# REGULARITY OF THE DENSITY OF SURFACE STATES

VADIM KOSTRYKIN AND ROBERT SCHRADER*

ABSTRACT. We prove that the integrated density of surface states of continuous or discrete Anderson-type random Schrödinger operators is a measurable locally integrable function rather than a signed measure or a distribution. This generalize our recent results on the existence of the integrated density of surface states in the continuous case and those of A. Chahrour in the discrete case. The proof uses the new $L^p$-bound on the spectral shift function recently obtained by Combes, Hislop, and Nakamura. Also we provide a simple proof of their result on the Hölder continuity of the integrated density of bulk states.

## 1. INTRODUCTION AND MAIN RESULTS

Recently Combes, Hislop, and Nakamura [8] proved a remarkable inequality for the $L^p$-norm of the spectral shift function. A generalization of this inequality was then found by Hundertmark and Simon [13]. As an application of this inequality Combes, Hislop, and Nakamura prove Hölder continuity of the integrated density of states for a wide class of random Schrödinger operators. In this article, using the $L^p$-bound, we will prove that the integrated density of surface states of continuous or discrete Schrödinger operators is a measurable, locally integrable function thus extending results of [9], [5], and [16]. Also we will provide a simple proof of the Hölder continuity of the integrated density of (bulk) states for some Anderson type models. It is based on the combination of the $L^p$-bound with the Birman-Solomyak formula for the spectral shift function [2]. Although the proof of the Hölder continuity has already been given in [8] and is based on the same ideas, our proof is simpler for the model we consider.

So we first consider random Schrödinger operators of the form $H_\omega = H_0 + V_\omega$ on $L^2(\mathbb{R}^\nu)$, $\nu \geq 2$ with $H_0 = -\Delta$ and $V_\omega$ being the random potential of Anderson type centered near a hypersurface in $\mathbb{R}^\nu$. More precisely we consider a decomposition $\mathbb{Z}^\nu = \mathbb{Z}^{\nu_1} \oplus \mathbb{Z}^{\nu_2}$ with $\nu_1 + \nu_2 = \nu$, $\nu_2 \geq 1$ and introduce random potentials of the form

$$V_\omega(x) = \sum_{\mathbf{j} \in \mathbb{Z}^{\nu_1}} \alpha_{\mathbf{j}}(\omega) f(x - \mathbf{j}), \tag{1}$$

where $\alpha_j(\omega)$ is a sequence of random i.i.d. variables on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ with common distribution $\kappa$, i.e. $\mathfrak{F}$ is a $\sigma$-algebra on $\Omega$, $\mathbb{P}$ a probability measure on $(\Omega, \mathfrak{F})$, and $\kappa(B) = \mathbb{P}\{\alpha_{\mathbf{j}} \in B\}$ for any Borel subset $B$ of $\mathbb{R}$. Let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$. The random variables $\{\alpha_{\mathbf{j}}(\omega)\}_{\mathbf{j} \in \mathbb{Z}^{\nu_1}}$ are supposed to form a stationary, metrically transitive random field, i.e. there are measure preserving ergodic transformations $\{T_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{Z}^{\nu_1}}$ such that $\alpha_{\mathbf{j}}(T_{\mathbf{k}}\omega) = \alpha_{\mathbf{j}-\mathbf{k}}(\omega)$ for all $\omega \in \Omega$. The single-site potential $f$ is supposed to be supported in the unit cube $\Delta_0$ centered at the origin, $\operatorname{supp} f \subseteq \Delta_0 = [-1/2, 1/2]^\nu$ and $f \in L^2(\mathbb{R}^\nu)$. Additionally if $\nu \geq 4$ the potential $f$ is supposed to belong to $L^r(\mathbb{R}^\nu)$ with some $r > \nu/2$. Throughout this article the constant $r$ will be assumed to be fixed. Instead of the integer lattice in (1) we can alternatively consider an arbitrary lattice as discussed in [16].

Finally we assume that $f$ is sign-definite, i.e. either $f > 0$ or $f < 0$ on sets of positive Lebesgue measure. Below and without loss of generality further we will consider the case $f \geq 0$ only since

---

*Date*: November 10, 2000.

1991 *Mathematics Subject Classification.* (2000 Revision) Primary 82B44; Secondary 60H25.

*Key words and phrases.* Random Schrödinger operators, surface states, spectral shift function, density of states.

* R.S. supported in part by DFG SFB 288 “Differentialgeometrie und Quantenphysik”.

the case $f \leq 0$ is completely similar. Also supp $\kappa$ is supposed to be bounded, i.e. there is $\alpha_- > -\infty$ and $\alpha_+ < \infty$ such that $\alpha_- \leq \alpha_{\mathbf{j}}(\omega) \leq \alpha_+$ for all $\mathbf{j} \in \mathbb{Z}^{\nu_1}$ and all $\omega \in \Omega$. Under these conditions the operator $H_\omega = H_0 + V_\omega$ defined in the form sense is self-adjoint on $\mathfrak{Q}(H_0)$ for all $\omega \in \Omega$. The assumptions on $V_\omega$ can be relaxed by requiring that the expectations of certain quantities are finite. The corresponding modifications are obvious and we will not dwell on them.

Let $A$ and $C$ be bounded self-adjoint operators and let $C$ be trace class. The spectral shift function $\xi(\cdot; A + C, A) \in L^1(\mathbb{R})$ is defined by the trace formula

$$\mathrm{tr}(\phi(A+C) - \phi(A)) = \int_{\mathbb{R}} \phi'(\lambda)\xi(\lambda; A+C, A)d\lambda \tag{2}$$

which is valid for a sufficiently wide class of continuous functions $\phi$ and $\|\xi\|_{L^1(\mathbb{R})} \leq \|C\|_{\mathcal{J}_1}$ where $\|\cdot\|_{\mathcal{J}_1}$ is the trace norm. For relative trace class perturbations the spectral shift function can be defined by means of the invariance principle (see e.g. [4, 25]). In particular if $A$ and $B$ are self-adjoint possibly unbounded but bounded below with common domain of definition and if $(B+a)^{-p} - (A+a)^{-p}$ is trace class for some $a > 0$ and $p > 1$ then

$$\xi(\lambda; B, A) = -\xi((\lambda+a)^{-p}; (B+a)^{-p}, (A+a)^{-p}). \tag{3}$$

It vanishes for all $\lambda < \inf\{\mathrm{spec}(B), \mathrm{spec}(A)\}$. A detailed account on the theory of the spectral shift function can be found in the review [4] and in the book [25]. For recent studies we refer to [14, 10] and references therein. Recently the spectral shift function found a number of applications in the theory of random Schrödinger operators [15], [16], [17], [5], [6], [23], [18], [8].

Let $\Lambda \subset \mathbb{R}^{\nu_1}$ be a rectangular box $[a_1, b_1] \times \ldots \times [a_{\nu_1}, b_{\nu_1}]$. We understand the limit $\Lambda \to \infty$ in the sense that $a_i \to -\infty$ and $b_i \to \infty$ for all $i = 1, \ldots, \nu_1$. For an arbitrary box $\Lambda$ we define

$$V_{\omega,\Lambda}(x) = \sum_{\substack{\mathbf{j} \in \mathbb{Z}^{\nu_1} \\ \mathbf{j} \in \Lambda}} \alpha_{\mathbf{j}}(\omega) f(\cdot - \mathbf{j}). \tag{4}$$

In [16] we proved that for any $g \in C_0^1(\mathbb{R})$ the limit

$$\lim_{\Lambda \to \infty} \frac{1}{\mathrm{meas}_{\nu_1}(\Lambda)} \int_{\mathbb{R}} g(\lambda)\xi(\lambda; H_0 + V_{\omega,\Lambda}, H_0)d\lambda =: \mu(g)$$

exists almost surely and is non-random. The linear functional $\mu(g)$ is related to the density of surface states $\mu_s(g)$ (see [9]) such that $\mu_s(g) = \mu(g')$ (with $g'$ being the derivative of $g$), where

$$\mu_s(g) = \lim_{\substack{\Lambda_1 \to \infty \\ \Lambda_2 \to \infty}} \frac{1}{\mathrm{meas}_{\nu_1}(\Lambda_1)} \mathrm{tr}\left[\chi_{\Lambda_1 \times \Lambda_2}(g(H_0 + V_{\omega,\Lambda_1}) - g(H_0))\right], \quad g \in C_0^2,$$

almost surely for arbitrary sequences of boxes $\Lambda_1 \subset \mathbb{R}^{\nu_1}$, $\Lambda_2 \subset \mathbb{R}^{\nu_2}$ tending to infinity. Englisch, Kirsch, Schröder, and Simon [9] analyzed the surface states occuring at the boundary between two Anderson-type crystals and proved that the distribution induced by the functional $\mu_s(g)$ (i.e. the density of surface states) is of order (at most) 3. This result applies almost verbatim also to interactions of type (1), so we have

$$\mu_s(g) = \int_{\mathbb{R}} g(\lambda) N_s'(\lambda) d\lambda, \qquad g \in C_0^3(\mathbb{R}),$$

where the distribution $N_s(\lambda)$ of order at most 2 is called the integrated density of surface states.

Using a slightly different approach from ours Chahrour [5] constructed the functional $\mu(g)$ for the case of discrete Schrödinger operators and showed that the integrated density of surface states $N_s(\lambda)$ is a distribution of order (at most) 1. In [6] he proved that for discrete Schrödinger operators with *nonrandom* periodic potentials $N_s(\lambda)$ is a measurable function.

Further in [16] we proved that the functional $\mu(g)$ induces a signed Borel measure $d\Xi(\lambda)$ such that for any $g \in C_0(\mathbb{R})$

$$\mu(g) = \int_{\mathbb{R}} g(\lambda) d\Xi(\lambda).$$

This result implies that "$N_s(\lambda)d\lambda$" is a $\sigma$-finite Borel measure.

We will now extend this result and prove

**Theorem 1.** *For continuous Schrödinger operators $H_\omega = H_0 + V_\omega$ with $V_\omega$ being defined by* (1) *the (signed) density of surface states measure $d\Xi(\lambda)$ is Lebesgue absolutely continuous.*

In the other words Theorem 1 states that the integrated density of surface states $N_s(\lambda)$ is a measurable locally integrable function.

However, it remains unclear whether $N_s(\lambda)$ possesses further regularity properties, e.g. whether it is a function of locally bounded variation such that "$dN_s(\lambda)$" defines a measure. In fact, it is difficult to control the smoothness of $N_s(\lambda)$ since it may oscillate rapidly due to the presence of alternating surface states and surface "holes".

The results of our article [16] extend almost verbatim (actually with several simplifications) to the case of discrete Schrödinger operators. More precisely we consider discrete Schrödinger operators with random potentials on a hypersurface,

$$(5)\quad (h_\omega u)(\mathbf{n}) = (h_0 u)(\mathbf{n}) + \widetilde{V}_\omega(\mathbf{n}_1)\delta(\mathbf{n}_2)u(\mathbf{n}), \quad (h_0 u)(\mathbf{n}) = \sum_{|\mathbf{j}|=1} u(\mathbf{n}-\mathbf{j}), \quad \mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2) \in \mathbb{Z}^{\nu_1} \oplus \mathbb{Z}^{\nu_2},$$

where $\delta(\mathbf{n}_2)$ is the Kronecker symbol and $\widetilde{V}_\omega(\mathbf{n}_1)$ a metrically transitive random field on $\mathbb{Z}^{\nu_1}$.

We will prove the following analogue of Theorem 1

**Theorem 2.** *For discrete Schrödinger operators of the form* (5) *the (signed) density of surface states measure $d\Xi(\lambda)$ is Lebesgue absolutely continuous.*

The proofs of Theorems 1 and 2 will be given in Section 2. Section 3 plays a complementary role. Its aim is to give a simple proof of the Combes-Hislop-Nakamura result on the Hölder continuity of the integrated density of (bulk) states for some random Schrödinger operators. The proof is based on the combination of the Combes-Hislop-Nakamura $L^p$-bound with the formula of Birman and Solomyak [2]. This combination is a generalization of Simon's spectral averaging method which was used to prove Lipshitz continuity of the integrated density of bulk states (Wegner's estimate) for some random Jacobi matrices [23].

**Acknowledgements.** We are indebted to J.M. Combes for useful discussions and for sending us the preliminary version of the preprint preprint [8].

## 2. Proofs of Theorems 1 and 2

The two main ingredients of our approach to prove Theorem 1 are the Banach-Alauglu theorem (see e.g. [20]) and the Combes-Hislop-Nakamura $L^p$-bound for the spectral shift function [8] (see also its generalization by Hundertmark and Simon in [13]).

It is generally known that the discrete case is much easier to handle than the continuous case. Indeed in the discrete case (Theorem 2) we actually do not need the $L^p$-bound for the spectral shift function and will use instead a well-known bound for finite rank perturbations. We note also that in the case of finite rank perturbations this bound is implied by the $L^p$-bound.

In the sequel we will use the following well-known lemma, which is a direct consequence of the Banach-Alauglu theorem.

**Lemma 3.** *Let $1 < q \le \infty$. Let the sequence of real valued functions $f_n \in L^q(a,b)$ satisfy*

$$\int_a^b |f_n(\lambda)|^q d\lambda \le C \quad (q<\infty) \qquad \text{or} \qquad \sup_{\lambda\in(a,b)} |f_n(\lambda)| \le C \quad (q=\infty)$$

*uniformly in $n$ for some $C<\infty$. If the sequence $f_n(\lambda)d\lambda$ converges weakly to a signed measure $d\mu(\lambda)$ then this measure is absolutely continuous.*

For reader's convenience we recall the proof. By the Banach-Alauglu theorem we can find a subsequence $f_{n(i)}$ of $f_n$ which converges in the weak-$*$ topology, i.e. there exists $f_\infty \in L^q(a,b) \subset$

$L^1(a,b)$ such that

$$\int f_{n(i)}(\lambda)g(\lambda)d\lambda \to \int f_\infty(\lambda)g(\lambda)d\lambda, \qquad g\in L^{q/(q-1)}(a,b).$$

Thus the measure $d\mu(\lambda) = f_\infty(\lambda)d\lambda$ is absolutely continuous.

We start with the proof of Theorem 2 which is much easier than the one for Theorem 1. For an arbitrary rectangular box $\Lambda\subset\mathbb{R}^{\nu_1}$ with integer-valued vertices we define

$$V_{\omega,\Lambda}(\mathbf{n}) = \begin{cases} \widetilde{V}_\omega(\mathbf{n}_1)\delta(\mathbf{n}_2), & \mathbf{n}_1\in\Lambda,\\ 0, & \text{otherwise.}\end{cases}$$

Adopting the results of our article [16] to the case of discrete Schrödinger operators we have

**Proposition 4.** *For any $g\in C_0^1(\mathbb{R})$ the limit*

$$\lim_{\Lambda\to\infty}\frac{1}{\mathrm{meas}_{\nu_1}(\Lambda)}\int_{\mathbb{R}} g(\lambda)\xi(\lambda;h_0+V_{\omega,\Lambda},h_0)d\lambda$$

*exists and defines a linear functional $\mu(g)$ on $C_0^1(\mathbb{R})$. This functional extends to all $g\in C_0(\mathbb{R})$ with the representation*

$$\mu(g) = \int g(\lambda)d\Xi(\lambda)$$

*and with $\Xi$ being a signed Borel measure.*

The main idea behind the proof of Proposition 4 is to consider the random fields $V_\omega^+(\mathbf{n})$ and $V_\omega^-(\mathbf{n})$ such that $V_\omega^+(\mathbf{n}) = \max\{V_\omega(\mathbf{n}),0\}$ and $V_\omega^-(\mathbf{n}) = \min\{V_\omega(\mathbf{n}),0\}$. It is straightforward to see that $V_\omega^+(\mathbf{n})$ and $V_\omega^-(\mathbf{n})$ are stationary, $\mathbb{Z}^{\nu_1}$-metrically transitive random fields. By the chain rule for the spectral shift function we have

$$\xi(\lambda;h_0+V_{\omega,\Lambda},h_0) = \xi(\lambda;h_0+V_{\omega,\Lambda}^+ + V_{\omega,\Lambda}^-, h_0+V_{\omega,\Lambda}^+) + \xi(\lambda;h_0+V_{\omega,\Lambda}^+,h_0). \tag{6}$$

The first term on the r.h.s. of (6) is non-positive and the second is non-negative. The next step is to prove the almost sure existence of the limits

$$\lim_{\Lambda\to\infty}\frac{1}{\mathrm{meas}_{\nu_1}(\Lambda)}\int_{\mathbb{R}} g(\lambda)\xi(\lambda;h_0+V_{\omega,\Lambda}^+,h_0)d\lambda =:\ \mu^+(g)$$

and

$$\lim_{\Lambda\to\infty}\frac{1}{\mathrm{meas}_{\nu_1}(\Lambda)}\int_{\mathbb{R}} g(\lambda)\xi(\lambda;h_0+V_{\omega,\Lambda}^- + V_{\omega,\Lambda}^+, h_0+V_{\omega,\Lambda}^+)d\lambda =:\ \mu^-(g)$$

for all $g\in C_0^1(\mathbb{R})$. But this follows from arguments used in [16] or [5]. The functionals $\mu^\pm(g)$ are sign-definite. By the Riesz representation theorem they define Borel measures $\Xi^\pm(\cdot)$. Moreover we have $\mu(g)=\mu^+(g)+\mu^-(g)$ and therefore $\Xi(\cdot)=\Xi^+(\cdot)+\Xi^-(\cdot)$, where $\Xi^+(\cdot)$ is a positive and $\Xi^-(\cdot)$ a negative Borel measure.

Now we note that $V_{\omega,\Lambda}$ is a finite rank perturbation,

$$\mathrm{Rank}\, V_{\omega,\Lambda} \le \mathrm{meas}_{\nu_1}(\Lambda).$$

Therefore we have

$$\frac{1}{\mathrm{meas}_{\nu_1}(\Lambda)}|\xi(\lambda;h_0+V_{\omega,\Lambda},h_0)| \le 1.$$

Applying Lemma 3 with $q=\infty$ and using the fact that $C_0^1$-functions are dense in $C_0$ from Proposition 4 we immediately obtain that the measure $d\Xi$ is absolutely continuous. This completes the proof of Theorem 2.

We turn to the proof of Theorem 1. We have the following analogue of Proposition 4 (see [16, Section 5.2]):

**Proposition 5.** *For any $g \in C_0^1(\mathbb{R})$ the limit*

$$\lim_{\Lambda\to\infty} \frac{1}{\mathrm{meas}_{\nu_1}(\Lambda)} \int_{\mathbb{R}} g(\lambda)\xi(\lambda; H_0 + V_{\omega,\Lambda}, H_0)d\lambda$$

*exists and defines a linear functional $\mu(g)$ on $C_0^1(\mathbb{R})$. This functional extends to all $g \in C_0(\mathbb{R})$ and admits the representation*

$$\mu(g) = \int g(\lambda)d\Xi(\lambda)$$

*with $\Xi$ being a signed Borel measure.*

Let $s_j(T)$ denote the singular values of a compact operator $T$. For any $0 < p < \infty$ define the functional $T \mapsto |T|_p$ by

$$|T|_p^p = \sum_j s_j(T)^p.$$

As well known, for $p \geq 1$ this functional defines a norm. The set of compact operators $T$ with finite $|T|_p$ we denote by $\mathfrak{J}_p$. In particular, $\mathfrak{J}_1$ is the space of all trace class operators and $\mathfrak{J}_2$ is the space of all Hilbert-Schmidt operators. If $T_i \in \mathfrak{J}_{p_i}$, $0 < p_i < \infty$, $i = 1,2$ then $T_1T_2 \in \mathfrak{J}_p$ with $p^{-1} = p_1^{-1} + p_2^{-1}$ and

$$|T_1T_2|_p \leq |T_1|_{p_1}|T_2|_{p_2}. \tag{7}$$

The proof of this inequality can be found in [3, Corollary 11.11] (actually there is a misprint there).

The proof of Theorem 1 heavily relies on the following lemma which is due to Combes, Hislop and Nakamura [8]. A generalization of this result can be found in [13].

**Lemma 6.** *Let $A$ be a bounded self-adjoint operator on a separable Hilbert space. Let the trace class operator $C$ be in $\mathfrak{J}_{1/p}$ for some $1 \leq p < \infty$. Then*

$$\|\xi(\cdot; A + C, A)\|_{L^p} \leq |C|_{1/p}^{1/p}.$$

In the case $p = 1$ this bound provides the well-known $L^1$-bound for the spectral shift function, $\|\xi(\cdot; A + C, A)\|_{L^1} \leq |C|_1$. The case $p = \infty$ is relevant in the case of finite rank perturbations, $\|\xi(\cdot; A + C, A)\|_{L^\infty} \leq \mathrm{Rank}\, C$.

To proceed, we recall the definition of the Birman-Solomyak spaces $l^p(L^q)$, $1 \leq p, q \leq \infty$. They are the sets of all measurable functions satisfying $\|\phi\|_{l^p(L^q)} < \infty$ with

$$\|\phi\|_{l^p(L^q)} = \left( \sum_{\mathbf{j}\in\mathbb{Z}^\nu} \left[ \int_{\Delta_0} |\phi(x + \mathbf{j})|^q dx \right]^{p/q} \right)^{1/p}$$

and $\Delta_0$ being the unit cube centered at the origin.

We will write $R_{\omega,\Lambda}(-c) = (H_0 + V_{\omega,\Lambda} + c)^{-1}$ and $R_0(-c) = (H_0 + c)^{-1}$ for the resolvents of the operators $H_0 + V_{\omega,\Lambda}$ and $H_0$ respectively. With the above assumptions on $f$ we prove

**Proposition 7.** *Let $k$ be an integer such that $k > (\nu - 1)/2$ if $\nu \geq 4$ and $k = 1$ if $\nu \leq 3$. Let $c$ be a sufficiently large positive number. Then for any $p > \nu/2(k+1)$ such that $p \leq 4$ if $\nu \leq 3$ and $p < 2r$ if $\nu \geq 4$ the difference $R_{\omega,\Lambda}(-c)^k - R_0(-c)^k$ satisfies the inequality*

$$|R_{\omega,\Lambda}(-c)^k - R_0(-c)^k|_p \leq C\, \mathrm{meas}_{\nu_1}(\Lambda)^{1/p} \tag{8}$$

*with $C$ being a constant independent of $\Lambda$ and $\omega$.*

*Remarks:* 1. The number $p$ can always be chosen to satisfy $p < 1$.

2. A result of this type was already proved by Combes, Hislop, and Nakamura in [8] (Proposition 5.1). The new ingredient in Proposition 7 is the volume dependence in the bound (8).

3. The fact that $R_{\omega,\Lambda}(-c)^k - R_0(-c)^k \in \mathfrak{J}_1$ for $k > (\nu-1)/2$, $\nu \geq 4$, and sufficiently large $c$ follows from Theorem XI.12 of Reed-Simon [21].

For any measurable function $W$ we define $W^{1/2}$ by $W^{1/2} = \mathrm{sign}W \cdot |W|^{1/2}$. For the proof of Proposition 7 and again with the assumptions on $f$ we need the following

**Lemma 8.** *Let $k \geq 1/2$ and $p > \nu/2k$, $p \geq 1$. Moreover, let $p \leq 4$ if $\nu \leq 3$ and $p < 2r$ if $\nu \geq 4$. Then there is a constant $C > 0$ depending on $\kappa$, $f$, $k$, and $\nu$ only such that*

$$|R_0(-c)^k |V_{\omega,\Lambda}|^{1/2}|_p \leq C\ \mathrm{meas}_{\nu_1}(\Lambda)^{1/p} \qquad \text{and} \qquad |V_{\omega,\Lambda}^{1/2} R_0(-c)^k|_p \leq C\ \mathrm{meas}_{\nu_1}(\Lambda)^{1/p}.$$

*Remark:* Note that both inequalities

$$\begin{aligned} \nu/2k < p < 2r \quad &\text{if} \quad \nu \geq 4, \\ \nu/2k < p \leq 4 \quad &\text{if} \quad \nu \leq 3. \end{aligned}$$

can always be satisfied. Indeed, for any $k \geq 1/2$ and $\nu \leq 3$ the inequality $\nu/2k < 4$ holds. Since $2r > \nu$ for $\nu \geq 4$ and any $k \geq 1/2$ we have $\nu/2k < 2r$.

*Proof.* We consider the operator $R_0(-c)^k|V_{\omega,\Lambda}|^{1/2}$. The operator $V_{\omega,\Lambda}^{1/2} R_0(-c)^k$ may be discussed similarly. Define the function

$$g(x) = (x^2 + c)^{-k}, \qquad x \in \mathbb{R}^\nu. \tag{9}$$

It is easy to verify that $g \in L^q(\mathbb{R}^\nu)$ and $g \in l^q(L^2)$ for any $q > \nu/2k$.

Suppose first that $\nu \leq 3$. From the assumption $f \in L^2(\mathbb{R}^\nu)$ and the support property of $f$ it follows that $f \in L^1(\mathbb{R}^\nu) \cap L^2(\mathbb{R}^\nu)$ and thus $f \in L^q(\mathbb{R}^\nu)$ for any $1 \leq q \leq 2$. If $\nu \geq 4$ then from the assumption that $f \in L^2(\mathbb{R}^\nu) \cap L^r(\mathbb{R}^\nu)$ for some $r > \nu/2$ and since $f$ has compact support it follows that $f \in L^q(\mathbb{R}^\nu)$ for any $1 \leq q < r$.

For the case $1 \leq p \leq 2$ we estimate as follows

$$\begin{aligned} \|V_{\omega,\Lambda}^{1/2}\|_{l^p(L^2)}^p &= \Big\| \sum_{\substack{\mathbf{j} \in \mathbb{Z}^{\nu_1} \\ \mathbf{j} \in \Lambda}} |\alpha_{\mathbf{j}}(\omega)|^{1/2} f(\cdot - \mathbf{j})^{1/2} \Big\|_{l^p(L^2)}^p \\ &= \sum_{\substack{\mathbf{j} \in \mathbb{Z}^{\nu_1} \\ \mathbf{j} \in \Lambda}} \left[ \int_{\Delta_0} |\alpha_{\mathbf{j}}(\omega)| f(x) dx \right]^{p/2} \\ &= \sum_{\substack{\mathbf{j} \in \mathbb{Z}^{\nu_1} \\ \mathbf{j} \in \Lambda}} |\alpha_j(\omega)|^{p/2} \|f\|_{L^1}^{p/2} \ \leq\ C_1\ \mathrm{meas}_{\nu_1}(\Lambda) \end{aligned} \tag{10}$$

with $C_1$ being some constant depending on $\kappa$, $f$, $p$, and $\nu$ only. Similarly for the case $2 \leq p < \infty$ we have

$$\begin{aligned} \|V_{\omega,\Lambda}^{1/2}\|_{L^p(\mathbb{R}^\nu)}^p &= \Big\| \sum_{\substack{\mathbf{j} \in \mathbb{Z}^{\nu_1} \\ \mathbf{j} \in \Lambda}} |\alpha_{\mathbf{j}}(\omega)|^{1/2} f(\cdot - \mathbf{j})^{1/2} \Big\|_{L^p(\mathbb{R}^\nu)}^p \\ &= \sum_{\substack{\mathbf{j} \in \mathbb{Z}^{\nu_1} \\ \mathbf{j} \in \Lambda}} \int_{\Delta_0} |\alpha_j(\omega)|^{p/2} f(x)^{p/2} dx \\ &\leq \sum_{\substack{\mathbf{j} \in \mathbb{Z}^{\nu_1} \\ \mathbf{j} \in \Lambda}} |\alpha_j(\omega)|^{p/2} \|f\|_{L^{p/2}(\mathbb{R}^\nu)}^{p/2} \ \leq\ C_2\ \mathrm{meas}_{\nu_1}(\Lambda) \end{aligned} \tag{11}$$

with $C_2$ depending again on $\kappa$, $f$, $p$, and $\nu$ only.

To estimate the norm of $R_0(-c)^k|V_{\omega,\Lambda}|^{1/2}$ we use the Birman-Solomyak inequality [1] (see also [22])

$$|w\, g(-i\nabla)|_q \leq C_q\ \|w\|_{l^q(L^2)}\ \|g\|_{l^q(L^2)}, \qquad 1 \leq q \leq 2$$

and the Seiler-Simon inequality (see [22])

$$|w\, g(-i\nabla)|_q \leq (2\pi)^{-\nu/q}\ \|w\|_{L^q}\ \|g\|_{L^q}, \qquad 2 \leq q < \infty.$$

Setting $w = |V_{\omega,\Lambda}|^{1/2}$ and $g$ given by (9) in these inequalities and then using the estimates (10) and (11) proves the lemma. $\square$

*Proof of Proposition 7.* First we consider the case $\nu \le 3$. By the resolvent equation

$$R_{\omega,\Lambda}(-c) - R_0(-c) = -R_0(-c)|V_{\omega,\Lambda}|^{1/2}(I + V_{\omega,\Lambda}^{1/2} R_0(-c)|V_{\omega,\Lambda}|^{1/2})^{-1} V_{\omega,\Lambda}^{1/2} R_0(-c).$$

Since the operator norm of $(I + V_{\omega,\Lambda}^{1/2} R_0(-c)|V_{\omega,\Lambda}|^{1/2})^{-1}$ is uniformly bounded, we obtain from (7) and Lemma 8

$$|R_{\omega,\Lambda}(-c) - R_0(-c)|_p \le C\ |R_0(-c)|V_{\omega,\Lambda}|^{1/2}|_{2p}\ |V_{\omega,\Lambda}^{1/2} R_0(-c)|_{2p} \le C \operatorname{meas}_{\nu_1}(\Lambda)^{1/p}.$$

We turn to the case $\nu \ge 4$. For any $k \in \mathbb{N}$ we have

$$\begin{aligned} R_{\omega,\Lambda}(-c)^k - R_0(-c)^k &= \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dc^{k-1}} [R_{\omega,\Lambda}(-c) - R_0(-c)] \\ &= \sum_{\substack{l+m+n=k-1 \\ l,m,n \in \mathbb{N}\cup\{0\}}} c_{lnm}\, R_0(-c)^{l+1} |V_{\omega,\Lambda}|^{1/2}\, K_m\, V_{\omega,\Lambda}^{1/2} R_0(-c)^{n+1}, \end{aligned}$$

with some coefficients $c_{lnm}$. The operators $K_m$ are given by

$$K_m = \frac{d^m}{dc^m} (I + V_{\omega,\Lambda}^{1/2} R_0(-c)|V_{\omega,\Lambda}|^{1/2})^{-1}, \quad m \in \mathbb{N} \cup \{0\}. \tag{12}$$

Applying Lemma 8 we obtain

$$\begin{aligned} R_0(-c)^{l+1}|V_{\omega,\Lambda}|^{1/2} &\in \mathfrak{J}_{p_1} \qquad \text{for} \qquad p_1 > \frac{\nu}{2(l+1)}, \\ V_{\omega,\Lambda}^{1/2} R_0(-c)^{n+1} &\in \mathfrak{J}_{p_2} \qquad \text{for} \qquad p_2 > \frac{\nu}{2(n+1)} \end{aligned}$$

with

$$\begin{aligned} |R_0(-c)^{l+1}|V_{\omega,\Lambda}|^{1/2}|_{p_1} &\le C \operatorname{meas}_{\nu_1}(\Lambda)^{1/p_1}, \\ |V_{\omega,\Lambda}^{1/2} R_0(-c)^{n+1}|_{p_2} &\le C \operatorname{meas}_{\nu_1}(\Lambda)^{1/p_2}. \end{aligned}$$

We turn to the discussion of the operators $K_m$. Obviously we have

$$\begin{aligned} &\frac{d}{dc} (I + V_{\omega,\Lambda}^{1/2} R_0(-c)|V_{\omega,\Lambda}|^{1/2})^{-1} \\ &= (I + V_{\omega,\Lambda}^{1/2} R_0(-c)|V_{\omega,\Lambda}|^{1/2})^{-1}\ V_{\omega,\Lambda}^{1/2} R_0(-c)^2 |V_{\omega,\Lambda}|^{1/2}\ (I + V_{\omega,\Lambda}^{1/2} R_0(-c)|V_{\omega,\Lambda}|^{1/2})^{-1}. \end{aligned} \tag{13}$$

Let $m, i \in \mathbb{N}$, $i \le m$ be given. By $M_{m,i}$ we denote the set of all multiindices $\underline{m} = (m_1, \ldots, m_i)$ with $m_1, \ldots, m_i \in \mathbb{N}$ satisfying the following conditions

$$\begin{aligned} m_1,\ \ldots,\ m_i\ &\ge 2, \\ m_1\ +\ \ldots\ +\ m_i &= m + i. \end{aligned}$$

Applying the formula (13) to (12) recursively we obtain that for any $m \ge 1$ the operator $K_m$ can be represented in the form

$$\begin{aligned} K_m = \sum_{i=1}^{m} \sum_{\underline{m} \in M_{mi}} c_{mi\underline{m}}\ &(I + V_{\omega,\Lambda}^{1/2} R_0(-c)|V_{\omega,\Lambda}|^{1/2})^{-1} \\ &\cdot \prod_{j=1}^{i} V_{\omega,\Lambda}^{1/2}\ R_0(-c)^{m_j}\ |V_{\omega,\Lambda}|^{1/2}\ (I + V_{\omega,\Lambda}^{1/2} R_0(-c)|V_{\omega,\Lambda}|^{1/2})^{-1} \end{aligned} \tag{14}$$

with $c_{mi\underline{m}}$ being some real numbers.

We prove now that for any $m \ge 1$

$$K_m \in \mathfrak{J}_{p_3}, \quad |K_m|_{p_3} \le C \operatorname{meas}_{\nu_1}(\Lambda)^{1/p_3} \quad \text{with} \quad p_3 > \frac{\nu}{2(m+1)}. \tag{15}$$

From Lemma 8 and using the inequality (7) we obtain

$$V_{\omega,\Lambda}^{1/2} R_0(-c)^{m_j} |V_{\omega,\Lambda}|^{1/2} \in \mathcal{J}_{q_j} \quad \text{with} \quad q_j > \frac{\nu}{2m_j}$$

and $q_j < 4$ for $\nu \leq 3$ and $q_j < 2r$ for $\nu \geq 4$. Moreover the inequality

$$\left| V_{\omega,\Lambda}^{1/2}\, R_0(-c)^{m_j}\, |V_{\omega,\Lambda}|^{1/2} \right|_{q_j} \leq C \operatorname{meas}_{\nu_1}(\Lambda)^{1/m_j}$$

holds. Thus

$$\prod_{j=1}^{i} V_{\omega,\Lambda}^{1/2}\, R_0(-c)^{m_j}\, |V_{\omega,\Lambda}|^{1/2} (I + V_{\omega,\Lambda}^{1/2} R_0(-c) |V_{\omega,\Lambda}|^{1/2})^{-1} \in \mathcal{J}_{\widetilde{q}_i}$$

with

$$\widetilde{q}_i > \frac{\nu}{2\sum_{j=1}^{i} m_j} = \frac{\nu}{2(m+i)}$$

and such that $\widetilde{q}_i < 4$ for $\nu \leq 3$ and $\widetilde{q}_i < 2r$ for $\nu \geq 4$. Moreover the estimate

$$\left| \prod_{j=1}^{i} V_{\omega,\Lambda}^{1/2}\, R_0(-c)^{m_j}\, |V_{\omega,\Lambda}|^{1/2}\, (I + V_{\omega,\Lambda}^{1/2} R_0(-c) |V_{\omega,\Lambda}|^{1/2})^{-1} \right|_{\widetilde{q}_i} \leq C \operatorname{meas}_{\nu_1}(\Lambda)^{1/\widetilde{q}_i}$$

holds. In the equation (14) the worst case occurs for $i = 1$. Thus the estimate (15) is proved.

From Lemma 7 it now follows that

$$R_{\omega,\Lambda}(-c)^k - R_0(-c)^k \in \mathcal{J}_p$$

with $p$ satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < \frac{2(l+1) + 2(n+1) + 2m + 2}{\nu} = \frac{2(k+2)}{\nu}$$

and the estimate (8) holds. Since the inequality $p > \nu/(2(k+2))$ is satisfied with any $p > \nu/(2(k+1))$ this completes the proof. $\square$

Now we are in the position to complete the proof of Theorem 1. We choose some $k > (\nu - 1)/2$ if $\nu \geq 4$ and set $k = 1$ if $\nu \leq 3$. Fix some $p$ satisfying $1 > p > \nu/2(k+1)$. Consider an arbitrary interval $(a, b)$ of the real line. Using the invariance principle for the spectral shift function (3) we estimate

$$\begin{aligned}
&\int_a^b \left| \frac{\xi(\lambda; H_0 + V_{\omega,\Lambda}, H_0)}{\operatorname{meas}_{\nu_1}(\Lambda)} \right|^{1/p} d\lambda \\
\leq\ & (\operatorname{meas}_{\nu_1}(\Lambda))^{-1/p} \int_a^b \left| \xi((\lambda + c)^{-k}; R_{\omega,\Lambda}(-c)^k, R_0(-c)^k) \right|^{1/p} d\lambda \\
=\ & (\operatorname{meas}_{\nu_1}(\Lambda))^{-1/p}\, k^{-1} \int_{(b+c)^{-k}}^{(a+c)^{-k}} \left| \xi(t; R_{\omega,\Lambda}(-c)^k, R_0(-c)^k) \right|^{1/p} t^{-1/k-1} dt \\
\leq\ & (\operatorname{meas}_{\nu_1}(\Lambda))^{-1/p}\, k^{-1} (b+c)^{k+1} \int_{\mathbb{R}} \left| \xi(t; R_{\omega,\Lambda}(-c)^k, R_0(-c)^k) \right|^{1/p} dt.
\end{aligned}$$

Now applying Lemma 6 we get

$$\int_a^b \left( \frac{\xi(\lambda; H_0 + V_{\omega,\Lambda}, H_0)}{\operatorname{meas}_{\nu_1}(\Lambda)} \right)^{1/p} d\lambda \leq C\ (\operatorname{meas}_{\nu_1}(\Lambda))^{-1/p}\, |R_{\omega,\Lambda}(-c)^k - R_0(-c)^k|_p.$$

By Proposition 7 the r.h.s. of this inequality is bounded uniformly in $\Lambda$ and $\omega \in \Omega$. Thus Lemma 3 with $q = p$ proves the absolute continuity of $d\Xi$.

## 3. HÖLDER CONTINUITY OF THE INTEGRATED DENSITY OF BULK STATES

Here we give a simple proof of the Hölder continuity of the integrated density of bulk states for some random Schrödinger operators based on the new $L^p$-bound of Combes, Hislop, and Nakamura and on the formula of Birman and Solomyak [2].

To be concrete, we consider the Holden-Martinelli model, where the single-site potential is the characteristic function of the unit cube,

$$H_\omega = -\Delta + \sum_{j \in \mathbb{Z}^\nu} \alpha_j(\omega)\, \chi(\cdot - j) \qquad \text{on} \quad L^2(\mathbb{R}^\nu).$$

The distribution $\kappa$ will be supposed to be absolutely continuous, $d\kappa = p(\alpha)d\alpha$, and compactly supported, i.e. $\operatorname{supp} p \subseteq [\alpha_-, \alpha_+]$. The integrated density of states has the following representation (see [19])

$$N(\lambda) = \mathbb{E}\left\{ \operatorname{tr}\left(\chi\, \mathsf{E}_{H_\omega}((-\infty, \lambda))\, \chi\right)\right\},$$

where $\mathsf{E}_{H_\omega}$ denotes the spectral projection corresponding to $H_\omega$. Lipshitz continuity of $N(\lambda)$ was proved in [7].

With $I = (\lambda_1, \lambda_2)$ and $A_\omega = H_\omega|_{\alpha_0(\omega)=0}$ we consider

$$\begin{aligned} N(\lambda_2) - N(\lambda_1) &= \mathbb{E}\left\{ \operatorname{tr}\left(\chi\, \mathsf{E}_{H_\omega}(I)\, \chi\right)\right\} \\ &= \mathbb{E}\left\{ \int_{\alpha_-}^{\alpha_+} d\alpha\, p(\alpha)\, \operatorname{tr}(\chi\, \mathsf{E}_{A_\omega + \alpha\chi}(I)\, \chi)\right\} \\ &\le \|p\|_\infty \mathbb{E}\left\{ \int_{\alpha_-}^{\alpha_+} d\alpha\, \operatorname{tr}\left(\chi\, \mathsf{E}_{A_\omega + \alpha\chi}(I)\, \chi\right)\right\}. \end{aligned} \tag{16}$$

Now we will use the Birman-Solomyak formula [2] (see also [24], [11]). The present formulation is from [24]. Let

$$L^p_{\text{unif,loc}}(\mathbb{R}^\nu) := \left\{ \phi \,\middle|\, \sup_x \int_{|x-y|\le 1} |\phi(y)|^p dy < \infty \right\}, \qquad p > \nu/2.$$

**Theorem 9.** *Let $A = -\Delta + W$ with $W \in L^p_{\text{unif,loc}}(\mathbb{R}^\nu)$ and $V \in l^1(L^2)$, $V \ge 0$. For any compact interval $I \subset \mathbb{R}$ the following relation is valid*

$$\int_I \xi(\lambda; A + \alpha_- V, A + \alpha_+ V) d\lambda = \int_{\alpha_-}^{\alpha_+} \operatorname{tr}\left(V^{1/2} \mathsf{E}_{A+sV}(I) V^{1/2}\right) ds.$$

Applying this theorem to (16) we obtain

$$N(\lambda_2) - N(\lambda_1) \le \|p\|_\infty \mathbb{E}\left\{ \int_{\lambda_1}^{\lambda_2} \xi(\lambda; A_\omega + \alpha_+\chi, A_\omega + \alpha_-\chi) d\lambda \right\}.$$

From the Hölder inequality it follows that for any $p > 1$ and any $\omega \in \Omega$

$$\int_{\lambda_1}^{\lambda_2} |\xi(\lambda; A_\omega + \alpha_+\chi, A_\omega + \alpha_-\chi)| d\lambda \le \left( \int_{\lambda_1}^{\lambda_2} \xi(\lambda; A_\omega + \alpha_+\chi, A_\omega + \alpha_-\chi)^p d\lambda \right)^{1/p} |\lambda_2 - \lambda_1|^{\frac{p-1}{p}}$$

Choose some $k > (\nu - 1)/2$ if $\nu \ge 4$ and $k \ge 1$ if $\nu \le 3$ so large that $k > \nu/2p - 1$. By the invariance principle for the spectral shift function (3) for any $\omega \in \Omega$ we have

$$\begin{aligned} &\int_a^b \xi(\lambda; A_\omega + \alpha_+\chi, A_\omega + \alpha_-\chi)^p d\lambda \\ &= \int_a^b \left|\xi((\lambda + c)^{-k}; (A_\omega + \alpha_+\chi + c)^{-k}, (A_\omega + \alpha_-\chi + c)^{-k})\right|^p d\lambda \\ &= k^{-1} \int_{(b+c)^{-k}}^{(a+c)^{-k}} \left|\xi(t; (A_\omega + \alpha_+\chi + c)^{-k}, (A_\omega + \alpha_-\chi + c)^{-k})\right|^p t^{-1/k-1} dt \\ &\le k^{-1}(b+c)^{k+1} \int_{\mathbb{R}} \left|\xi(t; (A_\omega + \alpha_+\chi + c)^{-k}, (A_\omega + \alpha_-\chi + c)^{-k})\right|^p dt. \end{aligned} \tag{17}$$

By Proposition 5.1 of [8] we have that

$$(A_\omega + \alpha_+\chi + c)^{-k} - (A_\omega + \alpha_-\chi + c)^{-k} \in \mathcal{J}_{1/p}. \tag{18}$$

Alternatively we can use our Proposition 7. For instance, for $k = 1$ we have

$$\begin{aligned}(A_\omega + \alpha_+\chi + c)^{-1} - (A_\omega + \alpha_-\chi + c)^{-1} &= \left[I + (H_0 + \alpha_-\chi + c)^{-1} V_\omega|_{\alpha_0(\omega)=0}\right]^{-1} \\ \cdot \left\{(H_0 + \alpha_+\chi + c)^{-1} - (H_0 + \alpha_-\chi + c)^{-1}\right\} &\left[I - V_\omega|_{\alpha_0(\omega)=0} (A_\omega + \alpha_+\chi + c)^{-1}\right].\end{aligned}$$

Since for sufficiently large $c$ the first and the last factors on the r.h.s. of this equality are bounded uniformly in $\omega \in \Omega$, the relation (18) follows from Proposition 7. Since $\chi \in L^\infty$ the additional restrictions $p \le 4$ if $\nu \le 3$ and $p < 2r$ if $\nu \ge 4$ can be omitted.

Thus from Lemma 6 it follows that the l.h.s. of (17) is bounded by a constant $C > 0$ uniformly in $\omega \in \Omega$. Finally this leads to the estimate

$$N(\lambda_2) - N(\lambda_1) \le C_p |\lambda_2 - \lambda_1|^{\frac{p-1}{p}}$$

for any $p > 1$, which proves the Hölder continuity of the integrated density of states. We can apply similar arguments to models more general than the Holden-Martinelli model. We will not dwell on this here.

## References


[1] M.S. Birman and M.Z. Solomyak, *Estimates of singular numbers of integral operators. III*, Vestnik Leningrad Univ. **24** (1969) 35 – 48 (Russian).

[2] M.S. Birman and M.Z. Solomyak, *Remarks on the spectral shift function*, J. Sov. Math. **3** (1975) 408 – 419.

[3] M.Sh. Birman and M.Z. Solomyak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, D. Reidel, Dordrecht, 1987.

[4] M.Sh. Birman and D.R. Yafaev, *The spectral shift function. Work by M.G.Krein and its further development*, St. Petersburg Math. J. **4** (1993) 833 – 870.

[5] A. Chahrour, *Densité intégrée d'états surfaciques et la fonction généralisée de déplacement spectral pour un opérateur de Schrödinger surfacique ergodicue*, Helv. Phys. Acta **72** (1999) 93 – 122.

[6] A. Chahrour, *On the spectrum of Schrödinger operator with periodic surface potential*, preprint (2000).

[7] J.M. Combes and P. Hislop, *Localization for some continuous, random Hamiltonians in d dimensions*, J. Funct. Anal. **124** (1994) 149 – 180.

[8] J.M. Combes, P. Hislop, S. Nakamura, *The $L^p$-theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random operators*, Commun. Math. Phys. (to appear).

[9] H. Englisch, W. Kirsch, M. Schröder, and B. Simon, *Random Hamiltonians ergodic in all but one direction*, Commun. Math. Phys. **128** (1990) 613 – 625.

[10] F. Gesztesy and K.A. Makarov, *The Ξ operator and its relation to Krein's spectral shift function*, J. Anal. Math. **81** (2000) 139 – 183.

[11] F. Gesztesy, K.A. Makarov, and A.K. Motovilov, *Monotonicity and concavity properties of the spectral shift function*, in F. Gesztesy, H. Holden, J. Jost, S. Paycha, M. Röckner, and S. Scarlatti (Eds.), Canadian Mathematical Society's Conference Proceedings Series, Canadian Math. Soc. and Amer. Math. Soc., Providence, RI. (to appear); available from `http://www.ma.utexas.edu/mp_arc/` as 99-339.

[12] I.C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space*, Amer. Math. Soc. Transl. of Math. Monographs 18, Providence, RI, 1969.

[13] D. Hundertmark and B. Simon, *An optimal $L^p$-bound on the Krein spectral shift function*, J. Anal. Math. (to appear); available from `http://www.ma.utexas.edu/mp_arc/` as 00-370.

[14] V. Kostrykin, *Concavity of eigenvalue sums and the spectral shift function*, J. Funct. Anal. **176** (2000) 100 – 114.

[15] V. Kostrykin and R. Schrader, *Scattering theory approach to random Schrödinger operators in one dimension*, Rev. Math. Phys. **11** (1999) 187 – 242.

[16] V. Kostrykin and R. Schrader, *The density of states and the spectral shift density of random Schrödinger operators*, Rev. Math. Phys. **12** (2000) 807 – 847.

[17] V. Kostrykin and R. Schrader, *Global bounds for the Lyapunov exponent and the integrated density of states of random Schrödinger operators in one dimension*, J. Phys. A: Math. Gen. **33** (2000) (to appear); available from `http://www.ma.utexas.edu/mp_arc/` as 00-226.

[18] S. Nakamura, *A remark on the Dirichlet-Neuman decoupling and the integrated density of states*, preprint (2000); available from `http://www.ma.utexas.edu/mp_arc/` as 00-121.

[19] L. Pastur and A. Figotin, *Spectra of Random and Almost-Periodic Operators*, Springer, Berlin, 1992.

[20] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis*, Academic Press, New York, 1972.

[21] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, III: Scattering Theory*, Academic Press, New York, 1979.

[22] B. Simon, *Trace Ideals and Their Applications*, Cambridge University Press, Cambridge, 1979.

[23] B. Simon, *Spectral analysis of rank one perturbations and applications*, in J. Feldman, R. Froese, and L. Rosen (eds.) *Mathematical Quantum Theory II: Schroedinger Operators*, Proceedings of the Canadian Mathematical Society annual seminar on mathematical quantum theory held in Vancouver, Canada, August 4-14, 1993, CRM Proceedings Lecture Notes, Vol. 8, Amer. Math. Soc., Providence, R.I., 1995, p. 109 – 149.

[24] B. Simon, *Spectral averaging and the Krein spectral shift*, Proc. Amer. Math. Soc. **126** (1998) 1409 – 1413.

[25] D.R. Yafaev, *Mathematical Scattering Theory. General Theory*, Amer. Math. Soc. Transl. of Math. Monographs Vol. 105, Providence, RI, 1992.

VADIM KOSTRYKIN, FRAUNHOFER-INSTITUT FÜR LASERTECHNIK, STEINBACHSTRASSE 15, D-52074, AACHEN, GERMANY

*E-mail address*: `kostrykin@t-online.de, kostrykin@ilt.fhg.de`

ROBERT SCHRADER, INSTITUT FÜR THEORETISCHE PHYSIK, FREIE UNIVERSITÄT BERLIN, ARNIMALLEE 14, D-14195 BERLIN, GERMANY

*E-mail address*: `schrader@physik.fu-berlin.de`