\tfrac{1}{Z}\int \gamma_q\tfrac{\partial}{\partial\gamma_q}\left\{\det\left[a_k\delta_{k,p}-\tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-\frac{1}{2}}\right\} e^{-\frac{v_0^2}{2}}dv_0 \prod_q e^{-\frac{1}{2}|\gamma_q|^2}d\gamma_q d\bar\gamma_q \\
&= 1-\tfrac{1}{2}\int \gamma_q \frac{\frac{\partial}{\partial\gamma_q}\det\left[a_k\delta_{k,p}-\frac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]}{\det\left[a_k\delta_{k,p}-\frac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]}dP(\gamma) \\
&= 1-\tfrac{1}{2}\sum_p \tfrac{-ig}{\sqrt{L^d}}\int \gamma_q \left[a_k\delta_{k,p}-\tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-1}_{p,p+q} dP(\gamma) \qquad (4.56)
\end{aligned}$$

Computing the inverse matrix element in (4.56) again in the two loop approximation, one arrives at

$$\langle|\gamma_q|^2\rangle = 1-\langle|\gamma_q|^2\rangle\tfrac{g^2}{2L^d}\sum_p \tfrac{1}{(a_p+\langle\sigma_p\rangle)(a_{p+q}+\langle\sigma_{p+q}\rangle)}$$

or

$$\langle|\gamma_q|^2\rangle = \frac{1}{1+\frac{g^2}{2}\int_{[0,2\pi M]^d}\frac{d^dp}{(2\pi)^d}\frac{1}{(a_p+\langle\sigma_p\rangle)(a_{p+q}+\langle\sigma_{p+q}\rangle)}} \qquad (4.57)$$

which has to be solved in conjunction with

$$\langle\sigma_k\rangle = g^2\int_{[0,2\pi M]^d}\tfrac{d^dp}{(2\pi)^d}\tfrac{\langle|\gamma_{k-p}|^2\rangle+\frac{1}{2}}{a_p+\langle\sigma_p\rangle} \qquad (4.58)$$

Introducing the rescaled quantities

$$\langle\sigma_k\rangle = M^2 s_{\frac{p}{M}}\,,\quad \langle|\gamma_q|^2\rangle=\lambda_{\frac{q}{M}}\,,\quad a_k=M^2\varepsilon_{\frac{k}{M}}\,,\ \ \varepsilon_k=\sum_{i=1}^d \sin^2\tfrac{k_i}{2}+\tfrac{m^2}{M^2} \qquad (4.60)$$

(4.57,58) read

$$\begin{aligned}
s_k &= M^{d-4}g^2\int_{[0,2\pi]^d}\tfrac{d^dp}{(2\pi)^d}\tfrac{\lambda_{k-p}+\frac{1}{2}}{\varepsilon_p+s_p} \qquad (4.61)\\
\lambda_q &= \frac{1}{1+M^{d-4}\frac{g^2}{2}\int_{[0,2\pi]^d}\frac{d^dp}{(2\pi)^d}\frac{1}{(\varepsilon_p+s_p)(\varepsilon_{p+q}+s_{p+q})}} \qquad (4.62)
\end{aligned}$$

Unfortunately we cannot check this result with the rigorously proven triviality theorem since $\langle\sigma_k\rangle$ and $\langle|\gamma_q|^2\rangle$ only give information on the 2-point function $S(x,y)$, (4.41), and on $\frac{g^2}{M^d}\sum_x\langle\varphi(x)^4\rangle = \sum_q\Lambda(q)$ where $\Lambda(q) = \langle|\gamma_q|^2\rangle - 1$. However, the triviality theorem [F,FFS] makes a statement on the connected 4-point function $S_{4,c}(x_1,x_2,x_3,x_4)$ at noncoinciding arguments, namely that this function vanishes in the continuum limit in dimension $d>4$.

We now include the higher loop terms of (1.3,4) and give an interpretation in terms of diagrams. The exact equations for $\langle G\rangle(k)$ and $\langle|\gamma_q|^2\rangle$ are

$$\begin{aligned}\langle G\rangle(k) &= \int\left[a_k\delta_{k,p}-\tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-1}_{kk}dP(\gamma) \;=\; \left\langle\tfrac{1}{a_k+\sigma_k}\right\rangle \\ \sigma_k &= -\tfrac{ig}{\sqrt{L^d}}\,v_0+\sum_{r=2}^{N^d}\left(\tfrac{ig}{\sqrt{L^d}}\right)^r\sum_{\substack{p_2\cdots p_r\neq k\\ p_i\neq p_j}}G_k(p_2)\cdots G_{kp_2\cdots p_{r-1}}(p_r)\,\gamma_{k-p_2}\gamma_{p_2-p_3}\cdots\gamma_{p_r-k}\end{aligned}\tag{4.63}$$

and

$$\begin{aligned}\langle|\gamma_q|^2\rangle &= 1+\tfrac{ig}{2\sqrt{L^d}}\sum_p\int\gamma_q\left[a_k\delta_{k,p}-\tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-1}_{p,p+q}dP(\gamma) \\ &\overset{p\to p_2}{=} 1+\tfrac{1}{2}\sum_{r=2}^{N^d}\left(\tfrac{ig}{\sqrt{L^d}}\right)^r\sum_{\substack{p_2\cdots p_r\neq p_2+q\\ p_i\neq p_j}}\Big\langle G(p_2)G_{p_2}(p_3)\cdots G_{p_2\cdots p_{r-1}}(p_r)G_{p_2\cdots p_r}(p_2+q)\times \\ &\qquad\qquad \gamma_{p_2-p_3}\cdots\gamma_{p_{r-1}-p_r}\gamma_{p_r-p_2-q}\gamma_{p_2+q-p_2}\Big\rangle\end{aligned}\tag{4.64}$$

For $r>2$, we obtain terms $\langle\gamma_{k_1}\cdots\gamma_{k_r}\rangle$ whose connected contributions

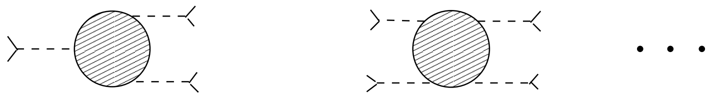

Figure 4: three and higher loop contributions

are, in terms of the electron or φ^4-lines, are at least six-legged. Since for the many-electron system and for the φ^4-model (for $d=4$) the relevant diagrams are two- and four-legged [FT,R], one may start with an approximation which ignores the connected r-loop contributions for $r>2$. This is obtained by writing

$$\left\langle\gamma_{k_1}\cdots\gamma_{k_n}\right\rangle \approx \left\langle\gamma_{k_1}\cdots\gamma_{k_n}\right\rangle_2 \tag{4.65}$$

where (the index '2' for 'retaining only two-loop contributions')

$$\left\langle\gamma_{k_1}\cdots\gamma_{k_{2n}}\right\rangle_2 := \sum_{\text{pairings }\sigma}\langle\gamma_{k_{\sigma 1}}\gamma_{k_{\sigma 2}}\rangle\cdots\langle\gamma_{k_{\sigma(2n-1)}}\gamma_{k_{\sigma 2n}}\rangle = \int\gamma_{k_1}\cdots\gamma_{k_{2n}}\,dP_2(\gamma) \tag{4.66}$$

if we define

$$dP_2(\gamma) := e^{-\sum_q \frac{|\gamma_q|^2}{\langle|\gamma_q|^2\rangle}} \prod_q \frac{d\gamma_q d\bar{\gamma}_q}{\pi\,\langle|\gamma_q|^2\rangle} \tag{4.67}$$

Substituting dP by dP_2 in (4.63,64), we obtain a model which differs from the original model only by irrelevant contributions and for which we are able to write down a closed set of equations for the two-legged particle correlation function $\langle G\rangle(k)$ and the two-legged squiggle correlation function $\langle|\gamma_q|^2\rangle$ by resumming all two-legged (particle and squiggle) subdiagrams. The exact equations of this model are

$$\langle G\rangle(k) = \int \left[a_k\delta_{k,p} - \tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-1}_{kk} dP_2(\gamma) \tag{4.68}$$

$$\langle|\gamma_q|^2\rangle = 1 + \tfrac{ig}{2\sqrt{L^d}}\sum_p \int \gamma_q \left[a_k\delta_{k,p} - \tfrac{ig}{\sqrt{L^d}}\gamma_{k-p}\right]^{-1}_{p,p+q} dP_2(\gamma) \tag{4.69}$$

and the resummation of the two-legged particle and squiggle subdiagrams is obtained by applying the inversion formula (1.3,4) to the inverse matrix elements in (4.68,69). A discussion similar to those of section 2 gives the following closed set of equations for the quantities $\langle G\rangle(k)$ and $\langle|\gamma_q|^2\rangle$:

$$\langle G\rangle(k) = \frac{1}{a_k + \langle\sigma_k\rangle}, \qquad \langle|\gamma_q|^2\rangle = \frac{1}{1+\langle\pi_q\rangle} \tag{4.70}$$

where

$$\langle\sigma_k\rangle = \tfrac{g^2}{2L^d}\sum_p \langle G\rangle(p) + \sum_{r=2}^{\ell}\left(\tfrac{ig}{\sqrt{L^d}}\right)^r \sum_{\substack{p_2\cdots p_r\neq k\\ p_i\neq p_j}} \langle G\rangle(p_2)\cdots\langle G\rangle(p_r)\,\langle\gamma_{k-p_2}\gamma_{p_2-p_3}\cdots\gamma_{p_r-k}\rangle_2 \tag{4.71}$$

$$\langle\pi_q\rangle = -\tfrac{1}{2}\sum_{r=2}^{\ell}\left(\tfrac{ig}{\sqrt{L^d}}\right)^r \sum_{s=3}^{r-1} \sum_{\substack{p_2\cdots p_r\neq p_2+q\\ p_i\neq p_j}} \Big(\delta_{q,p_{s+1}-p_s}\,\langle G\rangle(p_2)\cdots\langle G\rangle(p_r)\,\langle G\rangle(p_2+q)\times$$
$$\langle\gamma_{p_2-p_3}\cdots\widehat{\gamma}_{p_s-p_{s+1}}\cdots\gamma_{p_{r-1}-p_r}\gamma_{p_r-p_2-q}\rangle_2\Big) \tag{4.72}$$

In the last line we used that γ_q in (4.64) cannot contract to $\gamma_{p_2-p_3}$ or to $\gamma_{p_r-p_2-q}$. If the expectations of the γ-fields on the right hand side of (4.71,72) are computed according to (4.66), one obtains the expansion into diagrams. The graphs contributing to $\langle\sigma_k\rangle$ have exactly one string of particle lines, each line having $\langle G\rangle$ as propagator, and no particle loops (up to the tadpole diagram). Each squiggle corresponds to a factor $\langle|\gamma|^2\rangle$. The diagrams contributing to $\langle\pi\rangle$ have exactly one particle loop, the propagators being again the interacting two point functions, $\langle G\rangle$ for the particle lines and $\langle|\gamma|^2\rangle$ for the squiggles. In both cases there are no two-legged subdiagrams. However, although the equation $\langle|\gamma_q|^2\rangle = \frac{1}{1+\langle\pi_q\rangle}$ resums ladder or bubble diagrams (which is apparent from (4.57) or (4.26)) and more general four-legged particle subdiagrams if the terms for $r \geq 4$ in (4.72) are taken into account, the right hand side of (4.71,72) still contains diagrams with four-legged particle subdiagrams. Thus, the resummation of four-legged particle subdiagrams is only partially through the complete resummation of two-legged squiggle diagrams.

Finally observe that, in going from (4.68,69) to (4.70-72), we cut off the r-sum at some fixed order ℓ independent of the volume since we can only expect that the expansions are asymptotic ones, compare the discussion in section 2.

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