The kernel of Dirac operators on $\mathbb{S}^3$ and $\mathbb{R}^3$

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Abstract

In this paper we describe an intrinsically geometric way of producing magnetic fields on $\mathbb{S}^3$ and $\mathbb{R}^3$ for which the corresponding Dirac operators have a non-trivial kernel. In many cases we are able to compute the dimension of the kernel. In particular we can give examples where the kernel has any given dimension. This generalizes the examples of Loss and Yau [LY].

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1 Introduction

In [LY] Loss and Yau proved the existence of a magnetic field $\mathbf{B} = \nabla \times \mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ with the property that $\int_{\mathbb{R}^3} |\mathbf{B}|^2 < \infty$ and such that the Dirac operator

$$\sigma \cdot (-i\nabla - \mathbf{A}) \tag{1}$$

has a nonvanishing kernel in $L^2(\mathbb{R}^3; \mathbb{C}^2)$.

The significance of this result was its implications to the stability of matter (electrons and nuclei) coupled to classical electromagnetic fields. This model was studied in the series of papers [FLL, LY, LL]. The existence of a square integrable zero mode for the Dirac operator corresponding to a square integrable magnetic field implies that matter cannot be stable unless there is an upper bound on the fine structure constant.

Loss and Yau gave a very explicit construction of a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and of a corresponding zero mode, i.e. a solution to $[\sigma \cdot (-i\nabla - \mathbf{A})]\psi = 0$. They also discussed a general way of constructing a vector potential $\mathbf{A}$ for a given ψ so that ψ be in the kernel of the Dirac operator (1). However their methods gave only one element of the kernel for each magnetic field constructed. Moreover, the proofs are very computational and they somewhat left the origin of these zero modes unexplored. Further examples of zero modes were given later in [E] and [AMN1], based upon ideas from the Loss and Yau construction.

In this paper we discuss a more geometric way of constructing Dirac operators with a non trivial kernel on $\mathbb{R}^3$. More precisely, we describe a family of magnetic fields on the 3-sphere $\mathbb{S}^3$ for which we can give a characterization of the spectrum and in particular for some of these fields we can also calculate the dimension of the kernel. It is a well known fact (see [H] and Theorem 23 below) that the dimension of the kernel of the Dirac operator is a conformal invariant. Since $\mathbb{R}^3$ is conformally invariant to the 3-sphere with a point removed we can use the construction on $\mathbb{S}^3$ to learn about the kernel of Dirac operators on $\mathbb{R}^3$.

On a general Riemannian manifold one can define the Dirac operator if one has a *Spin* structure, a corresponding spinor bundle, and an appropriate covariant derivative (a *Spin* connection). The kernel of this (nonmagnetic) Dirac operator has been studied in [B]. If one is interested in Dirac operators with magnetic fields one must consider instead $Spin^c$ structures, $Spin^c$ spinor bundles and a $Spin^c$ connections. The magnetic field is then related to the curvature of the connection (see Definition 3). On $\mathbb{R}^3$ these structures reduce to the well known objects. The $Spin^c$ spinors are simply maps from $\mathbb{R}^3 \to \mathbb{C}^2$ and the Dirac operators are of the form (1).

On 2-dimensional manifolds and in general on even dimensional manifolds the Atiyah-Singer Index Theorem often gives nontrivial information on the index of the Dirac operator. In certain cases one knows from vanishing theorems that the index is equal to the dimension of the kernel. One example of such a result is the Aharonov-Casher Theorem (see Theorem 37) which holds for Dirac operators on $\mathbb{R}^2$ and $\mathbb{S}^2$. Characteristic for the index theorem is of course that the index is expressed in terms of topological quantities, whereas in general the dimension of the kernel is not a topological invariant. For odd-dimensional manifolds it is not easy to get information about the dimension of the kernel from index theorems. Given a Dirac operator (with magnetic field) on $\mathbb{S}^3$ we do not know in general how to say anything about its kernel.

In this paper we explain how, for certain magnetic fields on $\mathbb{S}^3$ one may, in a sense, separate variables and reduce the problem to a problem on $\mathbb{S}^2$, where one can use the Aharonov-Casher Theorem, and a problem on $\mathbb{S}^1$ which can be solved explicitly.

Our construction can be used on other manifolds than $\mathbb{S}^3$.

In Sect. 2 we discuss *Spin*^{c} structures and define the Dirac operator and magnetic fields. In Sect. 4 we discuss how the Dirac operator changes under conformal transformations. In Sect. 5 we describe how to lift *Spin*^{c} structures from 2 to 3-dimensional manifolds. One can of course not always do this. It requires that there is a map of the type known as a Riemannian submersion from the 3 to the 2-dimensional manifold. If such a map exists we give a lower bound on the kernel of the 3-dimensional Dirac operator using the index Theorem for the 2-dimensional Dirac operator (see Theorem 31 in Sect. 6). Finally in Sect. 8 we give the more detailed results for $\mathbb{S}^3$. In this case we use the Hopf map as the Riemannian submersion from $\mathbb{S}^3$ to $\mathbb{S}^2$. The Hopf map however has much stronger properties than just being a Riemannian submersion. These properties allow us to separate variables for the Dirac operator on $\mathbb{S}^3$. Our main results for $\mathbb{S}^3$ can be found in Theorem 35 and the remarks following it.

In particular, we show that one can construct Dirac operators on $\mathbb{R}^3$ and $\mathbb{S}^3$ having kernels of any given dimension. Examples of Dirac operators on $\mathbb{R}^3$ with degenerate kernels were recently given independently in [AMN2] for a subclass of the magnetic fields considered here. The exact degeneracy was however not proved there.

Our results were announced in [ES].

2 *Spin*^{c} bundles

1. DEFINITION (*Spin*^{c} spinor bundle). Let M be a 3-dimensional Riemannian manifold. A *Spin^c spinor bundle* Ψ over M is a 2-dimensional complex vector bundle over M with inner product and an isometry $\sigma : T^*M \to \Psi^{(2)}$, where

$$\Psi^{(2)} := \{A \in \mathrm{End}(\Psi) : A = A^*, \mathrm{Tr}A = 0\}.$$

The inner product on $\Psi^{(2)}$ is given by $(A, B) := \frac{1}{2}\mathrm{Tr}[AB]$. A spinor bundle over a 2-dimensional Riemannian manifold is defined in the same way except that then σ is only an injective partial isometry. The map σ is called the *Clifford multiplication* of the spinor bundle Ψ.

Note that if $A, B \in \Psi^2$ then $\{A, B\} := AB + BA = \mathrm{Tr}[AB]I = 2(A, B)I$ and therefore

$$\{\sigma(\alpha), \sigma(\beta)\} = 2(\alpha, \beta)I, \quad \text{for all } \alpha, \beta \in T^*M \tag{2}$$

where $(\cdot, \cdot)$ in the last equation denotes the metric (inner product) on T^*M. We have here used the convention that the Clifford multiplication is Hermitian rather than anti-Hermitian, which is the more common in the mathematics literature.

2. DEFINITION (***Spin*** **spinor bundle).** A *Spin^c* spinor bundle Ψ over a 2 or 3-dimensional manifold M is said to be a *Spin spinor bundle* if there exists an antilinear bundle isometry $\mathcal{C} : \Psi \to \Psi$ such that $(\eta, \mathcal{C}\eta) = 0$ and $\mathcal{C}^2\eta = -\eta$ for all $\eta \in \Psi$. In the physics literature the map $\mathcal{C}$ is often referred to as *charge conjugation.* An equivalent way to say that a *Spin^c* bundle is actually a Spin bundle is to say that the *determinant line bundle* $\Psi \wedge \Psi$ is trivial.

3. REMARK. We shall use mostly *Spin^c* spinor bundles in our results. For brevity, we shall refer to them simply as *spinor bundles. Spin* spinor bundles will be mentioned only in some additional remarks.

The following proposition shows that in 3-dimensions the spinor bundle gives a natural orientation of M.

4. PROPOSITION. *Let Ψ be a spinor bundle over a 3-dimensional manifold M with Clifford multiplication σ. If e^1, e^2, e^3 is an orthonormal basis in T_p^*M then $i\sigma(e^1)\sigma(e^2)\sigma(e^3) = \pm I$.*

Proof. Let $a = i\sigma(e^1)\sigma(e^2)\sigma(e^3)$. It follows immediately from (2) that a is Hermitian and commutes with $\sigma(e^j)$, $j = 1, 2, 3$. Hence a is a real scalar. Again from (2) it is clear that $\sigma(e^j)^2 = I$, for $j = 1, 2, 3$ and hence $a^2 = I$. $\square$

5. DEFINITION (Positive orientation). We say that e^1, e^2, e^3 is a positively oriented basis if $i\sigma(e^1)\sigma(e^2)\sigma(e^3) = -I$.

6. REMARK (Spinors over non-orientable manifolds). Proposition 4 shows that a spinor bundle with $\mathbb{C}^2$ fibers can exist only on orientable manifolds. However, one may also define spinor bundles over *non-orientable* 3-dimensional manifolds but in this case it should be a $\mathbb{C}^4$ bundle and the Clifford multiplication map should be an injective partial isometry satisfying (2).

7. PROPOSITION (Basis for Ψ gives basis for T_*M). *Let $\xi_\pm$ be a local orthonormal basis of spinor fields. Define the vectors e_1, e_2 such that*

$$\alpha(e_1) + i\alpha(e_2) = (\xi_-, \sigma(\alpha)\xi_+)$$

holds for all one-forms α. Then e_1, e_2 are orthonormal.

In the case when M is 3-dimensional also define e_3 such that

$$\alpha(e_3) = (\xi_+, \sigma(\alpha)\xi_+)$$

for all α. Then e_1, e_2, e_3 is a positively oriented orthonormal basis.

Proof. We treat the 3-dimensional case, the 2-dimensional case is similar. We prove instead that there is an orthonormal basis of forms e^1, e^2, e^3 for which e_1, e_2, e_3 is the dual basis. This follows if we show that $\sigma(e^1)$, $\sigma(e^2)$, $\sigma(e^3)$ are orthonormal in $\Psi^{(2)}$. Of course we define e^1, e^2, e^3 by $e^j(e_i) = \delta_{ij}$, for $i, j = 1, \ldots$. We then see from the definitions of e_1, e_2, e_3 that the matrices of $\sigma(e^1)$, $\sigma(e^2)$, $\sigma(e^3)$, in the basis $\xi_\pm$ are the standard Pauli matrices (note that $(\xi_+, \sigma(\alpha)\xi_+) = -(\xi_-, \sigma(\alpha)\xi_-)$ since $\sigma(\alpha)$ is traceless)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The orthonormality and positivity follows from simple matrix calculations.

□

8. DEFINITION (*Spin^c* Connection). A connection ∇ on a spinor bundle Ψ is said to be a *Spin^c* connection if for all tangent vectors $X \in T_*M$ we have

(i.) $X(\xi, \eta) = (\nabla_X \xi, \eta) + (\xi, \nabla_X \eta)$ for all sections ξ, η in Ψ.

(ii.) $[\nabla_X, \sigma(\alpha)] = \sigma(\nabla_X \alpha)$ for all one-forms α on M. Here $\nabla_X \alpha$ refers to the Levi-Civita connection acting on one-forms.

9. PROPOSITION (Local expression for ∇_X). *Let $\xi_\pm$ be a local orthonormal basis of spinor fields on a 3-dimensional manifold and let e_1, e_2, e_3 be the orthonormal basis defined in Prop. 7. Then for all vectors X*

$$\begin{pmatrix} (\xi_+, \nabla_X \xi_+) & (\xi_+, \nabla_X \xi_-) \\ & \\ (\xi_-, \nabla_X \xi_+) & (\xi_-, \nabla_X \xi_-) \end{pmatrix}$$

$$= i\tfrac{1}{2} \begin{pmatrix} (e_1, \nabla_X e_2) & -(e_3, \nabla_X e_2) - i(e_3, \nabla_X e_1) \\ & \\ -(e_3, \nabla_X e_2) + i(e_3, \nabla_X e_1) & -(e_1, \nabla_X e_2) \end{pmatrix} - i\alpha(X)I,$$

where α is the real local one-form given by

$$\alpha(X) = i\tfrac{1}{2}(\xi_+, \nabla_X \xi_+) + i\tfrac{1}{2}(\xi_-, \nabla_X \xi_-).$$

The same formulas are true in the 2-dimensional case if we simply replace e_3 by 0 everywhere.

Proof. Let e^1, e^2, e^3 be the dual basis to e_1, e_2, e_3. Using that $X(\omega(e_j)) = \nabla_X \omega(e_j) + \omega(\nabla_X e_j)$ for any one-form ω we find, with $\omega = e^j$, that

$$\begin{aligned}
(e_j, \nabla_X e_1) + i(e_j, \nabla_X e_2) &= e^j(\nabla_X e_1) + ie^j(\nabla_X e_2) \\
&= X(\xi_-, \sigma(e^j)\xi_+) - (\xi_-, \sigma(\nabla_X e^j)\xi_+) \\
&= (\nabla_X \xi_-, \sigma(e^j)\xi_+) + (\xi_-, \sigma(e^j)\nabla_X \xi_+).
\end{aligned}$$

Since $\sigma(e^3)\xi_\pm = \pm\xi_\pm$ (see the proof of Prop. 7) we find

$$(e_3, \nabla_X e_1) + i(e_3, \nabla_X e_2) = (\nabla_X \xi_-, \xi_+) - (\xi_-, \nabla_X \xi_+) = -2(\xi_-, \nabla_X \xi_+)$$

where we have also used that $0 = X(\xi_-, \xi_+) = (\nabla_X \xi_-, \xi_+) + (\xi_-, \nabla_X \xi_+)$. This also gives

$$(\xi_+, \nabla_X \xi_-) = -\overline{(\xi_-, \nabla_X \xi_+)} = \tfrac{1}{2}(e_3, \nabla_X e_1) - i\tfrac{1}{2}(e_3, \nabla_X e_2).$$

Using $\sigma(e^1)\xi_\pm = \xi_\mp$ we obtain

$$\begin{aligned}
(\xi_+, \nabla_X \xi_+) - (\xi_-, \nabla_X \xi_-) &= (\nabla_X \xi_-, \xi_-) + (\xi_+, \nabla_X \xi_+) \\
&= (e_1, \nabla_X e_1) + i(e_1, \nabla_X e_2) = i(e_1, \nabla_X e_2)
\end{aligned}$$

where we have used that $(e_1, \nabla_X e_1) = \frac{1}{2}X(e_1, e_1) = 0$. It only remains to show that α is real. This follows from $0 = X(\xi_\pm, \xi_\pm) = 2\mathrm{Re}(\xi_\pm, \nabla_X \xi_\pm)$.

In the 2-dimensional case we conclude from $(\nabla_X e^j, e^j) = 0$ that $\nabla_X e^1 = (\nabla_X e^1, e^2)e^2$ and $\nabla_X e^2 = (\nabla_X e^2, e^1)e^1$. Since $\sigma(e^1)\sigma(e^2)\xi_+ = i\xi_+$ we have

$$\begin{aligned}
\sigma(e^1)\sigma(e^2)\nabla_X \xi_+ &= \nabla_X(\sigma(e^1)\sigma(e^2)\xi_+) - \sigma(\nabla_X e^1)\sigma(e^2)\xi_+ - \sigma(e^1)\sigma(\nabla_X e^2)\xi_+ \\
&= i\nabla_X \xi_+ - \left((\nabla_X e^1, e^2) + (\nabla_X e^2, e^1)\right)\xi_+ = i\nabla_X \xi_+,
\end{aligned}$$

where we have used that $(\nabla_X e^1, e^2) + (\nabla_X e^2, e^1) = X(e^1, e^2) = 0$. Therefore $\nabla_X \xi_+$ is an eigenvector with eigenvalue i of the antihermitean operator $\sigma(e^1)\sigma(e^2)$. Since $\sigma(e^1)\sigma(e^2)\xi_- = -i\xi_-$ we conclude that $(\xi_-, \nabla_X \xi_+) = 0$ in the 2-dimensional case. The other statements in the 2-dimensional case follow as for the 3-dimensional case. $\square$

10. DEFINITION (Curvature tensor and magnetic form on Ψ). As usual the curvature tensor is defined by

$$\mathcal{R}_\Psi(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi,$$

where ξ is a spinor section and X, Y are vector fields on M. The *Magnetic 2-form* β on M is defined via the trace of $\mathcal{R}_\Psi$ as

$$\beta(X,Y) = i\tfrac{1}{2}\mathrm{Tr}[\mathcal{R}_\Psi(X,Y)]. \tag{3}$$

11. PROPOSITION. *The operator $i\mathcal{R}_\Psi(X,Y)$ is Hermitian on Ψ and β is a real closed two-form. Locally $\beta = d\alpha$, where α is the one-form defined in Proposition 9.*

Proof. In fact it follows from the definition of a *Spin*^{c} connection that

$$X\left(Y\left((\xi,\eta)\right)\right) = (\nabla_X\nabla_Y\xi, \eta) + (\xi, \nabla_X\nabla_Y\eta) + (\nabla_X\xi, \nabla_Y\eta) + (\nabla_Y\xi, \nabla_X\eta).$$

Hence

$$\begin{aligned}
&(\nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi, \eta) + (\xi, \nabla_X\nabla_Y\eta - \nabla_Y\nabla_X\eta)\\
&= \; X\left(Y\left((\xi,\eta)\right)\right) - Y\left(X\left((\xi,\eta)\right)\right)\\
&= \; [X,Y]\left((\xi,\eta)\right) = (\nabla_{[X,Y]}\xi, \eta) + (\xi, \nabla_{[X,Y]}\eta)
\end{aligned}$$

and thus $(\mathcal{R}_\Psi(X,Y)\xi, \eta) + (\xi, \mathcal{R}_\Psi(X,Y)\eta) = 0$.

That β is real is a consequence of $i\mathcal{R}_\Psi$ being Hermitian. That it is a 2-form, i.e, that it is antisymmetric is immediate from the definition. That $\beta = d\alpha$ will be shown in the proof of Theorem 12, hence it immediately follows that β is closed. $\square$

12. THEOREM (Spinor curvature in terms of Riem. curvature). *Let $\mathcal{R}$ denote the Riemann curvature tensor of M and let $\{e_i\}$ be any orthonormal basis of vector fields and $\{e^i\}$ the dual basis of one-forms. For all vectors X and Y we then have the identity*

$$\mathcal{R}_\Psi(X,Y) = \tfrac{1}{4}\sum_{ij}(e_i, \mathcal{R}(X,Y)e_j)\sigma(e^i)\sigma(e^j) - i\beta(X,Y)I$$

Proof. It is enough to check this identity at a point $p \in M$. It is also clear that the identity is independent of the choice of orthonormal basis of vectors. We may choose an orthonormal spinor basis $\xi_\pm$ such that $\nabla_X \xi_\pm = 0$ at p for all vectors X. The corresponding vectors e_i defined in Prop. 7 gives a local geodesic basis at p, i.e., $\nabla_{e_j} e_i = 0$. Moreover, the one-form α from Prop. 9 vanishes at the point p. Using the local expression for the $Spin^c$ connection given in Prop. 9 we find

$$\begin{aligned}\mathcal{R}_\Psi(X,Y)\xi_\pm &= \tfrac{i}{2}\left(\pm(e_1, \mathcal{R}(X,Y)e_2)\xi_\pm - \left((e_3, \mathcal{R}(X,Y)e_2) \mp i(e_3, \mathcal{R}(X,Y)e_1)\right)\xi_\mp\right) \\ &\quad -i\left(\nabla_X\alpha(Y) - \nabla_Y\alpha(X)\right)\xi_\pm\end{aligned}$$

That this agrees with the expression stated in the theorem follows easily from the expressions in Prop. 7. We see immediately that $\beta = d\alpha$. If we change the basis $\xi_\pm$, the one-form α changes by the addition of an exact one-form, hence $d\alpha$ is unchanged. $\square$

We end our discussion on $Spin^c$ structures by showing how these results are modified for *Spin* spinor bundles.

13. DEFINITION (*Spin* Connection). A $Spin^c$ connection ∇ on a spinor bundle which is also a *Spin* spinor bundle is said to be a *Spin* connection if ∇ commutes with the charge conjugation operator $\mathcal{C}$.

14. PROPOSITION (Uniqueness of *Spin* connections). *If ∇' and ∇'' are two $Spin^c$ connections on the same spinor bundle then there is a (real) one-form ω such that $(\nabla'_X - \nabla''_X)\xi = i\omega(X)\xi$, for all vector fields X and all spinor fields ξ. In particular, if ∇' and ∇'' are two Spin connections on the same Spin spinor bundle then $\nabla' = \nabla''$.*

Proof. It follows immediately from the definition of $Spin^c$ connections that $(\nabla'_X - \nabla''_X)$ commutes with multiplication by (complex) functions and with Clifford multiplication of one-forms. Hence $(\nabla'_X - \nabla''_X)$ is multiplication by a (complex) scalar. From (8) in the Definition 8 of $Spin^c$ connections it follows that it is a purely imaginary scalar. If ∇' and ∇'' are *Spin* connections then then multiplication by the imaginary scalar $(\nabla'_X - \nabla''_X)$ commutes with the antilinear map $\mathcal{C}$ hence the scalar is zero. □

15. PROPOSITION (*Spin* connections are non-magnetic). *If ∇ is a Spin connection then the corresponding magnetic 2-form vanishes.*

Proof. Indeed, if ξ is a unit spinor we have

$$\begin{aligned}
\beta(X,Y) &= i\tfrac{1}{2}\Big((\xi, \mathcal{R}_\Psi(X,Y)\xi) + (\mathcal{C}\xi, \mathcal{R}_\Psi(X,Y)\mathcal{C}\xi) \Big) \\
&= i\tfrac{1}{2}\Big((\xi, \mathcal{R}_\Psi(X,Y)\xi) + (\mathcal{R}_\Psi(X,Y)\xi, \xi) \Big) = 0.
\end{aligned}$$

since $\mathcal{C}$ is antilinear, it commutes with $\mathcal{R}_\Psi$ and satisfies $(\eta_1, \mathcal{C}\eta_2) = (\eta_2, \eta_1)$.

□

3 The Dirac and Laplace operators on Spinors

Given a *Spin*^{c} connection on a Spinor bundle Ψ we may define the first order Dirac operator and the second order Laplace operator on spinor sections.

16. DEFINITION (The Dirac operator). The Dirac operator $\mathcal{D} : \Gamma(\Psi) \to \Gamma(\Psi)$ is given by

$$\mathcal{D}\xi := -i \sum_j \sigma(e^j)\nabla_{e_j}\xi$$

where $\{e_j\}$ is an orthonormal basis of vectors and $\{e^j\}$ is the dual orthonormal basis of one forms. Here j runs from 1 to 2 or 3 depending on whether we are in the 2 or 3 dimensional case. It is straightforward to see that this definition is independent of the choice of basis $\{e_j\}$.

It would maybe have been more suggestive to write $\mathcal{D} = (-i)\sigma(\nabla)$.

The important observation about the Dirac operator is that it is symmetric.

17. THEOREM. *The Dirac operator is symmetric, i.e., for any two spinor fields ξ and η*

$$\int_M (\xi, \mathcal{D}\eta) = \int_M (\mathcal{D}\xi, \eta).$$

Proof. We compute the formal adjoint of $\mathcal{D}$,

$$\mathcal{D}^* = i \sum_j \nabla^*_{e_j}\sigma(e^j).$$

We now use that $\nabla^*_X = -\nabla_X - \mathrm{div}X = -\nabla_X - \sum_j(\nabla_{e_j}X, e_j)$. We then obtain

$$\mathcal{D}^* = -i\sum_j \sigma(e^j)\nabla_{e_j} - i\sum_j \sigma(\nabla_{e_j}e^j) - i\sum_j \mathrm{div}(e_j)\sigma(e^j).$$

Note that

$$\sum_j \nabla_{e_j} e^j = \sum_j \sum_k (\nabla_{e_j} e^j, e^k) e^k = -\sum_j \sum_k (\nabla_{e_j} e_k, e_j) e^k = -\sum_k \operatorname{div}(e_k) e^k.$$

and therefore the last two terms above cancel. $\square$

18. DEFINITION (Laplace operator on Ψ). The Laplace operator Δ on spinor fields is given by

$$-\Delta\xi = \left(\nabla^*_{e_1}\nabla_{e_1} + \nabla^*_{e_2}\nabla_{e_2} + \nabla^*_{e_3}\nabla_{e_3}\right)\xi.$$

We end this section by proving the famous Lichnerowicz formula.

19. THEOREM (Lichnerowicz formula). *On spinor fields ξ we have the Lichnerowicz formula*

$$\mathcal{D}^2\xi = -\Delta\xi + \tfrac{1}{4}R\xi + i\sum_{i<j}\beta(e_i, e_j)\sigma(e^i)\sigma(e^j)\xi,$$

where R is the scalar curvature and $\{e_i\}$ is any orthonormal basis of vector fields and $\{e^i\}$ is the corresponding dual basis of one-forms. Using the extension of the Clifford multiplication to higher order forms we could simply write the last term above as $i\sigma(\beta)$. In 3-dimensions this is identical to $-\sigma(*\beta)$, where the $*$ again refers to the Hodge dual.

Proof. We shall prove the Lichnerowicz formula at a point p. As usual we may choose an orthonormal basis $\{e_i\}$ such that $\nabla_{e_j} e_i(p) = 0$, for all i, j and thus also $\nabla_{e_j} e^i(p) = 0$, for all i, j, where $\{e^i\}$ is the dual basis. We then have at p that $\nabla^*_{e_i} = -\nabla_{e_i}$ and

$$\begin{aligned}
\mathcal{D}^2 &= -\sum_i \nabla^2_{e_i} - \sum_{i<j}\sigma(e^i)\sigma(e^j)\left[\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i}\right] \\
&= -\Delta - \sum_{i<j}\sigma(e^i)\sigma(e^j)\mathcal{R}_\Psi(e_i, e_j)
\end{aligned}$$

$$= \; -\Delta - \sum_{i<j} \sigma(e^i)\sigma(e^j)\left[\tfrac{1}{2}\sum_{k<l}(e_k, \mathcal{R}(e_i,e_j)e_l)\sigma(e^k)\sigma(e^l) - i\beta(e_i,e_j)\right]$$
$$= \; -\Delta - \sum_{i<j} \sigma(e^i)\sigma(e^j)\left[\tfrac{1}{2}(e_i, \mathcal{R}(e_i,e_j)e_j)\sigma(e^i)\sigma(e^j) - i\beta(e_i,e_j)\right]$$

where the last equality follows from the Bianchi identity. We therefore arrive at

$$\mathcal{D}^2 = -\Delta + \tfrac{1}{4}\sum_{ij}(e_i, \mathcal{R}(e_i,e_j)e_j) + i\sum_{i<j}\beta(e_i,e_j)\sigma(e^i)\sigma(e^j).$$

□

20. REMARK. We may of course extend $-\Delta$ and $\mathcal{D}$ defined in the sense of distributions to all L^2-sections $L^2(\Psi)$. Then $-\Delta$ and $\mathcal{D}$ are self-adjoint on the maximal domains $\{\psi \in L^2(\Psi) \;:\; \Delta\psi \in L^2(\Psi)\}$ and $\{\psi \in L^2(\Psi) \;:\; \mathcal{D}\psi \in L^2(\Psi)\}$, respectively.

4 Conformal transformations

We now consider how spin structures change under conformal transformations (see also [H]). Let g be the original metric on M and let $g_\Omega = \Omega^2 g$ be a conformal metric. Here $\Omega : M \to \mathbb{R}$ is a smooth non-vanishing function.

If Ψ is a spinor bundle over M and σ is the corresponding Clifford map with respect to the metric g then $\sigma_\Omega = \Omega^{-1}\sigma$ is Clifford map with respect to the metric g_Ω.

We are interested in how the connections change.

21. PROPOSITION (Conformal change of Levi-Civita connection). *Let ∇ be the Levi-Civita connection corresponding to the metric g and $\nabla^{(\Omega)}$ be the Levi-Civita connection corresponding to the metric g_Ω. Then for all* *one forms ω and all vectors X, Y we have*

$$\nabla^{(\Omega)}_X \omega = \nabla_X \omega - \Omega^{-1} X(\Omega)\omega + \Omega^{-1}(\omega, d\Omega) X^* - \Omega^{-1}\omega(X) d\Omega \qquad (4)$$

and

$$\nabla^{(\Omega)}_X Y = \nabla_X Y + \Omega^{-1} X(\Omega) Y + \Omega^{-1} Y(\Omega) X - \Omega^{-1}(X, Y) d\Omega^*, \qquad (5)$$

where $(\cdot, \cdot)$ refers to the inner product on one-forms and vectors corresponding to the metric g. Here X^* is the one-form corresponding to the vector X with respect to the metric g and likewise $d\Omega^*$ is the vector corresponding to the one-form $d\Omega$, i.e, $X^*(Y) = (X, Y)$ and $(d\Omega^*, Y) = d\Omega(Y) = Y(\Omega)$ for all vectors Y.

Proof. It is enough to prove (5) since (4) follows from the identity $\Omega^{-2}(\nabla^{(\Omega)}_X \omega)^* = \nabla^{(\Omega)}_X (\Omega^{-2}\omega^*)$. Here the Ω factors are due to the fact that $*$ is the duality in the g metric and not the g_Ω metric. Since $\nabla^{(\Omega)}$ clearly has all the properties of a connection we only have to check that

(i) it is torsion free, i.e.,

$$\nabla^{(\Omega)}_X Y - \nabla^{(\Omega)}_Y X = [X, Y]$$

for any vector fields X, Y ;

(ii) it is compatible with the metric g_Ω, i.e.,

$$X\left[(Y, Z)_\Omega\right] = \left(\nabla^{(\Omega)}_X Y, Z\right)_\Omega + \left(Y, \nabla^{(\Omega)}_X Z\right)_\Omega$$

for any vector fields X, Y, Z.

Both of these follow from simple calculations. $\square$

22. PROPOSITION (Conformal change of $Spin^c$ connection). *Let ∇ be a $Spin^c$ connection on a spinor bundle Ψ on M with Clifford map σ corresponding to the metric g. Then $\nabla^{(\Omega)}$ defined by*

$$\begin{aligned}\nabla_X^{(\Omega)} &= \nabla_X - \tfrac{1}{4}\Omega^{-1}\sigma(d\Omega)\sigma(X^*) + \tfrac{1}{4}\Omega^{-1}\sigma(X^*)\sigma(d\Omega) \\ &= \nabla_X + \tfrac{1}{4}\Omega^{-1}[\sigma(X^*), \sigma(d\Omega)]\end{aligned}$$

for any vector X, is a $Spin^c$ connection on the same spinor bundle with Clifford map $\sigma_\Omega = \Omega^{-1}\sigma$ corresponding to the metric g_Ω. Here again X^* refers to the one-form which is dual to the vector X relative to the metric g.

Proof. We have to prove (8) and (8) in Definition 8 of $Spin^c$ connections. The first relation (8) follows easily from the fact that the Clifford multiplication is Hermitian.

To prove (8) we note that the lift of $\nabla_X^{(\Omega)}$ to the Clifford multiplication is given by the following expression involving a double commutator

$$\nabla_X^{(\Omega)}(\sigma_\Omega(\omega)) = \nabla_X[\Omega^{-1}\sigma(\omega)] + \tfrac{1}{4}\Omega^{-2}\left[[\sigma(X^*), \sigma(d\Omega)], \sigma(\omega)\right]. \tag{6}$$

Using the commutator formula $[[A, B], C] = \{A, \{B, C\}\} - \{B, \{A, C\}\}$ we find

$$\begin{aligned}\left[[\sigma(X^*), \sigma(d\Omega)], \sigma(\omega)\right] &= \{\sigma(X^*), \{\sigma(d\Omega), \sigma(\omega)\}\} - \{\sigma(d\Omega), \{\sigma(X^*), \sigma(\omega)\}\} \\ &= 4(\omega, d\Omega)\sigma(X^*) - 4\omega(X)\sigma(d\Omega).\end{aligned}$$

Hence

$$\nabla_X^{(\Omega)}(\sigma_\Omega(\omega)) = \Omega^{-1}\sigma(\nabla_X\omega) - \Omega^{-2}X(\Omega)\sigma(\omega) + \Omega^{-2}(\omega, d\Omega)\sigma(X^*) - \Omega^{-2}\omega(X)\sigma(d\Omega).$$

We see immediately from (4) that this agrees with $\sigma_\Omega(\nabla_X^{(\Omega)}\omega) = \Omega^{-1}\sigma(\nabla_X^{(\Omega)}\omega)$.

□

23. THEOREM (Conformal change of the Dirac operator). *Let $\mathcal{D}$ and $\mathcal{D}_\Omega$ denote the Dirac operators corresponding to the Spin^c connections ∇ and $\nabla^{(\Omega)}$ as defined in Prop. 22. Then*

$$\mathcal{D}_\Omega = \begin{cases} \Omega^{-3/2}\mathcal{D}\Omega^{1/2}, & \textit{in the 2-dimensional case} \\ \Omega^{-2}\mathcal{D}\Omega, & \textit{in the 3-dimensional case .} \end{cases}$$

Proof. This follows from a simple calculation using Prop. 22. Indeed, we have

$$\mathcal{D}_\Omega = -i\sum_j \sigma_\Omega(e^j_\Omega)\nabla^{(\Omega)}_{e_j^\Omega}.$$

Here $e^j_\Omega = \Omega e^j$ and $e_j^\Omega = \Omega^{-1}e_j$ are respectively one-forms and vectors that are orthonormal with respect to the metric g_Ω if e^j and e_j are one-forms and vectors orthonormal with respect to g. Using $\sigma_\Omega = \Omega^{-1}\sigma$ we obtain from Prop. 22in dimension $n = 2$ or 3 that

$$\begin{aligned} \mathcal{D}_\Omega &= -i\sum_j \Omega^{-1}\sigma(e^j)\nabla^{(\Omega)}_{e_j} = \Omega^{-1}\mathcal{D} - \tfrac{i}{4}\sum_j \Omega^{-2}\sigma(e^j)[\sigma(e^j),\sigma(d\Omega)] \\ &= \Omega^{-1}\mathcal{D} - \sum_j \tfrac{i}{2}\Omega^{-2}\left(\sigma(d\Omega) - \tfrac{1}{2}\sigma(e^j)\{\sigma(e^j),\sigma(d\Omega)\}\right) \\ &= \Omega^{-1}\mathcal{D} - \tfrac{i}{2}\Omega^{-2}\left(n\sigma(d\Omega) - \sum_j \sigma(e^j)(e^j,d\Omega)\right) \\ &= \Omega^{-1}\mathcal{D} - i\frac{n-1}{2}\Omega^{-2}\sigma(d\Omega) = \Omega^{-(n+1)/2}\mathcal{D}\Omega^{(n-1)/2}. \end{aligned}$$

□

5 Riemannian submersions

Having studied how spin structures and Dirac operators change under conformal transformations we now turn to transformations between spaces of different dimensions.

We assume throughout this section that M is a 3-dimensional and N is a 2-dimensional Riemannian manifold. The natural type of transformations to study are Riemannian submersions $\phi : M \to N$. This means that $\phi_* : T_*M \to T_*N$ is a surjective partial isometry. For a discussion of Riemannian submersions and their relations to spin geometry see [GLP].

We show that it is possible to pull back spinor bundles, $Spin^c$ connections and Dirac operators from N to M along a Riemannian submersion.

We denote vol_N the volume form on N. Let $\nu = *\phi^*(vol_N)$, i.e. ν is the Hodge dual of the pull back of the volume form by the Riemannian submersion ϕ. In particular ν is a one-form on M. Note that any one-form on M is a linear combination of ν and the pull-back to M of a one-form on N. These properties are summarized in

24. PROPOSITION (The pull back of the volume form). *Let n be the dual vectorfield to $\nu = *\phi^*(vol_N)$. If f^1, f^2 is a (locally defined) oriented orthonormal basis in T^*N then ν, $\phi^*(f^1)$, $\phi^*(f^2)$ is a (locally defined) oriented orthonormal basis in T^*M. Let n, e_1, e_2 be the orthonormal vectors dual to the one-forms ν, $\phi^*(f^1)$, $\phi^*(f^2)$. Then $f_1 = \phi_*(e_1)$, $f_2 = \phi_*(e_2)$ is the dual basis to f^1, f^2 and we have*

(i) $([n, e_i], e_j) = 0$ *for* $i, j = 1, 2$

(ii) $(\nabla^M_{e_i} e_j, e_k) = (\nabla^N_{f_i} f_j, f_k)$ *for* $i, j, k = 1, 2$*, where* ∇^M *and* ∇^N *denote the Levi-Civita connections on* M *and* N *respectively.*

(iii) We have the identity

$$\begin{aligned}(\nu, *d\nu) &= d\nu(e_1, e_2) = -(n, [e_1, e_2]) = 2(\nabla^M_n e_1, e_2) \\ &= 2(\nabla^M_{e_1} n, e_2).\end{aligned}$$

Note that n is (locally) hypersurface orthogonal if and only if this quantity vanishes.

Proof. The characterization of ν is straightforward from the definition. It is also clear that $\phi_*(n) = 0$. If X is a vector on M orthogonal to n then since ϕ_* is an isometry on the subspace orthogonal to n we have $(\phi_*(e_j), \phi_*(X)) = (e_j, X) = \phi^*(f^j)(X) = f^j(\phi_*(X))$ and therefore $\phi_*(e_j)$ is dual to f^j.

If g is any function on N we have $n(g \circ \phi) = \phi_*(n)(g) = 0$ and $e_j(g \circ \phi) = \phi_*(e_j)(g) = (f_j(g)) \circ \phi$. Thus

$$\begin{aligned} \phi_*([n, e_j])(g) &= [n, e_j](g \circ \phi) = n(e_j(g \circ \phi)) - e_j(n(g \circ \phi)) \\ &= n(f_j(g) \circ \phi) - e_j(n(g \circ \phi)) = 0. \end{aligned}$$

Thus $\phi_*([n, e_j]) = 0$ and hence $([n, e_j], e_k) = 0$ for all $j, k = 1, 2.$.

Likewise we see that

$$\begin{aligned} \phi_*([e_i, e_j])(g) &= [e_i, e_j](g \circ \phi) = e_i(e_j(g \circ \phi)) - e_j(e_i(g \circ \phi)) \\ &= (f_i(f_j(g))) \circ \phi - (f_j(f_i(g))) \circ \phi = [f_i, f_j](g) \circ \phi. \end{aligned}$$

Hence $\phi_*([e_i, e_j]) = [f_i, f_j]$ and thus $([e_i, e_j], e_k) = ([f_i, f_j], f_k)$ if $i, j, k \in \{1, 2\}$. Since

$$(\nabla^M_{e_i} e_j, e_k) = \tfrac{1}{2}\Big(([e_i, e_j], e_k) - (e_i, [e_j, e_k]) + ([e_k, e_i], e_j)\Big)$$

and likewise for the covariant derivatives of the basis f_1, f_2 we see that $(\nabla^M_{e_i} e_j, e_k) = (\nabla^N_{f_i} f_j, f_k)$.

Since $([n, e_1], e_2) = ([n, e_2], e_1) = 0$ we have that

$$\begin{aligned} d\nu(e_1, e_2) &= \nabla^M_{e_1}\nu(e_2) - \nabla^M_{e_2}\nu(e_1) = (\nabla^M_{e_1} n, e_2) - (\nabla^M_{e_2} n, e_1) \\ &= (\nabla^M_n e_1, e_2) - (\nabla^M_n e_2, e_1) = 2(\nabla^M_n e_1, e_2). \end{aligned}$$

and

$$(n, [e_1, e_2]) = (n, \nabla^M_{e_1} e_2 - \nabla^M_{e_2} e_1) = -2(\nabla^M_{e_1} n, e_2) = -2(\nabla^M_n e_1, e_2).$$

$\square$

It is straightforward to check that a spinor bundle lifts to a spinor bundle under a Riemannian submersion as stated in the next proposition.

25. PROPOSITION (Lifting spinor bundles). *Let $\phi : M \to N$ be a Riemannian submersion. If Ψ_N is a spinor bundle on N with Clifford map σ_N then the induced bundle*

$$\Psi_M = \phi^*(\Psi_N) = \{(p, v) \in M \times \Psi_N : \pi(v) = \phi(p)\}$$

($\pi : \Psi_N \to N$ is the projection map of Ψ_N) is a spinor bundle on M with corresponding Clifford map defined by

$$\sigma_M(\phi^*\omega) = \sigma_N(\omega) \circ \phi \tag{7}$$

for all one forms ω on N and

$$\sigma_M(\nu) = -\left(i\sigma_N(f^1)\sigma_N(f^2)\right) \circ \phi \tag{8}$$

if f^1, f^2 is an oriented orthonormal frame in T^*N.

We recall that there is a natural connection on an induced bundle.

26. PROPOSITION (Induced connection). *If ∇ is any connection on Ψ_N then there is a unique connection $\phi^*(\nabla)$ on the induced bundle $\phi^*(\Psi_N)$ such that the chain rule $\phi^*(\nabla)_Y (\xi \circ \phi) = \left(\nabla_{\phi_*(Y)}\xi\right) \circ \phi$ is satisfied for all $Y \in T_*M$ and all sections ξ of Ψ_N.*

Proof. This result is standard in the theory of vector bundles. Locally on Ψ_N we may choose a basis $\xi_\pm$. Then $\xi_\pm \circ \phi$ is a local basis on $\phi^*(\Psi_N)$. Any section in $\phi^*(\Psi_N)$ may locally be written in terms of $\xi_\pm \circ \phi$. We may use this to extend $\phi^*(\nabla)$ to all sections of $\phi^*(\Psi_N)$. Note that the extension is unique. It is straightforward to see that this extension locally defines a connection and that the extension is independent of the choice of the local basis $\xi_\pm$. Hence the connection is globally defined on $\phi^*(\Psi_N)$ and it satisfies the chain rule. □

If ∇^N is a $Spin^c$ connection on Ψ_N it is however not necessarily true that the induced connection $\phi^*(\nabla^N)$ is a $Spin^c$ connection on $\Psi_M = \phi^*(\Psi_N)$. It is however easy to correct it such that it becomes a $Spin^c$ connection.

27. PROPOSITION (Lifting $Spin^c$ connections). *Let again $\phi : M \to N$ be a Riemannian submersion and $\nu = *\phi^*(vol_N)$. Let ∇^N be a $Spin^c$ connection on the spinor bundle Ψ_N. Let $\phi^*(\nabla^N)$ be the induced connection on $\Psi_M = \phi^*(\Psi_N)$. Then*

$$\nabla^M_X := \phi^*(\nabla^N)_X - \tfrac{1}{2}\sigma_M(\nu)\sigma_M(\nabla^M_X\nu) - \tfrac{i}{4}\nu(X)(\nu, *d\nu)\sigma_M(\nu) \qquad (9)$$

is a $Spin^c$ connection on Ψ_M.

Proof. As in the proof of Prop. 26 we see that it is enough to check the conditions (8) and (8), from the Definition 8 of $Spin^c$ connections, for spinor fields of the form $\xi \circ \phi$. The first condition (8) is clear since ∇^N is a $Spin^c$ connection.

To check (8) it is enough to consider either for the one form $\alpha = \nu$ or any locally defined one form of the form $\alpha = \phi^*(\omega)$, where ω is a locally defined one-form on N. We begin with the latter case. We first calculate $\nabla^M_X \phi^*(\omega)$. As in Prop. 24 let $e^1 = \phi^*(f^1)$, $e^2 = \phi^*(f^2)$ be local one forms such that

ν, e^1, e^2 is an orthonormal basis and let n, e_1, e_2 be the dual basis of vectors. From Prop. 24 we know that $(\nabla^M_{e_i} e^j, e^k) = (\nabla^N_{f_i} f^j, f^k)$ for $i, j, k \in \{1, 2\}$ and therefore $\phi^* \left[\nabla^N_{f_i}\omega\right]$ is the projection of $\nabla^M_{e_i}\phi^*(\omega)$ orthogonal to ν. We can therefore write in general

$$\begin{aligned} \nabla^M_X \phi^*(\omega) &= \nabla^M_{(X,n)n}\phi^*(\omega) + \nabla^M_{X-(X,n)n}\phi^*(\omega) \\ &= (X,n)\nabla^M_n\phi^*(\omega) + \left(\nu, \nabla^M_{X-(X,n)n}\phi^*(\omega)\right)\nu + \phi^*\left[\nabla^N_{\phi_*(X)}\omega\right] \\ &= \nu(X)\left(\nabla^M_n\phi^*(\omega) - \left(\nu, \nabla^M_n\phi^*(\omega)\right)\nu\right) - \left(\nabla^M_X\nu, \phi^*(\omega)\right)\nu \\ &\quad + \phi^*\left[\nabla^N_{\phi_*(X)}\omega\right]. \end{aligned}$$

Note now that since $(\phi^*(\omega), e^j) = (\omega, f^j) \circ \phi$ is constant in the direction n we have

$$\nabla^M_n\phi^*(\omega) = (\phi^*(\omega), e^1)\nabla^M_n e^1 + (\phi^*(\omega), e^2)\nabla^M_n e^2.$$

Moreover since $\nabla_n e^j$ is orthogonal to e^j we have

$$\begin{aligned} \nabla^M_n\phi^*(\omega) - \left(\nu, \nabla^M_n\phi^*(\omega)\right)\nu &= (\phi^*(\omega), e^1)(\nabla^M_n e^1, e^2)e^2 \\ &\quad + (\phi^*(\omega), e^2)(\nabla^M_n e^2, e^1)e^1 \\ &= \tfrac{1}{2}(\nu, *d\nu)\left((\phi^*(\omega), e^1)e^2 - (\phi^*(\omega), e^2)e^1\right), \end{aligned}$$

using Prop. 24. Thus since $[\sigma_M(\nu), \sigma_M(e^1)] = 2i\sigma_M(e^2)$ and $[\sigma_M(\nu), \sigma_M(e^2)] = -2i\sigma_M(e^1)$ we find that

$$\begin{aligned} \sigma_M\left(\nabla^M_X\phi^*(\omega)\right) &= -\tfrac{i}{4}\nu(X)(\nu, *d\nu)\left[\sigma_M(\nu), \sigma_M\left(\phi^*(\omega)\right)\right] - \left(\nabla^M_X\nu, \phi^*(\omega)\right)\sigma_M(\nu) \\ &\quad + \sigma_M\left(\phi^*\left[\nabla^N_{\phi_*(X)}\omega\right]\right). \end{aligned} \tag{10}$$

On the other hand using (9)

$$\begin{aligned} \nabla^M_X[\sigma_M(\phi^*(\omega))\xi \circ \phi] &= \nabla^N_{\phi^*(X)}[\sigma_N(\omega)\xi] \circ \phi \\ &\quad - \tfrac{1}{2}\sigma_M(\nu)\sigma_M(\nabla^M_X\nu)\sigma_M(\phi^*(\omega))\xi \circ \phi \\ &\quad - \tfrac{i}{4}\nu(X)(\nu, *d\nu)\sigma_M(\nu)\sigma_M(\phi^*(\omega))\xi \circ \phi. \end{aligned}$$

Hence

$$\begin{aligned}\left[\nabla^M_X, \sigma_M(\phi^*(\omega))\right] \xi \circ \phi &= \left(\sigma_N\left(\nabla^N_{\phi^*(X)}\omega\right)\xi\right)\circ\phi \\ &\quad - \tfrac{1}{2}\left[\sigma_M(\nu)\sigma_M(\nabla^M_X \nu), \sigma_M(\phi^*(\omega))\right]\xi\circ\phi \\ &\quad - \tfrac{i}{4}\nu(X)(\nu, *d\nu)\left[\sigma_M(\nu), \sigma_M(\phi^*(\omega))\right]\xi\circ\phi\end{aligned}$$

and we have that

$$\begin{aligned}\left[\nabla^M_X, \sigma_M(\phi^*(\omega))\right] &= \sigma_M\left(\phi^*\left(\nabla^N_{\phi^*(X)}\omega\right)\right) - \left(\nabla^M_X\nu, \phi^*(\omega)\right)\sigma_M(\nu) \\ &\quad -\tfrac{i}{4}\nu(X)(\nu, *d\nu)\left[\sigma_M(\nu), \sigma_M(\phi^*(\omega))\right] = \sigma_M\left(\nabla^M_X\phi^*(\omega)\right).\end{aligned}$$

The last identity follows from (10).

It remains to check that $\left[\nabla^M_X, \sigma_M(\nu)\right] = \sigma_M\left(\nabla^M_X\nu\right)$. This follows from repeated use of the identity just proved since from (8) we have $\sigma_M(\nu) = -i\sigma_M(\phi^*(f^1))\sigma_M(\phi^*(f^2))$. Thus if we again write $e^j = \phi^*(f^j)$, $j = 1, 2$ we get

$$\begin{aligned}\left[\nabla^M_X, \sigma_M(\nu)\right] &= -i\sigma_M(\nabla^M_X e^1)\sigma_M(e^2) - i\sigma_M(e^1)\sigma_M(\nabla^M_X e^2) \\ &= -i(\nabla^M_X e^1, \nu)\sigma_M(\nu)\sigma_M(e^2) - i(\nabla^M_X e^2, \nu)\sigma_M(e^1)\sigma_M(\nu) \\ &\quad -i(\nabla^M_X e^1, e^2) - i(\nabla^M_X e^2, e^1) \\ &= i(e^1, \nabla^M_X\nu)\sigma_M(\nu)\sigma_M(e^2) + i(e^2, \nabla^M_X\nu)\sigma_M(e^1)\sigma_M(\nu) \\ &= \sigma_M(\nabla^M_X\nu),\end{aligned}$$

where we have used that $(\nabla^M_X e^1, e^2) + (\nabla^M_X e^2, e^1) = 0$ and $i\sigma_M(e^1)\sigma_M(\nu) = \sigma_M(e^2)$ and $i\sigma_M(\nu)\sigma_M(e^2) = \sigma_M(e^1)$. □

28. PROPOSITION. *The magnetic 2-form of ∇^M, defined by (9) is the pull back of the magnetic 2-form of ∇^N.*

Proof. Let $\xi_\pm$ be a local basis for Ψ_N. Then $\xi_\pm \circ \phi$ is a local basis for Ψ_M. We know from Prop. 9 and 11 that if we define local one-forms α_N and α_M

on N and M respectively by

$$\alpha_M(X) \;=\; \tfrac{i}{2}\Big[\, \big(\xi_+ \circ \phi, \nabla^M_X(\xi_+ \circ \phi)\big) + \big(\xi_- \circ \phi, \nabla^M_X(\xi_- \circ \phi)\big) \Big], \quad X \in T_*M$$

and

$$\alpha_N(Y) \;=\; \tfrac{i}{2}\Big[(\xi_+, \nabla^N_Y \xi_+) + (\xi_-, \nabla^N_Y \xi_-)\Big], \quad Y \in T_*N$$

then $d\alpha_M$ and $d\alpha_N$ are the magnetic 2-forms on M and N respectively. We now show that $\phi^*(\alpha_N) = \alpha_M$ which implies the statement of the proposition.

Since ν is orthogonal to $\nabla^M_X \nu$ we have that the anticommutator

$$\{\sigma_M(\nu), \sigma_M(\nabla^M_X \nu)\} = 0$$

and hence $\sigma_M(\nu)\sigma_M(\nabla^M_X \nu)$ is traceless. Therefore

$$\begin{aligned} \alpha_M(X) &= \tfrac{i}{2}\Big[\, \big(\xi_+ \circ \phi, \phi^*(\nabla^N)_X(\xi_+ \circ \phi)\big) + \big(\xi_- \circ \phi, \phi^*(\nabla^N)_X(\xi_- \circ \phi)\big) \Big] \\ &= \tfrac{i}{2}\Big[(\xi_+, \nabla^N_{\phi_*(X)} \xi_+) + (\xi_-, \nabla^N_{\phi_*(X)} \xi_-)\Big] = \alpha_N(\phi_*(X)). \end{aligned}$$

□

We now turn to how the Dirac operator lifts.

29. LEMMA (Lifting the Dirac operator by a Riemannian submersion). *Let $\phi : M \to N$ be a Riemannian submersion and let $\nu = *\phi^*(vol_N)$. Let Ψ_N be a spinor bundle on N with $Spin^c$ connection ∇^N and let Ψ_M and ∇^M be the lifts described in Propositions 25 and 27. For the corresponding Dirac operators $\mathcal{D}_N$ on Ψ_N and $\mathcal{D}_M$ on Ψ_M we have for all sections ξ in Ψ_N that*

$$\mathcal{D}_M(\xi \circ \phi) = (\mathcal{D}_N \xi) \circ \phi + \tfrac{1}{2}\sigma_M\left(*d\nu - \tfrac{1}{2}(\nu, *d\nu)\nu\right)\sigma_M(\nu)\xi \circ \phi. \qquad (11)$$

Proof. We choose to write the Dirac operator locally as $\mathcal{D}_M = -i\sigma_M(e^1)\nabla^M_{e_1} - i\sigma_M(e^2)\nabla^M_{e_2} - i\sigma_M(\nu)\nabla^M_n$. We find from Prop. 27 that

$$\mathcal{D}_M\left(\xi \circ \phi\right) = (\mathcal{D}_N \xi) \circ \phi + A\xi \circ \phi,$$

where the matrix A is

$$\begin{aligned} A &= \tfrac{i}{2}\sigma_M(e^1)\sigma_M(\nu)\sigma_M(\nabla^M_{e_1}\nu) + \tfrac{i}{2}\sigma_M(e^2)\sigma_M(\nu)\sigma_M(\nabla^M_{e_2}\nu) \\ &\quad + \tfrac{i}{2}\sigma_M(\nabla^M_n\nu) - \tfrac{1}{4}(\nu, *d\nu). \end{aligned}$$

From Prop. 24 we have that $(\nabla^M_{e_i}\nu, e^i) = (\nabla^M_n e^i, e^i) = 0$ for $i = 1, 2$. Since also $(\nabla^M_{e_i}\nu, \nu) = 0$ we conclude therfore that

$$\nabla^M_{e_1}\nu = (\nabla^M_{e_1} n, e_2)e^2 = \tfrac{1}{2}(\nu, *d\nu)e^2,$$

and

$$\nabla^M_{e_2}\nu = -(\nabla^M_{e_1} n, e_2)e^1 = -\tfrac{1}{2}(\nu, *d\nu)e^1.$$

Thus since $\sigma_M(e^1)\sigma_M(e^2)\sigma_M(\nu) = i$ we have

$$\begin{aligned} A &= \tfrac{i}{4}(\nu, *d\nu)\left(\sigma_M(e^1)\sigma_M(\nu)\sigma_M(e^2) - \sigma_M(e^2)\sigma_M(\nu)\sigma_M(e^1)\right) \\ &\quad + \tfrac{i}{2}\sigma_M(\nabla^M_n\nu) - \tfrac{1}{4}(\nu, *d\nu) = \tfrac{i}{2}\sigma_M(\nabla^M_n\nu) + \tfrac{1}{4}(\nu, *d\nu). \end{aligned}$$

Hence we just proved that

$$\mathcal{D}_M(\xi\circ\phi) = (\mathcal{D}_N\xi)\circ\phi + \tfrac{i}{2}\sigma_M(\nabla^M_n\nu)\xi\circ\phi + \tfrac{1}{4}(\nu, *d\nu)\xi\circ\phi. \qquad (12)$$

Using $\sigma_M(e^1)\sigma_M(e^2)\sigma(\nu) = i$ note that

$$i\sigma_M(e^1)\xi\circ\phi = \sigma_M(e^2)\sigma_M(\nu)\xi\circ\phi \quad \text{and} \quad i\sigma_M(e^2)\xi\circ\phi = -\sigma_M(e^1)\sigma_M(\nu)\xi\circ\phi.$$

Thus we have that $\mathcal{D}_M(\xi\circ\phi) - (\mathcal{D}_N\xi)\circ\phi$ is equal to

$$\Big(\tfrac{1}{2}(\nabla^M_n\nu, e^1)\sigma_M(e^2) - \tfrac{1}{2}(\nabla^M_n\nu, e^2)\sigma_M(e^1) + \tfrac{1}{4}(\nu, *d\nu)\sigma_M(\nu)\Big)\sigma_M(\nu)\xi\circ\phi.$$

If we now use that

$$(e^1, *d\nu) = d\nu(e_2, n) = (\nabla^M_{e_2}\nu, \nu) - (\nabla^M_n\nu, e^2) = -(\nabla^M_n\nu, e^2)$$

and likewise that $(e^2, *d\nu) = (\nabla^M_n\nu, e^1)$ we obtain 11 . $\square$

Of course it would have been nicer if the last term in (11) had not been present. One may try to change the connection on M, by adding a one-form, in order to cancel the last term in (11). It is however not possible to cancel this term for all spinors, but only for spinors with spin pointing in a specific direction. The following theorem is a simple consequence of Lemma 29.

30. THEOREM. *We use the notation of Lemma 29. If we therefore define new $Spin^c$ connections*

$$\nabla^{M,\pm} := \nabla^M \mp \tfrac{i}{2}\left(*d\nu - \tfrac{1}{2}(\nu, *d\nu)\nu\right) \tag{13}$$

we obtain for the corresponding Dirac operators $\mathcal{D}_M^\pm$ that

$$\mathcal{D}_M^\pm(\xi \circ \phi) = (\mathcal{D}_N\xi) \circ \phi, \tag{14}$$

for spinors satisfying $\sigma_M(\nu)\xi \circ \phi = \pm\xi \circ \phi$. In particular, if $\xi \in \Psi_N$ is a normalized zero mode of $\mathcal{D}_N$ with a definite spin direction

$$S = \left(\xi, -i\sigma_N(f^1)\sigma_N(f^2)\xi\right) \in \{+,-\},$$

then $\xi \circ \phi \in \Psi_M$ is a zero mode of $\mathcal{D}_M^{sgn(S)}$.

6 Lower bound on the number of zero modes

Theorem 30 allows us to construct zero modes of Dirac operators on M with a certain connection starting from definite-spin zero modes of a Dirac operator $\mathcal{D}_N$ on N. On the other hand, the index theorem for $\mathcal{D}_N$ claims that

$$\operatorname{ind} \mathcal{D}_N = \frac{1}{2\pi}\int_N \beta_N,$$

where β_N is the magnetic two-form of ∇_N. This integer is the Chern number of the determinant line bundle of Ψ_N. The index is defined as

$$\begin{aligned}\operatorname{ind} \mathcal{D}_N \;&=\; \dim\{\xi \in \Gamma(\Psi_N) : \mathcal{D}_N\xi = 0,\ \sigma_M(\nu)\xi = \xi\} \\ &\quad - \dim\{\xi \in \Gamma(\Psi_N) : \mathcal{D}_N\xi = 0,\ \sigma_M(\nu)\xi = -\xi\}\ ,\end{aligned}$$

where we have abused notations and let $\sigma_M(\nu)$ act on sections in Ψ_N. The action is of course the one given in (8).

Theorem 30 and the index Theorem together give the following theorem.

31. THEOREM (Lower bound on the number of zero modes). *Let Ψ_N be a spinor bundle over a 2-dimensional manifold N, let ∇^N be a Spin^c connection and let $\mathcal{D}_N$ be the Dirac operator. We denote the magnetic two-form of ∇_N by β_N and let*

$$\Phi := \frac{1}{2\pi}\int_N \beta_N \quad \textit{and} \quad s := sgn(\Phi) \in \{+,-\}.$$

Consider a Riemannian submersion $\phi : M \to N$ from some 3-dimensional manifold M and let Ψ_M and ∇^M be the lifts of the spinor bundle and the connection as described in Propositions 25 and 27. Let $\nu := *\phi^*(vol_N)$ and define a new Spin^c connection

$$\widetilde{\nabla}^M := \nabla^M - i\tfrac{s}{2}\left(*d\nu - \tfrac{1}{2}(\nu, *d\nu)\nu\right)$$

on Ψ_M. The corresponding Dirac operator is denoted by $\widetilde{\mathcal{D}}_M = (-i)\sigma(\widetilde{\nabla}^M)$. Then

$$dim\ Ker\, \widetilde{\mathcal{D}}_M \geq \left|\frac{1}{2\pi}\int_N \beta_N\right| \tag{15}$$

32. REMARK. The magnetic two-form of $\widetilde{\mathcal{D}}_M$ is

$$\beta_M := \phi^*(\beta_N) + \tfrac{s}{2}\, d\left(*d\nu - \tfrac{1}{2}(\nu, *d\nu)\nu\right).$$

We do not have an independent characterization of all magnetic two-forms β on M which can be presented in this form.

33. REMARK. It is very interesting to investigate the cases of equality in (15). Our method loses equality in two different steps. Less seriously, we estimate the dimension of the kernel of $\mathcal{D}_N$ by the index, which is not sharp unless some vanishing theorem holds for $\mathcal{D}_N$. More importantly, we only considered those zero modes of $\widetilde{\mathcal{D}}_M$ whose spin direction is parallel with ν. We will show in the next section that in the special case of the Hopf map $\phi : \mathbb{S}^3 \to \mathbb{S}^2$ we have equality in (15).

Proof of Theorem 31. We recall that $\widetilde{\sigma} := -i\sigma_N(f^1)\sigma_N(f^2)$ commutes with $\mathcal{D}_N$. Let $\mathcal{D}_N^\pm$ be the restriction of $\mathcal{D}_N$ onto the subspaces $\widetilde{\sigma}\xi = \pm\xi$. Then by the index theorem

$$\dim \mathrm{Ker}\, \mathcal{D}_N^+ - \dim \mathrm{Ker}\, \mathcal{D}_N^- = \Phi = \frac{1}{2\pi}\int_N \beta_N,$$

which implies that

$$\max_\pm \dim \mathrm{Ker}\, \mathcal{D}_N^\pm \geq |\Phi|.$$

Notice that $\widetilde{\mathcal{D}}_M = \mathcal{D}_M^s$, where the sign superscript is $s = \mathrm{sgn}(\Phi) \in \{+,-\}$. Hence, using Theorem 30 we have

$$\dim \mathrm{Ker}\, \widetilde{\mathcal{D}}_M \geq \max_\pm \dim \mathrm{Ker}\, \mathcal{D}_N^\pm,$$

which completes the proof. □

7 Geometry of the Hopf map

We identify $\mathbb{S}^2$ with the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The standard metric on $\mathbb{S}^2$ can be written as $g_2 = \left(1 + \frac{1}{4}|w|^2\right)^{-2} dw\, d\bar{w}$. The 3-sphere we write as $\mathbb{S}^3 = \{(z_1, z_2) \ : \ |z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$ and let g_3 be the standard metric on $\mathbb{S}^3$. The Hopf map $\phi : \mathbb{S}^3 \to \mathbb{S}^2$ can then be written as $\phi(z_1, z_2) = 2z_1 z_2^{-1}$. We remark, that for our purposes the conjugate map $\phi'(z_1, z_2) = 2\bar{z}_1\bar{z}_2^{-1}$ could have been chosen as well. We summarize a few geometric properties of the Hopf map.

34. LEMMA (Hopf map is Riemannian submersion). *The Hopf map is a Riemannian submersion between the Riemannian manifolds $M = (\mathbb{S}^3, g_3)$ and $N = (\mathbb{S}^2, \frac{1}{4}g_2)$. Let $\nu = *\phi^*(vol_N)$ and let n be the vector field corresponding to the one-form ν. Then n is a geodesic vectorfield on $\mathbb{S}^3$, its integral curves are main circles of $\mathbb{S}^3$. Moreover we have the relation*

$$d\nu = -2 *\nu. \tag{16}$$

Proof. The vectorfield $v := (z_1, z_2)$ on $\mathbb{C}^2$ is orthogonal to $\mathbb{S}^3$. Define the following three vectorfields on $\mathbb{C}^2$; $u_1 := (i\bar{z}_2, -i\bar{z}_1)$, $u_2 := (\bar{z}_2, -\bar{z}_1)$ and $n := (iz_1, iz_2)$. It is easy to see that they are orthogonal to v, hence they are also vectorfields on $\mathbb{S}^3$. Moreover, they form a positively oriented orthonormal basis of $T_*\mathbb{S}^3$.

The integral curves $\chi(t) = e^{it}n$ of the vectorfield n are main circles in $\mathbb{S}^3$. Hence n is a geodesic vectorfield on $\mathbb{S}^3$. Moreover $\phi \circ \chi(t)$ is independent of t, so $\phi_*(n) = 0$ and ϕ is a fibration with $\mathbb{S}^1$ fibers, all having the same length 2π.

To check that $\phi : M \to N$ is a partial isometry, we compute the pushforwards of u_1, u_2 at any point (z_1, z_2):

$$\phi_*(u_1) = \left(i\bar{z}_2 \frac{\partial}{\partial z_1} - iz_2 \frac{\partial}{\partial \bar{z}_1} - i\bar{z}_1 \frac{\partial}{\partial z_2} + iz_1 \frac{\partial}{\partial \bar{z}_2} \right) \frac{2z_1}{z_2} = \frac{2i}{z_2^2}$$

and similarly $\phi_*(u_2) = 2z_2^{-2}$. It is clear that $\phi_*(u_1)$ and $\phi_*(u_2)$ are orthogonal and we check that their length is

$$\frac{1}{2}\left(1 + \frac{1}{4}|\phi(z_1, z_2)|^2\right)^{-1} \frac{2}{|z_2|^2} = 1.$$

Moreover, $vol_N = f^1 \wedge f^2$, where f^j is the the one-form on $\mathbb{S}^2$ dual to $\phi_*(u_j)$, hence $\nu = *\phi^*(vol_N)$ is dual to the vector n.

In order to compute $*d\nu$, we first compute the Levi-Civita connection on $\mathbb{S}^3$ in the basis u_1, u_2, n. For example

$$\nabla_n u_1 = p\left(iz_1\frac{\partial}{\partial z_1} - i\bar{z}_1\frac{\partial}{\partial \bar{z}_1} + iz_2\frac{\partial}{\partial z_2} - i\bar{z}_2\frac{\partial}{\partial \bar{z}_2}\right)(i\bar{z}_2,\ -i\bar{z}_1) = p(u_2) = u_2,$$

where $p : T_*\mathbb{C}^2 \to T_*\mathbb{S}^3$ is the canonical projection. Similarly, we obtain

$$\nabla_n u_2 = -u_1, \qquad \nabla_{u_1} n = -u_2, \qquad \nabla_{u_2} n = u_1, \tag{17}$$

$$\nabla_{u_1} u_2 = n, \qquad \nabla_{u_2} u_1 = -n, \qquad \nabla_n n = 0.$$

Finally, we obtain

$$(\nu, *d\nu) = d\nu(u_1, u_2) = (\nabla_{u_1} n, u_2) - (\nabla_{u_2} n, u_1) = -2$$

and

$$(u_1, *d\nu) = d\nu(u_2, n) = (\nabla_{u_2} n, n) - (\nabla_n n, u_2) = 0$$

$$(u_2, *d\nu) = -d\nu(u_1, n) = -(\nabla_{u_1} n, n) + (\nabla_n n, u_1) = 0,$$

and (16) follows. $\square$

8 Dirac operators on $\mathbb{S}^3$ and $\mathbb{R}^3$

We show that in certain cases the only zero modes of a three dimensional Dirac operator are those which were obtained in Theorem 30. Moreover, for these operators we are able to determine the full spectrum.

8.1 Spinor bundles on $\mathbb{S}^3$ and $\mathbb{S}^2$

We discuss $Spin^c$ spinor bundles on $\mathbb{S}^2$ and $\mathbb{S}^3$ in the Appendix. Here we just mention that on $\mathbb{S}^3$ there is only one (up to isomorphisms) $Spin^c$ bundle and it is just a trivial bundle. For any closed 2-form β there is a 1-form α with

$d\alpha = \beta$ since $H^2(\mathbb{S}^3, \mathbb{R}) = \{0\}$. Hence any β is the magnetic 2-form of some $Spin^c$ connection. This connection is given for example by the formula in Proposition 9.

Moreover, the connection is up to an overall gauge transformation uniquely determined by the magnetic 2-form, in particular the spectrum of the Dirac operator depends only on the magnetic 2-form. Indeed, if $\nabla^{M,1}$ and $\nabla^{M,2}$ both have magnetic two form β, then by Proposition 14 $\nabla^{M,1} - \nabla^{M,2} = i\omega$ with $d\omega = 0$. Since $H^1(\mathbb{S}^3, \mathbb{R}) = 0$, $\omega = df$ with some function $f \in C^\infty(M)$. Therefore $U\nabla^{M,1}U^* = \nabla^{M,2}$ with the unitary operator $U = e^{if}$, and the same relation holds for the corresponding Dirac operators. The spectrum is gauge invariant.

On $\mathbb{S}^2$ there are inequivalent $Spin^c$ bundles. For each $n \in \mathbb{Z}$ there is a $Spin^c$ bundle Ψ_n on $\mathbb{S}^2$, such that all $Spin^c$ connections on Ψ_n have the property that the corresponding magnetic 2-form integrated over $\mathbb{S}^2$ gives $2\pi n$ (Sections A.1, A.2). The bundles Ψ_n exhaust all $Spin^c$ bundles on $\mathbb{S}^2$ (Proposition 46). The integer n is the Chern number of Ψ_n (or rather that of the determinant line bundle of Ψ_n). Moreover any 2-form on $\mathbb{S}^2$ that integrates to $2\pi n$ for some $n \in \mathbb{Z}$ is the magnetic 2-form for some connection on Ψ_n (Proposition 48). Again the connection is uniquely determined by this 2-form, up to an overall gauge freedom and the spectrum of the Dirac operator depends only on the magnetic 2-form.

8.2 Spectrum of the Dirac operator on $\mathbb{S}^3$

Let $M = (\mathbb{S}^3, g_3)$, $N = (\mathbb{S}^2, \frac{1}{4}g_2)$ and ϕ be the Hopf map, $\nu := *\phi^*(vol_N)$. Let β_M be a two-form (magnetic field) on M such that $\beta_M = h*\nu$, $h \in C^\infty(M)$, i.e. the magnetic field is parallel with the pull-up volume form. Clearly $\beta_M = \phi^*(g\, vol_N)$ for some function $g \in C^\infty(N)$, $h = g \circ \phi$ (see Remark 36).

Let Ψ_M be the spinor bundle on M, which is unique up to isomorphism, and let ∇^M be a *Spin*^{c} connection on Ψ_M with magnetic two form β_M. The corresponding Dirac operator is $\mathcal{D}_M = -i\sigma(\nabla^M)$. Recall that β_M determines $\mathcal{D}_M$ up to a gauge transformation.

35. THEOREM (Spectrum of $\mathcal{D}_M$). *We define*

$$c := \left\langle \frac{1}{(2\pi)^2} \int_M \beta_M \wedge \nu \right\rangle \qquad \textit{and} \qquad m := \left[\frac{1}{(2\pi)^2} \int_M \beta_M \wedge \nu \right]$$

where for any $x \in \mathbb{R}$ we let $\langle x \rangle$ denote the unique number in $(-\frac{1}{2}, \frac{1}{2}]$ such that $x - \langle x \rangle \in \mathbb{Z}$. Let $[x] := x - \langle x \rangle$ be this integer.

For any $k \in \mathbb{Z}$ let

$$\beta_N(k) := (g - 2(c+k))\,(vol_N). \tag{18}$$

Then clearly (see Remark 36)

$$\frac{1}{2\pi} \int_N \beta_N(k) = m - k \in \mathbb{Z}, \tag{19}$$

hence on the spinor bundle Ψ_{m-k} with Chern number $m-k$ on N there exists a two dimensional Dirac operator $\mathcal{D}_{N,(k)}$ with magnetic two-form $\beta_N(k)$. Let $\Sigma_+(k)$ be the positive spectrum of $\mathcal{D}_{N,(k)}$.

(i.) The spectrum of $\mathcal{D}_M$ is given

$$Spec\, \mathcal{D}_M = \bigcup_{k \in \mathbb{Z}} \left(S_k \cup \left\{ \pm \sqrt{\lambda^2 + (k+c)^2} - \frac{1}{2} \ : \ \lambda \in \Sigma_+(k) \right\} \right) \tag{20}$$

where

$$S_k = \begin{cases} \{k + c - \frac{1}{2}\}, & \textit{if } m > k \\ \emptyset, & \textit{if } m = k \\ \{-k - c - \frac{1}{2}\}, & \textit{if } m < k\ . \end{cases}$$

(ii.) The multiplicity of an eigenvalue of $\mathcal{D}_M$ is equal to the number of ways it can be written as $\pm\sqrt{\lambda^2 + (k+c)^2} - \frac{1}{2}$ with $k \in \mathbb{Z}$ and $\lambda \in \Sigma_+(k)$

counted with multiplicity or as an element in S_k counted with multiplicity $|m-k|$.

(iii.) The eigenspace of $\mathcal{D}_M$ with eigenvalue in S_k contains spinors with definite spin value $sgn(m-k)$, (i.e, the eigenvalue of $\sigma(\nu)$).

36. REMARK. From the assumption $\beta_M = h * \nu$ it follows that $\beta_M = \phi^*(g\ vol_N)$ for some function $g \in C^\infty(N)$ and that

$$\frac{1}{(2\pi)^2}\int_M \beta_M \wedge \nu = \frac{1}{2\pi}\int_N g\ vol_N. \tag{21}$$

To see this, we compute $0 = d\beta_M = dh \wedge *\nu = n(h)\ vol_M$ using that $d * \nu = d\Big(\phi^*(vol_N)\Big) = 0$ from (16). Hence h is constant along the Hopf fibers, $h = g \circ \phi$, and therefore β_M is a pullback of a (not necessarily exact) two form on N. The integral relation (21) follows from using Fubini's theorem in local patches and the fact that the length of the fibers is 2π. From this relation (19) is straight forward. (Recall that $\int vol_N = \pi$).

The eigenvalues in S_k correspond to the zero eigenvalues of the 2-dimensional operator $\mathcal{D}_{N,(k)}$. For these we have used the following theorem of Aharonov and Casher [AC] to say exactly what the multiplicity is and exactly what the spin is of the eigenfunctions. For completeness we give a proof of this theorem in the Appendix.

37. THEOREM (Aharonov Casher theorem on $\mathbb{S}^2$). *Let ∇ be a covariant derivative on the spinor bundle Ψ_n on $\mathbb{S}^2$ with Chern number $n = \frac{1}{2\pi}\int \beta$, where β is the corresponding 2-form. Let $\mathcal{D}$ be the corresponding Dirac operator. Then the dimension of the space of harmonic spinors $ker\ \mathcal{D}$ is $|n|$. Moreover, $ker\ \mathcal{D} \subset \{\eta \mid \sigma(\nu)\eta = \eta\}$ if $n > 0$ and $ker\ \mathcal{D} \subset \{\eta \mid \sigma(\nu)\eta = -\eta\}$ if $n < 0$, where as before we have written $\sigma(\nu) = -i\sigma(f^1)\sigma(f^2)$ for any positively oriented (local) orthonormal basis f^1, f^2 of one-forms. $\sigma(\nu)$ is independent of the choice of one-form basis.*

38. REMARK. With the notation of Theorem 35 we see immediately that for the very special case of the eigenvalue in S_0 we can say that the multiplicity is precisely $|m|$. If $|c| < 1/2$ we cannot say this for the eigenvalues in S_k for $k \neq 0$ because it is possible that these could also be written as $\pm\sqrt{\lambda^2 + (k' + c)^2} - \frac{1}{2}$ for some $k' \neq k$ and $\lambda \neq 0$. The case when $c = \frac{1}{2}$ is of most interest to us and we formulate it as a theorem.

39. THEOREM (Dimension of the space of harmonic spinors on $\mathbb{S}^3$). *With the notation of Theorem 35 we consider the case when $c = 1/2$. In this case the eigenvalue 0 has multiplicity m if $m > 0$ and $-m-1$ if $m < -1$ otherwise 0 is not an eigenvalue. The eigenvalue -1 has multiplicity $-m$ if $m < 0$ and $m+1$ if $m > -1$.*

Proof. This is a simple consequence of Theorem 35. The eigenvalue in S_0 has multiplicity precisely $|m|$ and the eigenvalue in S_{-1} has eigenvalue precisely $|m+1|$. The theorem now follows from considering the different cases. $\square$

40. REMARK. Thus for any natural number we can construct a Dirac operator on $\mathbb{S}^3$ such that the dimension of the kernel is that given number. Using the conformal invariance of the dimension of the kernel one may arrive at a construction on $\mathbb{R}^3$ which is summarized below. One has to discuss the behavior at infinity, but this is not very difficult.

41. THEOREM (Zero modes on $\mathbb{R}^3$). *Let $\tau : \mathbb{R}^3 \to M \setminus \{p\}$ be the inverse of the stereographic projection onto $\mathbb{R}^3$ from the sphere $M = (\mathbb{S}^3, g_3)$ with a point removed. Let $\beta = \tau^*(h *\nu) = (h \circ \tau)\tau^*(*\nu)$ be the pullback of an arbitrary closed two form $h *\nu$ on $\mathbb{S}^3$. [In other words, $\beta = \widetilde{h}\ \tau^*(*\nu)$ is an arbitrary closed two form on $\mathbb{R}^3$ which is parallel with $\tau^*(*\nu)$ and the function $\widetilde{h} \circ \tau^{-1}$ extends to a regular function h on $\mathbb{S}^3$.] Then $h = g \circ \phi$ with some function $g \in C^\infty(\mathbb{S}^2)$ and $(\tau^{-1})^*\beta$ extends to a two form $\beta_M = (g \circ \phi) *\nu$ on*

M. Let $\mathcal{D}_{\mathbb{R}^3}$ be the Dirac operator on $\mathbb{R}^3$ with magnetic two form β and let $\mathcal{D}_M$ be the Dirac operator on M with magnetic two form β_M. Then

$$dim\ Ker\,\mathcal{D} = dim\ Ker\,\mathcal{D}_M\ ,$$

and dim $Ker\,\mathcal{D}_M$ is given in Theorem 39.

Proof. From Remark 36, we easily see that any two form of the form $\beta = \tau^*(h * \nu)$ is closed if and only if $h = g \circ \phi$ with some function g on $\mathbb{S}^2$. Then β_M defined as $(g \circ \phi) * \nu$ on M coincides with $(\tau^{-1})^*\beta$ on $M \setminus \{p\}$.

Next, we recall that the magnetic two form determines the Dirac operator on $\mathbb{S}^3$ and $\mathbb{R}^3$ up to bundle isomorphism and gauge transformation, hence $\mathcal{D}_{\mathbb{R}^3}$ and $\mathcal{D}_M$ in the theorem are well defined. They can be considered acting on the trivial bundle $\Psi_{\mathbb{R}^3} = \mathbb{R}^3 \times \mathbb{C}^2$ and $\Psi_M = M \times \mathbb{C}^2$, respectively.

The stereographic projection isometrically identifies $(M \setminus \{p\}, g_3)$ with $(\mathbb{R}^3, \Omega^2 ds^2)$, where $\Omega(x) := (1 + x^2)^{-1}$. For any normalized spinor $\xi \in$ Ker $\mathcal{D}_M$, $\mathcal{D}_M\xi = 0$, we have $\mathcal{D}_{\mathbb{R}^3}(\Omega\xi) = 0$ by Theorem 23. By elliptic regularity of $\mathcal{D}_M$, ξ is smooth, in particular $\langle \xi, \xi \rangle$ is bounded. Hence

$$\int_{\mathbb{R}^3} \langle \Omega\xi, \Omega\xi \rangle (vol_{\mathbb{R}^3}) \leq \max_M \langle \xi, \xi \rangle \int_{\mathbb{R}^3} \Omega^2 (vol_{\mathbb{R}^3}) < \infty$$

hence $\Omega\xi \in$ Ker $\mathcal{D}_{\mathbb{R}^3}$.

Conversely, if $\psi \in$ Ker $\mathcal{D}_{\mathbb{R}^3}$, then $\mathcal{D}_M\xi = 0$ away from p with $\xi := \Omega^{-1}\psi$. It follows from $\psi \in L^2(\Psi_{\mathbb{R}^3})$ that $\xi \in L^2(\Psi_M)$, since $\Omega \leq 1$. These two facts easily imply that ξ extends to p and $\mathcal{D}_M\xi = 0$ everywhere. In fact, consider a sequence of bounded cutoff functions $\chi_n \in C^\infty(M)$, $\chi_n(p) = 0$, $\chi_n \to 1$ such that $\|d\chi_n\|_{L^2(\Lambda^1(M))} \to 0$. Such sequence exists in three dimensions. Clearly $\mathcal{D}_M(\chi_n\xi) \to \mathcal{D}_M\xi$ and $\mathcal{D}_M(\chi_n\xi) = -i\sigma(d\chi_n)\xi \to 0$ as $n \to 0$, hence $\mathcal{D}_M\xi = 0$ on M in the sense of distributions. It then follows from elliptic regularity of $\mathcal{D}_M$ and $\xi \in L^2(\Psi_M)$ that ξ is smooth on M and $\xi \in$ Ker $\mathcal{D}_M$ $\square$

42. REMARK. Finally one may note that the first zero mode constructed by Loss and Yau in [LY] is the stereographic projection of the one one gets according to Theorem 39 with $m = 1$ and $c = 1/2$. This corresponds to choosing $g = 3$ in Theorem 35.

The proof of Theorem 35 is divided into subsections.

8.3 Rotationally symmetric eigenbasis

By the properties of ϕ and our assumption on β_M the rotation along the $\mathbb{S}^1$ fibers is a symmetry of the data, hence the generator of this rotation should commute with $\mathcal{D}_M$.

43. PROPOSITION. *Let n be the vector dual to ν and let us define*

$$Q := -i\nabla_n^M - \frac{1}{2}\sigma(\nu), \tag{22}$$

which is symmetric since $\operatorname{div} n = 0$. Then

$$[\mathcal{D}_M, Q] = 0 \tag{23}$$

(when acting on smooth sections), and we also have that

$$\{\mathcal{D}_M, \sigma(\nu)\} = 2Q - \sigma(\nu) \tag{24}$$

(on the domain of $\mathcal{D}_M$).

Proof. We use the basis u_1, u_2, n constructed in Section 7, and let u^1, u^2, ν be the dual basis. Sometimes we use the notation $u^3 = \nu$, $u_3 = n$ for brevity. We also drop the superscript M from ∇^M. Then

$$\{\mathcal{D}_M, \sigma(\nu)\} \quad = \quad (-i)\sum_{j=1}^{3} \left\{\sigma(u^j)\nabla_{u_j},\ \sigma(u^3)\right\}$$

$$
\begin{aligned}
&= (-i)\sum_{j=1}^{3}\left\{\sigma(u^j),\sigma(u^3)\right\}\nabla_{u_j} + (-i)\sum_{j=1}^{3}\sigma(u^j)\left[\nabla_{u_j},\sigma(u^3)\right] \\
&= -2i\nabla_n - i\sum_{j=1}^{3}\sigma(u^j)\sigma(\nabla_{u_j}u^3) = -2i\nabla_n - 2\sigma(\nu) = 2Q - \sigma(\nu).
\end{aligned}
$$

In the last step we used $-i\sigma(u^1)\sigma(u^2) = \sigma(u^3)$ and various $\nabla_{u_i}u^j$'s computed in (17).

To prove 23, we first compute

$$
\begin{aligned}
[\mathcal{D}_M, \sigma(\nu)] &= (-i)\sum_{j=1}^{3}\left[\sigma(u^j)\nabla_{u_j},\sigma(u^3)\right] \\
&= (-i)\sum_{j=1}^{3}\left[\sigma(u^j),\sigma(u^3)\right]\nabla_{u_j} + (-i)\sum_{j=1}^{3}\sigma(u^j)\left[\nabla_{u_j},\sigma(u^3)\right] \\
&= 2\sigma(u^1)\nabla_{u_2} - 2\sigma(u^2)\nabla_{u_1} - 2\sigma(\nu).
\end{aligned}
$$

Next, we compute

$$
\begin{aligned}
[\mathcal{D}_M, \nabla_n] &= (-i)\sum_{j=1}^{2}\sigma(u^j)\left[\nabla_{u_j},\nabla_n\right] + i\sum_{j=1}^{2}\left[\nabla_n,\sigma(u^j)\right]\nabla_{u_j} \\
&= (-i)\sum_{j=1}^{2}\sigma(u^j)\left[\nabla_{u_j},\nabla_n\right] + i\sigma(u^2)\nabla_{u_1} - i\sigma(u^1)\nabla_{u_2}.
\end{aligned}
$$

To compute the commutators, we use

$$
\left[\nabla_{u_j},\nabla_n\right] = R_\Psi(u_j,n) + \nabla_{[u_j,n]}
$$

and Theorem 12 to express R_Ψ in terms of the Riemannian curvature R and the magnetic two form β_M. Then R can be computed from $\nabla_{u_i}u_j$'s (17), and the result is

$$
\begin{array}{lll}
R(u_1,u_3)u_1 = -u_3 & R(u_1,u_3)u_2 = 0 & R(u_1,u_3)u_3 = u_1 \\
R(u_2,u_3)u_1 = 0 & R(u_2,u_3)u_2 = -u_3 & R(u_2,u_3)u_3 = u_2.
\end{array}
$$

By the assumption on the magnetic field in Theorem 35, $\beta_M(u_j, u_3) = 0$ for $j = 1, 2$. We obtain

$$R_\Psi(u_1, u_3) = -\tfrac{i}{2}\sigma(u^2)\ , \qquad R_\Psi(u_2, u_3) = \tfrac{i}{2}\sigma(u^1),$$

hence using $[u_1, u_3] = -2u_2$, $[u_2, u_3] = 2u_1$ we conclude that

$$(-i)\sum_{j=1}^{2}\sigma(u^j)\left[\nabla_{u_j}, \nabla_n\right] = 2i\sigma(u^1)\nabla_{u_2} - 2i\sigma(u^2)\nabla_{u_1} - i\sigma(u^3).$$

Therefore

$$[\mathcal{D}_M, \nabla_n] = i\sigma(u^1)\nabla_{u_2} - i\sigma(u^2)\nabla_{u_1} - i\sigma(u^3).$$

and combining this with $[\mathcal{D}_M, \sigma(\nu)]$ computed above, we arrive at (23). $\square$

Since $\mathcal{D}_M$ has a pure point spectrum, (23) implies that it has an eigenbasis consisting of eigenspinors of Q. One expects that these eigenspinors are actually pull-ups of some spinors in an appropriate spinor bundle on N. This is correct after a gauge transformation which we describe now.

For any $k \in \mathbb{Z}$ let us fix a spinor connection $\nabla^N = \nabla^{N,(k)}$ with a magnetic form $\beta_N(k)$ on the spinor bundle Ψ_{m-k} with Chern number $m-k$ on N. We identify Ψ_M with the lift of Ψ_{m-k}. Let $\nabla^{M,(k)}$ be the lift of $\nabla^{N,(k)}$ according to Proposition 27:

$$\nabla_X^{M,(k)} := \phi^*(\nabla^{N,(k)})_X - \frac{1}{2}\sigma(\nu)\sigma(\nabla_X\nu) + \frac{i}{2}\nu(X)\sigma(\nu), \tag{25}$$

where we also used (16). Let $\beta_{M,(k)} := \phi^*(\beta_N(k))$ be the magnetic form of $\nabla_X^{M,(k)}$. Finally we define

$$\widetilde{\nabla}^{M,(k)} := \nabla^{M,(k)} + i(c+k)\nu. \tag{26}$$

The magnetic two form of $\widetilde{\nabla}^{M,(k)}$ is

$$\begin{aligned}\beta_{M,(k)} - (c+k)d\nu &= \beta_{M,(k)} + 2(c+k)(*\nu) = \phi^*\Big(\beta_N(k) + 2(c+k)(vol_N)\Big)\\ &= \phi^*\Big(g(vol_N)\Big) = \beta_M\end{aligned}$$

by (18). It is now clear that $\widetilde{\nabla}^{M,(k)}$ and ∇^M are gauge equivalent since their magnetic fields are the same. Therefore there exists a function $f_k \in C^\infty(M)$, depending on k, such that

$$\widetilde{\nabla}^{M,(k)} = e^{if_k} \nabla^M e^{-if_k}.$$

Hence the spectrum of $\mathcal{D}_M$ and $\widetilde{\mathcal{D}}_{M,(k)} = -i\sigma(\widetilde{\nabla}^{M,(k)})$ are the same for any $k \in \mathbb{Z}$, and we will work with the latter operators.

Let

$$\widetilde{Q}^{(k)} := e^{if_k} Q e^{-if_k}$$

then by unitary transformation we get from (23) and (24) that

$$\left[\widetilde{\mathcal{D}}_{M,(k)},\ \widetilde{Q}^{(k)}\right] = 0 \tag{27}$$

$$\left\{\widetilde{\mathcal{D}}_{M,(k)},\ \sigma(\nu)\right\} = 2\widetilde{Q}^{(k)} - \sigma(\nu) \tag{28}$$

on smooth sections. Since $\widetilde{\mathcal{D}}_{M,(k)}$ has compact resolvent, each eigenspace is finite dimensional and by elliptic regularity consists of smooth sections. It then follows from (27) and (28) that there exists an eigenbasis of $\widetilde{\mathcal{D}}_{M,(k)}$ consisting of eigenspinors of $\widetilde{Q}^{(k)}$.

44. PROPOSITION. *The spectrum of Q belongs to the set $\mathbb{Z} + c$. Moreover, for any integer k, if*

$$\widetilde{Q}^{(k)}\chi = (k + c)\chi, \tag{29}$$

then there exists a section ξ of the spinor bundle Ψ_{m-k} on N with Chern number $m - k$ such that $\chi = \xi \circ \phi$.

Proof. Since Q and $\widetilde{Q}^{(k)}$ are unitarily equivalent for any k, it is enough to compute the spectrum of $\widetilde{Q}^{(k)}$. Let $\widetilde{Q}^{(k)}\chi = E\chi$. First we compute $\widetilde{Q}^{(k)}$ on any pull-up spinor $\eta \circ \phi$ with $\eta \in \Gamma(\Psi_{m-k})$. Notice that $\widetilde{Q}^{(k)} = -i\widetilde{\nabla}_n^{M,(k)} - \frac{1}{2}\sigma(\nu)$, hence

$$\begin{aligned} \widetilde{Q}^{(k)}(\eta \circ \phi) &= -i\nabla_n^{M,(k)}(\eta \circ \phi) + (k+c)(\eta \circ \phi) - \frac{1}{2}\sigma(\nu)(\eta \circ \phi) \\ &= (k+c)(\eta \circ \phi) \end{aligned} \tag{30}$$

using (25), (26) and that $\phi^*(\nabla^{N,(k)})_n(\eta \circ \phi) = 0$.

Next we choose a point $p \in \mathbb{S}^3$ where χ does not vanish. We choose an orthonormal basis $\{\xi^+, \xi^-\}$ in Ψ_{m-k} in a neighborhood V around the point $\phi(p)$ and we pull it up. This gives an orthonormal basis $\{\xi^+ \circ \phi, \xi^- \circ \phi\}$ in $\phi^{-1}(V)$, which is a tubular neighborhood of the circle fiber C going through p. In this neighborhood we can write the eigenspinor χ as $\chi = r_+(\xi^+ \circ \phi) + r_-(\xi^- \circ \phi)$ with some functions $r_\pm \in C^\infty(M)$. Then

$$\widetilde{Q}^{(k)}\chi = -i(nr_+)(\xi^+ \circ \phi) - i(nr_-)(\xi^- \circ \phi) + (k+c)\chi$$

and by $\widetilde{Q}^{(k)}\chi = E\chi$ and linear independence of $\xi_\pm \circ \phi$ we get that $nr_\pm = i(E - (k+c))r_\pm$. Hence $r_\pm$ must be of the form

$$r_\pm = r_\pm^{(0)} \exp\left[i\theta\left(E - (k+c)\right)\right],$$

where θ is the arclength parameter along C in the direction of n and $n(r_\pm^{(0)}) = 0$. Since the total length of C is 2π and at least one of r_+, r_- is not identically zero, we see that $E - (k+c) \in \mathbb{Z}$, hence $E \in \mathbb{Z} + c$.

Now the second statement in Proposition 44 is straight forward. If $E = k + c$ is an eigenvalue in (29), then $nr_+ = nr_- = 0$, i.e. $r_\pm$ are pull-up functions, $r_\pm = r_\pm^* \circ \phi$ with some $r_\pm^* \in C^\infty(N)$, hence $\chi = (r_+^*\xi_+) \circ \phi + (r_-^*\xi_-) \circ \phi$, and it is the pull-up of $\xi = r_+^*\xi_+ + r_-^*\xi_-$. $\square$

We summarize our result

45. THEOREM. *Let $\mathcal{D}_M\psi = e\psi$ and $Q\psi = \mu\psi$. Then $\mu = k + c$ with some $k \in \mathbb{Z}$. Fix this k, let $\widetilde{\psi} := e^{if_k}\psi$, then by unitarity*

$$\widetilde{\mathcal{D}}_{M,(k)}\widetilde{\psi} = e\widetilde{\psi} \qquad \text{and} \qquad \widetilde{Q}^{(k)}\widetilde{\psi} = \mu\widetilde{\psi}. \tag{31}$$

Then $\widetilde{\psi} = \xi \circ \phi$, with some section ξ of the spinor bundle Ψ_{m-k} on N with Chern number $m - k$. Moreover, for any section χ of Ψ_{m-k} we have

$$\widetilde{\mathcal{D}}_{M,(k)}(\chi \circ \phi) = \big(\mathcal{D}_{N,(k)}\chi\big) \circ \phi - \frac{1}{2}\chi \circ \phi + (k + c)\sigma(\nu)\chi \circ \phi\,. \tag{32}$$

Proof. All statements have been proven in Proposition 44 except the (32), which is a straight forward calculation. □

8.4 Proof of Theorem 35

Proof. Let e be an eigenvalue of $\mathcal{D}_M$ and consider the corresponding eigenspace, which is finite dimensional. In this subspace we find a simultaneous eigenbasis of Q, hence we consider spinors ψ with $\mathcal{D}_M\psi = e\psi$ and $Q\psi = \mu\psi$ for some μ. Then $\mu = k + c$ with some $k \in \mathbb{Z}$ and fix this k. Following Theorem 45, let $\widetilde{\psi} := e^{if_k}\psi$ and $\widetilde{\psi} = \xi \circ \phi$ with some $\xi \in \Psi_{m-k}$. From (32) we have

$$\Big(\widetilde{\mathcal{D}}_{M,(k)} + \frac{1}{2}\Big)(\xi \circ \phi) = \Big[\big(\mathcal{D}_{N,(k)} + (k + c)\sigma(\nu)\big)\,\xi\Big] \circ \phi\,. \tag{33}$$

Using (32) once more for $\chi = \big(\mathcal{D}_{N,(k)} + (k + c)\sigma(\nu)\big)\,\xi$ we obtain

$$\Big(\widetilde{\mathcal{D}}_{M,(k)} + \frac{1}{2}\Big)^2(\xi \circ \phi) = \Big[\big(\mathcal{D}_{N,(k)} + (k + c)\sigma(\nu)\big)^2\,\xi\Big] \circ \phi\,.$$

Notice that $\{\mathcal{D}_{N,(k)}, \sigma(\nu)\} = 0$, which easily follows from (24) and (30). In particular, the nonzero spectrum of $\mathcal{D}_{N,(k)}$ is symmetric and

$$\mathcal{D}^2_{N,(k)}\xi = \Big[\Big(e + \frac{1}{2}\Big)^2 - (k + c)^2\Big]\xi$$

i.e. ξ is an eigenspinor of $\mathcal{D}^2_{N,(k)}$ and we define

$$\lambda := \sqrt{\left(e+\frac{1}{2}\right)^2 - (k+c)^2}.$$

If $\lambda > 0$, then clearly $\lambda \in \Sigma_+(k)$ (recall that $\Sigma_+(k)$ is the positive spectrum of $\mathcal{D}_{N,(k)}$). The multiplicity of e in the subspace $\{\psi \ : \ Q\psi = (k+c)\psi\}$ is bounded by the multiplicity of λ in the set $\Sigma_+(k)$.

If $\lambda = 0$, then by Theorem 37 the eigenvalue 0 belongs to the spectrum of $\mathcal{D}_{N,(k)}$ if and only if $m \neq k$. In this case the multiplicity of the 0-eigenvalue is $|m-k|$, and the eigenspinor is contained in the subspace $\{\psi \ : \ \sigma(\nu)\psi = [\mathrm{sgn}(m-k)]\psi\}$. Hence, by (33) we obtain $e = [\mathrm{sgn}(m-k)](k+c) - \frac{1}{2}$ and the multiplicity of this eigenvalue of $\widetilde{\mathcal{D}}_{M,(k)}$ is at most $|m-k|$. This shows that Spec $\mathcal{D}_M$ is included in the union given in (20) with multiplicity.

For the converse statement, for any fixed $k \in \mathbb{Z}$ we start with an eigenspace of $\mathcal{D}_{N,(k)}$ with eigenvalue λ.

If $\lambda = 0$, then the same space is also an eigenspace of $\sigma(\nu)$ with eigenvalue $\mathrm{sgn}(m-k)$ by Theorem 37. Hence by (33) the lift of this eigenspace to Ψ_M is an eigenspace of $\widetilde{\mathcal{D}}_{M,(k)}$ with eigenvalue $e = [\mathrm{sgn}(m-k)](k+c) - \frac{1}{2}$.

If $\lambda > 0$, then for any element ξ of this eigenspace we form

$$\chi^{\pm} := \xi + \frac{-\lambda \pm \sqrt{\lambda^2 + (k+c)^2}}{k+c}\sigma(\nu)\xi$$

if $k+c \neq 0$ and $\chi^+ := \xi$, $\chi^- := \sigma(\nu)\xi$ if $k+c=0$. The sets $\{\chi_j^-\}$ and $\{\chi_j^+\}$ are both linearly independent as ξ's run through a linearly independent set $\{\xi_j\} \subset \mathrm{Ker}\ (\mathcal{D}_{N,(k)} - \lambda)$. For, if $\sum_j c_j\chi_j^+ = 0$ with some constants c_j then

$$0 = \left(1 + \lambda^{-1}\mathcal{D}_{N,(k)}\right)\sum_j c_j\chi_j^+ = 2\sum_j c_j\xi^j.$$

It is easy to check from (33) that $\chi^{\pm} \circ \phi$ is an eigenspinor of $\widetilde{\mathcal{D}}_{M,(k)}$ with eigenvalue $e = \pm\sqrt{\lambda^2 + (k+c)^2} - \frac{1}{2}$. This completes the proof of Theorem 35.

$\square$

A Appendix: Spinor bundles on $\mathbb{S}^2$ (and $\mathbb{S}^3$) and the Aharonov-Casher Theorem

We shall now construct spinor bundles Ψ on $\mathbb{S}^2$. First we choose coordinates. Let $\mathbb{S}^2_+ = \mathbb{S}^2 \setminus \{S\}$ and $\mathbb{S}^2_- = \mathbb{S}^2 \setminus \{N\}$, where N and S are the north and south poles respectively.

Consider the stereographic projections $z_\pm : \mathbb{S}^2_\pm \to \mathbb{C}$ defined by

$$\omega = \begin{cases} \left(\dfrac{-4\overline{z_-(\omega)}}{4+|z_-(\omega)|^2}, -\dfrac{4-|z_-(\omega)|^2}{4+|z_-(\omega)|^2} \right), & \text{for } \omega \in \mathbb{S}^2_- \\ \\ \left(\dfrac{4z_+(\omega)}{4+|z_+(\omega)|^2}, \dfrac{4-|z_+(\omega)|^2}{4+|z_+(\omega)|^2} \right), & \text{for } \omega \in \mathbb{S}^2_+ \end{cases},$$

where we have identified $\mathbb{S}^2 \subset \mathbb{C} \times \mathbb{R}$. Note that for $\omega \in \mathbb{S}^2_- \cap \mathbb{S}^2_+$ we have $z_-(\omega) = -4z_+(\omega)^{-1}$. With the above choice the maps $z_\pm$ are orientation preserving when we choose the standard orientations of $\mathbb{S}^2$ and $\mathbb{C}$ (strictly speaking z_- is a stereographic projection followed by a reflection). If we use the metric

$$ds^2 = (1 + \tfrac{1}{4}|z|^2)^{-2} dz\, d\overline{z}$$

on $\mathbb{C}$ both maps $z_\pm$ are also isometries.

A.1 Spinor bundles on $\mathbb{S}^2$ and $\mathbb{S}^3$

Corresponding to each $n \in \mathbb{Z}$ we define a spinor bundle Ψ_n on $\mathbb{S}^2$ by the following properties:

- There are open subsets $\Psi_n^\pm \subset \Psi_n$ such that $\Psi_n = \Psi_n^+ \cup \Psi_n^-$

- There are diffeomorphisms $\phi_\pm : \Psi_n^{(\pm)} \to \mathbb{S}^2_\pm \times \mathbb{C}^2$.

- If $\eta \in \Psi_n$ and $\phi_\pm(\eta) = (\omega_\pm, u_\pm)$ then

$$\omega_+ = \omega_- \quad \text{and} \quad u_- = \mathcal{U}_n(z_+(\omega_+))\mathcal{W}(z_+(\omega_+))u_+ \tag{34}$$

  where

$$\mathcal{U}_n(z) = \left(\frac{|z|}{z}\right)^n \quad \text{and} \quad \mathcal{W}(z) = \begin{pmatrix} z|z|^{-1} & 0 \\ 0 & \overline{z}|z|^{-1} \end{pmatrix} \in SU(2)$$

- If $\alpha = a(z)d\overline{z} + \overline{a(z)}dz$ is a real one-form on $\mathbb{C}$ then the Clifford multiplication σ on Ψ_n is defined by

$$\phi_\pm\left(\sigma\left(z_\pm^*(\alpha)\right)\eta\right) = \left(\omega,\ (1 + \tfrac{1}{4}|z_\pm(\omega)|^2)\begin{pmatrix} 0 & \overline{a(z_\pm(\omega))} \\ a(z_\pm(\omega)) & 0 \end{pmatrix} u_\pm\right),$$

  when $\phi_\pm(\eta) = (\omega, u_\pm)$. Here $z_\pm^*(\alpha)$ are the pull-backs of the one-form α to $\mathbb{S}^2_\pm$. Note that it is the Clifford multiplication relative to the metric ds^2 which is being used on $\mathbb{C}^2$.

It is fairly easy to check that this really defines a spinor bundle Ψ_n on $\mathbb{S}^2$. In particular, we notice that if $\tilde{\alpha}$ is a one-form on $\mathbb{S}^2$ then $\tilde{\alpha}\big|_{\mathbb{S}^2_\pm} = z_\pm^*(\alpha_\pm)$, where $\alpha_\pm = a_\pm(z)\overline{dz} + \overline{a_\pm(z)}dz$ satisfies $a_+(z) = 4a_-(-4z^{-1})\overline{z}^{-2}$. Thus we see that the Clifford multiplication transforms consistently between $\Psi^{(\pm)}$, i.e.,

$$\begin{aligned}(1 + \tfrac{1}{4}|z_-|^2)&\begin{pmatrix} 0 & \overline{a_-(z_-)} \\ a_-(z_-) & 0 \end{pmatrix} \\ &= (1 + \tfrac{1}{4}|z_+|^2)\mathcal{W}(z_+)\begin{pmatrix} 0 & \overline{a_+(z_+)} \\ a_+(z_+) & 0 \end{pmatrix}\mathcal{W}(z_+)^*\end{aligned}$$

where $z_\pm = z_\pm(\omega)$.

46. PROPOSITION. *If Ψ is a spinor bundle over $\mathbb{S}^2$ then Ψ is diffeomorphic to Ψ_n for some $n \in \mathbb{Z}$.*

Proof. We just sketch this standard argument. Since any vector bundle on $\mathbb{C}$ or on $\mathbb{S}^2_\pm$ is trivial we easily see that any spinor bundle on $\mathbb{S}^2$ is of the form described above with $\mathcal{U}_n(z_+(\omega))$ replaced by a general function $\mathcal{U} : \mathbb{S}^2 \setminus \{N, S\} \to U(1)$. Let $-n$ be the degree of the map $\mathcal{U}$ (i.e., the degree when we restrict to e.g. the equatorial circle). Then $\mathbb{S}^2 \setminus \{N, S\} \ni \omega \mapsto \mathcal{U}(\omega)\mathcal{U}_n^*(z_+(\omega)) \in U(1)$ is a map of degree 0. We may therefore find two functions $\mathcal{U}_\pm : \mathbb{S}^2 \setminus \{N, S\} \to U(1)$ with $\mathcal{U}_+$ equal 1 near N and likewise for S such that

$$\mathcal{U}(\omega)\mathcal{U}_n^*(z_+(\omega)) = \mathcal{U}_+(\omega)\mathcal{U}_-(\omega)^*.$$

If we now use $\mathcal{U}_\pm$ to change the coordinates on the fibers of Ψ over $\mathbb{S}^2_\pm$ respectively we see that the transformation matrix $\mathcal{U}$ will be replaced by $\mathcal{U}_n \circ z_+$. $\square$

47. REMARK. A similar argument immediately proves that on $\mathbb{S}^3$ there is only one *Spin*^{c} bundle. This bundle is in fact trivial, since we can find a global orthonormal frame $(e_1, e_2, e_3) := (u_1, u_2, n)$ on $\mathbb{S}^3$ (see Section 7). The Clifford multiplication on $\mathbb{S}^3 \times \mathbb{C}^2$ is defined by $\sigma(e^j) := \sigma_j$, where σ_j is the j-th Pauli matrix.

A.2 *Spin*^{c} connections on Ψ_n

We first describe *Spin*^{c} connections on $\mathbb{C}$ with respect to the metric $ds^2 = (1+\frac{1}{4}|z|^2)^{-2} dz d\overline{z}$. On $\mathbb{C}$ we consider the trivial spinor bundle, i.e., $\Psi = \mathbb{C}\times\mathbb{C}^2$.

Since the metric ds^2 is conformally equivalent to the standard metric $dzd\overline{z}$ with the conformal factor $\Omega(z) = (1 + \frac{1}{4}|z|^2)^{-1}$ we may use the results from Section 4. The spinor sections with respect to the standard metric on $\mathbb{C}$ are given in terms of a (real) one-form $\alpha = a(z)d\overline{z} + \overline{a(z)}dz$ as follows. The covariant derivative along a (real) vector field $X = \xi(z)\partial_z + \overline{\xi(z)}\partial_{\overline{z}}$ is

$$\xi(z)\Big(\partial_z - i\overline{a(z)}\Big) + \overline{\xi(z)}\Big(\partial_{\overline{z}} - ia(z)\Big). \tag{35}$$

We may calculate the covariant derivative of spinors in the conformal metric ds^2 from Prop 22 using

$$\sigma(X^*) = \tfrac{1}{2}\sigma\left(\overline{\xi(z)}dz + \xi(z)d\overline{z}\right) = \begin{pmatrix} 0 & \overline{\xi(z)} \\ \xi(z) & 0 \end{pmatrix}$$

and

$$\sigma(d\Omega) = -\tfrac{1}{2}(1 + \tfrac{1}{4}|z|^2)^{-2}\begin{pmatrix} 0 & \overline{z} \\ z & 0 \end{pmatrix}.$$

According to Prop 22 the covariant derivative of spinors in the metric ds^2 is

$$\nabla_X^\alpha = \xi(z)\Big(\partial_z - i\overline{a(z)}\Big) + \overline{\xi(z)}\Big(\partial_{\overline{z}} - ia(z)\Big) - \frac{i\mathrm{Im}\left(z\overline{\xi(z)}\right)}{(4+|z|^2)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now assume that we have a covariant derivative ∇ on Ψ_n. Let η be a section in Ψ_n and $\tilde{X}$ a vectorfield on $\mathbb{S}^2$. For $\omega \in \mathbb{S}^2_\pm$ write

$$\phi_\pm(\eta(\omega)) = (\omega, u_\pm(z_\pm(\omega))) \quad \text{and} \quad (z_\pm)_*\tilde{X}(\omega) = (\omega, X_\pm(z_\pm(\omega))).$$

Then there exists one-forms $\alpha_\pm$ on $\mathbb{C}$ such that

$$\phi_\pm\left(\nabla_{\tilde{X}}\eta(\omega)\right) = \Big(\omega, \nabla_{X_\pm}^{\alpha_\pm} u_\pm(z_\pm(\omega))\Big). \tag{36}$$

Note that by (34) we must have

$$\nabla_{X_-}^{\alpha_-} u_-(z_-(\omega)) = \left(\frac{|z_+(\omega)|}{z_+(\omega)}\right)^n \mathcal{W}(z_+(\omega))\nabla_{X_+}^{\alpha_+} u_+(z_+(\omega))$$

i.e.,

$$\nabla_{X_-}^{\alpha_-}\left[\left(\mathcal{U}_n\mathcal{W}u_+\right)(-4z^{-1})\right] = \mathcal{U}_n(-4z^{-1})\mathcal{W}(-4z^{-1})\left(\nabla_{X_+}^{\alpha_+}u_+\right)(-4z^{-1}) \tag{37}$$

With $X_\pm = 2\mathrm{Re}\,[\xi_\pm(z)\partial_z]$ we have the relation $\xi_+(z) = \frac{1}{4}z^2\xi_-(-4z^{-1})$. A straightforward calculation then shows that if $\alpha_\pm = 2\mathrm{Re}\,[a_\pm(z)d\overline{z}]$ then (37) implies that

$$a_-(z) = 4\overline{z}^{-2}a_+(-4z^{-1}) + i\tfrac{n}{2}\overline{z}^{-1}. \tag{38}$$

Conversely, for any choice of functions $a_\pm$ on $\mathbb{C}$ satisfying (38) the relation (36) will define a *Spin*^{c} connection on $\mathbb{S}^2$.

It is easy to see that $\beta_{|\mathbb{S}^2_\pm} = z^*_\pm(d\alpha_\pm)$. Using Stokes law we then have that

$$\int_{\mathbb{S}^2}\beta = \int_C z^*_-(\alpha_-) + z^*_+(\alpha_+) = \frac{in}{2}\int_{|z|=2}\frac{d\overline{z}}{\overline{z}} - \frac{dz}{z} = 2\pi n\ , \tag{39}$$

where C is the equatiorial curve oriented appropriately, corresponding to the circle $|z| = 2$ being oriented counterclockwise. Note that

$$\int_{|z|=2} 4\overline{z}^{-2}a_+(-4z^{-1})d\overline{z} = -\int_{|z|=2} a_+(z)d\overline{z}.$$

48. PROPOSITION. *For any closed 2-form β on $\mathbb{S}^2$ with $\int\beta = 2\pi n$ there is a Spin^c connection on Ψ_n such that β is the magnetic 2-form.*

Proof. We must show that there are one-forms $\alpha_\pm$ on $\mathbb{C}$ satisfying (38) such that $\beta_{|\mathbb{S}^2_\pm} = z^*_\pm(d\alpha_\pm)$. We construct them explicitly. As before we will write $\alpha_\pm = 2\mathrm{Re}[a_\pm(z)d\overline{z}]$ with $a_\pm$ defined below.

Let $\tilde{\beta}_\pm = (z_\pm^{-1})^*(\beta)$. We define

$$h_+(z) := \pi^{-1}\int_{\mathbb{C}} \log|z - z'|^2\tilde{\beta}_+(z')\ , \tag{40}$$

$$a_+(z) := \frac{i}{4}\partial_{\overline{z}}h_+(z)$$

and

$$a_-(z) := 4\overline{z}^{-2}a_+(-4z^{-1}) + i\tfrac{n}{2}\overline{z}^{-1}$$

according to (38). A simple calculation shows that $a_-(z)$ is smooth on $\mathbb{C}$, i.e. the singularities apparently present in (38) exactly cancel each other. For this argument we use that $\int_{\mathbb{C}} \beta_+ = 2\pi n$ and that β_+ is the pushforward of a (smooth) 2 form on $\mathbb{S}^2$.

From the definition of h_+ we see that $(\partial_{\overline{z}}\partial_z h(z))dz \wedge d\overline{z} = -2i\tilde{\beta}_+(z)$, which implies $d\alpha_+ = \tilde{\beta}_+$. Finally $d\alpha_- = \tilde{\beta}_-$ follows from the relation (38) and that $z_-^*(d\alpha_-) = z_+^*(d\alpha_+)$ on $\mathbb{S}^2_- \cap \mathbb{S}^2_+$.

□

A.3 The Dirac operator on Ψ_n

Let $\mathcal{D}$ be the Dirac operator corresponding to the *Spin*^{c} connection ∇ on a spinor bundle Ψ_n on $\mathbb{S}^2$ with Chern number n. We may then for any spinor field η in Ψ_n write

$$\phi_\pm(\tilde{\mathcal{D}}\eta(\omega)) = (\omega, \mathcal{D}_\pm u_\pm(z_\pm(\omega)))$$

for $\omega \in \mathbb{S}^2_\pm$. Here $\mathcal{D}_\pm$ are the Dirac operators on $\mathbb{C}$ corresponding to the metric ds^2. According to Theorem 23 we can express $\mathcal{D}_\pm$ in terms of the standard Dirac operators on $\mathbb{C}$ corresponding to the standard covariant derivative (35). The standard Dirac operators are

$$-2i \begin{pmatrix} 0 & \partial_z - i\overline{a_\pm(z)} \\ \partial_{\overline{z}} - ia_\pm(z) & 0 \end{pmatrix}. \tag{41}$$

Finally, we give the proof of the Aharonov Casher Theorem stated in Section 8.

Proof of Theorem 37. Let $\eta \in \ker\mathcal{D}$ and define $u_\pm : \mathbb{C} \to \mathbb{C}^2$ by $\phi_\pm\eta(\omega) = (\omega, u_\pm(z_\pm(\omega)))$, for $\omega \in \mathbb{S}^2_\pm$. Then $\mathcal{D}_\pm u_\pm = 0$. According to Theorem 23 we then have that $v_\pm(z) = \Omega(z)^{1/2}u_\pm(z)$, where as before $\Omega(z) = (1 + \frac{1}{4}|z|^2)^{-1}$, are in the kernel of the standard Dirac operators (41). By (34) the spinors $v_\pm$ satisfy the transformation property

$$\Omega(z)^{-1/2}v_-(z) = \Omega(-4z^{-1})^{-1/2}\mathcal{U}(-4z^{-1})\mathcal{W}(-4z^{-1})v_+(-4z^{-1})\ . \qquad (42)$$

Clearly the map from η to (v_-, v_+) with the above transformation property is a linear isomorphism.

It turns out to be fairly simple to characterize the elements $v_\pm$ in the kernel of the standard Dirac operators. Since σ_3 anticommutes with the standard Dirac operator, we can consider the cases $\sigma_3 v_\pm = v_\pm$ and $\sigma_3 v_\pm = -v_\pm$ separately. Assume for definiteness that $\sigma_3 v_\pm = v_\pm$, i.e.

$$[\partial_{\bar z} - ia_\pm(z)]v_\pm = 0\ . \qquad (43)$$

We write $v_\pm$ in the form

$$v_\pm(z) = f_\pm(z)e^{-\frac{1}{4}h_\pm(z)}$$

where h_+ is defined in (40) and

$$h_-(z) := h_+(-4z^{-1}) + 2n\log|z|^2\ .$$

One easily computes that $\partial_{\bar z}h_\pm(z) = ia_\pm(z)$, and using (43) we obtain that $\partial_{\bar z}f_\pm(z) = 0$, i.e. these are analytic functions. Finally (42) gives the following relation between f_- and f_+

$$f_-(z) = 2(-z)^{n-1}f_+(-4z^{-1})\ .$$

Hence f_- is an analytic function, bounded by a constant times $|z|^{n-1}$ at infinity. Then $n \geq 1$ and f_- is a polynomial of degree at most $n-1$. A basis in the kernel of the Dirac operator is obtained by choosing $f_-(z) = 1, z, \ldots z^{n-1}$, and the dimension is n.

Similar argument shows that if $\sigma_3 v_\pm = -v_\pm$, then $n \leq -1$, and the dimension of the space of such zero modes is $|n|$. In particular all zero modes have definite spin and only one of the two eigenspaces of $\sigma_3 = \sigma(\nu)$ can accomodate zero modes, depending on the sign of n. Recalling that n is the total flux divided by 2π, we have completed the proof of Theorem 37. $\square$

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