On the Fractal Dimension of the Visible Universe

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Estimates of the fractal dimension D of the set of galaxies in the universe, based on ever improving data sets, tend to settle on $D \approx 2$. This result raised a raging debate due to its glaring contradiction with astrophysical models that expect a homogeneous universe. A recent mathematical result indicates that there is no contradiction, since measurements of the dimension of the *visible* subset of galaxies is bounded from above by $D = 2$ even if the true dimension is anything between $D = 2$ and $D = 3$. We demonstrate this result in the context of a simple fractal model, and explain how to proceed in order to find a better estimate of the true dimension of the set of galaxies.

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The value of the (fractal) dimension D of the galaxy distribution in the universe is an important open question in cosmology. Steadily improving observations are available, giving scientists hope that enough data will allow finally to decide the highly debated issue of whether D is 3 or substantially lower (usually stated to be about 2). For example, in the recent book [1] on the "Discovery of Cosmic Fractals" it is emphatically declared that "The megafractals—the cosmic continents, archipelagos and islands—were the news brought home by the modern explorers of the cosmos, exotic, but truths nevertheless about the worlds overseas. Even if the fractal dimension and the maximum scale are still debated, megafractals cry for explanation. Their origin is the number one challenge for cosmological physics." What these authors refer to are mainly results of fractal analysis of the data sets of galaxies which indicate the fractal dimension D of the set of galaxies is about 2 [2–4]. In this Letter we build on a recent theorem of fractal mathematics [5] which indicates that these results may not be "truths nevertheless", but rather a reflection of an inherent impossibility to measure a dimension larger than 2. The true dimension may be anything between 2 and 3, and the upper number is not excluded, in agreement with standard astrophysical theories of a homogeneous universe [7]. In this way, "the number one challenge" may have been resolved in an unexpected and somewhat disappointing way, namely: when $D > 2$ *one cannot measure* D by observing the visible galaxies. In addition to explaining this result, we present a partial remedy by exploring certain aspects of the data analysis that may indicate the existence of a dimension larger than 2.

In essence the fractal analysis of any given atlas of galaxies is a simple matter, once one takes carefully into account the side issues described in [3] which deal with questions like limited angles, faint luminosities, and other observational issues. After worrying about all these details one ends up with a set of points, or coordinates, each of which stands for a galaxy, with redshift data used to determine its distance from us (the observers). Given such a set of points $\{\boldsymbol{X}_i\}_{i=1}^N$ in $\mathcal{R}^3$, we define the correlation integral $C(r)$ as the number of pairs of points of this set whose distance is smaller or equal to r,

$$C(r) = \frac{2}{N(N-1)} \sum_{i<j} \theta(r - |\boldsymbol{X}_i - \boldsymbol{X}_j|) \ , \qquad (1)$$

where $\theta(y)$ is the step function, unity for $y > 0$ and zero otherwise. For a fractal set of dimension D plotting $\log(C(r))$ versus $\log r$ results in a curve whose slope is the correlation dimension D_2 of the Grassberger-Procaccia algorithm [8] (see also [9, 10]). In general $D_2 \leq D$; for sets whose clustering is not singular one can expect that $D_2 = D$ [9]. For reasons related to angular restrictions and the like, in [2–4] the authors consider a quantity $\Gamma^*(r)$, which for a general fractal coincides with $C(r)/r^3$. Thus their plots should have slopes $D_2 - 3$. They find consistently $D_2 \approx 2$.

The question is then whether this is really an indication that the set of all galaxies is of dimension $D \approx 2$. We argue first that this may not be the case. In a recent paper [5] the following theorem was established: let F be a fractal set in R^3 with dimension $D > 2$. The *visible part* of the set F from a point P is the subset F_V of those points lit by a spotlight at P. *Then the part F_V that is visible to an observer can in general not have a dimension more than 2* [6].

We stress that this result is not about the projection of the set onto the celestial sphere, but about those observations in which the *distance* of each point (galaxy) is given along with its celestial coordinates (such data sets are called 3-dimensional catalogs). The meaning of the theorem is that it is in fact impossible to determine the dimension of the set of galaxies from measurements of the visible subset if the dimension of the full set is larger than 2. The basic reason for this impossibility is that galaxies "hide" behind each other when the dimension is above 2. This issue will not go away with improving the catalogs. Rather, it will become more and more important as better and better catalogs become available.

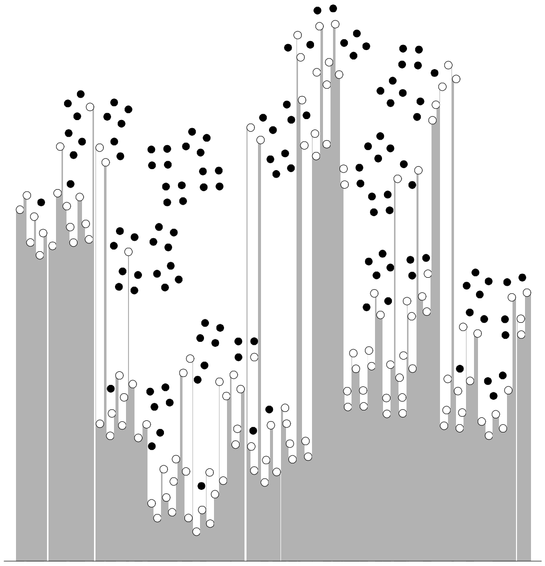

FIG. 1: A simple illustrative model for a fractal universe, drawn by a hierarchical construction at level 4, with the visible part in white, the invisible points in black. A disk is deemed visible as soon as any part of it is visible. Since the construction involves division by 4 and scale changes by $\lambda = 0.4$ the dimension is $D = -\log 4/\log\lambda \approx 1.51$.

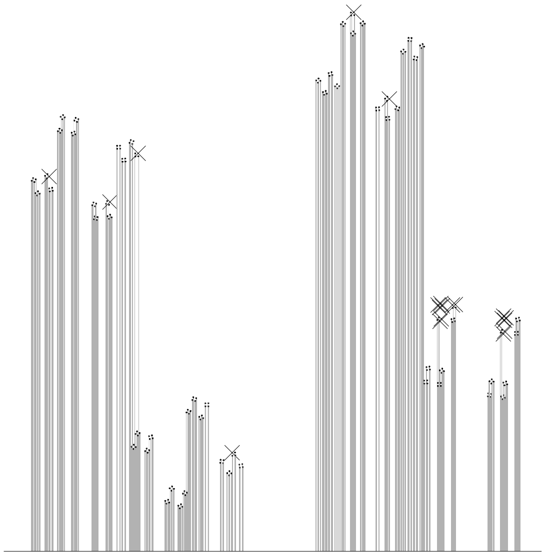

FIG. 2: A second model, as in Fig.1, but now with dimension $D = 0.85$. Although $D < 1$, there are still hidden disks, and to make them appear more clearly, we marked them with an $\times$.

We now illustrate some aspects of this problem, and in particular show that there might be some lower bound on the true dimension when taking into account finite size effects (which are absent in mathematical treatments of fractals, but are an evident necessity of any real-life experiment).

We first note that the catalogs provide measurement of the positions of galaxies away from us. In other words, we should consider a relatively small sphere around P and look with radial rays issuing from the sphere. In [5] it is shown that looking from a plane defines an equivalent problem, and we prefer that formulation. To further simplify the discussion in our examples we will consider a fractal embedded in 2 rather than 3 dimensions, illuminated by rays perpendicular to a randomly given baseline. In Fig. 1 we present a simple model of a fractal universe which is constructed hierarchically. At the nth level of the construction we see 4^n balls of size λ^n which are supposed to contain galaxies. At the $(n+1)$th level each ball is further subdivided to 4 balls of size λ^{n+1}. To avoid non-generic effects we rotate the new group of balls with a random angle at each step of construction. Fig. 1 shows the set of balls at the 4th level with $\lambda = 0.4$. The fractal dimension of this example is $D = -\log(4)/\log(\lambda) = 1.5129$. The figure represents the visible set F_V (from a random line) as the lighted balls, whereas the invisible set (the complement of F_V) is shown as black balls. One can understand the theorem of [5] in the following intuitive sense. The projection (the footprint of the gray zone of Fig. 1) of the fractal on the line has dimension 1 when $D > 1$ [11, 12]. On the other hand, for the hierarchical construction up to level k, the balls have size λ^k, and thus the projection of the visible part has dimension 1 as soon as there are at least $1/\lambda^k$ visible balls (assuming they hide all others). The boundary between the gray zone and the white zone in Fig.1 forms a graph of a function connecting the visible balls. In the kth level we call this function $f_k(x)$. Since the fractal set is constructed hierarchically, we expect a scaling relation $f_{k+1}(x) = \lambda^{-1} f_k(\lambda^{-1}x)$. This scaling relation guarantees that the graph cannot become too rough and will remain of dimension 1. To see this clearly think about the Weierstrass function $g(x) = \sum_{k=0}^{\infty} a^k \sin(b^k x)$. It is well known that the graph of this function is rough when $ab > 1$ and $a < 1$. Indeed, (cf. [13]) the Weierstrass function almost scales in the sense that

$$g(x) = a^{-1} g(bx) - a^{-1} \sin x \ . \tag{2}$$

The last term is smooth and its contribution to the dimension is negligible. Covering the graph with balls leads to the well known result $D = 2 - |\log a|/\log b$. But, in our analogous case, $a = \lambda$, $b = 1/\lambda$, leading to a 1-dimensional graph. Loosely speaking the stretching is not very strong in the y-direction, and the dimension of the graph (and hence of the visible set) remains 1. In fact, the same argument explains why in dimension $D < 1$ the visible part of a fractal has the same dimension as the fractal itself.

To illustrate these issues we consider first a fractal of dimension smaller than 2 (cf. Fig. 2). Here the visible and full sets will have the same dimension, as is demonstrated in Fig. 3, where we plot $\log C(r)$ vs. $\log r$ for both F and

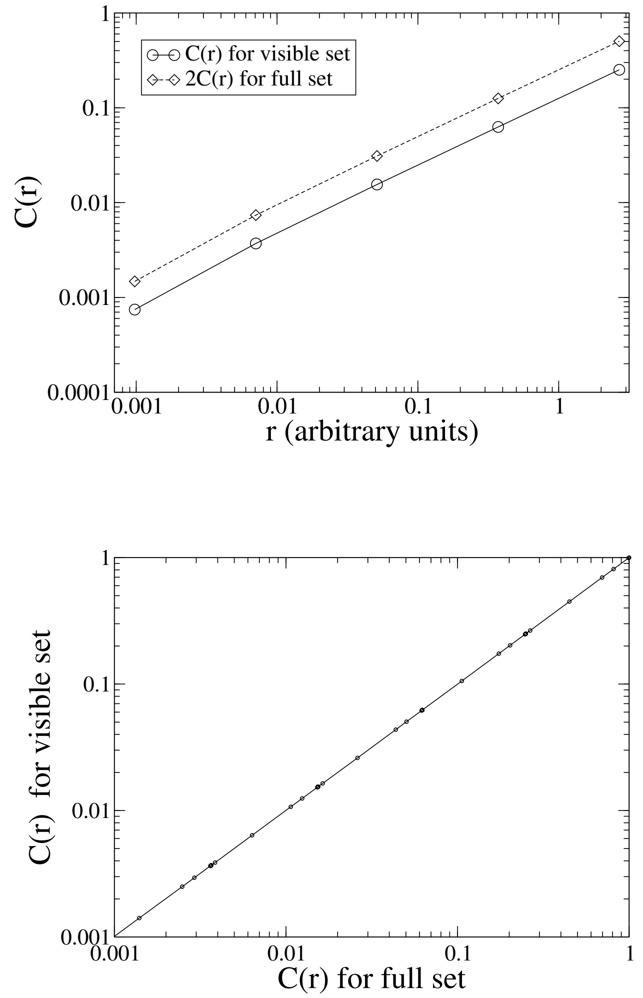

FIG. 3: Upper panel: The graphs of $C(r)$ for the visible part and the full fractal of dimension $D = 0.7$, at level 6. The top curve is the binned number of pairs of points whose pairwise distance falls in the bin (in equal bins on the logarithmic scale) for the full set (multiplied by 2 to shift the curve up). The lower curve is the same for the visible part. The least square fits for the measured dimensions are $D = 0.7095 \pm 0.0036$ and $D = 0.71023 \pm 0.0038$. Lower panel: $C(r)$ for the full fractal versus $C(r)$ for the visible part at level 6. A least square fit gives a slope of 1.0002. Note that this does *not at all* mean that all disks are visible!

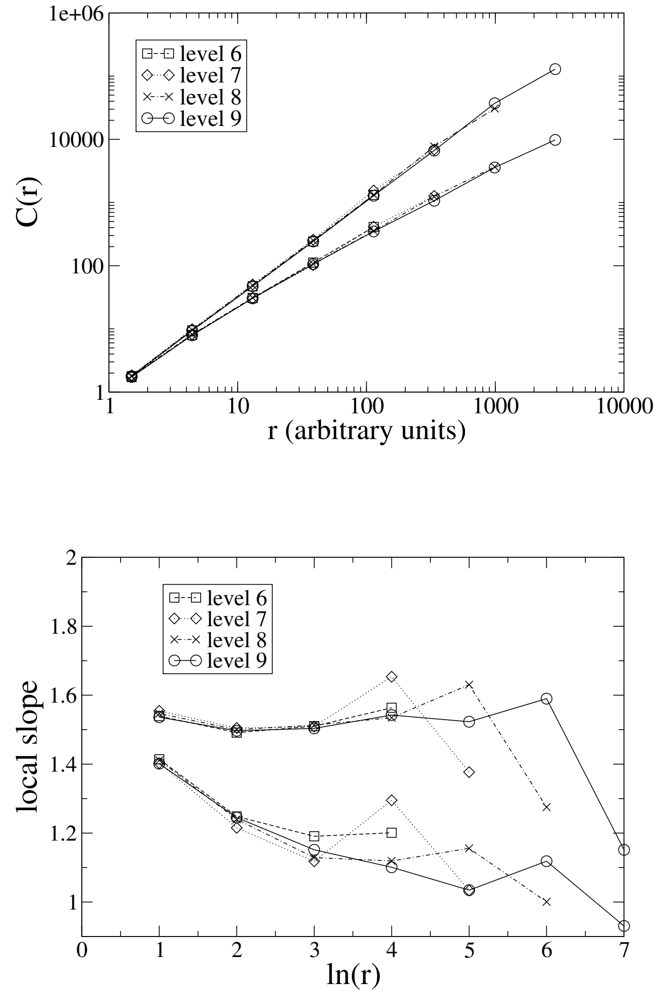

FIG. 4: Upper panel: Double logarithmic plot of $C(r)$ vs. r for the visible and full sets of dimension $D = 1.5$. Shown are measurements for 4 consecutive levels of hierarchical construction (sets with 4^6 to 4^9 points). The data are normalized to collapse at the lowest available scale. Lower panel: The pointwise slopes (dimensions) of the curves in the upper panel. Note that the full set is seen to have dimension $D = 1.5$, while the visible part tends asymptotically to dimension 1 as the length scale increases.

F_V (upper panel). Evidently the slope is the same for these sets. To demonstrate this fact further we present in the lower panel of Fig. 3 a plot of $\log C(r)$ for F vs. $\log C(r)$ for F_V. The slope of this line is unity, stressing the fact that the bulk of F is revealed in the visible subset F_V.

The results change qualitatively when the dimension of the set is higher than 2. In the upper panel of Fig. 4 we present the double logarithmic plot of the correlation integral vs distance for F and F_V of the set of dimension 1.5 of Fig. 1. Clearly they do not scale in the same way, with the visible set settling on dimension 1 when larger and larger r are taken into account. This is underlined again by the results shown in the lower panel where the pointwise slopes of the curves in the upper panel are shown. Obviously the correlation integral for F settles nicely on dimension $D \approx 1.5$, whereas the local slope of the correlation integral for F_V tends to $D \approx 1$ as r increases. We stress that subdividing the set further in the hierarchic construction will not cure the problem. Quite on the the contrary, it will make the visible set F_V a relatively smaller subset of the full set F. Unfortunately going deeper in the hierarchic construction is analogous to studying larger and deeper catalogs, so we cannot expect that newer and better data on the galaxy distributions may automatically cure the problem. We thus conclude that the results of the fractal analysis presented so far do not exclude a homogeneous universe with the fractal dimension of the full set of galaxies being as high as 3.

Lastly, we should investigate whether all is lost, or whether there is a way to probe the true dimension of the full set F from the knowledge of the visible set F_V. A modest way out is offered by the observation that the slopes of the curves in the upper panel of Fig. 4 *are very close* at *small* distances. This observation is underlined by the pointwise slopes of the curve in Fig. 4 at small distances. This is clearly a finite size effect which can be understood by looking again at Fig. 1. Due to the finite size of the smallest balls at this level of construction, many of the visible balls appear in groups of 4. This is due to the balls that were visible for the previous level of construction (4th in this case), mainly near the edge, but not only, which remain visible also after one step of refinement. With less degree of conviction one can also observe groups of 16 balls, or almost 16 balls, that are visible mainly near the visible edge. This finite size phenomenon will go away at the present small length scales when we subdivide many steps further, but will remain observable at the smallest available scales forever. This observation rationalizes why we get the "correct" dimension of the full set F from the smaller scales of the correlation integral.

These observations indicate that despite the mathematical impossibility of measuring the true dimension, its value may be gleaned from the behavior of the correlation integral at small scales. We stress that this possibility is not only due to close points (or galaxies) near the visible edge—also points that are far away from the observation line (or point P) contribute. Balls that are lighted at level n have high probability to give rise to a full set of lighted balls also in the next level, but not so for many subdivisions. Thus lighted balls will count the "right" dimension only with regard to small pairwise distances close to where they are. Once we try to count larger pairwise distances we unavoidable run to the problems explained above. Indeed, interestingly enough, it appears that the data analysis presented in [2–4] indicates a slight *increase* in the apparent dimension for smaller scales. We suggest that this increase may very well point to the true answer, namely, that the dimension of the set of galaxies is considerably larger than 2, and maybe even 3-dimensional in agreement with the expectations expressed in [7].

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[1] Y. Baryshev and P. Teerikorpi, *Discovery of Cosmic Fractals* (World Scientific, 2002).
[2] P. Coleman and L. Pietronero: "The Fractal Structure of the Universe", Phys. Rep. **213**, 311 (1992).
[3] F. Sylos Labini, M. Montuori and L. Pietronero: "Scale Invariance of Galaxy Clustering" Phys. Rep. **293**, 61 (1998)
[4] L. Pietronero, M. Bottaccio, M. Montuori, F. Sylos Labini, "Scaling in cosmic structures", Fractals, in press.
[5] E. Järvenpää, M. Järvenpää, P. MacManus and T.C. O'Neil, "Visible Parts and Dimensions" preprint 258, University of Jyväskylä (2002).
[6] In fact, there are really two possibilities of which the second is even more dramatic for any measurement than the first: Either the part F_V that is visible to an observer at P can in general not have a dimension greater than 2 or it may happen that the dimension of F_V depends in an essential way on the observation point P. All the known examples belong to the first class.
[7] J.P.E. Peebles: "The Standard Cosmological model" In *les Rencontres de Physique de la Vallée d'Aosta* (1998) ed. M. Greco
[8] P. Grassberger and I. Procaccia, Phys. Rev. Lett. **50**, 346 (1983).
[9] H.G.E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).
[10] J.-P. Eckmann and D. Ruelle, Rev. Mod. Phys. 57, 617 (1985).
[11] P. Mattila, Ann. Acad. Sci. Fenn. Math. A **1**, 227 (1975).
[12] K.J. Falconer, *The Geometry of Fractal Sets* (Cambridge 1985) Theorem 6.9b, p. 83.
[13] B. Hunt, Proc. Amer. Math. Soc. **126**, 791–800 (1998).