On the zero modes of Pauli operators*

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Two results are proved for nul $\mathbb{P}_A$, the dimension of the kernel of the Pauli operator $\mathbb{P}_A = \left\{\boldsymbol{\sigma}\cdot \left(\frac{1}{i}\boldsymbol{\nabla} + \vec{A}\right)\right\}^2$ in $[L^2(\mathbb{R}^3)]^2$: (i) for $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$, where $\vec{B} = \text{curl}\vec{A}$ is the magnetic field, nul $\mathbb{P}_{tA} = 0$ except for a finite number of values of t in any compact subset of $(0,\infty)$; (ii) $\left\{ \vec{B} : \text{ nul } \mathbb{P}_A = 0, \ |\vec{B}| \in L^{3/2}(\mathbb{R}^3) \right\}$ contains an open dense subset of $[L^{3/2}(\mathbb{R}^3)]^3$.

Key Words: Pauli operator, zero modes, magnetic field, Birman-Schwinger operator

1. INTRODUCTION

The Pauli operator is formally defined by

$$\mathbb{P}_A = \left\{\boldsymbol{\sigma}\cdot \left(\frac{1}{i}\boldsymbol{\nabla} + \vec{A}\right)\right\}^2 \equiv \sum_{j=1}^{3}\left\{\sigma_j \left(\frac{1}{i}\partial_j + A_j\right)\right\}^2 \tag{1}$$

where $\vec{A} = (A_1, A_2, A_3)$ is a vector potential which is such that *curl* $\vec{A} = \vec{B}$, the magnetic field, and $\boldsymbol{\sigma} = \vec{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$ is the triple of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

The expression (1) defines a non-negative self-adjoint operator in $[L^2(\mathbb{R}^3)]^2$; its precise definition will be given in §2.

Zero modes of $\mathbb{P}_A$ are the eigenvectors corresponding to an eigenvalue at zero. The existence of zero modes has profound implications to the stability

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of matter when $\mathbb{P}_A$, or the Dirac-Weyl operator $\boldsymbol{\sigma}\cdot\left(\frac{1}{i}\boldsymbol{\nabla}+\vec{A}\right)$, is used for the model, for the vanishing kinetic energy of zero modes means that their potential energy can not be controlled by their kinetic. For an account of this phenomenon and its consequences, we refer to [8, 10, 11]. Also, the importance of zero modes for the understanding of other deep physical problems is emphasized in [1]. A significant mathematical implication of zero modes is that there can't be an analogue of the Cwikel-Lieb-Rosenblum inequality for the number of negative eigenvalues of $\mathbb{P}_A+V$ in terms of some L^p norm of the scalar potential V, since any small negative perturbation V would produce negative eigenvalues, contrary to such an inequality if V is sufficiently small.

The first example of a magnetic field $\vec{B}$ which yields zero modes was the following constructed in [11]:

$$\vec{B}(\boldsymbol{x}) \;=\; \frac{12}{(1+r^2)^3}\,(2x_1x_3-2x_2,\; 2x_2x_3+2x_1,\; 1-x_1^2-x_2^2+x_3^2),$$

where $\boldsymbol{x}=(x_1,x_2,x_3)$ and $r=|\boldsymbol{x}|$. There are two features of the Loss/Yau example which are of particular relevance to us:

$$|\vec{B}(\boldsymbol{x})|\in L^p(\mathbb{R}^3)\quad\text{for any}\;\; p>\frac{3}{4}, \tag{3}$$

$$\frac{1}{3}(3+2l)\vec{B}(\boldsymbol{x}),\; l\in\mathbb{N},\qquad\text{also yields zero modes.} \tag{4}$$

We shall reserve comment on these till later. Other examples of zero modes, based on the construction of [11], are given in [6], [1]. In an attempt to explain the origin of zero modes, Erdős and Solovej in [7] give a more geometric viewpoint. Using the known behaviour of the Dirac operator under conformal transformations, and that $\mathbb{R}^3$ is conformally equivalent to a punctured sphere $\mathbb{S}^3$, they establish their zero modes on $\mathbb{S}^3$ as well as on $\mathbb{R}^3$ as pull-backs of zero modes on $\mathbb{S}^2$ under the Hopf map $\mathbb{S}^3\to\mathbb{S}^2$. It is also shown in [7] that arbitrary degeneracy is possible; examples of this may also be found in [2].

In even-dimensional manifolds, the Atiyah-Singer index theorem is a powerful tool for investigating the kernel of $\mathbb{P}_A$, since the deficiency of $\mathbb{P}_A$ can vanish, in which case the index is equal to nul $\mathbb{P}_A$, the nullity of $\mathbb{P}_A$ (i.e. the dimension of the kernel $\ker\mathbb{P}_A$). A celebrated example is the Aharonov-Casher Theorem in $\mathbb{R}^2$ and its analogue due to Avron and Tomaras in $\mathbb{S}^2$ (see [5]). In $\mathbb{R}^2$, this assert that for suitable $\vec{B}$ (e.g. $\vec{B}$ bounded and of compact support), the nullity of $\mathbb{P}_A$ is

$$\left\{\frac{1}{2\pi}\left|\int\limits_{\mathbb{R}^2} B(\boldsymbol{x})d\boldsymbol{x}\right|\right\}$$

where $\{y\}$ denotes the largest integer strictly less than y and $\{0\} = 0$; note that in $\mathbb{R}^2$, the magnetic field has only one component, and is thus a scalar field. Thus in $\mathbb{R}^2$, zero modes are abundant; they exist as long as the magnetic flux $\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} B(\boldsymbol{x})d\boldsymbol{x}$ takes values outside $[-1.1]$. The situation in $\mathbb{S}^2$ is very different. There are now zero modes if and only if the magnetic flux is an integer, a picture which is somewhat reminiscent of that in (4). In fact, this is typical of what happens on any compact manifold (of even or odd dimension), as shown by Anghel in [3].

Apart from the examples in [11, 7] mentioned above very little is known for $\mathbb{R}^3$, and indeed for non-compact manifolds of odd dimension, since it is not easy to obtain information from the Atiyah-Singer index theorem in this case. However, we prove that the situation in $\mathbb{R}^3$ is like that described above for compact manifolds, and is thus dramatically different to $\mathbb{R}^2$. Specifically, we prove in Theorems 4.1 and 4.2

- for $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$, nul $\mathbb{P}_{tA} = 0$ except for a finite number of values of t in any compact subset of $[0, \infty)$;
- $\left\{ \vec{B} : \text{ nul } \mathbb{P}_A = 0, \text{ curl } \vec{A} = \vec{B} \text{ and } |\vec{B}| \in L^{3/2}(\mathbb{R}^3) \right\}$ contains an open dense subset of $[L^{3/2}(\mathbb{R}^3)]^3$.

This explains why zero modes are so difficult to obtain. Note that the Loss-Yau example satisfies our hypothesis. The analogous result holds for $\mathbb{R}^n$ with $n > 3$.

2. PRELIMINARIES

We can write (1) as

$$\mathbb{P}_A = S_A + \vec{\sigma}\cdot\vec{B}, \quad \vec{B} = curl\, \vec{A}, \tag{5}$$

where S_A is the magnetic Schrödinger operator

$$S_A = \left(\frac{1}{i}\boldsymbol{\nabla} + \vec{A}\right)^2 \mathbb{I}_2 \equiv \sum_{j=1}^{3}\left(\frac{1}{i}\partial_j + A_j\right)^2 \mathbb{I}_2, \tag{6}$$

$\mathbb{I}_2$ being the 2×2 identity matrix and $\vec{\sigma}\cdot\vec{B}$ the Zeeman term. Note that a gauge transformation $\vec{A} \mapsto \vec{A} + df$ does not alter the nullity, and hence nul $\mathbb{P}_A$ is independent of the gauge. We denote $[L^2(\mathbb{R}^3)]^2$ by $\mathcal{H}$ and its standard inner-product and norm by $(\cdot\,,\cdot)$ and $\|\cdot\|$ respectively:

$$\|f\|^2 = \int\limits_{\mathbb{R}^3} |f(\mathbf{x})|^2 \, d\mathbf{x},$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{C}^2$. It will be assumed throughout that

$$A_j \in L^2_{loc}(\mathbb{R}^3), \quad j = 1, 2, 3. \tag{7}$$

We continue to denote by S_A the Friedrichs extension of (6) on $[C_0^\infty(\mathbb{R}^3)]^2$. It is a non-negative self-adjoint operator with no zero modes, and its form domain $\mathcal{Q}(S_A)$ is the completion of $[C_0^\infty(\mathbb{R}^3)]^2$ with respect to the norm given by

$$\|\varphi\|_{1,A} = \left\{ \left\| \left(\frac{1}{i}\mathbf{\nabla} + \vec{A} \right) \varphi \right\|^2 + \|\varphi\|^2 \right\}^{1/2}. \tag{8}$$

The operator realisation of $\mathbb{P}_A$ is given in the first lemma.

LEMMA 2.1. *Let* $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$. *Then the sesquilinear form*

$$p_A[\varphi, \psi] = (\mathbb{P}_A \varphi, \psi), \quad \varphi, \psi \in [C_0^\infty(\mathbb{R}^3)]^2 \tag{9}$$

is symmetric, closable and non-negative in $\mathcal{H}$. *The associated self-adjoint operator* $\mathbb{P}_A$ *has form domain* $\mathcal{Q}(S_A)$.

Proof. Given $\varepsilon > 0$, we may write $|\vec{B}| = B_1 + B_2$, where $\|B_1\|_{L^{3/2}(\mathbb{R}^3)} < \varepsilon$ and $\|B_2\|_{L^\infty(\mathbb{R}^3)} < C_\varepsilon$, for some constant C_ε depending on ε. Then

$$(\mathbb{P}_A \varphi, \varphi) = (S_A \varphi, \varphi) + ((\vec{\sigma} \cdot \vec{B}) \varphi, \varphi)$$

and

$$\begin{aligned} |((\vec{\sigma} \cdot \vec{B}) \varphi, \varphi)| &\le (B_1 \varphi, \varphi) + (B_2 \varphi, \varphi) \\ &\le \|B_1\|_{L^{3/2}(\mathbb{R}^3)} \|\varphi\|^2_{[L^6(\mathbb{R}^3)]^2} + C_\varepsilon \|\varphi\|^2 \\ &\le \varepsilon \gamma^2 \|\mathbf{\nabla} |\varphi|\|^2 + C_\varepsilon \|\varphi\|^2 \end{aligned}$$

by the Sobolev Embedding Theorem, with γ the norm of the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$,

$$\le \varepsilon \gamma^2 \left\| \left(\frac{1}{i}\mathbf{\nabla} + \vec{A} \right) \varphi \right\|^2 + C_\varepsilon \|\varphi\|^2$$

by the diamagnetic inequality (see [9, Thm 7.21]). The lemma follows from this. ∎

Hereafter, we shall always assume that

$$|\vec{B}| \in L^{3/2}(\mathbb{R}^3). \tag{10}$$

The operator

$$\mathbb{P} \ := \ \mathbb{P}_A + |\vec{B}| \tag{11}$$

may be defined as in Lemma 2.1, namely, the self-adjoint operator associated with the form

$$p[\varphi] \ \equiv \ p[\varphi,\varphi] \ = \ (\mathbb{P}\varphi,\varphi), \tag{12}$$

with form domain $\mathcal{Q}(S_A)$. As for S_A, $\mathbb{P}$ has no zero modes. Thus S_A and $\mathbb{P}$ are injective and have dense domains and ranges in $\mathcal{H}$. Furthermore, $\mathcal{D}(\mathbb{P}^{1/2}) = \mathcal{D}(S_A^{1/2}) = \mathcal{Q}(S_A)$.

The operator of prime interest is $\mathbb{P}_A$. We shall write it as $\mathbb{P}_A = \mathbb{P} - |\vec{B}|$, and then, initially, proceed along lines which are reminiscent of those described in [4] for proving the Cwikel-Lieb-Rosenbljum inequality for the Schrödinger operator. The problem is essentially reduced to one for an associated operator of Birman-Schwinger type. The following spaces feature prominently in the analysis.

- H_A^1 is the completion of $\mathcal{D}(S_A^{1/2})$ with respect to the norm

$$\|\varphi\|_{H_A^1} \ := \ \|S_A^{1/2}\varphi\|; \tag{13}$$

H_0^1 has norm $\|\varphi\|_{H_0^1} \ := \ \|\boldsymbol{\nabla}\varphi\|$.

- $\mathbb{H}_B^1$ is the completion of $\mathcal{D}(\mathbb{P}^{1/2})$ with respect to the norm

$$\|\varphi\|_{\mathbb{H}_B^1} \ := \ \|\mathbb{P}^{1/2}\varphi\|. \tag{14}$$

<u>Remarks</u>

1. $[C_0^\infty(\mathbb{R}^3)]^2$ is dense in H_A^1 and $\mathbb{H}_B^1$.
2. The space H_0^1 is not a subspace $\mathcal{H}$. However, for $\varphi \in [C_0^\infty(\mathbb{R}^3)]^2$, the Hardy inequality

$$\int_{\mathbb{R}^3} \frac{|\varphi(\boldsymbol{x})|^2}{|\boldsymbol{x}|^2} d\boldsymbol{x} \ \le \ 4 \int_{\mathbb{R}^3} |\boldsymbol{\nabla}\varphi(\boldsymbol{x})|^2 d\boldsymbol{x}$$

is valid, and this implies that H_0^1 may be identified with the function space

$$H_0^1 \ = \ \left\{ u \in [H_{loc}^1(\mathbb{R}^3)]^2 \ : \ \|u\|_{H_0^1}^2 + \| \, u/|\cdot| \, \|^2 < \infty \right\} \tag{15}$$

and $\|\cdot\|_{H_0^1}$ is equivalent to the norm defined by

$$\left(\|u\|_{H_0^1}^2 \ + \ \| \, u/|\cdot| \, \|^2 \right)^{1/2}.$$

3. For the spaces H^1_A and $\mathbb{H}^1_B$, which also do not lie in $\mathcal{H}$, we have the natural embedding

$$\mathbb{H}^1_B \hookrightarrow H^1_A. \tag{16}$$

Also, by the diamagnetic inequality, $\varphi \mapsto |\varphi|$ maps H^1_A continuously into H^1_0, which, in turn, is continuously embedded in $[L^6(\mathbb{R}^3)]^2$ by the Sobolev Embedding Theorem. In fact the spaces in (16) are isomorphic when (10) is satisfied.

For a magnetic potential $\vec{A}$ satisfying $|\vec{A}| \in L^3(\mathbb{R}^3)$, H^1_A can be shown to be continuously embedded in H^1_0. Such a choice of $\vec{A}$ is possible in view of the next lemma which is similar to Theorem A1 in Appendix A of [8].

LEMMA 2.2. *Let* $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$ *and define*

$$\vec{A}(\boldsymbol{x}) \;=\; \frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \frac{(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3} \times \vec{B}(\boldsymbol{y}) d\boldsymbol{y}. \tag{17}$$

Then $|\vec{A}| \in L^3(\mathbb{R}^3)$, *curl* $\vec{A} = \vec{B}$, *div* $\vec{A} = 0$ *in* $\mathcal{D}'$ *and*

$$\|\vec{A}\|_{L^3(\mathbb{R}^3)} \leq C \; \|\vec{B}\|_{L^{3/2}(\mathbb{R}^3)}$$

for some constant C.

Proof. The proof is similar to that in [8]. The following formal argument for deriving (17) is instructive, and will be helpful for obtaining the analogous result in $\mathbb{R}^n$ for $n > 3$.

The set of Hamiltonian quaternions $\mathbb{H}$ is the unitary $\mathbb{R}-$algebra generated by the symbols i, j, k with the relations

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k, \quad jk = -kj = i. \quad ki = -ik = j.$$

Multiplication is associative but obviously not commutative.

If we identify a magnetic field $\vec{B} = (B_1, B_2, B_3)$ and a magnetic potential $\vec{A} = (A_1, A_2, A_3)$ with purely imaginary quaternionic fields on $\mathbb{R}^3$

$$\boldsymbol{b} = B_1(\boldsymbol{x})i + B_2(\boldsymbol{x})j + B_3(\boldsymbol{x})k,$$

$$\boldsymbol{a} = A_1(\boldsymbol{x})i + A_2(\boldsymbol{x})j + A_3(\boldsymbol{x})k,$$

then the equation

$$\tilde{D}(\boldsymbol{a}) \;=\; \boldsymbol{b},$$

where $\tilde{D} = i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$, is equivalent to

$$curl\,\vec{A} = \vec{B}, \quad \text{and} \quad div\,\vec{A} = 0.$$

We can solve the equation $\tilde{D}(\boldsymbol{a}) \;=\; \boldsymbol{b}$ by the convolution of $\boldsymbol{b}$ with the Green's function of $\tilde{D}$. Since $\tilde{D}^2 = -\Delta$ then $\tilde{D}(G(\boldsymbol{x}))$ is the Green's function for $\tilde{D}$ if $G(\boldsymbol{x})$ is the Green's function $\frac{1}{4\pi}\frac{1}{|\boldsymbol{x}|}$ for $-\Delta$. The identity (17) is exactly this convolution of $\boldsymbol{b}$ with $\tilde{D}(G(\boldsymbol{x}))$. ▌

LEMMA 2.3. *Let* $|\vec{A}| \in L^3(\mathbb{R}^3)$, $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$. *Then*
(i) for all $f \in \mathbb{H}^1_B$,

$$\begin{aligned} \left|\, (\, [\vec{D}\cdot\vec{A} + \vec{A}\cdot\vec{D}]f, f)\,\right| &\leq 2\gamma\, \|\vec{A}\|_{L^3(\mathbb{R}^3)}\, \|f\|^2_{H^1_0} &(18)\\ &\leq 2\gamma\, \|\vec{A}\|_{L^3(\mathbb{R}^3)}\, \|f\|^2_{\mathbb{H}^1_B}, &(19)\end{aligned}$$

where $\vec{D} = \frac{1}{i}\boldsymbol{\nabla}$ *and* γ *is the norm of the Sobolev embedding* $H^1_0 \hookrightarrow L^6(\mathbb{R}^3)$;
(ii)

$$\mathbb{H}^1_B \hookrightarrow H^1_A \hookrightarrow H^1_0 \hookrightarrow [L^6(\mathbb{R}^3)]^2. \tag{20}$$

Proof. (i) Let $\varphi \in [C_0^\infty(\mathbb{R}^3)]^2$. Then

$$\begin{aligned} \left|\, (\, [\vec{D}\cdot\vec{A} + \vec{A}\cdot\vec{D}]\varphi, \varphi)\,\right| &= \left|\, 2\ \mathrm{Re}\sum_{j=1}^{3}(A_j\varphi, D_j\varphi)\,\right| \\ &\leq 2\|\vec{A}\|_{L^3(\mathbb{R}^3)}\, \|\varphi\|_{[L^6(\mathbb{R}^3)]^2}\, \|\boldsymbol{\nabla}\varphi\| \\ &\leq 2\gamma\, \|\vec{A}\|_{L^3(\mathbb{R}^3)}\, \|\boldsymbol{\nabla}\varphi\|^2 \end{aligned}$$

Thus (18) follows by continuity, and this implies (19) once (20) is established.
(ii) Let $\varphi \in [C_0^\infty(\mathbb{R}^3)]^2$ and $k > 1$. Then

$$\begin{aligned} k(S_{\vec{A}}\,\varphi, \varphi) &= (k-1)(-\Delta\varphi, \varphi) + \big(\{-\Delta + k[\vec{D}\cdot\vec{A} + \vec{A}\cdot\vec{D}] + k^2|\vec{A}|^2\}\,\varphi, \varphi\big) \\ &\quad + \big([k|\vec{A}|^2 - k^2|\vec{A}|^2]\varphi, \varphi\big) \\ &= (k-1)(-\Delta\varphi, \varphi) + (S_{k\vec{A}}\,\varphi, \varphi) - (k^2-k)\big(|\vec{A}|^2\varphi, \varphi\big) \\ &\geq (k-1)(-\Delta\varphi, \varphi) - (k^2-k)\big(|\vec{A}|^2\varphi, \varphi\big), \end{aligned}$$

whence

$$\begin{aligned}(k-1)\|\nabla\varphi\|^2 &\leq k\|\varphi\|^2_{H^1_A} + k(k-1)\gamma^2\ \|\vec{A}\|^2_{L^3(\mathbb{R}^3)}\ \|\ \nabla|\varphi|\ \|^2 \\ &\leq \left\{k + k(k-1)\gamma^2\ \|\vec{A}\|^2_{L^3(\mathbb{R}^3)}\right\}\ \|\varphi\|^2_{H^1_A}\end{aligned}$$

by the diamagnetic inequality. Thus (20) is established, and so (19). ▮

3. A BIRMAN-SCHWINGER OPERATOR

Set

$$p[\varphi] := (\mathbb{P}\varphi,\varphi), \quad b[\varphi] := (|\vec{B}|\varphi,\varphi) \tag{21}$$

on $[C_0^\infty(\mathbb{R}^3)]^2$, so that $p_A = p - b$. From Lemma 2.1 and the remark after (12), the operators $\mathbb{P}_A$, $\mathbb{P}$ associated with p_A, p respectively have the same form domain $\mathcal{Q}$, and this is $\mathcal{D}(\mathbb{P}^{1/2})$ with the graph norm

$$\left(\ \|\mathbb{P}^{1/2}\varphi\| + \|\varphi\|^2\ \right)^{1/2}. \tag{22}$$

Also $[C_0^\infty(\mathbb{R}^3)]^2$ is a form core. It follows that

$$\mathcal{Q} = \mathbb{H}^1_B\ \cap\ \mathcal{H} \tag{23}$$

with norm (22); the embedding $\mathbb{H}^1_B \hookrightarrow [L^6(\mathbb{R}^3)]^2$ guarantees the completeness, since convergent sequences in $\mathbb{H}^1_B$ therefore converge pointwise to their limits, almost everywhere.

From

$$0 \leq b[\varphi] \leq p[\varphi]$$

it follows that there exists a bounded self-adjoint operator $\mathcal{B}$ on $\mathbb{H}^1_B$ such that

$$b[\varphi] = (\mathcal{B}\ \varphi,\varphi)_{\mathbb{H}^1_B}, \quad \varphi \in \mathbb{H}^1_B. \tag{24}$$

For $\varphi \in \mathcal{R}(\mathbb{P}^{1/2})$, the range of $\mathbb{P}^{1/2}$,

$$\|\mathbb{P}^{-1/2}\ \varphi\|_{\mathbb{H}^1_B} = \|\varphi\| \tag{25}$$

and hence, since $\mathcal{D}(\mathbb{P}^{1/2})$ and $\mathcal{R}(\mathbb{P}^{1/2})$ are dense subspaces of $\mathbb{H}^1_B$, $\mathcal{H}$ respectively, $\mathbb{P}^{-1/2}$ extends to a unitary map

$$U\ :\ \mathcal{H} \longrightarrow \mathbb{H}^1_B, \quad U = \mathbb{P}^{-1/2}\ \text{ on }\ \mathcal{R}(\mathbb{P}^{1/2}). \tag{26}$$

Define

$$\mathcal{S} \;:=\; |\vec{B}|^{1/2}U \;:\; \mathcal{H} \longrightarrow \mathcal{H} \tag{27}$$

Note that for $u \in \mathbb{H}^1_B$,

$$\|\, |\vec{B}|^{1/2}u \,\|^2 \le \|\vec{B}\|_{L^{3/2}(\mathbb{R}^3)} \|u\|^2_{[L^6(\mathbb{R}^3)]^2} \le \text{const}\cdot \|u\|^2_{\mathbb{H}^1_B} \tag{28}$$

by (20).

Theorem 3.1.

$$\begin{aligned} \text{nul } \mathbb{P}_A &= \dim\{u :\; \mathcal{B}u = u,\; u \in \mathbb{H}^1_B \;\cap\; \mathcal{H}\} \\ &\le \text{nul } F, \end{aligned}$$

where $F = 1 - \mathcal{S}\mathcal{S}^*$.

Proof. Let $u, \varphi \in \mathcal{D}(\mathbb{P}^{1/2})$. Then

$$\begin{aligned} p_A[u,\varphi] &= p[u,\varphi] - b[u,\varphi] \\ &= (u - \mathcal{B}u, \varphi)_{\mathbb{H}^1_B}. \end{aligned}$$

Hence, $u \in \ker \mathbb{P}_A \subset \mathcal{D}(\mathbb{P}^{1/2})$ if and only if $\mathcal{B}u = u$ with $u \in \mathcal{H}$. Moreover, for any $f, g \in \mathcal{H}$

$$\big(\, \mathcal{S}f, \mathcal{S}g \,\big) \;=\; \big(\, \mathcal{B}\, Uf, Ug \,\big)_{\mathbb{H}^1_B},$$

whence

$$\big(\, [\mathcal{S}^*\mathcal{S} - 1]f, g \,\big) \;=\; \big(\, [\mathcal{B} - 1]\, Uf, Ug \,\big)_{\mathbb{H}^1_B}$$

The result follows since nul $[\mathcal{S}^*\mathcal{S} - 1] =$ nul $[\mathcal{S}\mathcal{S}^* - 1]$. ∎

The operator $\mathcal{S}\mathcal{S}^*$ is of Birman-Schwinger type. We have, in terms of (27)

$$\mathcal{S}\mathcal{S}^* \;=\; |\vec{B}|^{1/2}U^2|\vec{B}|^{1/2} \quad \text{on} \;\; \mathcal{D}(\mathbb{P}^{1/2}) \tag{29}$$

and this extends by continuity to a bounded operator on $\mathcal{H}$. To see (29), first observe that for $f \in \mathcal{R}(\mathbb{P}^{1/2})$, $g \in \mathcal{D}(\mathbb{P}^{1/2})$

$$\begin{aligned} \big(\, f, \mathcal{S}^*g \,\big) &= \big(\, \mathcal{S}f, g \,\big) \;=\; \big(\, |\vec{B}|^{1/2}Uf, g \,\big) \\ &= \big(\, Uf, |\vec{B}|^{1/2}g \,\big) \\ &= \big(\, \mathbb{P}^{-1/2}f, |\vec{B}|^{1/2}g \,\big); \end{aligned}$$

note that $|\vec{B}|^{1/2} g \in \mathcal{H}$ by (28) and since $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)$. Hence $|\vec{B}|^{1/2} g \in \mathcal{D}(\mathbb{P}^{-1/2})$ and $\mathbb{P}^{-1/2}|\vec{B}|^{1/2} g = \mathcal{S}^* g$. In other words

$$\mathcal{S}^* \;=\; \mathbb{P}^{-1/2}|\vec{B}|^{1/2} \quad \text{on} \;\; \mathcal{D}(\mathbb{P}^{1/2}), \tag{30}$$

whence (29).

LEMMA 3.1. *$\mathcal{S}\mathcal{S}^*$ is compact and*

$$\|\mathcal{S}\|^2 \;\leq\; \gamma^2 \; \|\vec{B}\|_{L^{3/2}(\mathbb{R}^3)} \tag{31}$$

where γ is the norm of $H_0^1 \hookrightarrow [L^6(\mathbb{R}^3)]^2$.

Proof. This is quite standard, but we give the short proof for completeness. We show that $|\vec{B}|^{1/2} : \mathbb{H}_B^1 \longrightarrow \mathcal{H}$ is compact. Let $\{\varphi_n\}$ be a sequence which converges weakly to zero in $\mathbb{H}_B^1$, and hence in H_0^1 by (20). Then, in particular $\|\varphi_n\|_{H_0^1} \leq k$, say. Given $\varepsilon > 0$, set $|\vec{B}| = B_1 + B_2$ where $B_1 \in C_0^\infty(\mathbb{R}^3)$ with support Ω_ε and $B_1 \leq k_\varepsilon$ say, and $\|B_2\|_{L^{3/2}(\mathbb{R}^3)} < \varepsilon$. Then

$$\begin{aligned} \| \, |\vec{B}|^{1/2}\varphi_n \, \|^2 \;&\leq\; k_\varepsilon \|\varphi_n\|^2_{[L^2(\Omega_\varepsilon)]^2} + \; \gamma^2 \; \|B_2\|_{L^{3/2}(\mathbb{R}^3)} \; \|\varphi_n\|^2_{H_0^1} \\ &\leq\; k_\varepsilon \|\varphi_n\|^2_{[L^2(\Omega_\varepsilon)]^2} + \; \gamma^2 \; \varepsilon \; \|\varphi_n\|_{\mathbb{H}_B^1}. \end{aligned}$$

The first term on the right-hand side tends to zero as $n \to \infty$ by the Rellich Theorem. Consequently $|\vec{B}|^{1/2} : \mathbb{H}_B^1 \longrightarrow \mathcal{H}$ is compact and hence so is $\mathcal{S} = |\vec{B}|^{1/2} U$.

The inequality (31) follows from (28). ∎

4. THE MAIN RESULT

For $t \in (0, \infty)$, replace $\vec{A}$ by $t\vec{A}$ and denote the corresponding operators by $\mathbb{P}_t$, $\mathcal{S}_t$ and F_t. It follows from (31) that

$$\|\mathcal{S}_t\|^2 \;\leq\; \gamma^2 \; t \; \|\vec{B}\|_{L^{3/2}(\mathbb{R}^3)} \longrightarrow 0$$

as $t \to 0$. Hence, $F_t = 1 - \mathcal{S}_t\mathcal{S}_t^*$ is such that, for some $t_0 > 0$,

$$\text{nul } F_t \;=\; 0, \quad t \in (0, t_0). \tag{32}$$

We proceed to prove that $\{\mathcal{S}_t\mathcal{S}_t^*\}$ is a real analytic family.

LEMMA 4.1. *Let $s \in (0,\infty)$ be fixed, and suppose that $|\vec{A}| \in L^3(\mathbb{R}^3)$. Then, there exists a neighbourhood $N(s)$ of s such that*

$$\frac{1}{t}\mathcal{S}_t\mathcal{S}_t^* \;=\; \frac{1}{s}\mathcal{S}_s\mathcal{S}_s^* + \sum_{n=1}^{\infty}(t-s)^n K_n, \quad t \in N(s), \tag{33}$$

where the K_n are bounded operators on $\mathcal{H}$.

Proof. For $\psi, \varphi \in \left[C_0^\infty(\mathbb{R}^3)\right]^2$

$$\left(\, \mathbb{P}_t\psi, \varphi \,\right) \;=\; \left(\, \left[\, -\Delta + t(\vec{D}\cdot\vec{A} + \vec{A}\cdot\vec{D}) + t^2|\vec{A}|^2 + t(\vec{\sigma}\cdot\vec{B} + |\vec{B}|) \,\right]\psi, \varphi \,\right)$$

and

$$\left(\, \left[\, \mathbb{P}_t - \mathbb{P}_s \,\right]\psi, \varphi \,\right) \;=\; (t-s)\left(\, Q\psi, \varphi \,\right)$$

where

$$Q \;=\; \vec{D}\cdot\vec{A} + \vec{A}\cdot\vec{D} + (t+s)|\vec{A}|^2 + \vec{\sigma}\cdot\vec{B} + |\vec{B}|.$$

From (19) and since

$$\begin{aligned}\left(\, |\vec{A}|^2\psi, \psi \,\right) \;&\le\; \|\vec{A}\|^2_{L^3(\mathbb{R}^3)}\; \|\psi\|^2_{[L^6(\mathbb{R}^3)]^2} \\ &\le\; \gamma^2\; \|\vec{A}\|^2_{L^3(\mathbb{R}^3)}\; \|\psi\|^2_{\mathbb{H}^1_B}\end{aligned}$$

and

$$\left(\, |\vec{B}|\; \psi, \psi \,\right) \;\le\; \gamma^2\|\vec{B}\|_{L^{3/2}(\mathbb{R}^3)}\; \|\psi\|^2_{\mathbb{H}^1_B}$$

we have

$$|\left(\, Q\psi, \psi \,\right)| \;\le\; c\|\psi\|^2_{\mathbb{H}^1_B}$$

for some constant c, and so $R = \mathbb{P}_s^{-1/2} Q \mathbb{P}_s^{-1/2}$ satisfies

$$|\left(\, R\psi, \psi \,\right)| \;\le\; c\|\psi\|^2$$

and extends to an operator in $\mathcal{L}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$. Thus, there exists a neighbourhood $N(s)$ of s such that for $t \in N(s)$

$$\left\{\, 1 + (t-s)R \,\right\}^{-1} \;=\; \sum_{n=0}^{\infty}(t-s)^n(-R)^n$$

in $\mathcal{H}$. It follows that

$$\mathbb{P}_t \;=\; \mathbb{P}_s + (t-s)Q \;=\; \mathbb{P}_s^{1/2}[1 + (t-s)R]\mathbb{P}_s^{1/2},$$

and

$$|\vec{B}|^{1/2}\mathbb{P}_t^{-1}|\vec{B}|^{1/2} = |\vec{B}|^{1/2}\mathbb{P}_s^{-1/2}\left[\ 1+(t-s)R\ \right]^{-1}\mathbb{P}_s^{-1/2}|\vec{B}|^{1/2};$$

note that nul $\mathbb{P}_t = 0$ for any t. For $f \in \mathcal{R}(\mathbb{P}_s^{1/2})$

$$\begin{aligned} s \parallel |\vec{B}|^{1/2}\mathbb{P}_s^{-1/2} f \parallel^2 &= s\ (\ |\vec{B}|\mathbb{P}_s^{-1/2}f, \mathbb{P}_s^{-1/2}f\ ) \\ &\leq (\ \mathbb{P}_s^{-1/2}f, \mathbb{P}_s^{-1/2}f\ )_{\mathbb{H}^1_{sB}} \\ &= \|f\|^2. \end{aligned}$$

Hence $|\vec{B}|^{1/2}\mathbb{P}_s^{-1/2}$, and $\mathbb{P}_s^{-1/2}|\vec{B}|^{1/2}$, are bounded on $\mathcal{H}$. We may therefore write

$$|\vec{B}|^{1/2}\mathbb{P}_t^{-1}|\vec{B}|^{1/2} = \sum_{n=1}^{\infty}(t-s)^n K_n \ + \ |\vec{B}|^{1/2}\mathbb{P}_s^{-1}|\vec{B}|^{1/2},$$

where

$$K_n = |\vec{B}|^{1/2}\mathbb{P}_s^{-1/2}(-R)^n\mathbb{P}_s^{-1/2}|\vec{B}|^{1/2},$$

and the series lies in $\mathcal{L}(\mathcal{H})$ for $t \in N(s)$. The preceeding argument implies that with $T_t = |\vec{B}|^{1/2}\mathbb{P}_t^{-1/2}$, $|T_t^*|^2$ has an extension in $\mathcal{L}(\mathcal{H})$. It follows from (30) that $T_t^* = \mathcal{S}_t^*$, and this yields the lemma. ▌

We are now in a position to apply the argument of Anghel in [3]. For $[a,b] \subset (0,\infty)$, set

$$d_t = \text{nul } F_t, \qquad d_{\min} = \min_{t\in[a,b]} d_t.$$

LEMMA 4.2. *The map $t \mapsto d_t$ is upper semi-continuous.*

Proof. The kernel of F_t is finite-dimensional, and we have the orthogonal decomposition

$$\mathcal{H} = \ker F_t \oplus (\ker F_t)^{\perp}.$$

With respect to this decomposition, we can represent F_t as

$$F_t = \begin{pmatrix} 0 & 0 \\ 0 & D_t \end{pmatrix},$$

where $D_t : (\ker F_t)^{\perp} \longrightarrow (\ker F_t)^{\perp}$ and

$$\|D_t\| \geq c_t > 0. \tag{34}$$

We are required to prove that, for any t, there exists a neighbourhood $N(t)$ such that $d_{t'} \leq d_t$ for all $t' \in N(t)$. We can write

$$F_{t'} \;=\; \begin{pmatrix} L_{t'} & M_{t'} \\ M_{t'}^* & D_t + C_{t'} \end{pmatrix}$$

where $L_{t'} : \ker F_t \longrightarrow \ker F_t$, $\quad C_{t'} : (\ker F_t)^\perp \longrightarrow (\ker F_t)^\perp$ are bounded self-adjoint operators and $M_{t'} : (\ker F_t)^\perp \longrightarrow \ker F_t$ is bounded. As $t' \to t$, we know from Lemma 4.1 that $L_{t'}$, $M_{t'}$ and $C_{t'} \to 0$ in norm. Choose a neighbourhood $N(t)$ of t such that $\|C_{t'}\| < c_t$ for $t' \in N(t)$, where c_t is the constant in (34). Then $D_t + C_{t'}$ is invertible for all $t' \in N(t)$. The operator

$$\mathcal{A} \;=\; \begin{pmatrix} \mathbb{I} & -M_{t'}(D_t + C_{t'})^{-1} \\ 0 & (D_t + C_{t'})^{-1} \end{pmatrix},$$

where $\mathbb{I}$ is the identity, is a bounded injection on $\mathcal{H}$, and we have

$$\mathcal{A}\cdot F_{t'} \;=\; \begin{pmatrix} L_{t'} - M_{t'}(D_t + C_{t'})^{-1} M_{t'}^* & 0 \\ (D_t + C_{t'})^{-1} M_{t'}^* & \mathbb{I} \end{pmatrix}. \tag{35}$$

It follows that

$$\begin{aligned} d_{t'} &= \mathrm{nul}(\mathcal{A}\cdot F_{t'}) \\ &= \dim\{ \ker[L_{t'} - M_{t'}(D_t + C_{t'})^{-1} M_{t'}^*] \\ &\qquad \cap \ker[(D_t + C_{t'})^{-1} M_{t'}^*] \,\} \\ &= \mathrm{nul}\,[L_{t'}^2 + M_{t'} M_{t'}^*] \qquad\qquad (36) \\ &\leq d_t, \end{aligned}$$

whence the lemma. ∎

Theorem 4.1. *For any $c \in (0,\infty)$, $d_t = 0$, and hence* nul $\mathbb{P}_{tA} = 0$, *on $[0,c]$ except at a finite number of points.*

Proof. We already know from Theorem 3.1 and (32) that nul $\mathbb{P}_{tA} \leq d_t = 0$ in $(0,t_0)$. It is therefore sufficient to prove the theorem for $[a,c]$, where $0 < a < t_0$. Define

$$\begin{aligned} J \;=\; \{\, t \in [a,c] \;:\; & \text{there exists a neighbourhood } N_t \\ & \text{of } t, \text{ such that } d_{t'} = 0 \text{ in } N_t' = N_t \setminus \{t\} \,\}. \end{aligned}$$

The theorem will follow if we prove that $J = [a,c]$, in view of the compactness of $[a,c]$. We shall prove that J is both open and closed. Since $a < t_0$, we know that $J \neq \emptyset$.

It is clear from Lemma 4.2 that J is open. To prove that it is closed, let $\{t_k\}$ be a sequence in J and $\lim t_k = t$; we may assume that $d_{t_k} = 0$. In the notation of the proof of Lemma 4.2, set

$$Q_{t'} \;=\; L_{t'}^2 + M_{t'} M_{t'}^*.$$

Then, from (36)

$$\begin{aligned} d_t &= \text{rank } Q_{t'} + \text{nul } Q_{t'} \qquad (37)\\ &= \text{rank } Q_{t'} + d_{t'} \qquad (38) \end{aligned}$$

and

$$d_t \;=\; \text{rank } Q_{t_k}. \qquad (39)$$

If we can prove that rank $Q_{t'} = d_t$ for all t' in some deleted neighbourhood N' of t, it will follow from (38) that $t \in J$, as required.

Since rank $Q_{t_k} = d_t$, then any minor $\text{Min}_{t'}$ of $Q_{t'}$ of order greater than d_t must vanish when $t' = t_k$. Hence, since $t' \mapsto \text{Min}_{t'}$ is analytic, $\text{Min}_{t'} = 0$ in some neighbourhood N of t, and so rank $Q_{t'} \le d_t$ in N. By (39) there exist a minor of $Q_{t'}$ of order d_t which does not vanish on some subsequence of $\{t_k\}$, and hence can have a zero only at $t' = t$ within some neighbourhood N of t. Consequently, $d_t \ge$ rank $Q_{t'} \ge d_t$ for $t' \in N' = N \setminus \{t\}$, and, the theorem is proved. ▌

THEOREM 4.2. *The set* $\{\ \vec{B}:\ \text{nul } \mathbb{P}_A = 0,\ \text{curl } \vec{A} = \vec{B}$ *and* $|\vec{B}| \in L^{3/2}(\mathbb{R}^3)\ \}$ *contains an open dense subset of* $[L^{3/2}(\mathbb{R}^3)]^3$.

Proof. Let $\mathcal{S}$ in (27) be denoted by $\mathcal{S}_B$ and set $F_B = 1 - \mathcal{S}_B \mathcal{S}_B^*$. We shall prove that

$$\{\ \vec{B}:\ \text{nul } F_B = 0 \ \text{ and } \ |\vec{B}| \in L^{3/2}(\mathbb{R}^3)\ \} \qquad (40)$$

is an open subset of $[L^{3/2}(\mathbb{R}^3)]^3$; the theorem will then follow from Theorem 3.1 since the density of (40) is a consequence of Theorem 4.1.

For $\varepsilon > 0$, let $\vec{B}, \vec{B}_0$ be magnetic fields which satisfy $\|\vec{B} - \vec{B}_0\|_{L^{3/2}(\mathbb{R}^3)} < \varepsilon$. Then, if $\vec{A}$, $\vec{A}_0$ are the associated vector potentials given in Lemma 2.2, $\|\vec{A} - \vec{A}_0\|_{L^3(\mathbb{R}^3)} < c\,\varepsilon$ for some $c > 0$. It follows as in the proof of Lemma 4.1 that, with $\mathbb{P} = \mathbb{P}_A + |\vec{B}|$ and $\mathbb{P}_0 = \mathbb{P}_{A_0} + |\vec{B}_0|$,

$$\mathbb{P} - \mathbb{P}_0 \;=\; \mathbb{V},$$

where, for $\varphi \in [C_0^\infty(\mathbb{R}^3)]^2$,

$$\begin{aligned}(\ \mathbb{V}\varphi, \varphi \ ) \quad &= \quad (\ [\vec{D}\cdot(\vec{A}-\vec{A}_0) + (\vec{A}-\vec{A}_0)\cdot \vec{D}]\varphi, \varphi \ ) \\ &\qquad + (\ [\ |\vec{A}|^2 - |\vec{A}_0|^2 \ ]\varphi, \varphi \ ) \\ &\qquad + (\ [\ \vec{\sigma}\cdot(\vec{B}-\vec{B}_0) + |\vec{B}| - |\vec{B}_0| \ ]\varphi, \varphi \ )\end{aligned}$$

and

$$\begin{aligned}\big| \ (\ \mathbb{V}\varphi, \varphi \ ) \ \big| \quad &\leq \quad c \ \big[\|\vec{A}-\vec{A}_0\|_{L^3(\mathbb{R}^3)} + \|\vec{B}-\vec{B}_0\|_{L^{3/2}(\mathbb{R}^3)} \big] \ \|\varphi\|^2_{H_0^1} \\ &\leq \quad c' \ \varepsilon \ \|\varphi\|^2_{\mathbb{H}^1_{B_0}}\end{aligned}$$

on using Lemmas 2.2 and 2.3 and Hölder's inequality. Moreover, $\mathbb{U} = \mathbb{P}_0^{-1/2} \ \mathbb{V} \ \mathbb{P}_0^{-1/2}$ satisfies

$$\big| \ (\ \mathbb{U}\varphi, \varphi \ ) \ \big| \ \leq \ c' \ \varepsilon \ \|\varphi\|^2$$

and

$$|\vec{B}|^{1/2} \ \mathbb{P}^{-1} \ |\vec{B}|^{1/2} \ = \ |\vec{B}|^{1/2} \ \mathbb{P}_0^{-1/2} \ \big[\ 1 + \mathbb{U} \ \big]^{-1} \ \mathbb{P}_0^{-1/2} \ |\vec{B}|^{1/2}$$

for ε sufficiently small. Also, as $\varepsilon \to 0$,

$$|\vec{B}|^{1/2} \ \mathbb{P}_0^{-1/2} \ - \ |\vec{B}_0|^{1/2} \ \mathbb{P}_0^{-1/2} \ \longrightarrow \ 0$$

in $\mathcal{L}(\mathcal{H})$. It follows that, as $\varepsilon \to 0$, $F_B \to F_{B_0}$ in $\mathcal{L}(\mathcal{H})$, and that, as in the proof of Lemma 4.2, the map

$$\vec{B} \ \longmapsto \ \text{nul } F_B$$

is upper semi-continuous. The set (40) is therefore open and the theorem is proved. ∎

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