A Weiner-Hopf Approximation Technique for a Multiple Plate Diffraction Problem*

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Abstract

An approximation method is derived for the computation of the acoustic field between a series of parallel plates, subject to a time periodic incident field. The method is based on the Wiener-Hopf method of factorization, with computations involving orthogonal bases of functions that are analytic in the complex half-plane.

1 Introduction

In this article we present an extension of the classical Weiner-Hopf technique to the solution of the acoustic field between a series of parallel plates of varying heights and spacings. This problem has received considerable attention throughout the years. An elegant solution for two, semi-infinite plates of equal height, was obtained by Jones in [9], (see also [12]). In [10], Jones presented the solution for three uniformly spaced, semi-infinite soft plates, of equal heights. The solution for two semi-infinite plates with different heights was given by Abrahams and Wickham in [1]. They extended their work in [2] to model scattering of water waves from opposing vertical barriers. In [3] Alkumru considered the case of electromagnetic diffraction from three uniformly spaced, semi-infinite *thick* plates, of equal heights where the plates have non-zero thickness.

The difficulty in extending the Weiner-Hopf technique to the general N-plate setting has been in obtaining a suitable product factorization of the $N \times N$ matrix arising in the method. In [11] Meister, Rottbrand and Speck presented an operator splitting approach to construct a suitable factorization of the associated Fourier symbol, and showed that the Weiner-Hopf system was invertible provided the plates were sufficiently separated.

The multi-plate diffraction problem has recently been considered as a model for predicting radiowave propagation in urban microcell environments [5][6][7][8][13]. (In the case the magnetic (or electric)

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field of the incoming signal is orientated perpendicular to the direction of propogation of the plane wave, the problem may be recast as the scalar Helmholtz equation.) Several different approaches have been applied to this problem: empirical, semi-empirical, theoretical, ray-tracing and those employing the uniform theory of diffraction [4][7][8][6].

This paper is organized as follows. In section 2 we derive the approximating system of equations for the coupled field. The approximation approach to solving these equations is then presented in section 3. An example demonstrating the technique is given in section 4.

2 Mathematical Model

The problem investigated in this paper is that of a plane wave field impinging upon a system of N semi-infinite plates, each parallel to the xz-plane and arranged one behind the other along the positive y-axis (figure 1). The height of the edges of the plates above the xy-plane may vary from

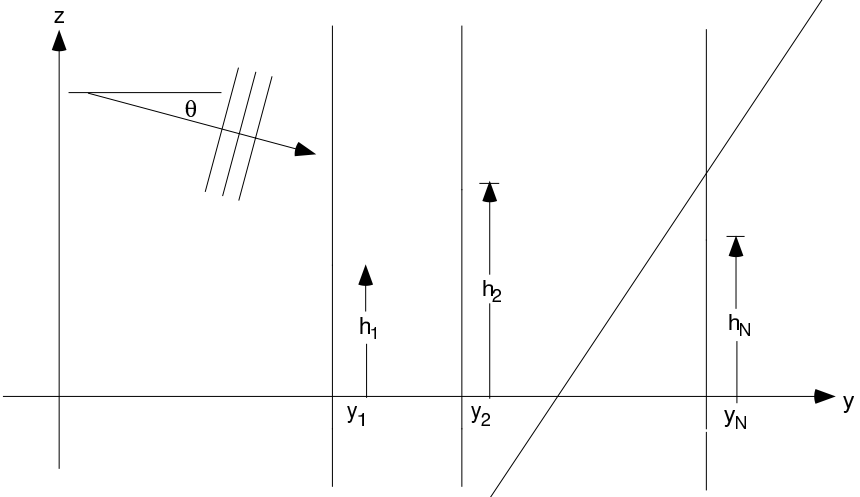

Figure 1: Plane waves incident on a set of parallel plates.

plate to plate. We assume that the vector normal to the wavefronts of the incident field is parallel to the yz-plane and makes an angle θ with the positive y-axis.

Assuming an $e^{j\omega t}$ time dependence, the incident field is specified by

$$\phi_{inc}(y,z) = e^{-jk(y\cos\theta + z\sin\theta)}.$$

We partition the yz-plane into $N+1$ subdomains defined by

$$\begin{aligned} D_0 &= \{(y,z) : -\infty < y < y_1,\ -\infty < z < \infty\}, \\ D_n &= \{(y,z) : y_n < y < y_{n+1},\ -\infty < z < \infty\}, \qquad n = 1,\ldots,N-1, \end{aligned}$$

and

$$D_N = \{(y,z) : y_N < y < \infty,\ -\infty < z < \infty\}.$$

If we let $\psi_n(y,z)$ denote the total wave field in domain D_n, then $\psi_n(y,z)$ must satisfy the two-dimensional Helmholtz equation,

$$\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0, \qquad (y,z)\in D_n\,, \tag{1}$$

for each $n=0,\ldots,N$. In addition, we impose the following interface conditions:

$$\psi_{n-1}(y_n-0,z)=\psi_n(y_n+0,z), \qquad h_n<z<\infty, \quad n=1,\ldots,N, \tag{2}$$

$$\frac{\partial\psi_{n-1}(y_n-0,z)}{\partial y}=\frac{\partial\psi_n(y_n+0,z)}{\partial y}, \qquad h_n<z<\infty, \quad n=1,\ldots,N, \tag{3}$$

$$\frac{\partial\psi_{n-1}(y_n-0,z)}{\partial y}=0, \qquad -\infty<z<h_n, \quad n=1,\ldots,N, \tag{4}$$

and

$$\frac{\partial\psi_n(y_n+0,z)}{\partial y}=0, \qquad -\infty<z<h_n, \quad n=1,\ldots,N. \tag{5}$$

Conditions (2) and (3) arise from requiring the wavefield and its normal derivative to be continuous along the interfaces between regions above the plates. Conditions (4) and (5) constitute a hard boundary condition on both sides of each plate. Note that together conditions (3), (4), and (5) imply that

$$\frac{\partial\psi_{n-1}(y_n-0,z)}{\partial y}=\frac{\partial\psi_n(y_n+0,z)}{\partial y}, \qquad -\infty<z<\infty, \quad n=1,\ldots,N\,. \tag{6}$$

In addition to the interface conditions we impose a constraint that disallows waves propagating in from infinity assuming an $e^{j\omega t}$ time dependence, that is, a radiation condition. For the index of refraction we use $k=k_1-jk_2$ with $0<k_2<<1$, and requiring solutions to be bounded. Calculations are simplified if we represent the total wave field in each domain as a sum of the incident field and a residual field $\phi_n(y,z)$,

$$\psi_n(y,z)=\phi_n(y,z)+\phi_{inc}(y,z), \qquad n=0,\ldots,N.$$

Denote the Fourier transform of $\phi_n(y,z)$ with respect to z by $\Phi_n(y,\lambda)$,

$$\Phi_n(y,\lambda)=\int_{-\infty}^{\infty}\phi_n(y,z)e^{-j\lambda z}\,dz. \tag{7}$$

It follows from (1) and the radiation conditions that

$$\Phi_0(y,\lambda) = L_1(\lambda)e^{\sqrt{\lambda^2-k^2}\,y}, \tag{8}$$

$$\Phi_n(y,\lambda) = L_{n+1}(\lambda)e^{\sqrt{\lambda^2-k^2}\,y}+R_n(\lambda)e^{-\sqrt{\lambda^2-k^2}\,y}, \quad n=1,\ldots,N-1\,, \tag{9}$$

and

$$\Phi_N(y,\lambda) = R_N(\lambda)e^{-\sqrt{\lambda^2-k^2}\,y}. \tag{10}$$

The branch cut for $\sqrt{\lambda-k}$ is taken to be $\{\lambda:\lambda=k-j\tau,\ 0<\tau<\infty\}$ and the branch cut for $\sqrt{\lambda+k}$ is taken to be $\{\lambda:\lambda=k+j\tau,\ 0<\tau<\infty\}$. It follows that $\mathcal{R}e(\sqrt{\lambda^2-k^2})>0$ if $|\mathcal{R}e(\lambda)|>\mathcal{R}e(k)$.

Fourier transforming each of the interface conditions (6) with respect to z and substituting (8)-(10) into the resulting equations yields the following relationships between the left and right propagating mode amplitude functions,

$$\begin{aligned}
L_1(\lambda)e^{\gamma y_1} &= L_2(\lambda)e^{\gamma y_1} - R_1(\lambda)e^{-\gamma y_1}, && (11)\\
L_{n+1}(\lambda)e^{\gamma y_{n+1}} - R_n(\lambda)e^{-\gamma y_{n+1}} &= L_{n+2}(\lambda)e^{\gamma y_{n+1}} - R_{n+1}(\lambda)e^{-\gamma y_{n+1}}, \quad n=1,\ldots,N-2, && (12)\\
L_N(\lambda)e^{\gamma y_N} - R_{N-1}(\lambda)e^{-\gamma y_N} &= -R_N(\lambda)e^{-\gamma y_N}, && (13)
\end{aligned}$$

where $\gamma := \sqrt{\lambda^2 - k^2}$.

The total wave field in each domain D_n expressed in terms of Fourier integral representations of the $\phi_n(y,z)$ are

$$\begin{aligned}
\psi_0(y,z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} L_1(\lambda)e^{\gamma y}e^{j\lambda z}\,d\lambda \;+\; \phi_{inc}(y,z), && (14)\\
\psi_n(y,z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[L_{n+1}(\lambda)e^{\gamma y} + R_n(\lambda)e^{-\gamma y}\right] e^{j\lambda z}\,d\lambda \;+\; \phi_{inc}(y,z), \quad n=1,\ldots,N-1, && (15)\\
\psi_N(y,z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_N(\lambda)e^{-\gamma y}e^{j\lambda z}\,d\lambda + \phi_{inc}(y,z). && (16)
\end{aligned}$$

If we now apply interface conditions (2) to (14)-(16) and use the relationships (11)-(13) we find that

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} R_1(\lambda)e^{-\gamma y_1}e^{j\lambda h_1}e^{j\lambda z}\,d\lambda = 0, \quad z>0, && (17)\\
\frac{1}{2\pi}\int_{-\infty}^{\infty} \left[L_{n+1}(\lambda) - L_n(\lambda)\right] e^{\gamma y_n}e^{j\lambda h_n}e^{j\lambda z}\,d\lambda = 0, \quad z>0, \quad n=1,\ldots,N-1, && (18)\\
\frac{1}{2\pi}\int_{-\infty}^{\infty} \left[R_{n+1}(\lambda) - R_n(\lambda)\right] e^{-\gamma y_{n+1}}e^{j\lambda h_{n+1}}e^{j\lambda z}\,d\lambda = 0, \quad z>0, \quad n=1,\ldots,N-1, && (19)
\end{aligned}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} L_N(\lambda)e^{\gamma y_N}e^{j\lambda h_N}e^{j\lambda z}\,d\lambda = 0, \quad z>0. \qquad (20)$$

Any argument similar to that of [12] pg.101, see appendix B, shows that

$$\begin{aligned}
&R_1(\lambda)e^{-\gamma y_1}e^{j\lambda h_1},\\
&\left[L_{n+1}(\lambda) - L_n(\lambda)\right] e^{\gamma y_n}e^{j\lambda h_n}, \quad n=1,\ldots,N-1,\\
&\left[R_{n+1}(\lambda) - R_n(\lambda)\right] e^{-\gamma y_{n+1}}e^{j\lambda h_{n+1}}, \quad n=1,\ldots,N-1,
\end{aligned}$$

and

$$L_N(\lambda)e^{\gamma y_N}e^{j\lambda h_N},$$

are analytic functions of λ for $\mathcal{I}m(\lambda) > k_2\sin\theta$ and are $O(|\lambda|^{-1})$ as $|\lambda|\to\infty$. We therefore set

$$\begin{aligned}
E_1^{(+)}(\lambda) &:= R_1(\lambda)e^{-\gamma y_1}e^{j\lambda h_1}, && (21)\\
F_n^{(+)}(\lambda) &:= \left[L_n(\lambda) - L_{n+1}(\lambda)\right] e^{\gamma y_n}e^{j\lambda h_n}, \quad n=1,\ldots,N-1, && (22)\\
E_{n+1}^{(+)}(\lambda) &:= \left[R_{n+1}(\lambda) - R_n(\lambda)\right] e^{-\gamma y_{n+1}}e^{j\lambda h_{n+1}}, \quad n=1,\ldots,N-1, && (23)\\
F_N^{(+)}(\lambda) &:= L_N(\lambda)e^{\gamma y_N}e^{j\lambda h_N}, && (24)
\end{aligned}$$

where the superscript plus signs indicate that the functions are analytic in the upper half space and tend to zero as $|\lambda| \to \infty$.

The interface conditions (4)-(5) applied to (14)-(16) yield the coupled system of $2N$ integral equations

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\lambda^2-k^2}L_1(\lambda)e^{\gamma y_1}e^{j\lambda h_1}e^{j\lambda z}\,d\lambda = jk\cos\theta\, e^{-jk(y_1\cos\theta+(z+h_1)\sin\theta)},\ z<0, \tag{25}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\lambda^2-k^2}\left[L_{n+1}(\lambda)e^{\gamma y_n}-R_n(\lambda)e^{-\gamma y_n}\right]e^{j\lambda h_n}e^{j\lambda z}\,d\lambda = jk\cos\theta\, e^{-jk(y_n\cos\theta+(z+h_n)\sin\theta)},\ z<0,\ \ n=1,\ldots,N-1, \tag{26}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\lambda^2-k^2}\left[L_{n+1}(\lambda)e^{\gamma y_{n+1}}-R_n(\lambda)e^{-\gamma y_{n+1}}\right]e^{j\lambda h_{n+1}}e^{j\lambda z}\,d\lambda = jk\cos\theta\, e^{-jk(y_{n+1}\cos\theta+(z+h_{n+1})\sin\theta)},\ z<0,\ \ n=1,\ldots,N-1, \tag{27}$$

and

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\lambda^2-k^2}R_N(\lambda)e^{-\gamma y_N}e^{j\lambda h_N}e^{j\lambda z}\,d\lambda = jk\cos\theta\, e^{-jk(y_N\cos\theta+(z+h_N)\sin\theta)},\ \ z<0. \tag{28}$$

Define

$$g_1(z) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\lambda^2-k^2}L_1(\lambda)e^{\gamma y_1}e^{j\lambda h_1}e^{j\lambda z}\,d\lambda, & z>0,\\ 0, & z<0, \end{cases} \tag{29}$$

$$g_{n+1}(z) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\lambda^2-k^2}\left[L_{n+1}(\lambda)e^{\gamma y_{n+1}}-R_n(\lambda)e^{-\gamma y_{n+1}}\right]e^{j\lambda h_{n+1}}e^{j\lambda z}\,d\lambda, & z>0,\\ 0, & z<0, \end{cases} \tag{30}$$

for $n=1,\ldots,N-1$. Similarly, for $n=1,\ldots,N-1$, define

$$h_n(z) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\lambda^2-k^2}\left[L_{n+1}(\lambda)e^{\gamma y_n}-R_n(\lambda)e^{-\gamma y_n}\right]e^{j\lambda h_n}e^{j\lambda z}\,d\lambda, & z>0,\\ 0, & z<0, \end{cases} \tag{31}$$

and

$$h_N(z) = \begin{cases} -\frac{1}{2\pi}\int_{-\infty}^{\infty}\sqrt{\lambda^2-k^2}R_N(\lambda)e^{-\gamma y_N}e^{j\lambda h_N}e^{j\lambda z}\,d\lambda, & z>0,\\ 0, & z<0. \end{cases} \tag{32}$$

We now Fourier transform (25)-(28) to obtain the system of equations

$$\sqrt{\lambda^2-k^2}\,L_1(\lambda)e^{\gamma y_1}e^{j\lambda h_1} = -\frac{k\cos\theta}{\lambda+k\sin\theta}e^{-jk(y_1\cos\theta+h_1\sin\theta)}+G_1^{(-)}(\lambda), \tag{33}$$

$$\sqrt{\lambda^2-k^2}\,\left[L_{n+1}(\lambda)e^{\gamma y_n}-R_n(\lambda)e^{-\gamma y_n}\right]e^{j\lambda h_n} = -\frac{k\cos\theta}{\lambda+k\sin\theta}+e^{-jk(y_n\cos\theta+h_n\sin\theta)}H_n^{(-)}(\lambda),\ \ n=1,\ldots,N-1, \tag{34}$$

$$\sqrt{\lambda^2-k^2}\,\left[L_{n+1}(\lambda)e^{\gamma y_{n+1}}-R_n(\lambda)e^{-\gamma y_{n+1}}\right]e^{j\lambda h_{n+1}} =$$

$$-\frac{k\cos\theta}{\lambda+k\sin\theta}e^{-jk(y_{n+1}\cos\theta+h_{n+1}\sin\theta)}+G_{n+1}^{(-)}(\lambda),\quad n=1,\ldots,N-1, \tag{35}$$

and

$$\sqrt{\lambda^2-k^2}\,\left[-R_N(\lambda)e^{-\gamma\,y_N}\right]e^{j\lambda h_N}\;=\;-\frac{k\cos\theta}{\lambda+k\sin\theta}e^{-jk(y_N\cos\theta+h_N\sin\theta)}+H_N^{(-)}(\lambda), \tag{36}$$

where the functions $G_n^{(-)}(\lambda)$ and $H_n^{(-)}(\lambda)$, the Fourier transforms of $g_n(z)$ and $h_n(z)$, respectively, for $n=1,\ldots,N$, are holomorphic in the lower half plane $\mathcal{I}m(\lambda)<k_2$, see appendix C, and are $O(|\lambda|^{-1/2})$ as $|\lambda|\to-\infty$.

In order for the transforms (33)-(36) to exist it is necessary that $\mathcal{I}m(\lambda)>-k\sin\theta$. Consequently, inversion contours for (33)-(36) must pass below the branch point at $\lambda=-k$ and above the branch point at $\lambda=k$ and the pole at $\lambda=-k\sin\theta$.

Introducing the notation $\mathbf{R}(\lambda)=[R_1(\lambda),\ldots,R_N(\lambda)]^T$, $\mathbf{L}(\lambda)=[L_1(\lambda),\ldots,L_N(\lambda)]^T$, $a_n(\lambda)=e^{\sqrt{\lambda^2-k^2}\,y_n}$,

$$\mathbf{A}(\lambda)=\begin{bmatrix} a_1(\lambda) & 0 & \cdots & 0\\ a_1(\lambda) & a_2(\lambda) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ a_1(\lambda) & a_2(\lambda) & \cdots & a_N(\lambda)\end{bmatrix}, \tag{37}$$

$$\mathbf{B}(\lambda)=\begin{bmatrix} a_1^{-1}(\lambda) & a_2^{-1}(\lambda) & \cdots & a_N^{-1}(\lambda)\\ 0 & a_2^{-1}(\lambda) & \cdots & a_N^{-1}(\lambda)\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a_N^{-1}(\lambda)\end{bmatrix}, \tag{38}$$

and

$$\mathbf{P}(\lambda)=\begin{bmatrix} e^{j\lambda h_1} & 0 & \cdots & 0\\ 0 & e^{j\lambda h_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & e^{j\lambda h_N}\end{bmatrix}, \tag{39}$$

we find from equations (21)-(24) that

$$\mathbf{L}(\lambda)=\mathbf{B}(\lambda)\mathbf{P}^{-1}(\lambda)\mathbf{F}^{(+)}(\lambda), \tag{40}$$

and

$$\mathbf{R}(\lambda)=\mathbf{A}(\lambda)\mathbf{P}^{-1}(\lambda)\mathbf{E}^{(+)}(\lambda). \tag{41}$$

Equations (40) and (41) are then used to eliminate $\mathbf{R}(\lambda)$ and $\mathbf{L}(\lambda)$ from equations (33)-(36) to yield the system

$$\sqrt{\lambda^2-k^2}\left\{\left[\mathbf{I}+\mathbf{P}(\lambda)\mathbf{C}(\lambda)\mathbf{P}^{-1}(\lambda)\right]\mathbf{F}^{(+)}(\lambda)-\mathbf{P}(\lambda)\mathbf{C}^T(\lambda)\mathbf{P}^{-1}(\lambda)\mathbf{E}^{(+)}(\lambda)\right\}\;=\;-\frac{k\cos\theta}{\lambda+k\sin\theta}\mathbf{b}+\mathbf{G}^{(-)}(\lambda)\,, \tag{42}$$

$$\sqrt{\lambda^2-k^2}\left\{\mathbf{P}(\lambda)\mathbf{C}(\lambda)\mathbf{P}^{-1}(\lambda)\mathbf{F}^{(+)}(\lambda)-\left[\mathbf{I}+\mathbf{P}(\lambda)\mathbf{C}^T(\lambda)\mathbf{P}^{-1}\right]\mathbf{E}^{(+)}(\lambda)\right\} \;=\; -\frac{k\cos\theta}{\lambda+k\sin\theta}\mathbf{b}+\mathbf{H}^{(-)}(\lambda)\,, \quad (43)$$

where

$$\mathbf{F}^{(+)}(\lambda)=\begin{bmatrix} F_1^{(+)} \\ \vdots \\ F_N^{(+)} \end{bmatrix}, \quad \mathbf{E}^{(+)}(\lambda)=\begin{bmatrix} E_1^{(+)} \\ \vdots \\ E_N^{(+)} \end{bmatrix}, \quad \mathbf{G}^{(-)}(\lambda)=\begin{bmatrix} G_1^{(-)} \\ \vdots \\ G_N^{(-)} \end{bmatrix}, \quad \mathbf{H}^{(-)}(\lambda)=\begin{bmatrix} H_1^{(-)} \\ \vdots \\ H_N^{(-)} \end{bmatrix}, \quad (44)$$

$$\mathbf{b}=\begin{bmatrix} e^{-jk(y_1\cos\theta+h_1\sin\theta)} \\ \vdots \\ e^{-jk(y_N\cos\theta+h_N\sin\theta)} \end{bmatrix}, \quad (45)$$

and

$$\mathbf{C}(\lambda)=\begin{bmatrix} 0 & a_1(\lambda)a_2^{-1}(\lambda) & a_1(\lambda)a_3^{-1}(\lambda) & \cdots & a_1(\lambda)a_N^{-1}(\lambda) \\ 0 & 0 & a_2(\lambda)a_3^{-1}(\lambda) & \cdots & a_2(\lambda)a_N^{-1}(\lambda) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-1}(\lambda)a_N^{-1}(\lambda) \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (46)$$

If we now define the transformations

$$\hat{\mathbf{G}}^{(-)}(\lambda)=\frac{1}{\sqrt{\lambda+k}}\mathbf{G}^{(-)}(\lambda), \qquad \hat{\mathbf{H}}^{(-)}(\lambda)=\frac{1}{\sqrt{\lambda+k}}\mathbf{H}^{(-)}(\lambda), \quad (47)$$

then equations (42) and (43) may be written as

$$\sqrt{\lambda-k}\left\{\left[\mathbf{I}+\mathbf{P}(\lambda)\mathbf{C}(\lambda)\mathbf{P}^{-1}(\lambda)\right]\mathbf{F}^{(+)}(\lambda)-\mathbf{P}(\lambda)\mathbf{C}^T(\lambda)\mathbf{P}^{-1}(\lambda)\mathbf{E}^{(+)}(\lambda)\right\} \;=\; -\frac{k\cos\theta}{(\lambda+k\sin\theta)\sqrt{\lambda+k}}\mathbf{b}+\hat{\mathbf{G}}^{(-)}(\lambda)\,, \quad (48)$$

$$\sqrt{\lambda-k}\left\{\mathbf{P}(\lambda)\mathbf{C}(\lambda)\mathbf{P}^{-1}(\lambda)\mathbf{F}^{(+)}(\lambda)-\left[\mathbf{I}+\mathbf{P}(\lambda)\mathbf{C}^T(\lambda)\mathbf{P}^{-1}\right]\mathbf{E}^{(+)}(\lambda)\right\} \;=\; -\frac{k\cos\theta}{(\lambda+k\sin\theta)\sqrt{\lambda+k}}\mathbf{b}+\hat{\mathbf{H}}^{(-)}(\lambda)\,. \quad (49)$$

Factoring the first term on the right hand side of (48) and (49) into a sum of two functions, one holomorphic in the upper half plane and the other holomorphic in the lower half plane, we rewrite (48) and (49) as

$$\sqrt{\lambda-k}\left\{\left[\mathbf{I}+\mathbf{P}(\lambda)\mathbf{C}(\lambda)\mathbf{P}^{-1}(\lambda)\right]\mathbf{F}^{(+)}(\lambda)-\mathbf{P}(\lambda)\mathbf{C}^T(\lambda)\mathbf{P}^{-1}(\lambda)\mathbf{E}^{(+)}(\lambda)\right\} \;+$$

$$\frac{\cos\theta}{\lambda+k\sin\theta}\frac{\sqrt{k}}{\sqrt{1-\sin\theta}}\mathbf{b} \;=\; \frac{k\cos\theta}{\lambda+k\sin\theta}\left[\frac{1}{\sqrt{k(1-\sin\theta)}}-\frac{1}{\sqrt{\lambda+k}}\right]\mathbf{b}+\hat{\mathbf{G}}^{(-)}(\lambda)\,, \quad (50)$$

$$\sqrt{\lambda-k}\left\{\mathbf{P}(\lambda)\mathbf{C}(\lambda)\mathbf{P}^{-1}(\lambda)\mathbf{F}^{(+)}(\lambda)-\left[\mathbf{I}+\mathbf{P}(\lambda)\mathbf{C}^{T}(\lambda)\mathbf{P}^{-1}\right]\mathbf{E}^{(+)}(\lambda)\right\} \;+$$

$$\frac{\cos\theta}{\lambda+k\sin\theta}\frac{\sqrt{k}}{\sqrt{1-\sin\theta}}\mathbf{b} \;=\; \frac{k\cos\theta}{\lambda+k\sin\theta}\left[\frac{1}{\sqrt{k(1-\sin\theta)}}-\frac{1}{\sqrt{\lambda+k}}\right]\mathbf{b}+\hat{\mathbf{H}}^{(-)}(\lambda). \quad (51)$$

As the functions on the LHS of (50),(51) are analytic in the upper half plane and go to zero as $|\lambda|\to\infty$, and the functions on the RHS of (50),(51) are analytic in the lower half plane and go to zero as $|\lambda|\to\infty$, then from analytic continuation and Liouville's theorem we conclude that

$$\sqrt{\lambda-k}\left\{\left[\mathbf{I}+\mathbf{P}(\lambda)\mathbf{C}(\lambda)\mathbf{P}^{-1}(\lambda)\right]\mathbf{F}^{(+)}(\lambda)-\mathbf{P}(\lambda)\mathbf{C}^{T}(\lambda)\mathbf{P}^{-1}(\lambda)\mathbf{E}^{(+)}(\lambda)\right\} \;=$$

$$-\frac{\cos\theta}{\lambda+k\sin\theta}\frac{\sqrt{k}}{\sqrt{1-\sin\theta}}\mathbf{b}\,, \quad (52)$$

$$\sqrt{\lambda-k}\left\{\mathbf{P}(\lambda)\mathbf{C}(\lambda)\mathbf{P}^{-1}(\lambda)\mathbf{F}^{(+)}(\lambda)-\left[\mathbf{I}+\mathbf{P}(\lambda)\mathbf{C}^{T}(\lambda)\mathbf{P}^{-1}\right]\mathbf{E}^{(+)}(\lambda)\right\} \;=$$

$$-\frac{\cos\theta}{\lambda+k\sin\theta}\frac{\sqrt{k}}{\sqrt{1-\sin\theta}}\mathbf{b}. \quad (53)$$

3 Numerical Approximation

3.1 Functions holomorphic in a half-plane

Let $\alpha>0$ and consider the system of functions

$$\omega_m(\eta)=\sqrt{\frac{\alpha}{\pi}}\frac{(\eta-j\alpha)^{m-1}}{(\eta+j\alpha)^m}, \qquad \omega_{-m}(\eta)=\sqrt{\frac{\alpha}{\pi}}\frac{(\eta+j\alpha)^{m-1}}{(\eta-j\alpha)^m}, \qquad m=1,2,3,\ldots. \quad (54)$$

Restricting the domain of the functions to the real axis, the system (54) is a complete orthonormal system for $\mathcal{L}_2(-\infty,\infty)$ equipped with the inner product

$$\langle f,g\rangle=\int_{-\infty}^{\infty}f(\eta)\bar{g}(\eta)\,d\eta.$$

Furthermore, functions which are holomorphic in the upper half plane with boundary values in $\mathcal{L}_2(-\infty,\infty)$ have a unique representation

$$F(\eta)=\sum_{m=1}^{\infty}f_m\omega_m(\eta)$$

where the Fourier coefficients f_n are obtained from the boundary values of F on the real axis,

$$f_m=\int_{-\infty}^{\infty}F(\eta)\bar{\omega}_m(\eta)\,d\eta. \qquad m=1,2,3,\ldots.$$

Illustrated in figures 2 and 3, are the real and imaginary parts of $\omega_2(\eta)$ and $\omega_7(\eta)$, respectively.

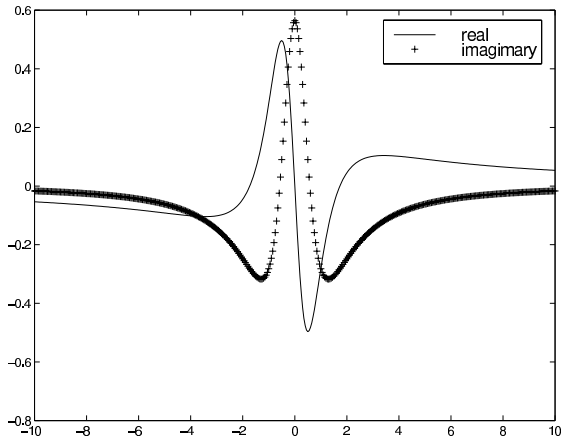

Figure 2: Real and Imaginary parts of $\omega_2(\eta)$.

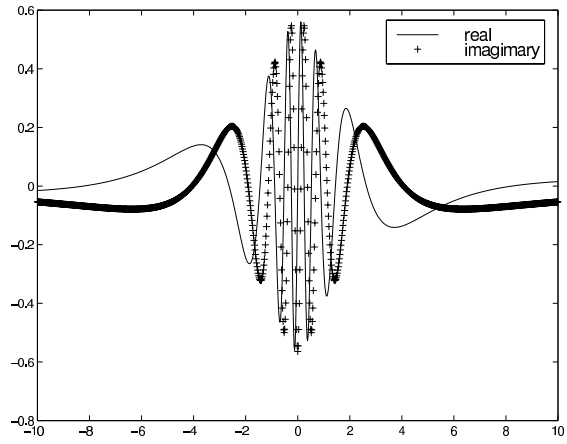

Figure 3: Real and Imaginary parts of $\omega_7(\eta)$.

3.2 Projection onto functions analytic in a half-plane

With k expressed in complex polar coordinates as $k = \rho_k w_k$ where $\rho_k = |k|$ and $w_k = e^{j \arg(k)}$, and using the scalings $\lambda = \rho_k \eta$, $\tilde{\mathbf{E}}^{(+)}(\eta) = \rho_k \, \mathbf{E}^{(+)}(\rho_k \eta)$, $\tilde{\mathbf{F}}^{(+)}(\eta) = \rho_k \, \mathbf{F}^{(+)}(\rho_k \eta)$, $\tilde{\mathbf{P}}(\eta) = \mathbf{P}(\rho_k \eta)$, and $\tilde{\mathbf{C}}(\eta) = \mathbf{C}(\rho_k \eta)$, then (52)-(53) may be rewritten as

$$\left[\mathbf{I} + \tilde{\mathbf{P}}(\eta)\tilde{\mathbf{C}}(\eta)\tilde{\mathbf{P}}^{-1}(\eta)\right] \tilde{\mathbf{F}}^{(+)}(\eta) - \tilde{\mathbf{P}}(\eta)\tilde{\mathbf{C}}^{T}(\eta)\tilde{\mathbf{P}}^{-1}(\eta)\tilde{\mathbf{E}}^{(+)}(\eta) \;=\; \frac{-\sqrt{w_k}\cos\theta}{(\eta + w_k \sin\theta)\sqrt{\eta - w_k}\sqrt{1-\sin\theta}}\mathbf{b}, \quad (55)$$

$$\tilde{\mathbf{P}}(\eta)\tilde{\mathbf{C}}(\eta)\tilde{\mathbf{P}}^{-1}(\eta)\tilde{\mathbf{F}}^{(+)}(\eta) - \left[\mathbf{I} + \tilde{\mathbf{P}}(\eta)\tilde{\mathbf{C}}^{T}(\eta)\tilde{\mathbf{P}}^{-1}(\eta)\right] \tilde{\mathbf{E}}^{(+)}(\eta) \;=\; \frac{-\sqrt{w_k}\cos\theta}{(\eta + w_k \sin\theta)\sqrt{\eta - w_k}\sqrt{1-\sin\theta}}\mathbf{b}. \quad (56)$$

We seek to solve for the approximations

$$\tilde{\mathbf{E}}^{(+)}(\eta) = \sum_{m=1}^{M} \mathbf{e}_m \omega_m(\eta), \quad (57)$$

$$\tilde{\mathbf{F}}^{(+)}(\eta) = \sum_{m=1}^{M} \mathbf{f}_m \omega_m(\eta). \quad (58)$$

Substituting (57)-(58) into (55) and (56) followed by taking inner products on both sides of (55) and (56) with $\bar{\omega}_\nu(\eta)$, $\nu = 1, 2, 3, \ldots, M$ yields the finite-dimensional system

$$\mathbf{f}_\nu + \sum_{m=1}^{M} Q_{\nu m}\mathbf{f}_m - \sum_{m=1}^{M} R_{\nu m}\mathbf{e}_m = \frac{-\sqrt{w_k}\cos\theta}{\sqrt{1-\sin\theta}}\, s_\nu \, \mathbf{b}\,, \qquad \nu = 1, \ldots, M\,, \quad (59)$$

and

$$\sum_{m=1}^{M} Q_{\nu m}\mathbf{f}_m - \mathbf{e}_\nu - \sum_{m=1}^{M} R_{\nu m}\mathbf{e}_m = \frac{-\sqrt{w_k}\cos\theta}{\sqrt{1-\sin\theta}}\, s_\nu \, \mathbf{b}\,, \qquad \nu = 1, \ldots, M\,, \quad (60)$$

where

$$Q_{\nu m} = \int_{-\infty}^{\infty} \tilde{\mathbf{P}}(\eta)\tilde{\mathbf{C}}(\eta)\tilde{\mathbf{P}}^{-1}(\eta)\omega_m(\eta)\bar{\omega}_\nu(\eta)\, d\eta \ ,$$

$$R_{\nu m} = \int_{-\infty}^{\infty} \tilde{\mathbf{P}}(\eta)\tilde{\mathbf{C}}^T(\eta)\tilde{\mathbf{P}}^{-1}(\eta)\omega_m(\eta)\bar{\omega}_\nu(\eta)\, d\eta \ ,$$

and

$$s_\nu = \int_{-\infty}^{\infty} \frac{\bar{\omega}_\nu(\eta)}{(\eta + w_k \sin\theta)\sqrt{\eta - w_k}}\, d\eta \ .$$

Note: $Q_{\nu m}$ and $R_{\nu m}$ denote $N \times N$ matrices. In obtaining (59)-(60) we have used the fact that $\langle \omega_m, \omega_\nu \rangle = \delta_{m\nu}$.

Subtracting (60) from (59) shows that

$$\mathbf{e}_\nu = -\mathbf{f}_\nu, \qquad \nu = 1, \ldots, M. \tag{61}$$

Using (61) in (59) yields the system of equations

$$\mathbf{f}_\nu + \sum_{m=1}^{M} S_{\nu m}\mathbf{f}_m = \frac{-\sqrt{w_k}\cos\theta}{\sqrt{1-\sin\theta}}\, s_\nu\, \mathbf{b} \ , \qquad \nu = 1, \ldots, M \ , \tag{62}$$

where

$$S_{\nu m} = Q_{\nu m} + R_{\nu m} = \int_{-\infty}^{\infty} \tilde{\mathbf{P}}(\eta)\left[\tilde{\mathbf{C}}(\eta) + \tilde{\mathbf{C}}^T(\eta)\right]\tilde{\mathbf{P}}^{-1}(\eta)\omega_m(\eta)\bar{\omega}_\nu(\eta)\, d\eta \ . \tag{63}$$

4 Numerical Illustration of the Method

In this section we present an example illustrating the method presented above. We consider the case of three parallel plates located at $y_1 = 0$, $y_1 = 2$, and $y_3 = 2$, with heights $h_1 = 6$, $h_2 = 5$, and $h_3 = 7$, respectively (see figure 4). The values for k and θ used were $k = 20 - j0.05$ and $\theta = -\pi/6$. Presented in figures 5–9 is $|\psi_n(y,z)|/|\phi_{inc}(y,z)|$, plotted over a region containing the three plates, and also for each of the regions between and outside the plates.

Note that the plot in figure 6 is consistent with the sum of the incoming wave plus the reflected wave with the maximum and minimum values occurring where expected. The plot in figure 9 is also consistent with what is expected for a signal in a "shadow" region.

Appendix

A Behavior of $\phi_n(y,z)$

Following the discussion of Noble [12], pg. 100, we have the following behavior for $\phi_n(y,z)$:

$$\lim_{r\to 0^+}\phi_n(y_i,h_i+r)=O(1)\ ,\ \lim_{r\to 0^+}\frac{\partial\phi_n}{\partial y}(y_i,h_i+r)=O(r^{-1/2})\ ,\ \text{ for } n=0,\ldots,N, \text{ and } i=n,n+1\,. \tag{64}$$

For any fixed $y, -\infty<y<\infty$,

$$|\phi|<C_1e^{-k_2z}\ \text{ as } z\to\infty\,,\ \text{and } |\phi|<C_2e^{-k_2\sin\theta\, z}\ \text{ as } z\to-\infty\,. \tag{65}$$

B Behavior of $(L_{n+1}-L_n)e^{\sqrt{\lambda^2-k^2}\,y_n}e^{j\lambda h_n}$, $(R_n-R_{n-1})e^{-\sqrt{\lambda^2-k^2}\,y_n}e^{j\lambda h_n}$

In view of $\Phi_n(y,\lambda)$ defined by (7), we introduce the following notation:

$$\Phi_n^+(y_i):=\Phi_n^+(y_i,\lambda)=\int_{h_i}^{\infty}\phi_n(y_n,z)e^{-j\lambda z}\,dz\ =\ e^{-j\lambda h_i}\int_0^{\infty}\phi_n(y_n,z+h_i)e^{-j\lambda z}\,dz\ \ ,\ i=n,n+1\,, \tag{66}$$

$$\Phi_n^-(y_i):=\Phi_n^-(y_i,\lambda)=\int_{-\infty}^{h_i}\phi_n(y_n,z)e^{-j\lambda z}\,dz\ =\ e^{-j\lambda h_i}\int_{\infty}^{0}\phi_n(y_n,z+h_i)e^{-j\lambda z}\,dz\ \ ,\ i=n,n+1\,. \tag{67}$$

From (65) it follows that $\Phi_n^+(y_i)$ is analytic for $Re(\lambda)<k_2$, and $\Phi_n^-(y_i)$ is analytic for $Re(\lambda)>k_2\sin\theta$.

In addition, from the Abelian theorems, [12] pg. 36, we have that

$$\Phi_n^+(y_i)\ =\ O(|\lambda|^{-1})\ \ ,\ \ \Phi_n^-(y_i)\ =\ O(|\lambda|^{-1})\quad \text{as } |\lambda|\to\infty. \tag{68}$$

Comparing (7)-(10) with (66),(67) we have:

$$\begin{aligned}\Phi_0^+(y_1)+\Phi_0^-(y_1)&=L_1(\lambda)e^{\sqrt{\lambda^2-k^2}\,y_1}\\ \Phi_1^+(y_1)+\Phi_1^-(y_1)&=L_2(\lambda)e^{\sqrt{\lambda^2-k^2}\,y_1}+R_1(\lambda)e^{-\sqrt{\lambda^2-k^2}\,y_1}\end{aligned} \tag{69}$$

$$\left.\begin{aligned}\Phi_{n-1}^+(y_n)+\Phi_{n-1}^-(y_n)&=L_n(\lambda)e^{\sqrt{\lambda^2-k^2}\,y_n}+R_{n-1}(\lambda)e^{-\sqrt{\lambda^2-k^2}\,y_n}\\ \Phi_n^+(y_n)+\Phi_n^-(y_n)&=L_{n+1}(\lambda)e^{\sqrt{\lambda^2-k^2}\,y_n}+R_n(\lambda)e^{-\sqrt{\lambda^2-k^2}\,y_n}\end{aligned}\right\}\ n=2,\ldots,N-1, \tag{70}$$

$$\begin{aligned}\Phi_{N-1}^+(y_N)+\Phi_{N-1}^-(y_N)&=L_N(\lambda)e^{\sqrt{\lambda^2-k^2}\,y_N}+R_{N-1}(\lambda)e^{-\sqrt{\lambda^2-k^2}\,y_N}\\ \Phi_n^+(y_n)+\Phi_n^-(y_n)&=R_N(\lambda)e^{-\sqrt{\lambda^2-k^2}\,y_N}\ .\end{aligned} \tag{71}$$

Using the fact that $\phi_{n-1}(y_n,z)=\phi_n(y_n,z)$ for $z>h_n$, $n=1,\ldots,N$, we have $\Phi_{n-1}^+(y_n)=\Phi_n^+(y_n)$. Hence, subtracting the equations in (69),(70), and (71) yields

$$\begin{aligned}\Phi_1^-(y_1)-\Phi_0^-(y_1)&=(L_2(\lambda)-L_1(\lambda))e^{\sqrt{\lambda^2-k^2}\,y_1}+R_1(\lambda)e^{-\sqrt{\lambda^2-k^2}\,y_1}\ ,\\ \Phi_n^-(y_n)-\Phi_{n-1}^-(y_n)&=(L_{n+1}(\lambda)-L_n(\lambda))e^{\sqrt{\lambda^2-k^2}\,y_n}+(R_n(\lambda)-R_{n-1}(\lambda))e^{-\sqrt{\lambda^2-k^2}\,y_n}\ ,\\ \Phi_N^-(y_N)-\Phi_{N-1}^-(y_N)&=-L_N(\lambda)e^{\sqrt{\lambda^2-k^2}\,y_N}+(R_N(\lambda)-R_{N-1}(\lambda))e^{-\sqrt{\lambda^2-k^2}\,y_N}\ .\end{aligned}$$

Next, using (11),(12),(13), implies

$$\begin{aligned} \Phi_1^-(y_1) - \Phi_0^-(y_1) &= 2(L_2(\lambda) - L_1(\lambda))e^{\sqrt{\lambda^2-k^2}\,y_1} \\ &= 2R_1(\lambda)e^{-\sqrt{\lambda^2-k^2}\,y_1} \ , \end{aligned} \tag{72}$$

$$\begin{aligned} \Phi_n^-(y_n) - \Phi_{n-1}^-(y_n) &= 2(L_{n+1}(\lambda) - L_n(\lambda))e^{\sqrt{\lambda^2-k^2}\,y_n} \\ &= 2(R_n(\lambda) - R_{n-1}(\lambda))e^{-\sqrt{\lambda^2-k^2}\,y_n} \ , \end{aligned} \tag{73}$$

$$\begin{aligned} \Phi_N^-(y_N) - \Phi_{N-1}^-(y_N) &= -2L_N(\lambda)e^{\sqrt{\lambda^2-k^2}\,y_N} \\ &= 2(R_N(\lambda) - R_{N-1}(\lambda))e^{-\sqrt{\lambda^2-k^2}\,y_N} \ . \end{aligned} \tag{74}$$

Thus, in view of the behavior of $\Phi_n^-(y_i)$ described above, we have that the function on the RHS of (72),(73),(74), are all analytic for $Re(\lambda) > k_2 \sin\theta$, and are $O(|\lambda|^{-1})$ as $|\lambda| \to \infty$.

C Behavior of $G_n^-(\lambda)\, H_n^-(\lambda)$.

We discuss the case of $H_n^-(\lambda)$. Below $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, and $'$ differentiation with respect to y.

From (31) observe that

$$h_n(z) = \mathcal{F}^{-1}\left(\Phi_n^{+\,\prime}(y_n)\right) = \begin{cases} \frac{\partial \phi_n(y_n,z)}{\partial y} & , \quad z > h_n \\ 0 & , \quad z < h_n \end{cases} \ . \tag{75}$$

In view of the behavior of $\frac{\partial \phi_n(y_n,z)}{\partial y}$, (64), and the Abelian theorems, [12] pg. 36, we have that $H_n^-(\lambda)$ is analytic for $Re(\lambda) < k_2$ and has $O(|\lambda|^{-1/2})$ behavior as $|\lambda| \to \infty$.

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