WEYL FUNCTION AND SPECTRAL PROPERTIES OF SELF-ADJOINT EXTENSIONS

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We characterize the spectra of self-adjoint extensions of a symmetric operator with equal deficiency indices in terms of boundary values of their Weyl functions. A complete description is obtained for the point and absolutely continuous spectrum while for the singular continuous spectrum additional assumptions are needed. The results are illustrated by examples.

1 Introduction

Let A be a closed symmetric operator on a separable Hilbert space $\mathfrak{H}$. If the operator A has equal deficiency indices $n_{\pm}(A) = \dim(\mathfrak{H} \ominus \operatorname{ran}(A \pm iI))$, then the symmetric operator A has a lot of self-adjoint extensions. These self-adjoint extensions can be labeled by the so-called Weyl function $M(\cdot)$. This function was introduced by Weyl [29, 30, 31] in order to investigate Sturm-Liouville problems. However, later it became clear that the Weyl function makes sense not only for the Sturm-Liouville problem but also for an arbitrary closed symmetric operator with equal deficiency indices. For this purpose it was necessary to generalize the method of boundary conditions used in the theory of Sturm-Liouville problems to arbitrary closed symmetric operators with equal deficiency indices.

This generalization is based on the concept of a boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* being an abstract generalization of the Green's identity (see [21] and references therein). Here $\mathcal{H}$ is a separable Hilbert space with $\dim(\mathcal{H}) = n_{\pm}(A)$ and Γ_0 and Γ_1 are linear mappings from $\operatorname{dom}(A^*)$ to $\mathcal{H}$ so that Green's identity is satisfied, see below (2.8). One naturally associates with this triple two self-adjoint extensions $A_i = A^* \upharpoonright \ker(\Gamma_i)$, $i = 0, 1$.

On this basis in [13] the notion of Weyl function was generalized to closed symmetric operators with equal deficiency indices, see also [14, 15]. In the next section we give a brief introduction to the abstract theory of boundary triples and Weyl functions.

If A is a simple closed symmetric operator in $\mathfrak{H}$, then the Weyl function $M(\cdot)$ corresponding to a boundary triple Π determines the pair $\{A, A_0\}$ (as well as the pair $\{A, A_1\}$) up to unitary equivalence. Therefore the problem arises to describe the spectrum and certain spectral subsets as point, singular, singular continuous and absolutely continuous spectrum of self-adjoint extensions of A in terms of the Weyl function. Of course, since in the abstract sense self-adjoint extensions of A and Weyl functions are identical objects the spectral information has to be encoded in the Weyl function. Indeed, Lemma 3.2 shows that it is so. However, under practical aspects this correspondence is useless since the reconstruction of the spectral properties is very implicit.

Our aim is to describe spectral subsets by the boundary values of Weyl function at the real axis. So the problem is the following: Let $M(\cdot)$ be the Weyl function of a certain self-adjoint extension A_0 of A. Introducing the associated scalar Weyl function $M_h(\cdot) := (M(\cdot)h, h)$, $h \in \mathcal{H}$, is it possible to localize the different spectral subsets of A_0 knowing the boundary values $M_h(x + i0)$, $x \in \mathbb{R}$, of the associated scalar Weyl function for a suitable set $\mathcal{T}$ of elements of $\mathcal{H}$. It turns out that the point and absolutely continuous spectrum can be exactly localized in this way, see Theorem 3.8 and Proposition 4.2. We also mention Theorem 4.3 and Corollary 4.4 which indicate sufficient conditions for A_0 to be purely absolutely continuous within an arbitrary interval (a, b).

With respect to the singular continuous spectrum we get in general an incomplete information. That means, we are only able to indicate a subset in which the singular continuous spectrum is contained, see Theorem 3.8. However, under certain circumstances we are able to localize the singular continuous spectrum too, see Theorem 4.3 and Corollary 4.4. We note in this connection that in our paper [8] applying Theorem 4.3 and Corollary 4.4 we construct self-adjoint extensions A_0 with arbitrary pure singular continuous spectrum within the gap (α, β) of a symmetric operator A.

The paper is organized as follows. In Section 2 we define and recall the notions of R-function, boundary triple, Weyl function and γ-field which are important ingredients for the next sections. In Section 3 we investigate the relation between the boundary values of the Weyl function and the spectral properties of the self-adjoint extensions. Finally, in Section 4 we prove a refinement of the results for the absolutely continuous spectrum and indicate sufficient conditions which allow to localize exactly the singular continuous spectrum. In Section 5 we give examples which illustrate the abstract observations of the sections above. In particular we consider here an infinite-dimensional Dirac operator $A_0 = A_0^* = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \otimes \frac{d}{dx} + \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix}$ in $L^2(\mathbb{R}, \mathcal{H})$, $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$, with an arbitrary nonnegative bounded operator T in $\mathcal{K}$. Applying Theorem 4.6 and Corollary 4.7 we establish that A_0 is purely absolutely continuous. We finish the paper with some general remarks in Section 6.

In a forthcoming paper we apply our general results from Sections 3 and 4 to inverse spectral theory (in the sense of [2], [6]-[11]) of a symmetric operator with several gaps.

In conclusion the reader is referred to recent publications [18, 19] which are closed to our paper. In [18, 19] a spectral analysis of matrix-valued Herglotz (Nevanlinna) functions as well as their application to the well-known Aronszajn-Donoghue theory is considered.

2 Preliminaries

2.1 R-functions

Let $\mathcal{H}$ be a separable Hilbert space. Recall that an operator function $F(\cdot)$ with values in $[\mathcal{H}]$ is said to be a Herglotz or Nevanlinna function [1, 5, 18, 19, 21] or R-function [24] if it is holomorphic in $\mathbb{C}_+$ and for every $z \in \mathbb{C}_+$ the operator $F(z)$ in $\mathcal{H}$ is dissipative, i.e. $\Im\mathrm{m}(F(z)) := (F(z) - F(z)^*)/2i \geq 0$. In the following we prefer the notion R-function. The class of R-functions with values in $[\mathcal{H}]$ is denoted by $(R_\mathcal{H})$. If $F(\cdot) \in (R_\mathcal{H})$, then there exist bounded self-adjoint operators C_0, $C_1 \geq 0$, a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, a self-adjoint operator L in $\mathcal{K}$, a bounded non-negative operator $R \geq 0$ with $R|\mathcal{K} \ominus \mathcal{H} = 0$ such that

$$F(z) = C_0 + C_1 z + R^{1/2}(I_\mathcal{K} + zL)(L - z)^{-1}R^{1/2}|\mathcal{H}, \quad z \in \mathbb{C}_+, \tag{2.1}$$

see [25]. Denoting by $E_L(\cdot)$ the spectral measure of the self-adjoint operator L one immediately obtains from (2.1) the representation

$$F(z) = C_0 + C_1 z + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\Sigma_F(t), \quad z \in \mathbb{C}_+, \tag{2.2}$$

where $\Sigma_F(\cdot)$ is an operator-valued Borel measure on $\mathbb{R}$ given by

$$d\Sigma_F(t) = (1+t^2)R^{1/2}dE_L(\cdot)R^{1/2}, \qquad t \in \mathbb{R}. \tag{2.3}$$

Obviously, the measure $\Sigma_F(\cdot)$ is self-adjoint and obeys

$$\int_{-\infty}^{+\infty} \frac{1}{1+t^2} d\Sigma_F(t) \in [\mathcal{H}]. \tag{2.4}$$

In contrast to spectral measures of self-adjoint operators it is not necessarily true that $\mathrm{ran}(\Sigma(\delta_1))$ is orthogonal to $\mathrm{ran}(\Sigma(\delta_2))$ for disjoint Borel sets δ_1 and δ_2. However, the measure $\Sigma_F(\cdot)$ is uniquely determined by the R-function $F(\cdot)$. The integral in (2.2) is understood in the strong sense. In the following the operator-valued measure $\Sigma_F(\cdot)$ is called the spectral measure of $F(\cdot)$.

Usually, with the operator-valued measure $\Sigma_F(\cdot)$ one associates an distribution operator-valued function $\Sigma_F(\cdot)$ which is defined by

$$\Sigma_F(t) = \begin{cases} \Sigma_F([0,t)) & : \quad t > 0 \\ 0 & : \quad t = 0 \\ -\Sigma_F([t,0)) & : \quad t < 0. \end{cases} \tag{2.5}$$

It turns out that the distribution function $\Sigma_F(\cdot)$ is strongly left continuous and satisfies the conditions

$$\Sigma_F(t) = \Sigma_F(t)^*, \quad \Sigma_F(s) \leq \Sigma_F(t), \quad -\infty < s \leq t < \infty. \tag{2.6}$$

In the following the distribution function $\Sigma_F(\cdot)$ is called the spectral function of $F(\cdot)$.

We note that the spectral function $\Sigma_F(\cdot)$ of $F(\cdot)$ can be obtained by the Stieltjes transformation:

$$\frac{1}{2}(\Sigma_F(t+0)+\Sigma(t)) - \frac{1}{2}(\Sigma(s+0)+\Sigma(s)) = \tag{2.7}$$
$$w-\lim_{y\downarrow 0}\frac{1}{\pi}\int_s^t \Im\mathrm{m}(F(x+iy))dx, \quad t,s\in\mathbb{R}$$

where it is used that the spectral function is strongly left continuous.

2.2 Boundary triples and self-adjoint relations

In the following A will always denote a closed symmetric operator with equal deficiency indices $n_+(A) = n_-(A)$. Without loss of generality we can assume that A is simple. This means that A has no self-adjoint parts.

Our approach to the inverse spectral theory of self-adjoint extensions is based on the concept of boundary triples (see [21] and references therein) which may be considered as an abstract version of the second Green's formula.

DEFINITION 2.1 A triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings $\Gamma_i : \mathrm{dom}(A^*) \longrightarrow \mathcal{H}, \quad i = 0,1$, is called a boundary triple for the adjoint operator A^* of A if:

(i) The second Green's formula takes place

$$(A^*f,g) - (f,A^*g) = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \quad f,g \in \mathrm{dom}(A^*), \tag{2.8}$$

(ii) the mapping $\Gamma := \{\Gamma_0, \Gamma_1\} : \mathrm{dom}(A^*) \longrightarrow \mathcal{H}\oplus\mathcal{H}$ is surjective

REMARK 2.2 (1) The boundary triple for the adjoint A^* of the symmetric operator A is not unique.

(2) It is easily seen that for each self-adjoint extension $\tilde{A}$ of A there exists a boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that $\tilde{A} = A^*\restriction \ker(\Gamma_0)$ is valid.

(3) Moreover (see [15]) for each transversal pair $A_0, A_1 \in Ext_A$ (that means $\mathrm{dom}(A_0)\cap \mathrm{dom}(A_1) = \mathrm{dom}(A)$ and $\mathrm{dom}(A_0) + \mathrm{dom}(A_1) = \mathrm{dom}(A^*)$) there exists a boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that condition the $A_i = A^*\restriction\ker(\Gamma_i)$, $i = 0,1$, is satisfied.

(4) For each boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of A^* and each bounded self-adjoint operator B in $\mathcal{H}$ also $\Pi_B = \{\mathcal{H}, \Gamma_0^B, \Gamma_1^B\}$ is a boundary triple for A^* where $\Gamma_1^B := \Gamma_0, \quad \Gamma_0^B := B\Gamma_0 - \Gamma_1$. We set $A_0^B := A^*\restriction\ker(\Gamma_0^B)$.

Once a boundary triple for A^* is fixed one can give explicitly a one-to-one correspondence between the set Ext_A of self-adjoint extensions of A and the set of self-adjoint linear relations Θ in $\mathcal{H}$. To explain this we recall the definition of self-adjoint relations:

DEFINITION 2.3 (i) A closed linear relation Θ in $\mathcal{H}$ is a closed subspace Θ of $\mathcal{H} \oplus \mathcal{H}$.

(ii) The closed linear relation Θ is symmetric if $(g_1, f_2) - (f_1, g_2) = 0$, for all $\{f_1, g_1\}, \{f_2, g_2\} \in \Theta$.

(iii) The closed linear relation Θ is self-adjoint if it is maximal symmetric.

EXAMPLE 2.4 (1) Let B be a self-adjoint operator in $\mathcal{H}$, not necessarily bounded. Then the graph $G(B)$ of B is a self-adjoint relation in $\mathcal{H}$.

(2) The subspaces $\Theta_0 := \{0\} \times \mathcal{H}$, $\Theta_1 := \mathcal{H} \times \{0\}$, of $\mathcal{H} \times \mathcal{H}$ are self-adjoint relations in $\mathcal{H}$. Obviously, Θ_0 is not the graph of any operator.

The following notation is fixed.

DEFINITION 2.5 Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^*.

(i) For every self-adjoint relation Θ in $\mathcal{H}$ we put

$$D^\Theta := \{f \in \mathrm{dom}(A^*) : \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}, \quad A^\Theta := A^* | D^\Theta. \tag{2.9}$$

(ii) In particular, we set $A_i := A^{\Theta_i}$, $i = 0, 1$, if Θ_i, $i = 0, 1$, are given by Example 2.4(2).

(iii) If $\Theta = G(B)$, where B is an operator on $\mathcal{H}$, then we set $A^B := A^\Theta$.

REMARK 2.6 If B be is a self-adjoint operator on $\mathcal{H}$, then a simple computation shows that $A^B = A^* |\ker(\Gamma_1 - B\Gamma_0) = A_0^B$.

PROPOSITION 2.7 ([14, 21]) *Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^*. Then for every self-adjoint relation Θ in $\mathcal{H}$ the operator A^Θ, given by Definition 2.5, is a self-adjoint extension of A. The mapping $\Theta \mapsto A^\Theta$ from the set of self-adjoint relations in $\mathcal{H}$ onto the set Ext_A of self-adjoint extensions of A is bijective.*

2.3 Weyl functions

It is well known that Weyl functions are an important tool in the direct and inverse spectral theory of singular Sturm-Liouville operators. In [13, 14, 15] the concept of Weyl function was generalized to an arbitrary symmetric operator A with infinite deficiency indices $n_+(A) = n_-(A)$. In this subsection we recall basic facts about Weyl functions.

DEFINITION 2.8 ([13, 14]) Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for the operator A^*. The Weyl function of A corresponding to the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is the unique mapping $M(\cdot) : \rho(A_0) \longrightarrow [\mathcal{H}]$ satisfying

$$\Gamma_1 f_z = M(z)\Gamma_0 f_z, \quad f_z \in \mathcal{N}_z, \quad z \in \rho(A_0), \tag{2.10}$$

where $\mathcal{N}_z = \ker(A^* - zI)$.

It is well known (see [13, 14]) that the above implicit definition of the Weyl function is correct and the Weyl function $M(\cdot)$ is a R-function obeying $0 \in \rho(\Im\mathrm{m}(M(i)))$.

The Weyl function immediately provides some information about the "spectral properties" of self-adjoint extensions of the symmetric operator A. To explain this let us recall the notion of the inverse of a closed linear relation.

DEFINITION 2.9 A closed linear relation Θ in $\mathcal{H}$ is called boundedly invertible if the inverse relation $\Theta^{-1} := \{\{g, f\} \in \mathcal{H} \times \mathcal{H} : \{f, g\} \in \Theta\}$ is the graph of a bounded operator defined on $\mathcal{H}$. In particular, we say $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(\Theta)$ if the closed linear relation $\Theta - \lambda I = \{\{f, g - \lambda f\} : \{f, g\} \in \Theta\}$ is boundedly invertible.

Using that definition the following proposition holds.

PROPOSITION 2.10 [13, 14] *Let A be a simple closed densely defined symmetric operator in $\mathfrak{H}$. Suppose that $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triple for A^*, $M(\cdot)$ is the corresponding Weyl function, Θ a self-adjoint relation in $\mathcal{H}$ and $\lambda \in \rho(A_0)$. Then the following holds:*

(i) $\lambda \in \rho(A^\Theta)$ *if and only if* $0 \in \rho(\Theta - M(\lambda))$.

(ii) $\lambda \in \sigma_\tau(A^\Theta)$ *if and only if* $0 \in \sigma_\tau(\Theta - M(\lambda))$, $\tau = p, c$.

If A is a simple symmetric operator, then the Weyl function $M(\cdot)$ determines the pair $\{A, A_0\}$ up to unitary equivalence (see [15, 23]). This fact allows one to reformulate the spectral properties of A_0 in terms of $M(\cdot)$.

We shall often say that $M(\cdot)$ is the Weyl function of the pair $\{A, A_0\}$ or, simply, of A_0. Though the Weyl function $M(\cdot)$ of a boundary triple is unique there are a lot of Weyl functions corresponding in the above sense to a pair $\{A, A_0\}$. However it is not difficult to prove [14] that two such functions $M_1(\cdot)$ and $M_2(\cdot)$ with values in $[\mathcal{H}_1]$ and $[\mathcal{H}_2]$ are connected via

$$M_2(z) = K^* M_1(z) K + D \tag{2.11}$$

where $D = D^* \in [\mathcal{H}_2]$ and $K \in [\mathcal{H}_2, \mathcal{H}_1]$ is boundedly invertible.

2.4 γ-fields

With each boundary triple we can associate a so-called γ-field.

DEFINITION 2.11 Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^*. The γ-field $\gamma(\cdot)$ corresponding to Π is defined by

$$\gamma(z) := (\Gamma_0 | \mathcal{N}_z)^{-1} : \mathcal{H} \longrightarrow \mathcal{N}_z, \quad z \in \rho(A_0). \tag{2.12}$$

One can easily check that

$$\gamma(z) = (A_0 - z_0)(A_0 - z)^{-1} \gamma(z_0), \quad z, z_0 \in \rho(A_0) \tag{2.13}$$

and consequently $\gamma(\cdot)$ is a γ-field in the sense of [23]. The γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ are related by

$$M(z) - M(z_0)^* = (z - \bar{z}_0) \gamma(z_0)^* \gamma(z), \quad z, z_0 \in \rho(A_0). \tag{2.14}$$

The relation (2.14) means the $M(\cdot)$ is a Q-function of a pair $\{A, A_0\}$ [23].

Further, we note that if A is simple, then $\mathcal{N}_z$, $z \in \rho(A_0)$, is generating with respect to A_0, otherwise, A would not be simple. Since $\mathrm{ran}(\gamma(z)) = \mathcal{N}_z$, $z \in \rho(A_0)$, one gets that $\mathrm{ran}(\gamma(z))$, $z \in \rho(A_0)$, is generating with respect to A_0 too. We will use this fact in the following.

3 Weyl function and spectrum

Let us recall some definitions and introduce some notations. Let μ be a Borel measure on $\mathbb{R}$. A support of μ is a set S such that $\mu(\mathbb{R}\setminus S) = 0$. We note that $S \subseteq \tilde{S}$ implies that $\tilde{S}$ is a support too. Measures μ and ν on $\mathbb{R}$ are called orthogonal if some of their supports are disjoint. The topological support $S(\mu)$ of μ is the smallest closed set which is a support of μ. It is clear that $S(\mu)$ is the intersection of all closed supports of μ. It is obvious that two equivalent measures have the same topological support but the converse is not generally true. According to the Lebesgue-Jordan decomposition $\mu = \mu_s + \mu_{ac}, \qquad \mu_s = \mu_{pp} + \mu_{sc}$, where μ_s, μ_{pp}, μ_{sc} and μ_{ac} are the corresponding singular, pure point, singular continuous and absolutely continuous measures of μ, respectively. We set $S_\tau(\mu) := S(\mu_\tau)$, $\tau = s, pp, sc, ac$.

The sets $S_s(\mu)$, $S_{pp}(\mu)$, $S_{sc}(\mu)$, $S_{ac}(\mu)$ are closed and called singular, pure point, singular continuous and absolutely continuous supports of μ. We note that the closed supports $S_s(\mu)$, $S_{pp}(\mu)$, $S_{ac}(\mu)$ and $S_{sc}(\mu)$ are not generally mutually disjoint. To obtain mutually disjoint supports we introduce the following (not necessarily closed) sets:

$$
\begin{aligned}
S'_s(\mu) &:= \{t \in \mathbb{R} : d\mu(t)/dt \text{ exists and } d\mu(t)/dt = \infty\} && (3.1)\\
S'_{pp}(\mu) &:= \{t \in \mathbb{R} : \mu(\{t\}) \neq 0\}, && (3.2)\\
S'_{sc}(\mu) &:= \{t \in \mathbb{R} : d\mu(t)/dt \text{ exists, } d\mu(t)/dt = \infty \text{ and } \mu(\{t\}) = 0\} && (3.3)\\
S'_{ac}(\mu) &:= \{t \in \mathbb{R} : d\mu(t)/dt \text{ exists and } 0 < d\mu(t)/dt < \infty\} && (3.4)
\end{aligned}
$$

where the distribution function $\mu(\cdot)$ is similar to (2.5) defined. By De la Vallée Poussin's Decomposition Theorem (see [28]) it turns out that $S'_s(\mu)$, $S'_{pp}(\mu)$ and $S'_{sc}(\mu)$ are supports of μ_s, μ_{pp}, μ_{sc} and μ_{ac}. Obviously, the sets $S'_s(\mu)$, $S'_{ac}(\mu)$ as well as $S'_{pp}(\mu)$, $S'_{sc}(\mu)$ and $S'_{ac}(\mu)$ are mutually disjoint. Since the sets $S'_\tau(\mu)$, $\tau = s, pp, sc$, are of Lebesgue measure zero and mutually disjoint we find that for any Borel set $\mathcal{X} \subseteq \mathbb{R}$ one has

$$
\mu(\mathcal{X} \cap S'_\tau(\mu)) = \mu_\tau(\mathcal{X}), \quad \tau = s, pp, sc, ac. \tag{3.5}
$$

Therefore we call in the sequel the sets $S'_s(\mu)$, $S'_{pp}(\mu)$, $S'_{sc}(\mu)$ and $S'_{ac}(\mu)$ singular, pure point, singular continuous and absolutely continuous supports of μ, respectively. We note that

$$
S_{pp}(\mu) = \overline{S'_{pp}(\mu)} \quad \text{and} \quad S_\tau(\mu) \subseteq \overline{S'_\tau(\mu)} \subseteq S(\mu),\ \tau = s, sc, ac. \tag{3.6}
$$

In general, it is not possible to replace inclusions by equalities.

Let now $\Sigma(\cdot)$ be a measure with values in $[\mathcal{H}]$. The measure $\Sigma(\cdot)$ admits a Lebesgue-Jordan decomposition $\Sigma = \Sigma^s + \Sigma^{ac}$, $\Sigma^s = \Sigma^{pp} + \Sigma^{sc}$. As above the notation $S_s(\Sigma) := S(\Sigma^s)$, $S_{pp}(\Sigma) := S(\Sigma^{pp})$, $S_{sc}(\Sigma) := S(\Sigma^{sc})$ and $S_{ac}(\Sigma) := S(\Sigma_{ac})$ stands for the singular, pure point, singular continuous and absolutely continuous closed supports of the measure $\Sigma(\cdot)$, respectively. Moreover, we set

$$S_p(\Sigma) = \{t \in \mathbb{R} : \Sigma(\{t\}) \neq 0\}. \tag{3.7}$$

Obviously, we have $S_p(\Sigma) \subseteq S_{pp}(\Sigma)$ and $\overline{S_p(\Sigma)} = S_{pp}(\Sigma)$.

With each operator-valued measure $\Sigma(\cdot)$ we can associate a scalar measure $\Sigma_h(\cdot) := (\Sigma(\cdot)h, h)$, $h \in \mathcal{H}$. In the following we are interested in the problem whether the spectral properties of the operator-valued measure $\Sigma(\cdot)$ can be characterized by a family of scalar measures. To this end let $\mathcal{T} = \{h_k\}_{k=1}^N$, $1 \leq N \leq +\infty$, be a total set in $\mathcal{H}$. With $\mathcal{T}$ we associate the family $\{\Sigma_{h_k}(\cdot)\}_{k=1}^N$ of scalar measures. Let us introduce the following sets:

$$\begin{aligned} S'_s(\Sigma;\mathcal{T}) &:= \cup_{k=1}^N S'_s(\Sigma_{h_k}), && (3.8)\\ S'_{pp}(\Sigma;\mathcal{T}) &:= \cup_{k=1}^N S'_{pp}(\Sigma_{h_k}), && (3.9)\\ S'_{sc}(\Sigma;\mathcal{T}) &:= \cup_{k=1}^N S'_{sc}(\Sigma_{h_k}) \setminus S'_{pp}(\Sigma), && (3.10)\\ S'_{ac}(\Sigma;\mathcal{T}) &:= \cup_{k=1}^N S'_{ac}(\Sigma_{h_k}) \setminus S'_s(\Sigma). && (3.11) \end{aligned}$$

By definition the sets $S'_s(\Sigma;\mathcal{T})$ and $S'_{ac}(\Sigma;\mathcal{T})$ are mutually disjoint. The same holds for $S'_{pp}(\Sigma;\mathcal{T})$, $S'_{sc}(\Sigma;\mathcal{T})$ and $S'_{ac}(\Sigma;\mathcal{T})$.

LEMMA 3.1 *Let $\mathcal{H}$ be a separable Hilbert space and let $\Sigma(\cdot)$ be an operator-valued measure in $\mathcal{H}$. Further, let $\mathcal{T} = \{h_k\}_{k=1}^N$, $1 \leq N \leq +\infty$, be a total set in $\mathcal{H}$. Then the sets $S'_s(\Sigma;\mathcal{T})$, $S'_{pp}(\Sigma;\mathcal{T})$, $S'_{sc}(\Sigma;\mathcal{T})$ and $S'_{ac}(\Sigma;\mathcal{T})$ are singular, pure point, singular continuous and absolutely continuous supports of $\Sigma(\cdot)$, respectively, i.e.*

$$\Sigma(\mathcal{X} \cap S'_\tau(\Sigma;\mathcal{T})) = \Sigma^\tau(\mathcal{X}), \quad \tau = s, pp, sc, ac, \tag{3.12}$$

for any Borel set $\mathcal{X} \subseteq \mathbb{R}$. In particular, the following relations hold

$$S_p(\Sigma) = S'_{pp}(\Sigma;\mathcal{T}) \text{ and } S_\tau(\Sigma) \subseteq \overline{S'_\tau(\Sigma;\mathcal{T})} \subseteq S(\Sigma),\ \tau = s, sc, ac. \tag{3.13}$$

Proof: By the Lebesgue-Jordan decomposition one easily gets that for each $h \in \mathcal{H}$ we have

$$(\Sigma^\tau(\mathcal{X})h, h) = \Sigma_{h,\tau}(\mathcal{X}), \quad \tau = s, pp, sc, ac, \tag{3.14}$$

for any Borel set $\mathcal{X} \subseteq \mathbb{R}$, where $\Sigma_{h,\tau}(\cdot)$ arises from the Lebesgue-Jordan decomposition of the scalar measure $\Sigma_h(\cdot)$.

Let $\tau = s$. Since $\mathrm{mes}(S'_s(\Sigma;\mathcal{T})) = 0$ we get

$$(\Sigma(\mathcal{X} \cap S'_s(\Sigma;\mathcal{T}))h_k, h_k) = \Sigma_{h_k}(\mathcal{X} \cap S'_s(\Sigma;\mathcal{T})) = \Sigma_{h_k,s}(\mathcal{X} \cap S'_s(\Sigma;\mathcal{T})) \tag{3.15}$$

for any $h_k \in \mathcal{T}$. Using (3.14), (3.15) and

$$\begin{aligned} &\Sigma_{h_k,s}(\mathcal{X} \cap S'_s(\Sigma;\mathcal{T})) = && (3.16)\\ &\quad \Sigma_{h_k,s}(\mathcal{X} \cap S'_s(\Sigma;\mathcal{T}) \cap S'_s(\Sigma_{h_k})) = \Sigma_{h_k,s}(\mathcal{X} \cap S'_s(\Sigma_{h_k})) = \Sigma_{h_k,s}(\mathcal{X}) \end{aligned}$$

we find $(\Sigma(\mathcal{X}\cap S'_s(\Sigma;\mathcal{T}))h_k,h_k)=(\Sigma^s(\mathcal{X})h_k,h_k)$ for any $h_k\in\mathcal{T}$. Since $\mathcal{T}$ is total we finally obtain $\Sigma(\mathcal{X}\cap S'_s(\Sigma;\mathcal{T}))=\Sigma^s(\mathcal{X})$ for any Borel set $\mathcal{X}\subseteq\mathbb{R}$. Similarly, we prove the statements for $\tau=pp,sc,ac$.

Let $x\in S'_{pp}(\Sigma;\mathcal{T})$. Then there is $h_k\in\mathcal{T}$ such that $x\in S'_{pp}(\Sigma_{h_k})$. Hence $\Sigma_{h_k}(\{x\})=(\Sigma(\{x\})h_k,h_k)\neq 0$ which yields $\Sigma(\{x\})\neq 0$ or $x\in S_p(\Sigma)$, i.e. $S''_{pp}(\Sigma;\mathcal{T})\subseteq S_p(\Sigma)$. Conversely, if $x\in S_p(\Sigma)$, then there is a $h\in\mathcal{H}$ such that $\Sigma_h(\{x\})\neq 0$. Hence there exists a $h_k\in\mathcal{T}$ such that $\Sigma_{h_k}(\{x\})=(\Sigma(\{x\})h_k,h_k)\neq 0$. If this is not the case, then for each $h_k\in\mathcal{T}$ one has $\Sigma_{h_k}(\{x\})=(\Sigma(\{x\})h_k,h_k)=0$. Since $\mathcal{T}$ is total this yields $(\Sigma(\{x\})h,h)=\Sigma_h(\{x\})=0$ for each $h\in\mathcal{H}$, contrary to the assumption. However, if there is a $h_k\in\mathcal{T}$ such that $\Sigma_{h_k}(\{x\})\neq 0$, then $x\in S'_{pp}(\Sigma;\mathcal{T})$, i.e. $S_p(\Sigma)\subseteq S''_{pp}(\Sigma;\mathcal{T})$. Hence $S_p(\Sigma)=S''_{pp}(\Sigma;\mathcal{T})$.

Further, from (3.12) we get $S_\tau(\Sigma)\subseteq\overline{S'_\tau(\Sigma;\mathcal{T})}$, $\tau=s,pp,sc,ac$. Taking (3.6) into account we get $S'_\tau(\Sigma_{h_k})\subseteq S(\Sigma_{h_k})$, $\tau=s,sc,ac$, for each $h_k\in\mathcal{T}$. Since $S(\Sigma_{h_k})\subseteq S(\Sigma)$ for each $h_k\in\mathcal{T}$ we get $S'_\tau(\Sigma;\mathcal{T})\subseteq S(\Sigma)$, $\quad\tau=s,sc,ac$, which immediately proves (3.13) $\quad\triangle$

Let $M(\cdot)$ be the Weyl function of A with respect to the boundary triple $\{\mathcal{H},\Gamma_0,\Gamma_1\}$. Since each Weyl function is an R-function one gets the representation (2.2). Taking (2.13) and (2.14) into account we obtain that $C_1=0$ which leads to the representation

$$M(z)=C_0+\int_{\mathbb{R}}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)d\Sigma(t),\quad z\in\mathbb{C}. \tag{3.17}$$

In the following we are interested in spectral properties of A_0 characterized by the operator-valued measures $\Sigma(\cdot)$. Important is the following lemma.

LEMMA 3.2 *Let A be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ with $n_+(A)=n_-(A)$. Further, let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(\cdot)$.*

If $E_{A_0}(\cdot)$ is the spectral measure of $A_0:=A^*\restriction\ker(\Gamma_0)$ ($\in Ext_A$) and $\Sigma(\cdot)$ that of the integral representation (3.17) of the Weyl function $M(\cdot)$, then the measures $E_{A_0}(\cdot)$ and $\Sigma(\cdot)$ are equivalent. In particular, one has $\sigma_\tau(A_0)=S_\tau(\Sigma)$, $\quad\tau=s,p,pp,sc,ac$.

Proof: By (2.14) one obtains

$$\begin{aligned}&\Im\mathrm{m}(M(x+iy)h,h)=\\&\qquad y(\gamma^*(x+iy)\gamma(x+iy)h,h)=y\|\gamma(x+iy)h\|^2.\end{aligned} \tag{3.18}$$

Further, it follows from (2.13) that

$$\gamma(x+iy)h=[I+(x+i(y-1))(A_0-x-iy)^{-1}]\gamma(i)h. \tag{3.19}$$

Combining (3.18) and (3.19) one gets

$$\begin{aligned}\Im\mathrm{m}(M(x+iy)h,h)&=y\|(A_0-i)(A_0-x-iy)^{-1}\gamma(i)h\|^2\\&=y\int_{-\infty}^{\infty}\frac{|t-i|^2}{|t-x-iy|^2}d(E_{A_0}(t)\gamma(i)h,\gamma(i)h)\\&=y\int_{-\infty}^{\infty}\frac{1+t^2}{(t-x)^2+y^2}d(E_{A_0}(t)\gamma(i)h,\gamma(i)h).\end{aligned} \tag{3.20}$$

On the other hand we obtain from (3.17) that

$$\Im\mathrm{m}(M(x+iy)h,h) = y\int_{-\infty}^{\infty}\frac{d(\Sigma(t)h,h)}{(t-x)^2+y^2}. \tag{3.21}$$

Applying Stieltjes inversion formula (2.7) to (3.20) and (3.21) one gets

$$d\Sigma_h(t) := d(\Sigma(t)h,h) = (1+t^2)d(E_{A_0}(t)\gamma(i)h,\gamma(i)h). \tag{3.22}$$

Therefore, the measures $\Sigma_h(\cdot)$ and $(E_{A_0}(\cdot)\gamma(i)h,\gamma(i)h)$ are equivalent for each $h\in\mathcal{H}$.

If $E_{A_0}(\mathcal{X}) = 0$ for some Borel set $\mathcal{X}\subseteq\mathbb{R}$, then $(E_{A_0}(\mathcal{X})\gamma(i)h,\gamma(i)h) = 0$ for any $h\in\mathcal{H}$. Hence $(\Sigma(\mathcal{X})h,h) = 0$ for any $h\in\mathcal{H}$ which yields $\Sigma(\mathcal{X}) = 0$. Conversely, if $\Sigma(\mathcal{X}) = 0$ for some Borel set $\mathcal{X}\subseteq\mathbb{R}$, then $(\Sigma(\mathcal{X})h,h) = 0$ for any $h\in\mathcal{H}$. Therefore $(E_{A_0}(\mathcal{X})\gamma(i)h,\gamma(i)h) = 0$ for any $h\in\mathcal{H}$. Hence $E_{A_0}(\mathcal{X})\gamma(i)h = 0$. On the other hand, from (2.13) one gets

$$\begin{aligned}E_{A_0}(\mathcal{X}))\gamma(z)h &= E_{A_0}(\mathcal{X})[I+(z-i)(A_0-z)^{-1}]\gamma(i)h\\ &= [I+(z-i)(A_0-z)^{-1}]E_{A_0}(\mathcal{X})\gamma(i)h = 0.\end{aligned}$$

for $z\in\mathbb{C}_+\cup\mathbb{C}_-$. Since the operator A is simple the set $\{\gamma(z)\mathcal{H} : z\in\mathbb{C}_+\cup\mathbb{C}_-\} = \mathfrak{H}$ is generating with respect to A_0. Therefore $E_{A_0}(\mathcal{X})\mathfrak{H} = 0$. The remaining assertions are obvious. $\triangle$

REMARK 3.3 Actually we have proved a little bit more. In fact, the proof shows that for each $h\in\mathcal{H}$ the measures $(E_{A_0}(\cdot)\gamma(i)h,\gamma(i)h)$ and $\Sigma_h(\cdot)$ are equivalent.

Using Lemma 3.1 and 3.2 we can made the following conclusions on the spectrum of the extension $A_0 = A_0^*$ of a simple symmetric operator A.

THEOREM 3.4 *Let A be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ with $n_+(A) = n_-(A)$. Further, let $\Pi = \{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(\cdot)$.*

If $E_{A_0}(\cdot)$ is the spectral measure of $A_0 := A^*\upharpoonright\ker\Gamma_0(\in Ext_A)$ and $\Sigma(\cdot)$ that of the integral representation (3.17) of the Weyl function $M(\cdot)$, then for each total set $\mathcal{T} = \{h_k\}_{k=1}^N$, $1\le N\le+\infty$, in $\mathcal{H}$ the sets $S'_s(\Sigma;\mathcal{T})$, $S'_{pp}(\Sigma;\mathcal{T})$, $S'_{sc}(\Sigma;\mathcal{T})$ and $S'_{ac}(\Sigma;\mathcal{T})$ are singular, pure point, singular continuous and absolutely continuous supports of $E_{A_0}(\cdot)$, respectively, i.e, we have

$$E_{A_0}(\mathcal{X}\cap S'_\tau(\Sigma;\mathcal{T})) = E^\tau_{A_0}(\mathcal{X}),\quad \tau = s,pp,sc,ac, \tag{3.23}$$

for each Borel set $\mathcal{X}\subseteq\mathbb{R}$. In particular, the relations $\sigma_p(A_0) = S'_{pp}(\Sigma;\mathcal{T})$ and $\sigma_\tau(A_0)\subseteq\overline{S'_\tau(\Sigma;\mathcal{T})}\subseteq\sigma(A_0)$, $\tau = s,sc,ac$, hold.

Proof: Since by Lemma 3.1 the sets $S'_\tau(\sigma;\mathcal{T})$, $\tau = s,pp,sc,ac$, are singular, pure point, singular continuous and absolutely continuous supports of $\Sigma(\cdot)$ one immediately gets from Lemma 3.2 that the same sets are supports of $E_{A_0}(\cdot)$ of the same type, i.e., (3.23) holds. If $x\in S'_{pp}(\Sigma;\mathcal{T})$, then there is at least one $k = 1,2,\ldots,N$ such that $(E_{A_0}(\{x\}\gamma(i)h_k,\gamma(i)h_k)\neq$

0. Hence $x \in \sigma_p(A_0)$. Conversely, if $x \in \sigma_p(A_0)$, then due to the fact that $\gamma(i)\mathcal{T}$ is generating for $E_{A_0}(\cdot)$ there is at least one $k = 1, 2, \ldots, N$ such that $(E_{A_0}(\{x\})\gamma(i)h_k, \gamma(i)h_k) \neq 0$. Hence $x \in S'_{pp}(\Sigma; \mathcal{T})$ which proves $\sigma_p(A_0) = S'_{pp}(\Sigma; \mathcal{T})$.

The relations $\sigma_\tau(A_0) \subseteq \overline{S'_\tau(\Sigma; \mathcal{T})} \subseteq \sigma(A_0)$, $\tau = s, sc, ac$, are consequences of Lemma 3.1 and Lemma 3.2. △

However, trying to apply the theory developed above to the operator-valued measure $\Sigma(\cdot)$ of the integral representation of the Weyl function $M(\cdot)$ it turns out that information on the measure $\Sigma(\cdot)$ is hardly available. More information is accessible from the boundary behaviour at the real axis of the Weyl function. So our next aim is to characterize the spectral properties of the operator-valued measure $\Sigma(\cdot)$ using the boundary behaviour of the Weyl function $M(\cdot)$. A first step is to develop a corresponding theory for scalar measures μ which satisfy

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < +\infty. \tag{3.24}$$

Let us associate with μ the Poisson integral

$$V(z) = \int_{\mathbb{R}} \frac{y d\mu(t)}{(t-x)^2 + y^2}, \quad z = x + iy \in \mathbb{C}_+, \tag{3.25}$$

which defines a positive harmonic function in $\mathbb{C}_+$. Conversely, it is well known (see [20, 22]) that each positive harmonic function $V_1(z)$ in $\mathbb{C}_+$ admits the representation $V_1(z) = ay + V(z)$ with $a \geq 0$ and $V(z)$ of the form (3.24) and (3.25). Below we summarize some well-known facts (see [3, 4, 16, 17]) on positive harmonic functions.

PROPOSITION 3.5 *Let μ be a positive Radon measure obeying (3.24) and let $V(z)$ be a positive harmonic function in $z = x + iy \in \mathbb{C}_+$ defined by (3.25). Then one has:*

(i) *For any $x \in \mathbb{R}$ the limit $V(x + i0) := \lim_{y \downarrow 0} V(x + iy)$ exists and is finite if and only if the symmetric derivative $D\mu(x)$,*

$$D\mu(x) = \lim_{\varepsilon \to 0} \frac{\mu(x+\varepsilon) - \mu(x-\varepsilon)}{2\varepsilon}, \tag{3.26}$$

exists and is finite. In this case one has

$$V(x + i0) = \pi D\mu(x). \tag{3.27}$$

(ii) *If the symmetric derivative $D\mu(x)$ exists and is infinite, then $V(z) \to +\infty$ as $z \to\succ x$.*

(iii) *For each $x \in \mathbb{R}$ one has $\Im\mathrm{m}(z-x)V(z) \to \mu(\{x\})$ as $z \to\succ x$.*

(iv) *$V(z)$ converges to a finite constant as $z \to\succ x$ if and only if the derivative $d\mu(t)/dt$ exists at $t = x$ and is finite.*

The symbol $\to\succ$ means that the limit $\lim_{r \downarrow 0} V(x + re^{i\theta})$, $x \in \mathbb{R}$, exists uniformly in $\theta \in [\varepsilon, \pi - \varepsilon]$ for each $\varepsilon \in (0, \pi/2)$.

Proposition 3.5 allows us to introduce for measures satisfying (3.24) the following sets ($z = x + iy$):

$$\begin{aligned} S''_s(\mu) &:= \{x \in \mathbb{R} : V(z) \to \infty \text{ as } z \to\succ x\} && (3.28)\\ S''_{pp}(\mu) &:= \{x \in \mathbb{R} : \Im\mathrm{m} \lim_{z \to\succ x} (z - x)V(z) > 0\} && (3.29)\\ S''_{sc}(\mu) &:= \{x \in \mathbb{R} : V(z) \to \infty \text{ and } (z - x)V(z) \to 0 \text{ as } z \to\succ x\} && (3.30)\\ S''_{ac}(\mu) &:= \{x \in \mathbb{R} : V(x + i0) \text{ exists and } 0 < V(x + i0) < \infty\} && (3.31) \end{aligned}$$

Obviously, the sets $S''_s(\mu)$ and $S''_{ac}(\mu)$ as well as $S''_{pp}(\mu)$, $S''_{sc}(\mu)$ and $S''_{ac}(\mu)$ are mutually disjoint. By Proposition 3.5 one immediately gets that

$$S'_{pp}(\mu) = S''_{pp}(\mu) \quad \text{and} \quad S'_\tau(\mu) \subseteq S''_\tau(\mu) \subseteq S(\mu),\ \tau = s, sc, ac. \tag{3.32}$$

Indeed, the relation $S'_{pp}(\mu) = S''_{pp}(\mu)$ is a consequence of (iii). By (ii) we get $S'_s(\mu) \subseteq S''_s(\mu)$. Similarly, we prove $S'_{sc}(\mu) \subseteq S''_{sc}(\mu)$ using (ii) and (iii). Finally, the relation $S'_{ac}(\mu) \subseteq S''_{ac}(\mu)$ follows from (i).

We note that it can happen that $S'_{sc}(\mu) \neq \emptyset$ and the inclusion $S'_{sc}(\mu) \subseteq S''_{sc}(\mu)$ is strict even if $\mu_{sc} = 0$.

Furthermore, we note that from (3.5) and the inclusion $S'_\tau(\mu) \subseteq S''_\tau(\mu)$, $\tau = s, pp, sc, ac$, we find that

$$\mu(\mathcal{X} \cap S''_\tau(\mu)) = \mu_\tau(\mathcal{X}), \quad \tau = s, pp, sc, ac, \tag{3.33}$$

for any Borel set $\mathcal{X} \subseteq \mathbb{R}$.

Now we are going to characterize the spectral parts of the extension A_0 by means of boundary values of the Weyl function $M(\cdot)$. Using the integral representation (3.17) of the Weyl function we easily get that

$$V_h(z) := \int_{\mathbb{R}} \frac{y}{(x - t)^2 + y^2} d\Sigma_h(t) = \Im\mathrm{m}(M_h(z)), \quad z \in \mathbb{C},\ h \in \mathcal{H}, \tag{3.34}$$

where

$$M_h(z) := (M(z)h, h), \quad z \in \mathbb{C},\ h \in \mathcal{H}. \tag{3.35}$$

The function $M_h(\cdot)$ is a scalar R-function. Since $M_h(\cdot)$ arises from the Weyl function we call it the associated scalar Weyl function. By the fact that the function $V_h(\cdot)$ is the imaginary part of the associated scalar Weyl function $M_h(\cdot)$ and the theory developed we can relate the boundary behaviour at the real axis of the imaginary part of associated scalar Weyl functions with the spectral properties of the self-adjoint extension A_0. To this end in addition to (3.8) - (3.11) we introduce the sets

$$\begin{aligned} S''_s(\Sigma; \mathcal{T}) &:= \cup_{k=1}^N S''_s(\Sigma_{h_k}), && (3.36)\\ S''_{pp}(\Sigma; \mathcal{T}) &:= \cup_{k=1}^N S''_{pp}(\Sigma_{h_k}), && (3.37)\\ S''_{sc}(\Sigma; \mathcal{T}) &:= \cup_{k=1}^N S''_{sc}(\Sigma_{h_k}) \setminus S''_{pp}(\Sigma), && (3.38)\\ S''_{ac}(\Sigma; \mathcal{T}) &:= \cup_{k=1}^N S''_{ac}(\Sigma_{h_k}) \setminus S''_s(\Sigma) && (3.39) \end{aligned}$$

By definition the sets $S''_s(\Sigma;\mathcal{T})$ and $S''_{ac}(\Sigma;\mathcal{T})$ are disjoint. The same holds for $S''_{pp}(\Sigma;\mathcal{T})$, $S''_{sc}(\Sigma;\mathcal{T})$ and $S''_{ac}(\Sigma;\mathcal{T})$. Furthermore, we denote that the sets $S''_\tau(\Sigma;\mathcal{T})$, $\tau = s, pp, sc$, have Lebesgue measure zero, i.e $\mathrm{mes}(S''_\tau(\Sigma;\mathcal{T})) = 0$, $\tau = s, pp, sc$. It turns out that the sets $S'_\tau(\Sigma;\mathcal{T})$ in Theorem 3.4 can be replaced by the sets $S''_\tau(\Sigma;\mathcal{T})$.

THEOREM 3.6 *Let A be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ with $n_+(A) = n_-(A)$. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(\cdot)$.*

If $E_{A_0}(\cdot)$ is the spectral measure of $A_0 := A^* \upharpoonright \ker\Gamma_0 (\in Ext_A)$ and $\Sigma(\cdot)$ that of the integral representation (3.17) of $M(\cdot)$, then for each total set $\mathcal{T} = \{h_k\}_{k=1}^N$, $1 \le N \le +\infty$, in $\mathcal{H}$ the sets $S''_s(\Sigma;\mathcal{T})$, $S''_{pp}(\Sigma;\mathcal{T})$, $S''_{sc}(\Sigma;\mathcal{T})$ and $S''_{ac}(\Sigma;\mathcal{T})$ are singular, pure point, singular continuous and absolutely continuous supports of $E_{A_0}(\cdot)$, respectively, i.e, we have

$$E_{A_0}(\mathcal{X} \cap S''_\tau(\Sigma;\mathcal{T})) = E^\tau_{A_0}(\mathcal{X}), \quad \tau = s, pp, sc, ac, \tag{3.40}$$

for each Borel set $\mathcal{X} \subseteq \mathbb{R}$. In particular, it holds $\sigma_p(A_0) = S''_{pp}(\Sigma;\mathcal{T})$ and $\sigma_\tau(A_0) \subseteq \overline{S''_\tau(\Sigma;\mathcal{T})} \subseteq \sigma(A_0)$, $\tau = s, sc, ac$.

Proof: We note that $\mathrm{mes}(S''_s(\Sigma_h)) = 0$ for each $h \in \mathcal{H}$. This is a consequence of the fact that for a.e. $x \in \mathbb{R}$ the symmetric derivative $D\Sigma_h(x)$ exists and is finite. By Proposition 3.5(i) this yields that for a.e. $x \in \mathbb{R}$ the limit $V_h(x + i0)$ exists and is finite. Therefore $\mathrm{mes}(S''_s(\Sigma;\mathcal{T})) = 0$. Using this and following the line of reasoning of Theorem 3.4 we easily prove (3.40).

Since $S'_{pp}(\Sigma_{h_k}) = S''_{pp}(\Sigma_{h_k})$ for each $h_k \in \mathcal{T}$ one gets $S'_{pp}(\Sigma;\mathcal{T}) = S''_{pp}(\Sigma;\mathcal{T})$. Applying Theorem 3.4 we find $\sigma_p(A_0) = S''_{pp}(\Sigma;\mathcal{T})$. Since $S'_\tau(\Sigma_{h_k}) \subseteq S''_\tau(\Sigma_{h_k})$, $\tau = s, sc, ac$, for each h_k one gets

$$\cup_{k=1}^N S'_\tau(\Sigma_{h_k}) \subseteq \cup_{k=1}^N S''_\tau(\Sigma_{h_k}), \quad \tau = s, sc, ac, \tag{3.41}$$

which yields $S'_\tau(\Sigma;\mathcal{T}) \subseteq S''_\tau(\Sigma;\mathcal{T})$, $\tau = s, sc, ac$. Applying Theorem 3.4 we find $\sigma_\tau(\Sigma;\mathcal{T}) \subseteq \overline{S''_\tau(\Sigma;\mathcal{T})}$, $\tau = s, sc, ac$. Otherwise, we have $S''_\tau(\Sigma_{h_k}) \subseteq S(\Sigma_{h_k}) \subseteq S(\Sigma)$ or $\overline{S''_\tau(\Sigma;\mathcal{T})} \subseteq S(\Sigma)$, $\tau = s, sc, ac$. Using Lemma 3.2 we complete the proof. △

Theorem 3.6 characterizes the spectral properties of A_0 by boundary values of the imaginary part of the associated scalar Weyl function. Our next aim is to use for that purpose the boundary values of the associated scalar Weyl function themselves. This step relies on the following proposition for scalar R-functions.

PROPOSITION 3.7 *Let $\Phi(\cdot)$ be a scalar R-function. Then for almost all $x \in \mathbb{R}$ the limit $\Phi(x + i0) := \lim_{y\downarrow 0} \Phi(x + i0)$ exists and is finite and, moreover, in this case one has $\Phi(x + i0) = \lim_{z\to\succ x} \Phi(z)$.*

Let us introduce the following sets:

$$\Omega_s(\Phi) := \{x \in \mathbb{R} : |\Phi(z)| \to +\infty \text{ as } z \to\succ x\}, \tag{3.42}$$

$$\Omega_{pp}(\Phi) := \{x \in \mathbb{R} : \lim_{z\to\succ x} (z - x)\Phi(z) \ne 0\}, \tag{3.43}$$

$$\Omega_{sc}(\Phi) := \{x \in \mathbb{R} : |\Phi(z)| \to +\infty \text{ and } (z - x)\Phi(z) \to 0 \text{ as } z \to\succ x\} \tag{3.44}$$

$$\Omega_{ac}(\Phi) := \{x \in \mathbb{R} : 0 < \Im\mathrm{m}\Phi(x + i0) < +\infty\} \tag{3.45}$$

In accordance with (2.2) the scalar R-function $\Phi(\cdot)$ admits the representation

$$\Phi(z) = C_0 + C_1 z + \int_{-\infty}^{+\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu(t), \quad z \in \mathbb{C}_+, \tag{3.46}$$

where $C_0, C_1 \in \mathbb{R}$, $C_1 \geq 0$ and the Borel measure $\mu(\cdot)$ obeys (3.24). Therefore, one gets $V(z) = \Im\mathrm{m}\Phi(z)$, $z \in \mathbb{C}_+$, where $V(\cdot)$ is defined by (3.25). Using that and Propositions 3.5 and 3.7 one immediately gets that

$$S''_{pp}(\mu) = \Omega_{pp}(\Phi) \quad \text{and} \quad S''_{\tau}(\mu) \subseteq \Omega_\tau(\Phi) \subseteq S(\mu), \ \tau = s, sc, \tag{3.47}$$

but $\Omega_{ac}(\Phi) \subseteq S''_{ac}(\mu) \subseteq S(\mu)$. However we mentioned that $\Omega_{ac}(\Phi) = S''_{ac}(\mu) \cap \Omega_f(\Phi)$ where

$$\Omega_f(\Phi) = \{x \in \mathbb{R} : \Phi(x+i0) \text{ exists and is finite}\}. \tag{3.48}$$

Obviously the sets $\Omega_s(\Phi)$, $\Omega_{pp}(\Phi)$, $\Omega_{sc}(\Phi)$ and $\Omega_{ac}(\Phi)$ are mutually disjoint and, moreover, are singular, pure point, singular continuous and absolutely continuous supports of $\mu(\cdot)$, i.e.

$$\mu(\mathcal{X} \cap \Omega_\tau(\Phi)) = \mu_\tau(\mathcal{X}), \quad \tau = s, pp, sc, ac, \tag{3.49}$$

for any Borel set $\mathcal{X} \subseteq \mathbb{R}$.

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(\cdot)$. Further, let $\mathcal{T} = \{h_k\}_{k=1}^N$, $1 \leq +\infty$, be a total set in $\mathcal{H}$. We set

$$\begin{aligned}
\Omega_s(M;\mathcal{T}) &:= \cup_{k=1}^N \Omega_s(M_{h_k}), & (3.50)\\
\Omega_{pp}(M;\mathcal{T}) &:= \cup_{k=1}^N \Omega_{pp}(M_{h_k}), & (3.51)\\
\Omega_{sc}(M;\mathcal{T}) &:= \cup_{k=1}^N \Omega_s(M_{h_k}) \setminus \Omega_{pp}(M;\mathcal{T}), & (3.52)\\
\Omega_{ac}(M;\mathcal{T}) &:= \cup_{k=1}^N \Omega_{ac}(M_{h_k}) \setminus \Omega_s(M;\mathcal{T}). & (3.53)
\end{aligned}$$

Obviously, the sets $\Omega_s(M;\mathcal{T})$ and $\Omega_{ac}(M;\mathcal{T})$ as well as the sets $\Omega_{pp}(M;\mathcal{T})$, $\Omega_{sc}(M;\mathcal{T})$ and $\Omega_{ac}(M;\mathcal{T})$ are mutually disjoint. We note that the sets $\Omega_\tau(M;\mathcal{T})$, $\tau = s, pp, sc$ have Lebesgue measure zero, i.e. $\mathrm{mes}(\Omega_\tau(M;\mathcal{T})) = 0$, $\tau = s, pp, sc$.

THEOREM 3.8 *Let A be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ with $n_+(A) = n_-(A)$. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(\cdot)$ and let $E_{A_0}(\cdot)$ be the spectral measure of the self-adjoint extension A_0 of A.*

If $\mathcal{T} = \{h_k\}_{k=1}^N$, $1 \leq N \leq +\infty$, is a total set in $\mathcal{H}$, then the sets $\Omega_s(M;\mathcal{T})$, $\Omega_{pp}(M;\mathcal{T})$, $\Omega_{sc}(M;\mathcal{T})$ and $\Omega_{ac}(M;\mathcal{T})$ are singular, pure point, singular continuous and absolutely continuous supports of $E_{A_0}(\cdot)$, respectively, i.e, we have

$$E_{A_0}(\mathcal{X} \cap \Omega_\tau(M;\mathcal{T})) = E^\tau_{A_0}(\mathcal{X}), \quad \tau = s, pp, sc, ac, \tag{3.54}$$

for each Borel set $\mathcal{X} \subseteq \mathbb{R}$. In particular, it holds $\sigma_p(A_0) = \Omega_{pp}(M;\mathcal{T})$ and $\sigma_\tau(A_0) \subseteq \overline{\Omega_\tau(M;\mathcal{T})} \subseteq \sigma(A_0)$ for $\tau = s, sc, ac$.

The proof is similar to that of Theorem 3.6. So we omit it. We note that the inclusions $\sigma_s(A_0) \subseteq \overline{\Omega_s(M;\mathcal{T})}$ and $\sigma_{sc}(A_0) \subseteq \overline{\Omega_{sc}(M;\mathcal{T})}$ of Theorem 3.8 may be strict even if $\sigma_{sc}(A_0)$ is empty.

4 Singular and absolutely continuous spectra

The relations above exactly characterizes only the point spectrum. However, a refinement of the procedure above shows that the absolutely continuous spectrum can be exactly obtained from the boundary behaviour of the Weyl function at the real axis too.

Let $\mu(\cdot)$ be a Borel measure on $\mathbb{R}$ and let $\mathcal{X} \subseteq \mathbb{R}$ be a Borel set. The set

$$\mathrm{cl}_{ac}(\mathcal{X}) = \{x \in \mathbb{R} : \mathrm{mes}((x-\epsilon, x+\epsilon) \cap \mathcal{X}) > 0 \quad \forall \epsilon > 0\} \tag{4.1}$$

is called the absolutely continuous closure of the set $\mathcal{X}$. Obviously, the set $\mathrm{cl}_{ac}(\mathcal{X})$ is always closed and one has $\mathrm{cl}_{ac}(\mathcal{X}) \subseteq \overline{\mathcal{X}}$.

LEMMA 4.1 *Let $\Phi(\cdot)$ be a scalar R-function which has the representation (3.46). Then $S_{ac}(\mu) = \mathrm{cl}_{ac}(\Omega_{ac}(\Phi))$.*

Proof: If $x \notin \mathrm{cl}_{ac}(\Omega_{ac}(\Phi))$, then there is an $\epsilon > 0$ such that $\mathrm{mes}((x-\epsilon, x+\epsilon) \cap \Omega_{ac}(\Phi)) = \emptyset$. Taking (3.49) into account we find

$$\mu_{ac}((x-\epsilon, x+\epsilon)) = \mu_{ac}((x-\epsilon, x+\epsilon) \cap \Omega_{ac}(\Phi)) = 0. \tag{4.2}$$

Hence $x \notin S(\mu_{ac}) = S_{ac}(\mu)$ which yields $S_{ac}(\mu) \subseteq \mathrm{cl}_{ac}(\Omega_{ac}(\Phi))$. Conversely, if $x \notin S_{ac}(\mu)$, then there is an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \cap S_{ac}(\mu) = \emptyset$, then $\mu_{ac}((x-\epsilon, x+\epsilon)) = 0$. Using

$$\begin{aligned} &\mu_{ac}((x-\epsilon, x+\epsilon)) = \\ &\qquad \mu_{ac}((x-\epsilon, x+\epsilon) \cap \Omega_{ac}(\Phi)) = \int_{(x-\epsilon,x+\epsilon)\cap\Omega_{ac}(\Phi)} \frac{d\mu(t)}{dt} dt = 0 \end{aligned} \tag{4.3}$$

and Proposition 3.5(i) and (iv) one gets

$$\mu_{ac}((x-\epsilon, x+\epsilon)) = \frac{1}{\pi} \int_{(x-\epsilon,x+\epsilon)\cap\Omega_{ac}(\Phi)} \Im\mathrm{m}(\Phi(t+i0)) dt = 0 \tag{4.4}$$

Hence $\Im\mathrm{m}(\Phi(t+i0)) = 0$ for a.e. $t \in (x-\epsilon, x+\epsilon) \cap \Omega_{ac}(\Phi)$. However, by definition of the set $\Omega_{ac}(\Phi)$ one has $\Im\mathrm{m}(\Phi(t+i0)) > 0$ for all $t \in \Omega_{ac}(\Phi)$ which implies $\mathrm{mes}((x-\epsilon, x+\epsilon) \cap \Omega_{ac}(\Phi)) = 0$. Hence $x \notin \mathrm{cl}_{ac}(\Omega_{ac}(\Phi))$ or, equivalently, $\mathrm{cl}_{ac}(\Omega_{ac}(\Phi)) \subseteq S_{ac}(\mu)$. $\triangle$

Using this lemma Theorem 3.8 admits with respect to the absolutely continuous part the following refinement.

PROPOSITION 4.2 *Let A be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ with $n_+(A) = n_-(A)$. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(\cdot)$.*

If $\mathcal{T} = \{h_k\}_{k=1}^N$, $1 \leq N \leq +\infty$, is a total set in $\mathcal{H}$, then the absolutely continuous spectrum of the self-adjoint extension A_0 of A is given by

$$\sigma_{ac}(A_0) = \overline{\cup_{k=1}^N \mathrm{cl}_{ac}(\Omega_{ac}(M_{h_k}))}. \tag{4.5}$$

Proof: By Lemma 4.1 one has $\mathrm{cl}_{ac}(\Omega_{ac}(M_{h_k})) = S_{ac}(\Sigma_{h_k})$ for $k = 1, 2, ..., N$. Since $S_{ac}(\Sigma_{h_k}) \subseteq S_{ac}(\Sigma) = \sigma_{ac}(A_0)$ we find $S_{ac}(\Sigma_{h_k}) \subseteq \sigma_{ac}(A_0)$ for $k = 1, 2, ..., N$. Hence

$$\overline{\cup_{k=1}^{N}\mathrm{cl}_{ac}(\Omega_{ac}(M_{h_k}))} \subseteq \sigma_{ac}(A_0). \tag{4.6}$$

Conversely, if $x \notin \overline{\cup_{k=1}^{N}\mathrm{cl}_{ac}(\Omega_{ac}(M_{h_k}))}$, then there is a $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \cap \mathrm{cl}_{ac}(\Omega_{ac}(M_{h_k})) = \emptyset$ for each $k = 1, 2, ..., N$. Hence $(x-\epsilon, x+\epsilon) \cap S_{ac}(\Sigma_{h_k}) = \emptyset$ for $k = 1, 2, ..., N$ or, equivalently, $\Sigma_{h_k,ac}((x-\epsilon, x+\epsilon)) = 0$ for $k = 1, 2, ..., M$. Since $\mathcal{T}$ is a total set one easily gets that $\Sigma_{h,ac}((x-\epsilon, x+\epsilon)) = 0$ for each $h \in \mathcal{H}$. Therefore $x \notin S_{ac}(\Sigma)$ which yields $x \notin S_{ac}(E_{A_0})$. Hence $\sigma_{ac}(A_0) \subseteq \overline{\cup_{k=1}^{N}\mathrm{cl}_{ac}(\Omega(M_{h_k}))}$ which proves (4.5). △

Theorem 3.8 and Proposition 4.2 immediately have several implications which are summarized in the following theorem.

THEOREM 4.3 *Let A be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ with $n_+(A) = n_-(A)$. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(\cdot)$.*

If $\mathcal{T} = \{h_k\}_{k=1}^{N}$, $1 \le N \le +\infty$, is a total set in $\mathcal{H}$, then for the self-adjoint extension A_0 of A the following conclusions are valid:

(i) *The self-adjoint extension A_0 of A has no point spectrum within the interval (a, b), i.e. $\sigma_{pp}(A_0) \cap (a, b) = \emptyset$, if and only if for each $k = 1, 2, ..., N$ one has*

$$\lim_{y\downarrow 0} yM_{h_k}(x+iy) = 0 \tag{4.7}$$

for all $x \in (a, b)$. In this case the following relation holds

$$\sigma(A_0) \cap (a,b) = \sigma_c(A_0) \cap (a,b) = \left(\overline{\cup_{k=1}^{N}\Omega_{sc}(M_{h_k})} \cup \overline{\cup_{k=1}^{N}\Omega_{ac}(M_{h_k})}\right) \cap (a,b). \tag{4.8}$$

(ii) *The self-adjoint extension A_0 of A has no singular continuous spectrum within the interval (a, b), i.e. $\sigma_{sc}(A_0) \cap (a, b) = \emptyset$, if for each $k = 1, 2, ..., N$ the set $\Omega_{sc}(M_{h_k}) \cap (a, b)$ is countable, in particular, if $(a, b) \setminus \Omega_{ac}(M_{h_k})$ is countable.*

(iii) *The self-adjoint extension A_0 of A has no absolutely continuous spectrum within the interval (a, b), i.e. $\sigma_{ac}(A_0) \cap (a, b) = \emptyset$, if and only if for each $k = 1, 2, ..., N$ the condition*

$$\Im\mathrm{m}(M_{h_k}(x+i0)) = 0 \tag{4.9}$$

holds for a.e. $x \in (a, b)$. In his case we have

$$\sigma(A_0) \cap (a,b) = \sigma_s(A_0) \cap (a,b) = \overline{\Omega_s(M;\mathcal{T})} \cap (a,b). \tag{4.10}$$

Proof: (i) If condition (4.7) is satisfied for all $x \in (a, b)$ and all $k = 1, 2, ..., N$, then a simple computation shows that $\lim_{z\to\succ x}(z-x)M_{h_k}(z) = 0$ holds for all $x \in (a, b)$ and each $k = 1, 2, ..., N$ too. Therefore $\Omega_{pp}(M_{h_k}) \cap (a, b) = \emptyset$ for $k = 1, 2, ..., N$ which yields $\Omega_{pp}(M;\mathcal{T}) \cap (a, b) = \emptyset$. Theorem 3.8 implies $\sigma_p(A_0) \cap (a, b) = \emptyset$ which yields $\sigma_{pp}(A_0) \cap (a, b) = \emptyset$. Conversely, if $\sigma_{pp}(A_0) \cap (a, b) = \emptyset$, then $\sigma_p(A_0) \cap (a, b) = \emptyset$. Again by Theorem 3.8 we find $\Omega_{pp}(M;\mathcal{T})\cap(a,b)=\emptyset$ Therefore $\Omega_{pp}(M_{h_k})\cap(a,b)=\emptyset$ for each $k=1,2,...,N$. However, this implies that $\lim_{z\to\succ x}(z-x)M_{h_k}(z)=0$ which yields $\lim_{y\to 0} yM_{h_k}(x+iy)=0$ for all $x\in(a,b)$ and each $k=1,2,...,N$.

The first part of relation (4.8) is a consequence of $\sigma(A_0)=\sigma_{pp}(A_0)\cup\sigma_c(A_0)$ and $\sigma_{pp}(A_0)\cap(a,b)=\emptyset$. The second part of relation (4.8) is a consequence of Theorem 3.8 which shows that

$$\sigma_\tau(A_0)\subseteq\overline{\Omega_\tau(M;\mathcal{T})}=\overline{\cup_{k=1}^N\Omega_\tau(M_{h_k})}\subseteq\sigma(A_0),\quad \tau=sc,ac \tag{4.11}$$

and $\sigma_c(A_0)=\sigma_{sc}(A_0)\cup\sigma_{ac}(A_0)$. Both facts imply that

$$\begin{aligned}&\sigma_c(A_0)\cap(a,b)\subseteq\\ &\quad\left(\overline{\cup_{k=1}^N\Omega_{sc}(M_{h_k})}\;\cup\;\overline{\cup_{k=1}^N\Omega_{ac}(M_{h_k})}\right)\cap(a,b)\subseteq\sigma(A_0)\cap(a,b)=\sigma_c(A_0)\cap(a,b)\end{aligned} \tag{4.12}$$

which proves (4.8).

(ii) By (3.32) and (3.47) we get that $S'_{sc}(\Sigma_{h_k}):=S'(\Sigma_{h_k,sc})\subseteq S''_{sc}(\Sigma_{h_k})\subseteq\Omega_{sc}(M_{h_k})$. Therefore, if $\Omega_{sc}(M_{h_k})\cap(a,b)$ is countable, then so is $S''_{sc}(\Sigma_{h_k})\cap(a,b)$. This yields that the singular continuous measure $\Sigma_{h_h,sc}(\cdot)$ is supported within the interval (a,b) on a countable set. However, this implies that $\Sigma_{h_k,sc}((a,b))=0$ for each $k=1,2,...,N$. Since $\mathcal{T}$ is a total set this has the consequence that for each $k=1,2,...,N$ and every $h\in\mathcal{H}$ one has $\Sigma_{h,sc}((a,b))=0$ which yields $\Sigma^{sc}((a,b))=0$. Therefore by Lemma 3.2 one gets $E^{sc}_{A_0}((a,b))=0$ which proves $\sigma_{sc}(A_0)\cap(a,b)=\emptyset$

If $(a,b)\setminus\Omega_{ac}(M_{h_k})$ is countable, then by $\Omega_{sc}(M_{h_k})\subseteq(a,b)\setminus\Omega_{ac}(M_{h_k})$ the set $\Omega_{sc}(M_{h_k})$ is countable too which completes the proof (ii).

(iii) If for each $k=1,2,...,N$ the condition (4.9) holds for a.e. $x\in(a,b)$, then for each $\epsilon>0$ one has $\mathrm{mes}((x-\epsilon,x+\epsilon)\cap\Omega_{ac}(M_{h_k})\cap(a,b))=0$. Hence $\mathrm{cl}_{ac}(\Omega_{ac}(M_{h_k}))\cap(a,b))=\emptyset$. Taking Proposition 4.2 into account we find $\sigma_{ac}(A_0)\cap(a,b)=\emptyset$. Conversely, if $\sigma_{ac}(A_0)\cap(a,b)=\emptyset$, then by Proposition 4.2 for each $k=1,2,...,N$ we have $\mathrm{cl}_{ac}(\Omega_{ac}(M_{h_k}))\cap(a,b)=\mathrm{cl}_{ac}(\Omega_{ac}(M_{h_k})\cap(a,b))=\emptyset$ which verifies condition (4.9) for a.e. $x\in(a,b)$.

Using $\sigma(A_0)\cap(a,b)=\sigma_s(A_0)$ and $\sigma_s(A_0)\subseteq\overline{\Omega_s(M;\mathcal{T})}\subseteq\sigma(A_0)$, which was proved in Theorem 3.8, we easily verify (4.10). △

COROLLARY 4.4 *Under the assumptions of Theorem 4.3 the following conclusions additionally are true*

(iv) *The spectrum of A_0 is purely singular continuous within the interval (a,b), i.e. $\sigma(A_0)\cap(a,b)=\sigma_{sc}(A_0)\cap(a,b)$, if for each $k=1,2,...,N$ condition (4.7) is valid for all $x\in(a,b)$ and condition (4.9) is satisfied for a.e. $x\in(a,b)$. In this case the relation*

$$\sigma(A_0)\cap(a,b)=\sigma_{sc}(A_0)\cap(a,b)=\overline{\cup_{k=1}^N\Omega_{sc}(\Sigma_{h_k})}\cap(a,b) \tag{4.13}$$

holds.

(v) *The spectrum of A_0 is purely absolutely continuous within the interval (a,b), i.e. $\sigma(A_0)\cap(a,b)=\sigma_{ac}(A_0)\cap(a,b)$, if for each $k=1,2,...,\ldots$ condition (4.7) is valid for all $x\in(a,b)$ and additionally the set $\Omega_{sc}(M_k)\cap(a,b)$ is countable, in particular, if $(a,b)\setminus\Omega_{ac}(M_k)$ is countable. We have*

$$\sigma(A_0)\cap(a,b)=\sigma_{ac}(A_0)\cap(a,b)=\overline{\cup_{k=1}^N\Omega_{ac}(\Sigma_{h_k})}\cap(a,b) \tag{4.14}$$

Proof: (iv) Since $\sigma(A_0) = \sigma_{pp}(A_0) \cup \sigma_{sc}(A_0) \cup \sigma_{ac}(A_0)$ we obtain from (i) and (iii) that $\sigma(A_0) \cap (a,b) = \sigma_{sc}(A_0) \cap (a,b)$. Further, Theorem 3.8 implies that $\sigma_{sc}(A_0) \subseteq \overline{\cup_{k=1}^N \Omega_{sc}(\Sigma_{h_k})} \subseteq \sigma(A_0)$ which yields

$$\sigma_{sc}(A_0) \cap (a,b) \subseteq \overline{\cup_{k=1}^N \Omega_{sc}(\Sigma_{h_k})} \cap (a,b) \subseteq \sigma(A_0) \cap (a,b). \tag{4.15}$$

Using $\sigma_{sc}(A_0) \cap (a,b) = \sigma(A_0) \cap (a,b)$ we verify (4.13).

(v) From (i) and (ii) we obtain $\sigma(A_0) \cap (a,b) = \sigma_{ac}(A_0) \cap (a,b)$. From Theorem 3.8 we get $\sigma_{ac}(A_0) \subseteq \overline{\cup_{k=1}^N \Omega_{ac}(\Sigma_{h_k})} \subseteq \sigma(A_0)$ which proves (4.14). △

Next let us indicate sufficient conditions which guarantee that the self-adjoint extension A_0 is purely absolutely continuous on the interval (a,b).

LEMMA 4.5 *Let A be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ with $n_+(A) = n_-(A)$. Further, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(\cdot)$ and γ-field $\gamma(\cdot)$. Assume that there is a non-negative non-decreasing function $\Psi(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x\to+\infty} \Psi(x) = +\infty$. If for some $h \in \mathcal{H}$ the condition*

$$c_\Psi(h) := \sup_{0<y<1} \int_a^b V_h(x+iy)\Psi\big(V_h(x+iy)\big)dy < \infty, \tag{4.16}$$

is satisfied, then

$$E_{A_0}(a,b)\gamma(z)h \in \mathfrak{H}^{ac}(A_0) \qquad \text{for each} \quad z \in \mathbb{C}_+ \cup C_- \tag{4.17}$$

Proof: Let us assume that the interval (a,b) is finite. By de la Valee Poussin Theorem [26] (see also [22]) the condition (4.16) implies that the set $\mathfrak{V} := \{V_h(x+iy) : 0 < y < 1\} \subseteq L^1((a,b))$ of non-negative functions has equicontinuous absolutely continuous integrals, that is

$$\lim_{\mathrm{mes}(E)\downarrow 0} \int_E V_h(x+iy)\, dx = 0 \tag{4.18}$$

uniformly in $y \in (0,1)$. It follows from (4.16) that the set $\mathfrak{V}$ is bounded in $L^1((a,b))$. This fact and (4.18) yield (see [12]) the sequentially weak compactness of $\mathfrak{V}$ in $L^1((a,b))$. Therefore, for each sequence $\{y_n\}_{n=1}^\infty$ tending to zero there exists a subsequence $\{y_{n_k}\}_{k=1}^\infty$ such that $V_h(x+iy_{n_k})$ weakly converges in $L^1((a,b))$ as $k \to \infty$, that is, there is an element $f_h \in L^1((a,b))$ such that for each $\varphi \in L^\infty((a,b))$ one has

$$\lim_{k\to\infty} \int_a^b V_h(x+iy_{n_k})\varphi(x)dx = \int_a^b f_h(x)\varphi(x)dx. \tag{4.19}$$

By (3.34) and Stieltjes inversion formula (2.7) we find

$$\begin{gathered}\frac{1}{2}\left(\Sigma_h(t+0) - \Sigma_h(t)\right) - \frac{1}{2}\left(\Sigma_h(a+0) + \Sigma_h(a)\right) = \\ \lim_{y\downarrow 0} \frac{1}{\pi} \int_a^t V_h(x+iy)dx.\end{gathered} \tag{4.20}$$

Taking (4.19) into account we finally get the relation

$$\frac{1}{2}\left(\Sigma_h(t+0)-\Sigma_h(t)\right)-\frac{1}{2}\left(\Sigma_h(a+0)+\Sigma_h(a)\right)= \tag{4.21}$$
$$\lim_{k\to\infty}\int_a^t V_h(x+iy_{n_k})dx=\frac{1}{\pi}\int_a^t f_h(x)dx$$

Thus, the measure $\Sigma_h(\cdot)$ is purely absolutely continuous on the interval (a,b), i.e. $S_s(\Sigma_h)\cap(a,b)=\emptyset$. By Remark 3.3 $(E_{A_0}(\cdot)\gamma(i)h,\gamma(i)h)$ and $\Sigma_h(\cdot)$ are equivalent measures. Hence the measure $(E_{A_0}(\cdot)\gamma(i)h,\gamma(i)h)$ is purely absolutely continuous on (a,b) too which immediately implies (4.17) for $z=i$. Taking into account (2.13) we immediately obtain (4.17) for each $z\in\mathbb{C}_+\cup\mathbb{C}_-$. If the interval (a,b) is infinite, then the relation (4.17) holds for every finite interval (a',b') contained in (a,b). This yields $E_{A_0}((a',b'))\gamma(z)h\in\mathfrak{H}^{ac}(A_0)$ for $z\in\mathbb{C}_+\cup\mathbb{C}_-$ for each finite interval (a',b'). Tending the finite interval $(a';b')$ to (a,b) we complete the proof. $\triangle$

THEOREM 4.6 *Let A be a simple densely defined closed symmetric operator on a separable Hilbert space $\mathfrak{H}$ with $n_+(A)=n_-(A)$. Further, let $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triple of A^* with Weyl function $M(\cdot)$.*

If there is a total set $\mathcal{T}=\{h_k\}_{k=1}^N$, $1\le N\le+\infty$, in $\mathcal{H}$ and a non-negative non-decreasing function $\Psi(\cdot):\mathbb{R}_+\to\mathbb{R}_+$ obeying $\lim_{x\to+\infty}\Psi(x)=+\infty$ such that for each $h_k\in\mathcal{T}$ condition (4.16) is satisfied, then the spectrum of A_0 is purely absolutely continuous on the interval (a,b), i.e. $\sigma(A_0)\cap(a,b)=\sigma_{ac}(A_0)\cap(a,b)$.

Proof: Since for each $h_k\in\mathcal{T}$ condition (4.16) is satisfied we get $E_{A_0}((a,b))\gamma(i)h_k\in\mathfrak{H}^{ac}(A_0)$. Since A_0 is simple and $\mathcal{T}$ is total the set $\gamma(i)\mathcal{T}=\{\gamma(i)h_k\}_{k=1}^N$ is generating for A_0. This implies that the set $E_{A_0}((a,b))\gamma(i)\mathcal{T}=\{E_{A_0}((a,b))\gamma(i)h_k\}_{k=1}^N$ is generating in the subspace $E_{A_0}((a,b))\mathfrak{H}$. Since $E_{A_0}((a,b))\gamma(i)h_k\in\mathfrak{H}^{ac}(A_0)$ for each $k=1,2,\ldots,N$ we get $E_{A_0}((a,b))\mathfrak{H}\subseteq\mathfrak{H}^{ac}(A_0)$. Therefore $E^s_{A_0}((a,b))=0$ which yields that $\sigma_s(A_0)\cap(a,b)=\emptyset$. $\triangle$

COROLLARY 4.7 *If there is a total set $\mathcal{T}=\{h_k\}_{k=1}^N$ such that for each $k=1,2,\ldots,N$ either*

$$C_p(h_k):=\sup_{y\in(0,1)}\int_a^b V_{h_k}(x+iy)^p dx<+\infty,\quad 1<p<+\infty, \tag{4.22}$$

or

$$C_\infty(h_k):=\sup_{y\in(0,1)}\sup_{x\in(a,b)}V_{h_k}(x+iy)<+\infty, \tag{4.23}$$

then the self-adjoint extension A_0 is purely absolutely continuous on the interval (a,b), i.e. $\sigma(A_0)\cap(a,b)=\sigma_{ac}(A_0)\cap(a,b)$.

Proof: If condition (4.22) is satisfied, then we set $\Psi(x)=x^{p-1}$, $x\in\mathbb{R}_+$. Applying Theorem 4.6 we complete this part of the proof.

If condition (4.23) is valid, then we first prove the result for a finite interval (a,b). In this case we find $V_{h_k}(x+iy)\le C_\infty(h_k)$ for $x\in(a,b)$ and $y\in(0,1)$. Therefore we get $V_{h_k}(x+iy)^p \le C^p_\infty(h_k)$, $\quad 1<p<\infty$, for $x\in(a,b)$ and $y\in(0,1)$ which shows that condition (4.22) is satisfied. Applying the first part we obtain that for finite intervals (a,b) the self-adjoint operator A_0 is purely absolutely continuous on (a,b). If (a,b) is infinite, then we approximate (a,b) by finite intervals. $\triangle$

REMARK 4.8 (1) One observes that $C_1(h) < +\infty$ holds true for an arbitrary Weyl function. Indeed, this is consequence of Stieltjes inversion formula (2.7).

(2) If (4.16) takes place for $(a,b)=\mathbb{R}$ and $\Psi(x)=\ln^+ x$, then one has $V_h(x+i0)\ln^+(V_h(x+i0))\in L^1(\mathbb{R})$ and by Zygmund Theorem [22] $\Re e(M_h(x+i0))\in L^1(\mathbb{R},(1+x^2)^{-1}dx)$. Thus $M_h(\lambda)(\lambda+i)^{-2}\in H^1(\mathbb{C}_+)$.

5 Examples

Let us illustrate the previous abstract theory by simple examples.

EXAMPLE 5.1 Let A be the symmetric operator in $L^2(\mathbb{R})$ defined by

$$(Af)(x) = -i\frac{d}{dx}f(x), \quad f\in \mathrm{dom}(A)=\{f\in W^1_2(\mathbb{R}) : f(0)=0\}. \tag{5.1}$$

It is clear that $n_\pm(A)=1$ and $\mathrm{dom}(A^*) = W^1_2(\mathbb{R}_-)\oplus W^1_2(\mathbb{R}_+)$. We set

$$\Gamma_0 f = \frac{1}{\sqrt{2}}[f(+0)+f(-0)] \quad \text{and} \quad \Gamma_1 f = \frac{i}{\sqrt{2}}[f(+0)-f(-0)], \tag{5.2}$$

for $f\in\mathrm{dom}(A^*)$. Taking Definition 2.1 into account one easily verifies that $\Pi=\{\mathbb{C},\Gamma_0,\Gamma_1\}$ is a boundary triple for A^*. We note that $A_0 := A^*\restriction\ker(\Gamma_0)$ has the domain $\mathrm{dom}(A_0)=\{f\in W^1_2(\mathbb{R}_-)\oplus W^1_2(\mathbb{R}_+) : f(+0)=-f(-0)\}$. Furthermore, one has that $f_\lambda(x)=e^{i\lambda x}\chi_\pm(x)$, $x\in\mathbb{R}$ and $\lambda\in\mathbb{C}_\pm$, where $\chi_\pm(\cdot):=\chi_{\mathbb{R}_\pm}(\cdot)$ is the indicator function of $\mathbb{R}_\pm$, belongs to $\ker(A^*-\lambda I)$. Using that and Definition 2.8 one gets that the corresponding Weyl function $M(\cdot)$ is given by $M(\lambda)=i$ for $\lambda\in\mathbb{C}_+$ and $M(\lambda)=-i$ for $\lambda\in\mathbb{C}_-$.

Each self-adjoint operator B on $\mathcal{H}=\mathbb{C}$ is simply a real number θ. Taking Remark 2.2(4) into account we obtain that $\Pi_\theta=\{\mathbb{C},\Gamma^\theta_0,\Gamma^\theta_1\}$, $\Gamma^\theta_0=\theta\Gamma_0-\Gamma_1$ and $\Gamma^\theta_1=\Gamma_1$, performs a new boundary triple for A^*. The self-adjoint extension $A^\theta_0 := A^*\restriction\ker(\Gamma^\theta_0)$ has the domain

$$\mathrm{dom}(A^\theta_0)=\{f\in W^1_2(\mathbb{R}_-)\oplus W^1_2(\mathbb{R}_+) : (i-\theta)f(+0)=(i+\theta)f(-0)\}. \tag{5.3}$$

The corresponding Weyl function $M_\theta(\cdot)$ is given by

$$M_\theta(\lambda)=\begin{cases} (\theta-i)^{-1} & : \quad \lambda\in\mathbb{C}_+ \\ (\theta+i)^{-1} & : \quad \lambda\in\mathbb{C}_- \end{cases}. \tag{5.4}$$

Using the Weyl functions we are going to investigate the spectrum of the self-adjoint extension A_0 and A^θ_0, $\theta\in\mathbb{R}$. Since the Hilbert space $\mathcal{H}=\mathbb{C}$ is one dimensional we choose $\mathcal{T}=\{h\}$ with $h=1$. Doing so we find $\Omega_s(M;\mathcal{T})=\Omega_s(M_\theta;\mathcal{T})=\emptyset$ and $\Omega_{ac}(M;\mathcal{T})=\Omega_{ac}(M_\theta;\mathcal{T})=\mathbb{R}$. Applying Theorem 3.8 we immediately find $\sigma_s(A_0)=\sigma_s(A^\theta_0)=\emptyset$, $\theta\in\mathbb{R}$. Further, taking Theorem 4.2 into account we get $\sigma_{ac}(A_0)=\sigma_{ac}(A^\theta_0)=\mathbb{R}$, $\theta\in\mathbb{R}$. Hence the operators A_0 and A^θ_0, $\theta\in\mathbb{R}$, are purely absolutely continuous.

EXAMPLE 5.2 Let A be a symmetric Dirac operator on $\mathbb{R}$ defined on $\mathrm{dom}(A) = \left\{\vec{f} = (f_1,\ f_2)^t \in W_2^1(\mathbb{R}, \mathbb{C}^2) : \vec{f}(0) = 0\right\}$ by

$$(A\vec{f})(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx}\vec{f}(x) + \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \vec{f}, \quad \vec{f} \in \mathrm{dom}(A), \tag{5.5}$$

where $a > 0$ and $(\cdot)^t$ means that one has to take the transposed expression. It is easily seen (and well-known) that $n_\pm(A) = 2$ and $\mathrm{dom}(A^*) = W_2^1(\mathbb{R}_-, \mathbb{C}^2) \oplus W_2^1(\mathbb{R}_+, \mathbb{C}^2)$. Setting

$$\Gamma_0\vec{f} = (f_2(-0),\ f_1(+0))^t \quad \text{and} \quad \Gamma_1\vec{f} = (f_1(-0),\ f_2(+0))^t \tag{5.6}$$

for $\vec{f} \in \mathrm{dom}(A^*)$ one has that $\Pi = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ performs a boundary triple for A^*. The self-adjoint extension $A_0 := A^* | \ker(\Gamma_0)$ has the domain

$$\mathrm{dom}(A_0) = \{\vec{f} \in W_2^1(\mathbb{R}_-) \oplus W_2^1(\mathbb{R}_+) : f_2(-0) = 0, \quad f_1(+0) = 0\}. \tag{5.7}$$

By $\sqrt{z}$, $z \in \mathbb{C}$, we denote the branch of the complex square root with cut along the non-negative real axis. We set $k(\lambda) := \sqrt{\lambda - a}\sqrt{\lambda + a}$, $\lambda \in \mathbb{C}$. Obviously, $k(\cdot)$ is analytic in $\mathbb{C} \setminus [-a, a]$. One has $\Im\mathrm{m}(k(\lambda)) \geq 0$ for $\lambda \in \mathbb{C}_+$ and $\Im\mathrm{m}(k(\lambda)) \leq 0$ for $\lambda \in \mathbb{C}_-$. We find $k(x) = \sqrt{x^2 - a^2}$ for $x \geq a$ and $k(x) = -\sqrt{x^2 - a^2}$ for $x \leq -a$. Solving the equation $A^*\vec{f} = \lambda\vec{f}$ one gets

$$\vec{f}_\lambda^{\mp}(x) := \begin{pmatrix} \pm i \frac{\sqrt{\lambda + a}}{\sqrt{\lambda - a}} e^{\mp ik(\lambda)x} \\ e^{\mp ik(\lambda)x} \end{pmatrix} \chi_\mp(x), \quad \lambda \in \mathbb{C}_+ \tag{5.8}$$

Using the definition of the Weyl function we find

$$M(\lambda) = \mathrm{diag}\left(i\frac{\sqrt{\lambda + a}}{\sqrt{\lambda - a}}\, i\frac{\sqrt{\lambda - a}}{\sqrt{\lambda + a}} \right), \quad \lambda \in \mathbb{C}_+. \tag{5.9}$$

Since $\mathcal{H} = \mathbb{C}^2$ is two-dimensional we choose $\mathcal{T} = \{h_1, h_2\}$ with $h_1 = (1, 0)^t$ and $h_2 = (0, 1)^t$. This leads to the associated scalar Weyl functions $M_{h_1}(\lambda) = i\frac{\sqrt{\lambda+a}}{\sqrt{\lambda-a}}$ and $M_{h_2}(\lambda) = i\frac{\sqrt{\lambda-a}}{\sqrt{\lambda+a}}$, $\lambda \in \mathbb{C}_+$. One immediately gets: $\Omega_s(M_{h_1}) = \{a\}$, $\Omega_s(M_{h_2}) = \{-a\}$, $\Omega_{sc}(M_{h_1}) = \{a\}$, $\Omega_{sc}(M_{h_2}) = \{-a\}$, $\Omega_{pp}(M_{h_1}) = \Omega_{pp}(M_{h_2}) = \emptyset$ and

$$\Omega_{ac}(M_{h_1}) = \Omega_{ac}(M_{h_2}) = \Omega_{ac}(M, \mathcal{T}) = (-\infty, -a) \cup (a, +\infty). \tag{5.10}$$

By Theorem 3.8 we find $\sigma_p(A_0) = \emptyset$. Further, from Proposition 4.2 we get $\sigma_{ac}(A_0) = (-\infty, -a] \cup [a, +\infty)$. Since the sets $\Omega_{sc}(M_{h_1})$ and $\Omega_{sc}(M_{h_2})$ are finite Theorem 4.3(ii) shows that $\sigma_{sc}(A_0) = \emptyset$. So the operator A_0 is absolutely continuous with $\sigma(A_0) = \sigma_{ac}(A_0) = (-\infty, -a] \cup [a, +\infty)$.

EXAMPLE 5.3 Let $T = T^*$ be a bounded nonnegative operator on a Hilbert space $\mathcal{K}$, $\dim \mathcal{K} = \infty$. Let A be a symmetric Dirac type operator defined on $\mathrm{dom}(A) = \left\{\vec{f} = (f_1,\ f_2)^t \in W_2^1(\mathbb{R}, \mathcal{H}) : \vec{f}(0) = 0\right\}$ by

$$(A\vec{f})(x) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \frac{d}{dx}\vec{f}(x) + \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix} \vec{f}(x),\ \vec{f} \in \mathrm{dom}(A), \tag{5.11}$$

where $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ and I stands for the identity operator on $\mathcal{K}$. One has $n_\pm(A) = \infty$ and $\mathrm{dom}(A^*) = W_2^1(\mathbb{R}_-, \mathcal{H}) \oplus W_2^1(\mathbb{R}_+, \mathcal{H})$.

Let us check that the operator A is a simple symmetric operator. Indeed, in the opposite case A admits a decomposition $A = C_1 \oplus C_2$ with self-adjoint $C_1 = C_1^*$ and a simple symmetric operator C_2. By Gelfand-Kostyuchenko Theorem (see[5]) C_1 has a complete system $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ of generalized eigenfunctions (distributions). Hence each $\varphi_{\lambda_0}(\lambda_0 \in \Lambda)$ is a (generalized) eigenfunction of A. It follows from the corresponding fact on scalar ordinary differential equations (see[5]) applied to $(\varphi_{\lambda_0}(x), e)$ with $e \in \mathcal{H}$ that φ_{λ_0} is weakly C^∞ - solution of the homogeneous initial value problem

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \varphi'_{\lambda_0}(x) + \begin{pmatrix} T - \lambda I & 0 \\ 0 & -T - \lambda I \end{pmatrix} \varphi_{\lambda_0}(x) = 0, \quad \varphi_{\lambda_0}(0) = 0.$$

By Cauchy Uniqueness Theorem we get $\varphi_{\lambda_0} = 0$. Thus A is simple.

Next we define Γ_0 and Γ_1 in accordance with (5.6) which leads to the boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^*. Following the line of reasoning of Example 5.2 one gets the Weyl function

$$M(\lambda) = \mathrm{diag}(M_+(\lambda),\ M_-(\lambda)), \quad \lambda \in \mathbb{C}_+, \tag{5.12}$$

with $M_+(\lambda) := i\frac{\sqrt{\lambda+T}}{\sqrt{\lambda-T}}$ and $M_-(\lambda) := i\frac{\sqrt{\lambda-T}}{\sqrt{\lambda+T}}$. Here $\sqrt{\lambda+T}$ and $\sqrt{\lambda-T}$ are defined by

$$\sqrt{\lambda \pm T} = \int_0^{\|T\|} \sqrt{\lambda \pm t}\, dE_T(t). \tag{5.13}$$

and the branch of $\sqrt{z}$ is chosen like in Example 5.2.

If $x \in \rho(T) := \mathbb{R} \setminus \sigma(T)$ and $h = k_+ \oplus 0$, then we easily find that

$$\begin{gathered} M_h(x+i0) = (M_+(x+i0)k_+, k_+) = \\ i\int_0^{|x|} \frac{\sqrt{|x+t|}}{\sqrt{|x-t|}} d(E_T(t)k_+, k_+) + \int_{|x|}^{\|T\|} \frac{\sqrt{|x+t|}}{\sqrt{|x-t|}} d(E_T(t)k_+, k_+) \end{gathered} \tag{5.14}$$

which yields

$$0 \le V_h(x+i0) = \Im\mathrm{m}(M_h(x+i0)) = \int_0^{|x|} \frac{\sqrt{|x+t|}}{\sqrt{|x-t|}} d(E_T(t)k_+, k_+). \tag{5.15}$$

Let $\epsilon := \inf \sigma(T) \ge 0$. If $x \in (-\epsilon, \epsilon)$, then $V_h(x+i0) = 0$. If $x \in \rho(T) \setminus [-\epsilon, \epsilon]$, then there is always a $k_+ \in \mathcal{K}$ such that $V_h(x+i0) > 0$. Choosing $h = 0 \oplus k_-$, $k_- \in \mathcal{K}$, and repeating the considerations we get $V_h(x+i0) = 0$ for $x \in (-\epsilon, \epsilon)$. If $-x \in \rho(T) \setminus [-\epsilon, \epsilon]$, then there is always a $k_- \in \mathcal{K}$ such that $V_h(x+i0) > 0$. Applying Proposition 4.2 we get that $\sigma_{ac}(A_0) = (-\infty, \epsilon] \cup [\epsilon, +\infty)$. In particular, we have $\sigma_{ac}(A_0) = \mathbb{R}$ if $\epsilon = 0$.

Finally, let us show that A_0 is purely absolutely continuous. We have

$$\begin{gathered} V_h(x+iy) = \\ \Im\mathrm{m}(M_h(x+iy)) = \int_0^{\|T\|} \Im\mathrm{m}\left(\frac{\sqrt{x+iy+t}}{\sqrt{x+iy-t}}\right) d(E_T(t)k_+, k_+) \end{gathered} \tag{5.16}$$

where $h = k_+ \oplus 0$. Applying the Hölder inequality one gets from (5.16)

$$|V_h(x+iy)| \le \int_0^{\|T\|} \left|\frac{\sqrt{x+iy+t}}{\sqrt{x+iy-t}}\right| d(E_T(t)k_+,k_+) \tag{5.17}$$
$$\le \|k_+\|^{2(p-1)} \int_0^{\|T\|} \frac{((x+t)^2+y^2)^{p/4}}{((x-t)^2+y^2)^{p/4}}\, d(E_T(t)k_+,k_+),$$

$1 < p < +\infty$. Let $(a,b) \subseteq \mathbb{R}$ be an arbitrary finite interval. One has

$$\int_a^b dx\, |V_h(x+iy)|^p \le \|k_+\|^{2(p-1)} \int_0^{\|T\|} d(E_T(t)k_+,k_+) \int_a^b dx\, \frac{((x+t)^2+y^2)^{p/4}}{((x-t)^2+y^2)^{p/4}}. \tag{5.18}$$

If $0 < y \le 1$, then there is a constant $C_1 > 0$ such that $(x+t)^2 + y^2 \le C_1$ for $t \in [0, \|T\|]$ and $x \in [a,b]$. Hence

$$\int_a^b dx\, |V_h(x+iy)|^p \le C_1^{p/4}\|k_+\|^{2(p-1)} \int_0^{\|T\|} d(E_T(t)k_+,k_+) \int_a^b dx\, \frac{1}{|x-t|^{p/2}}. \tag{5.19}$$

If $1 < p < 2$, then we have $C_p := \sup_{t\in[0,\|T\|]} \int_a^b dx\, |x-t|^{-p/2} < +\infty$, which yields

$$\sup_{0<y<1} \int_a^b dx\, |V_h(x+iy)|^p \le C_1^{p/4} C_p \|k_+\|^{2p} \tag{5.20}$$

We obtain a similar result if we choose $h = 0 \oplus k_-$, $k_- \in \mathcal{K}$. Applying Corollary 4.7 we find that the spectrum of A_0 is purely absolutely continuous within the interval (a,b). Since (a,b) is arbitrary we have that A_0 is purely absolutely continuous, i.e $\sigma_s(A_0) = \emptyset$.

One can easily generalize this example to the case of operators of the form $A_0 = -i\begin{pmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{pmatrix} \otimes \frac{d}{dx} + \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$, $\lambda_1, \lambda_2 \in \mathbb{R}$, considered in $L^2(\mathbb{R}, \mathcal{H})$, $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$.

6 Remarks

(1) The approach above allows only to restore the point spectrum and the absolutely continuous spectrum, see Theorem 3.8 and Proposition 4.2. The singular continuous spectrum can be recovered only under additional assumptions, see Theorem 4.3. It would be nice to obtain the singular continuous spectrum without additional assumptions using only boundary values of the associated scalar Weyl function.

(2) In [8] applying Theorem 4.3 we find a self-adjoint extension $\tilde{A}$ of a symmetric operator A with gap $(a,b) \subseteq \mathbb{R}$ which has a given purely singular continuous spectrum within the gap (a,b). The singular continuous spectrum can be arbitrary, in particular, it can be a Cantor set of Lebesgue measure zero. This generalizes results of [2, 6, 7, 9, 10, 11] on the spectrum of self-adjoint extensions of symmetric operators with gap. The approach used in these papers is quite different from the Weyl function approach used above. Nevertheless, from the abstract point of view it is clear that both approaches are related. Since the Weyl function of the self-adjoint extension A^Θ is given by $(\Theta - M(\cdot))^{-1}$ this

shows that the problem to find self-adjoint extensions with given spectrum can be embedded into a "perturbation problem for R-functions". So the theorems of [2, 6, 7, 9, 10, 11] can be regarded as "perturbation theorems for R-functions". However, it would be nice to prove these "perturbation theorems for R-functions" more directly in order to see what is going on.

(3) Taking Lemma 4.5 into account one proves the following result.

THEOREM 6.1 *Let $\tilde{A}$ be a self-adjoint operator in $\mathfrak{H}$ and Φ be a nonnegative continuous function on $\mathbb{R}_+$, satisfying* $\lim_{x\to\infty}\Psi(x)=\infty$. *If for some $h\in\mathfrak{H}$ the condition (4.16) holds with $V_h(x+iy):=\big((\tilde{A}-x-iy)^{-1}h,h\big)$, then $E_{\tilde{A}}((a,b))\{(\tilde{A}-z)^{-1}h : z\in \mathbb{C}_+\cup\mathbb{C}_-\}\subseteq\mathfrak{H}^{ac}(\tilde{A})$.*

Theorem 6.1 with $\Phi(x)=x^{p-1}$, $1<p<+\infty$ is well-known, see [27]. Thus one may consider Theorem 4.6 as a generalization of the known results to R- or Weyl functions.

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