Off–Diagonal Jacobi Matrices as a Model for Spectral Transition

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Abstract

We introduce a class of Jacobi matrices which model a deterministic (sparse) disorder in the sense that the perturbation of the Laplacean consists of a (direct) sum of fixed off–diagonal two by two matrices placed at sites whose distances from one another grow exponentially. We prove that the spectrum is the set $[-2,2]$ and there is a transition from (singular) continuous spectrum, for small "coupling", to (dense) pure point spectrum, for large "coupling", if the corresponding Prüfer angles are uniformly distributed (u.d.). We then prove that the latter sequence is u.d. almost everywhere for a certain range of parameters which is a result of independent interest.

1 Introduction and Summary

A basic open problem in the theory of disordered systems is the Anderson transition from localized to extended states, demonstrated for the Cayley tree in a remarkable paper [K]. Up to now, there is no proof of a spectral transition from absolutely continuous (a.c.) to dense pure point (p.p.) in lattice models with random or deterministic disorder, when a "coupling constant" and/or disorder vary. Following a suggestion of Howland [H], the

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existence of a spectral transition is addressed in this paper for a class of off–diagonal Jacobi matrices using the powerful methods of Last and Simon [LS] and Kiselev, Last and Simon [KLS], in which the nature of the spectrum is related to properties of transfer matrices.

We consider off–diagonal Jacobi matrices

$$J = \begin{pmatrix} 0 & p_0 & 0 & 0 & \cdots \\ p_0 & 0 & p_1 & 0 & \cdots \\ 0 & p_1 & 0 & p_2 & \cdots \\ 0 & 0 & p_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \tag{1.1}$$

with $p_n = p_n(\kappa)$, $n = 0, 1, 2, \ldots$, real–valued functions of a parameter κ which plays the role of a "coupling constant". Without loss of generality, $0 \leq \kappa \leq 1$ and we assume that $p_n : [0, 1] \longrightarrow \mathbb{R}_+$ is a continuous function for each j. Furthermore, as far as "sparseness" condition is concerned, $(p_n)_{n\geq 0}$ is required to be of the form

$$p_n = \begin{cases} 1 - r_j & \text{if} \quad n = a_j \in A \\ 1 & \text{if} \quad n \notin A \end{cases} , \tag{1.2}$$

with

$$r_j(0) = 0 \qquad \text{and} \qquad r_j(1) = 1 \, , \tag{1.3}$$

where $A = \{a_j\}_{j\geq 1}$ is a set of strictly positive integers such that

$$a_j - a_{j-1} \geq 2 \ , \qquad j = 2, 3, \ldots \ , \tag{1.4}$$

and

$$\lim_{j\to\infty} \frac{a_{j+1}}{a_j} = \beta > 1 \, , \tag{1.5}$$

where β is the "sparseness parameter".

A concrete example satisfying (1.3) is given by

$$r_j = \frac{\kappa}{1 + (1 - \kappa)\, \zeta_j} \tag{1.6}$$

with $\zeta_j \geq 0$. If $\zeta_j \to \infty$ as $j \to \infty$, then

$$r_j(\kappa) \to 0 \, , \tag{1.7}$$

for all $\kappa \in [0, 1)$ and $r_j(1) = \kappa$. If $\zeta_j = 0$ for all j, then

$$r_j(\kappa) = \kappa \, . \tag{1.8}$$

In both cases (either $\zeta_j \to \infty$ or $\zeta_j = 0$), as κ varies in the interval $(0,1)$, J interpolates continuously two opposite situations: J has purely a.c. spectrum at $\kappa = 0$ and purely point spectrum at $\kappa = 1$. Note that, in both cases, $p_n(1) = 0$ for all $n \in A$ and J becomes a direct sum of finite matrices with dense p.p. spectrum if (1.5) holds. This is Howland's remark [H], which shows off–diagonal Jacobi matrices as natural examples in which a spectral transition might be expected.

We shall call the class of Jacobi matrices satisfying (1.2)-(1.5) Howland's class $\mathcal{H}$, and the subset of those satisfying (1.7), $\mathcal{H}_0$. Note that $\mathcal{H}\backslash\mathcal{H}_0$ includes those in which (1.8) holds.

Let J_0 be the free Jacobi matrix

$$J_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

The spectrum $\sigma(J_0) = \sigma_{ac}(J_0) = [-2,2]$ is purely absolutely continuum. If $J \in \mathcal{H}_0$, $Q = J - J_0$ is a compact perturbation of J_0 and thus, by Weyl's invariance theorem, the essential spectrum of J remains the same as that of J_0, $\sigma_{\rm ess}(J) = \sigma_{\rm ac}(J_0) = [-2,2]$. The essential spectrum of J in this subclass is described as follows.

Theorem 1.1 *Let $J \in \mathcal{H}_0$. Then $\sigma_{\rm ess}(J)$ is purely a.c. if $\sum_{k=1}^{\infty} |r_k|^2 < \infty$ and purely s.c. if $\sum_{k=1}^{\infty} |r_k|^2 = \infty$ and "sparseness parameter" $\beta = \infty$.*

Theorem 1.1 may be proven along the lines of Theorem 1.7 of Kiselev-Last-Simon [KLS]. It has the following

Corollary 1.2 *Let J be as in Theorem 1.1. If $\sum_{k=1}^{\infty} |r_k|^2 < \infty$, then*

$$\sigma_{\rm ess}(J) = \begin{cases} \sigma_{\rm ac}(J) & \text{if} \quad 0 \le \kappa < 1 \\ \sigma_{\rm pp}(J) & \text{if} \quad \kappa = 1 \end{cases} .$$

If $\sum_{k=1}^{\infty}|r_k|^2=\infty$, *then*

$$\sigma_{\mathrm{ess}}(J)=\begin{cases}\sigma_{\mathrm{ac}}(J) & \text{if} \quad \kappa=0\\ \sigma_{\mathrm{sc}}(J) & \text{if} \quad 0<\kappa<1\\ \sigma_{\mathrm{pp}}(J) & \text{if} \quad \kappa=1\end{cases}.$$

Hence, the spectral transition in the subclass $\mathcal{H}_0$ occurs only at the end points of the range of the "coupling constants", at least under the sparseness condition (1.5).

Corollary 1.2 shows that, in order to occur a transition at a nontrivial value of κ, J has to be so that (1.7) does not hold (i.e. $r_n \nrightarrow 0$) and it is likely to be from s.c. to (dense) p.p., with a.c. spectrum just for $\kappa=0$. Our main result is as follows

Theorem 1.3 *Let $J\in\mathcal{H}\backslash\mathcal{H}_0$ with $r_j=1-p$ for all $j\in\mathbb{N}$, $p\in(0,1)$ and $\beta\in\mathbb{N}$ such that $p^2(\beta-1)>c_2$ holds with $c_2=1+O(p^2)$ as given in (5.5). Then, apart from a set with Lebesgue measure* 0,

$$\left\{\lambda\in[-2,2]:\left(\frac{p}{1-p^2}\right)^2(\beta-1)\left(4-\lambda^2\right)>1\right\} \tag{1.9}$$

is contained in the essential support Σ_{sc} of the s.c. spectral measure and

$$\left\{\lambda\in[-2,2]:\left(\frac{p}{1-p^2}\right)^2(\beta^3-1)(4-\lambda^2)<1\right\}. \tag{1.10}$$

is contained in the support Σ_{pp} of the (dense) p.p. measure.

Remark 1.4

1. *Condition $p^2(\beta-1)>c_2$ is of technical nature (see Remark 3 in Section 5). Theorem 1.3 is expected to hold for all $p\in(0,1)$ and $\beta\in\mathbb{N}$, $\beta>1$.*

2. *Take e.g. $\beta=2$: (1.9) holds for p sufficiently close to 1, i.e. for small "coupling" $\kappa=1-p$, and λ never in a neighborhood U near the edges of the band ± 2, while (1.10) holds for p sufficiently close to zero ("large" κ) and λ always close to the band edges (see Figure 1). These features are also expected in the Anderson model, with the continuous part of the spectrum being, however, absolutely continuous.*

3. *The set of zero Lebesgue measure in (1.9) may be p.p. and the one in (1.10) may be s.c.*

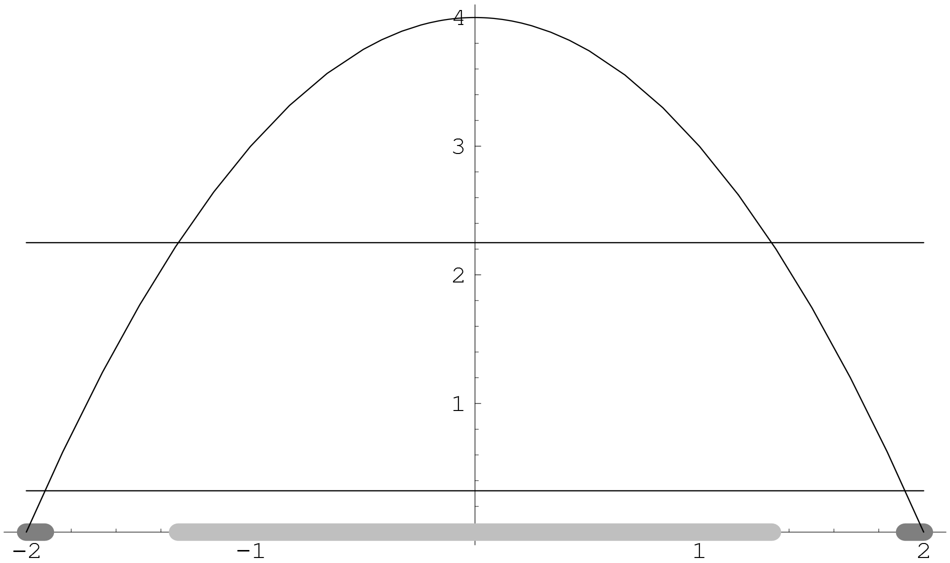

Figure 1: Singular continuous (light gray) and pure point (medium gray) spectrum for $\beta = 2$ and $p = 1/2$.

When $r_n \nrightarrow 0$, i.e. (1.8), the perturbation is not compact, which is a common feature with the theory of (random) disordered systems. This is the case we consider in the rest of the paper. Weyl's invariance principle is thus not applicable, and it is necessary to determine the spectrum. This is done in Section 2, where we also introduce the necessary notation. In Section 3 we show that a special choice of Prüfer (modulus and angle) variables is particularly convenient in the off–diagonal case. In Section 4 we recall the basic results of Last and Simon [LS] that we need, and state and prove our main Lemma, which roughly states that the spectral transition, from s. c. at small coupling to (dense) p. p. at large coupling, occurs whenever the corresponding Prüfer angles are uniformly distributed (u.d.). In Section 5 Theorem 1.3 is established by proving the latter assumption employing an extended version of Koksma's Theorem. A conclusion with open problems is presented in Section 6.

2 Some Notation and a Theorem on the Spectrum

A bounded Jacobi matrix J is a matrix of a essentially self–adjoint operators H on a separable Hilbert space $\mathscr{H}$: $J_{ij} = (\varphi_i, H\varphi_j)$, where $\{\varphi_j\}_{j\geq 0}$ is an orthonormal base of $\mathscr{H}$. The spectral measure $\mu = d\rho$ associated to J is given by the 00–element of the resolvent matrix [A, S]

$$(J - \lambda I)^{-1}_{00} = \left(\varphi_0, (H-\lambda)^{-1}\varphi_0\right) = \int_{-\infty}^{\infty} \frac{1}{x-\lambda} d\rho(x)\,. \tag{2.1}$$

We introduce, for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, a 2×2 transfer matrix

$$T(n, n-1; \lambda) = \begin{pmatrix} \dfrac{\lambda}{p_n} & \dfrac{-p_{n-1}}{p_n} \\ 1 & 0 \end{pmatrix}\,, \tag{2.2}$$

associated to the $l_2(\mathbb{N})$–solutions u of $(J - \lambda I)\,u = 0$, and set

$$T(n;\lambda) := T(n,n-1;\lambda)\,T(n-1,n-2;\lambda)\cdots T(0,-1;\lambda)\,, \tag{2.3}$$

with $p_{-1} = 1$. If $J \in \mathcal{H}$, in view of (1.2) and (1.4), there are three situations to be considered: for $n \notin \{a_j, a_j+1\}_{j\geq 1}$, $T(n,n-1;\lambda)$ is equal to the free transfer matrix

$$T_0(\lambda) = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}\,; \tag{2.4}$$

for $n = a_j + 1$ for some $j \geq 1$,

$$T(n,n-1;\lambda) = T_+(a_j;\lambda) := \begin{pmatrix} \lambda & -1 + r_j \\ 1 & 0 \end{pmatrix}\,; \tag{2.5}$$

and for $n = a_j$ for some $j \geq 1$,

$$T(n,n-1;\lambda) = T_-(a_j;\lambda) := \begin{pmatrix} \dfrac{\lambda}{1-r_j} & \dfrac{-\lambda}{1-r_j} \\ 1 & 0 \end{pmatrix}\,. \tag{2.6}$$

Let $\lambda = 2\cos\varphi$ where $\varphi \in (0,\pi)$ (see Theorem 2.1 for a proof that this parameterization is exhaustive), and

$$U = \begin{pmatrix} 0 & \sin\varphi \\ 1 & -\cos\varphi \end{pmatrix} \tag{2.7}$$

be the nonsingular matrix with

$$U^{-1} = \begin{pmatrix} \cot\varphi & 1 \\ \csc\varphi & 0 \end{pmatrix}\,.$$

The matrix U has a remarkable property of transforming $T_0(\lambda)$ into a pure rotation

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = U T_0(\lambda) U^{-1}\,. \tag{2.8}$$

It will be most convenient to introduce, for each $a_j \in A$, the matrices $P_{+-}(a_j)$ given by

$$T_{+-}^{U}(a_j) = R(\varphi)\, P_{+-}(a_j) R(\varphi) \tag{2.9}$$

where $T_{+-}^{U}(a_j) = UT_{+}(a_j)\, T_{-}(a_j)\, U^{-1}$. From (2.3) - (2.9) and, since a product of rotations is a rotation, we have

$$UT(n;\lambda)U^{-1} = R\left((n-a_k)\varphi\right) P_{+-}(a_k) R\left((a_k - a_{k-1})\varphi\right) \cdots P_{+-}(a_1) R\left((a_1)\varphi\right) \tag{2.10}$$

which will be useful in Section 3.

According to (1.2) and (1.8), in the equations above

$$r_j = 1 - p_{a_j} \tag{2.11}$$

where $\left(p_{a_j}\right)_{j\geq 1} \equiv 0$ refers to p. p. case, and $\left(p_{a_j}\right)_{j\geq 1} \equiv 1$ to the unperturbed (a. c.) case. We have

Theorem 2.1 *If $J \in \mathcal{H}$ and $0 \leq r_j \leq 1$ for all $j \in \mathbb{N}$, i. e. $p_{a_j} \in [0,1]$, then*

$$\sigma(J) = [-2, 2] \;\; ,$$

where $\sigma(A)$ denotes the spectrum of A.

Proof. Let $J \in \mathcal{H}$ with $(p_n)_{\geq 0}$ so that $0 \leq p_{a_j} \leq 1$, $\forall j$. To prove $\sigma(J) \subseteq [-2,2]$, let $u = (u_n)_{n\geq 0} \in l_2(\mathbb{N})$. We have

$$(u, Ju) = \sum_{n=0}^{\infty} (u_n\, u_{n+1})\, h_n \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} \tag{2.12}$$

where

$$h_n = \begin{pmatrix} 0 & p_n \\ p_n & 0 \end{pmatrix} .$$

Inserting

$$h_n = \lambda_n^{+}\, P_n^{+} + \lambda_n^{-}\, P_n^{-}$$

into (2.12), where $P_n^{\pm}$ projects onto the eigendirection associated to the eingenvalue $\lambda_n^{\pm} = \pm p_n$, we have

$$-2\,\overline{\lambda} \leq \frac{(u, Ju)}{(u,u)} \leq 2\,\overline{\lambda}$$

where $\overline{\lambda} = \sup_n \lambda_n^{+} = \sup_n p_n = 1$. Hence

$$\sigma(J) \subseteq \left[-2\,\overline{\lambda}, 2\,\overline{\lambda}\right] = [-2, 2] \; . \tag{2.13}$$

Note that, because J is a symmetric off–diagonal matrix, the spectrum is symmetric and $\inf_n \lambda_n^- = -\sup_n \lambda_n^+ = -\overline{\lambda}$. In order to prove the opposite inclusion

$$\sigma(J) \supseteq [-2, 2] \;, \tag{2.14}$$

we use Weyl's criterion (see Theorem VII.12 of [S1]): if A is a bounded self–adjoint operator in separable Hilbert space $\mathscr{H}$, λ is a point of the spectrum $\sigma(A)$ of A if and only if there exist a sequence $(\psi_n)_{n\geq 0}$ in $\mathscr{H}$, with $\|\psi_n\| = 1$, such that

$$\lim_{n\to\infty} \|(A-\lambda)\,\psi_n\| = 0\,. \tag{2.15}$$

Let $\lambda = 2\cos\varphi$, $\varphi \in [0, \pi]$, and define $\psi_n = (1/\sqrt{n})\,(e^{i\varphi}, e^{i2\varphi}, \ldots, e^{in\varphi}, 0, \ldots)$. Clearly $\psi_n \in l_2(\mathbb{N})$ and we claim that

$$\|(J-\lambda)\,\psi_n\| \leq c\frac{\ln n}{\sqrt{n}} \tag{2.16}$$

holds with $c = c(\beta)$ independent of n for any $J \in \mathcal{H}$. In order to prove (2.16), notice that $(J-\lambda)\,\psi_n$ consists of the action on ψ_n of a sum of local matrices, each bounded in norm by one. Two of them involve the extremes $e^{i\varphi}$ and $e^{in\varphi}$, and there are $O(\ln n)$ non–diagonal matrices, each acting just on two successive components of ψ_n. The $O(\ln n)$ comes from the sequence $(a_j)_{j\geq 1}$ satisfying the sparseness condition (1.5), $a_j \sim \beta^j$, with at most r points a_j within $[1, n]$, where r is such that $\beta^r \leq n$ or, equivalently,

$$r \leq \frac{\ln n}{\ln \beta}\;.$$

This proves (2.16) and therefore (2.14), concluding the proof of Theorem 2.1.

□

3 Prüfer Variables

We introduce the so called (modified) Prüfer variables for the off–diagonal Jacobi matrices in consideration. For $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, let $T(n, n-1; \lambda)$ be the 2×2 transfer matrix given by (2.2) and note that, if $\mathbf{u}_n = \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$ is a vector in $\mathbb{C}^2$ composed by the components of a $l_2(\mathbb{N})$–solution $u = (u_0, u_1, \ldots)$ of $(J-\lambda)\,u = 0$, then

$$\mathbf{u}_{n+1} = T(n, n-1; \lambda)\,\mathbf{u}_n$$

holds for $n = 0, 1, \ldots$, with $\mathbf{u}_0 = \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix}$ satisfying some initial data.

Writing

$$T_0^k = \underbrace{T_0\, T_0 \cdots T_0}_{k-\text{times}}$$

and

$$T_{+,-}(a;\lambda) = T_+(a;\lambda)\, T_-(a;\lambda)\,,$$

where T_0 and $T_\pm$ is given by (2.4)-(2.6), equation (2.3) can be written as

$$T\,(n;\lambda) = T_0^{n-a_k-1}\, T_{+,-}(a_k)\, T_0^{a_k-a_{k-1}-2}\, T_{+,-}(a_{k-1})\, \cdots\, T_{+,-}(a_1)\, T_0^{a_1-1}\,, \tag{3.1}$$

from which equation (2.10) can be obtained following the indications of Section 2. Inserting

$$R(-\varphi)UT_+(a)U^{-1} = \begin{pmatrix} p_a & 0 \\ (1-p_a)\cot\varphi & 1 \end{pmatrix}$$

and

$$UT_-(a)U^{-1}R(-\varphi) = \begin{pmatrix} 1 & 0 \\ (1/p_a - 1)\cot\varphi & 1/p_a \end{pmatrix}$$

into (2.9) gives

$$P_{+-}(a) = \begin{pmatrix} p_a & 0 \\ \alpha_a \cot\varphi & 1/p_a \end{pmatrix} \tag{3.2}$$

where

$$\alpha_a = \frac{1}{p_a} - p_a.$$

We now use equation (2.10), where R and $P_{+,-}$ are given by (2.8) and (3.2), respectively, to define the Prüfer variables $(R_k, \theta_k)_{k\geq 0}$ as follows. Let

$$\mathbf{v}_k = (R_{k-1}\cos\theta_k, R_{k-1}\sin\theta_k) \tag{3.3}$$

and

$$\widetilde{\mathbf{v}}_k = \left(R_k \cos\widetilde{\theta}_k, R_k \sin\widetilde{\theta}_k\right) \tag{3.4}$$

be given by

$$\mathbf{v}_k = R\left((a_k - a_{k-1})\varphi\right)\widetilde{\mathbf{v}}_{k-1} \tag{3.5}$$

and

$$\widetilde{\mathbf{v}}_k = P_{+-}\left(a_k\right)\mathbf{v}_k\,, \tag{3.6}$$

with initial condition $\widetilde{\mathbf{v}}_0 = \mathbf{v}_0 = (\cos\theta_0, \sin\theta_0)$. With these notation equation (2.10) reads

$$\widetilde{\mathbf{v}}_k = UT\,(a_k;\lambda)\,U^{-1}\mathbf{v}_0$$

and we have

$$R_k^2 = \|\mathbf{v}_{k+1}\|^2 = \|\widetilde{\mathbf{v}}_k\|^2 = R_{k-1}^2 \left\| \begin{pmatrix} p_{a_k} & 0 \\ \alpha_{a_k}\cot\varphi & 1/p_{a_k} \end{pmatrix} \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix} \right\|^2$$

and

$$\theta_{k+1} = \arg\left(P_{+-}(a_k)\mathbf{v}_k\right) + (a_{k+1} - a_k)\,\varphi$$

which yield

$$\frac{R_k^2}{R_{k-1}^2} = p_{a_k}^2 \cos^2\theta_k + \left(\alpha_{a_k}\cot\varphi\cos\theta_k + \frac{1}{p_{a_k}}\sin\theta_k\right)^2. \tag{3.7}$$

and

$$\theta_{k+1} = \tan^{-1}\left(\frac{1}{p_{a_k}^2}\tan\theta_k + \frac{\alpha_{a_k}}{p_{a_k}}\cot\varphi\right) + (a_{k+1} - a_k)\,\varphi\,. \tag{3.8}$$

The Euclidean norm of $UT\,(n;\lambda)\,U^{-1}$, given by

$$\left\|UT\,(n;\lambda)\,U^{-1}\mathbf{v}_0\right\|^2 = \left\|UT\,(a_N;\lambda)\,U^{-1}\mathbf{v}_0\right\|^2 = R_N^2, \tag{3.9}$$

for any n such that $a_{N+1} < n \leq a_N$, can thus be written as

$$\begin{aligned} R_N^2 &= \frac{R_N^2}{R_{N-1}^2}\frac{R_{N-1}^2}{R_{N-2}^2}\cdots\frac{R_1^2}{R_0^2} \\ &= \left(\exp\left\{\frac{1}{N}\sum_{k=1}^{N}\left(f\left(p_{a_k},\theta_k\right) - \ln p_{a_k}^2\right)\right\}\right)^N, \end{aligned} \tag{3.10}$$

where

$$f(p,\theta) = \ln\left(p^4\cos^2\theta + \left(\sin\theta + \left(1 - p^2\right)\cot\varphi\cos\theta\,\right)^2\right) \tag{3.11}$$

with the Prüfer angles $(\theta_k)_{k\geq 0}$ satisfying the recursion relation

$$\theta_{k+1} = g(p_{a_k},\theta_k) + (a_{k+1} - a_k)\,\varphi \tag{3.12}$$

where

$$g(p,\theta) = \tan^{-1}\left(\frac{1}{p^2}\left(\tan\theta + \cot\varphi\right) - \cot\varphi\right). \tag{3.13}$$

4 The Main Lemma

In this section we prove the main Lemma connecting uniform distribution of the Prüfer angles with the spectral transition. We use the powerful theorems of Y. Last and B. Simon [LS] relating the nature of the spectrum of J to the properties of the transfer matrix $T(n;\lambda)$: the essential support Σ_{ac} of the a.c. part μ_{ac} of spectral measure μ is given by

$$\Sigma_{ac} = \left\{ \lambda : \liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \|T(n;\lambda)\|^2 < \infty \right\} \tag{4.1}$$

and, consequently, the essential support Σ_s of the singular spectrum is given by

$$\Sigma_s = \left\{ \lambda : \lim_{j\to\infty} \|T(n_j;\lambda)\| = \infty \text{ for a subsequence } \{n_j\} \text{ with } \lim_{j\to\infty} n_j = \infty \right\} \tag{4.2}$$

By Theorems 1.6 and 1.7 of [LS] the eigenvalue equation $(J-\lambda I)u = 0$ has no $l_2(\mathbb{N})$–solutions if

$$\sum_{n=0}^{\infty} \|T(n;\lambda)\|^{-2} = \infty \tag{4.3}$$

and **at least one** $l_2(\mathbb{N})$–solution if

$$\sum_{n=0}^{\infty} \|T(n;\lambda)\|^2 \left(\sum_{k=n}^{\infty} \|T(k;\lambda)\|^{-2} \right)^2 < \infty\,. \tag{4.4}$$

We say that the Prüfer angles $(\theta_k)_{k\geq 1}$ defined by (3.3) are uniformly distributed modulo π if, for any interval $I = (a,b) \subset [0,\pi]$

$$\lim_{N\to\infty} \frac{\operatorname{card}\left(\{k : \theta_k \in I,\ 1 \leq k \leq N\}\right)}{N} = |I| \equiv b-a \tag{4.5}$$

where $\operatorname{card}(S)$, the cardinality of the set S, is the number of elements in S.

For simplicity, and without loss of generality, we fix here and henceforth $p_{a_k} = p$ for all k, $0 < p < 1$, and take $A = (a_k)_{k\geq 1}$ so that

$$a_k - a_{k-1} = \beta^k \tag{4.6}$$

i. e., $a_k = a_k(\beta) = \beta^{k+1}/(\beta-1)$, $\beta > 1$. In this case (3.10) and (3.12) read

$$R_N^2 = \left(\frac{1}{p^2} \exp\left\{ \frac{1}{N} \sum_{k=1}^{N} f(\theta_k) \right\} \right)^N \tag{4.7}$$

with

$$\theta_k = g(\theta_{k-1}) + \beta^k \varphi \tag{4.8}$$

where $f(\theta) = f(p, \theta)$ is given by (3.11) and $g(\theta) = g(p, \theta)$ by (3.13).

Let us assume that the sequence $(\theta_k)_{k\geq 1}$ given by (4.8) is uniformly distributed modulo π (abbreviated u.d. mod π)[1]. We have (see Corollary 1.1 of [KN]):

Theorem 4.1 *A sequence $\omega = (\theta_k)_{k\geq 1}$ of real numbers is u.d. mod π if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\theta_k) = \frac{1}{\pi} \int_0^{\pi} f(\theta)\, d\theta\,, \tag{4.9}$$

holds for every Riemann–integrable periodic function f of period π.

We are now ready to state and prove

Lemma 4.2 (Main) *If the Prüfer angles $(\theta_k)_{k\geq 1}$ are u.d. mod π for almost every $\varphi \in (0, \pi)$, then the conclusions of Theorem 1.3 hold.*

Proof. We must verify that the conditions on p, β and $\lambda = 2\cos\varphi$ which define (1.9) and (1.10) imply (4.3) and (4.4), respectively.

By assumption (4.9) the large N–behavior of R_N is governed by (4.7) with the sum in the exponent substituted by the integral (4.9). In order to employ Theorem 4.1 it is necessary to ascertain that f is a Riemann–integrable periodic function of period π, and compute its integral. By simple trigonometric identities (3.11) may be written in the form

$$f(\theta) = \ln\left(a + b\cos 2\theta + c\sin 2\theta\right) \tag{4.10}$$

where

$$\begin{aligned} 2a &= \left(1 - p^2\right)^2 \cot^2\varphi + 1 + p^4 \\ 2b &= \left(1 - p^2\right)^2 \cot^2\varphi - 1 + p^4 \end{aligned}$$

and

$$c = \left(1 - p^2\right) \cot\varphi\,,$$

which shows that f is periodic with period π. An explicitly computation gives

$$\begin{aligned} a + b\cos 2\theta + c\sin 2\theta &\geq \min_{\theta}\left(a + b\cos 2\theta + c\sin 2\theta\right) \\ &= a - \sqrt{b^2 + c^2} = a - \sqrt{a^2 - p^4} > 0 \end{aligned} \tag{4.11}$$

[1] The result of Section 5 holds for all initial data θ_0 but, according Theorem 2.3 of [KLS], it is enough to hold for two distinct values.

since

$$0 < b^2 + c^2 = a^2 - p^4 \tag{4.12}$$

if $0 < p < 1$, and this implies that $f(\theta)$ is Riemann–integrable. From [PBM], **29** p. 546 and (4.12), we have

$$\begin{aligned}
\frac{1}{\pi}\int_0^{\pi} f(\theta)\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \ln\left(a + b\cos x + c\sin x\right)\, dx \\
&= \ln\left(\frac{a + \sqrt{a^2 - b^2 - c^2}}{2}\right) \\
&= \ln\left(\frac{(1-p^2)^2\csc^2\varphi}{4} + p^2\right)
\end{aligned}$$

which, together with (4.7) and (4.9), yields

$$C_1\left(\frac{(1-p^2)^2\csc^2\varphi}{4p^2} + 1\right)^N \le R_N^2 \le C_2\left(\frac{(1-p^2)^2\csc^2\varphi}{4p^2} + 1\right)^N \tag{4.13}$$

for some positive constants C_1 and C_2 uniformly in N. Consequently, using

$$\begin{aligned}
\sum_{m\ge n} \|T(m;\lambda)\|_U^{-2} &\le \sum_{k\ge N} \|T(a_k;\lambda)\|_U^{-2} \sum_{a_k\le m<a_{k+1}} 1 \\
&\le \sum_{k\ge N} \beta^{k+1} R_k^{-2}\,,
\end{aligned}$$

with $a_N \le n < a_{N+1}$, and equations (A.2) and (A.3) of Appendix A, we have

$$\frac{\beta}{C\,C_2}\, S_N \le \sum_{m\ge n} \|T(m;\lambda)\|^{-2} \le \frac{\beta C}{C_1}\, S_N$$

where

$$S_N = \sum_{k=N}^{\infty} \beta^k \left(\frac{(1-p^2)^2\csc^2\varphi}{4p^2} + 1\right)^{-k}$$

converges if

$$4p^2(\beta - 1) < (1-p^2)^2\csc^2\varphi \tag{4.14}$$

and diverges if

$$4p^2(\beta - 1) > (1-p^2)^2\csc^2\varphi\,. \tag{4.15}$$

In view of (4.3) and (4.2), $\lambda \in \sigma(J) \subseteq [-2,2]$ belongs to the essential support of the s.c. spectrum of J provided (4.15) holds which, in view of $4\csc^{-2}\varphi = 4 - \lambda^2$, yields (1.9).

Now, assuming that (4.14) holds, in view of (4.4), $\lambda \in \sigma(J)$ is an eigenvalue of J if

$$S = \sum_{N=1}^{\infty} \beta^N R_N^2 S_N^2 \leq C' \sum_{N=1}^{\infty} \left(\frac{4p^2 \beta^3}{(1-p^2)^2 \csc^2 \varphi + 4p^2} \right)^N$$

converges, i.e.,

$$\frac{p^2}{(1-p^2)^2} (\beta^3 - 1)(4 - \lambda^2) < 1 \tag{4.16}$$

which yields (1.10) and concludes the proof of Theorem 4.2.

□

Remark 4.3

1. *The conditions (4.15) for divergence of S_N and (4.16) for convergence of S will not change if J belongs to the class $\mathcal{H}/\mathcal{H}_0$ with $\lim_{k\to\infty} a_k/a_{k-1} = \beta$ and $\lim_{k\to\infty} p_{a_k} = p$.*

5 Uniform Distribution of Prüfer Angles

We shall now establish (4.9) for $(\theta_k)_{k\geq 1}$ a sequence of numbers $\theta_k = \theta_k(\beta, p, \varphi, \theta_0)$ generated by the recursion (4.8). The proof that this sequence is u.d. mod π applies general metric theorems (see e. g. [KN]) which yields almost everywhere convergence with respect to the parameter dependence of numerical sequences. We shall fix $p \in (0,1)$, $\beta \in \mathbb{N}$ greater than 1, $\theta_0 \in (0,\pi)$ and consider the sequence $\theta_k = \theta_k(\varphi)$, $k = 1, 2, \ldots$, defined for each parameter φ in an interval I. In order to illustrate the technique to be employed, $\theta_k = \theta_k(\beta)$ will be considered as a function of sparseness parameter β and we shall investigate whether $(\theta_k)_{k\geq 1}$ is u.d. mod π in the limit $p \to 0$ with θ_0, $\varphi \in (0,\pi)$ fixed.

As $p \to 0$, (3.13) converges to the step function

$$\lim_{p\to 0} g(p,\theta) = \begin{cases} -\pi/2 & \text{if} \quad -\pi/2 < \theta < \varphi - \pi/2 \\ \pi/2 & \text{if} \quad \varphi - \pi/2 < \theta < \pi/2 \end{cases}$$

and (4.8) is simply given by

$$\theta_k = \pm\frac{\pi}{2} + \beta^k \varphi \tag{5.1}$$

as long as

$$\tan\theta_k + \cot\varphi \neq 0\,, \tag{5.2}$$

i. e., as long as $\theta_k(\varphi) \notin \{\varphi + (2m+1)\pi/2,\, m \in \mathbb{N}\} \equiv \mathcal{M}(\varphi)$.

Each point of $\mathcal{M}$ is a linear function of φ with slope 1 . Using the fact that $g'(\theta) \geq 0$ (see (5.7) below), it follows from

$$\frac{\partial \theta_k}{\partial \varphi} = g'(\theta_{k-1}) \frac{\partial \theta_{k-1}}{\partial \varphi} + \beta^k ,$$

and an inductive argument that $\partial \theta_k / \partial \varphi \geq \beta^k$ and θ_k is a monotone increasing function of φ . Hence, as φ varies over $(0, \pi)$, $\theta_k(\varphi)$ intercepts each linear function $\varphi + (2m+1)\pi/2$ of $\mathcal{M}$ in at most one point. For each k, $\Omega_k = \{\varphi \in (0, \pi) : \theta_k(\varphi) \in \mathcal{M}(\varphi)\}$ is a denumerable (finite) set and (5.1) holds for all $k \in \mathbb{N}$ and all $\varphi \in (0, \pi)$ except for a denumerable set $\Omega = \bigcup_k \Omega_k$.

Now, given a sequence of numbers $(u_k)_{k \geq 1}$let $(S_n)_{n \geq 1}$ be defined by

$$S_n = \frac{1}{n} \sum_{k=1}^{n} e^{2iu_k} .$$

According to Weyl criterion, the sequence $(u_k)_{k \geq 1}$ is u.d. mod π if and only if $\lim\limits_{n \to \infty} S_n = 0$ .If $u_k = u_k(x)$ is defined for each x lying in some interval $[a, b] \subset \mathbb{R}$, then $S_n = S_n(x)$ is defined for each $x \in [a, b]$. Denoting by $\|f\|_{L_2([a,b])} = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}$ the L_2 norm, the general metric result is as follows (see Theorem 4.2 of [KN], for a proof)

Theorem 5.1 *If the series*

$$\sum_{n=1}^{\infty} \frac{1}{n} \|S_n\|^2_{L_2([a,b])}$$

converges then the sequence $\{u_k\}$ is u.d. mod π for almost all $x \in [a, b]$, i. e. $\{u_k\}$ is u.d. mod π for every x lying in $[a, b]$ apart from a set with Lebesgue measure 0.

Returning to the sequence $(\theta_k(\beta))_{k \geq 1}$ in the limit $p \to 0$, except by the set Ω of values φ of Lebesgue measure 0 that (5.1) do not hold, $\theta_k = \pm \pi/2 + \beta^k \varphi$ is continuously differentiable with respect to β and

$$\theta'_k - \theta'_m = \beta^{k-1} \left(k - \beta^{-k+m} m \right) \varphi \geq 1 \tag{5.3}$$

increases monotonously for any $k > m$. These are sufficient to apply the following powerful result.

Theorem 5.2 (Koksma's Metric Theorem) *Let $(u_k(x))_{k \geq 1}$ be a sequence of real valued continuously differentiable functions defined in an interval $[a, b]$. If for any two positive integers $k \neq m$, $u'_k - u'_m$ is a monotone function of x such that $|u'_k - u'_m| \geq K > 0$ holds uniformly in k, m and $x \in [a, b]$, then $(u_k)_{k \geq 1}$ is u.d. mod π for almost all $x \in [a, b]$.*

Theorem 5.2 cannot be employed for $p \in (0,1)$ in the present form. However, for convenience, the proof of Koksma's theorem will be included in order to identify the problems and the ways to circumvent them.

Proof. By definition,

$$\begin{aligned}\|S_n\|^2_{L_2([a,b])} &= \frac{1}{n^2}\sum_{k,m=1}^{n}\int_a^b e^{2i(u_k(x)-u_m(x))}\,dx \\ &\leq \frac{|b-a|}{n}+\frac{2}{n^2}\sum_{k=2}^{n}\sum_{m=1}^{k-1}\left|\int_a^b e^{2i(u_k(x)-u_m(x))}\,dx\right| .\end{aligned}$$

Under the hypothesis, the second mean value theorem holds and there exist c , with $a<c<b$, such that

$$\begin{aligned}\left|\int_a^b e^{2i(u_k(x)-u_m(x))}\,dx\right| &= \left|\frac{1}{2i}\int_a^b \frac{de^{2i(u_k(x)-u_m(x))}}{u_k'(x)-u_m'(x)}\right| \\ &= \left|\frac{1}{2i}\left(\frac{1}{u_k'(a)-u_m'(a)}\int_a^c de^{2i(u_k(x)-u_m(x))}+\right.\right. \\ &\quad \left.\left.+\frac{1}{u_k'(b)-u_m'(b)}\int_c^b de^{2i(u_k(x)-u_m(x))}\right)\right| \\ &\leq \frac{1}{|u_k'(a)-u_m'(a)|}+\frac{1}{|u_k'(b)-u_m'(b)|} . \end{aligned} \tag{5.4}$$

By ordering $\{u_k'(x)\}_{k=1}^n$ such that $u_1'(x)<u_2'(x)<\cdots<u_n'(x)$ holds for all x, the difference of any two consecutive numbers $\left|u_k'(x)-u_{k-1}'(x)\right|>K$ and we have

$$\sum_{m=1}^{k-1}\frac{1}{|u_k'(x)-u_m'(x)|}<2\sum_{m=1}^{n}\frac{1}{mK}<\frac{2}{K}\ln 3n .$$

Putting all together, yields

$$\|S_n\|^2_{L_2([a,b])}<\frac{|b-a|}{n}+\frac{4}{nK}\ln 3n$$

which combined with Theorem 5.1 concludes the proof of the theorem.

□

Remark 5.3

1. *Since a_k-a_{k-1} is an integer number (the distance between two nonzero elements of $J-J_0$ along the second diagonal) β, under the parameterization (4.6), has to be an integer. Hence, Koksma's theorem, which holds almost everywhere, cannot ensure that $(\theta_k(\beta))_k$ is u.d. mod π for $\beta=2,3,\ldots$. The parameter dependence of θ_k from here on will be switched from β to φ.*

2. *To apply Theorem 5.2, $\theta'_k(\varphi) - \theta'_m(\varphi)$ has to be a monotone function for each interval $[a, b]$ independent of k and m. But this is not true for the sequence (4.8) for $0 < p < 1$. As we shall see, the number of monotone intervals of $\theta'_k - \theta'_m$ in $I \subset (0, \pi)$, with $k < m$ fixed, is proportional to β^k and this number is compensated by the constant K of Theorem 5.2 which, in our case, has the same dependence as in (5.3).*

We now need some properties of (3.13).

Proposition 5.4 *The function g given by (3.13) maps $[-\pi/2, \pi/2]$ into itself and has three fixed point solutions: $\theta^s_\pm = \pm\dfrac{\pi}{2}$ are two stable solutions and $\theta^u = \varphi - \dfrac{\pi}{2}$, $\varphi \in (0, \pi)$, is unstable. g is monotone increasing with $g'(\theta^s_\pm) = p^2$, $g'(\theta^u) = 1/p^2$ and there are two inflection points $\theta^i_\pm$ such that*

$$g'(\theta^i_-) = c_1 p^2 \leq g'(\theta) \leq \frac{c_2}{p^2} = g'(\theta^i_+) \tag{5.5}$$

holds for $c_{1(2)} = 1 + O(p^2)$. In addition, $\theta^i_+ \to \theta^u$, $\theta^i_- \to \varphi$ (or $\varphi - \pi$) as $p \to 0$ and $g'(\theta) \leq O(1)$ for $\theta \in I \backslash \left[\theta^i_- - p, \theta^i_+ + p\right]$.

Proof. Writing $g(\theta) = \tan^{-1}(\zeta/p^2)$ where

$$\zeta(\theta) = \tan\theta + (1 - p^2)\cot\varphi, \tag{5.6}$$

it follows from

$$g'(\theta) = \frac{p^2}{p^4 + \zeta^2}\sec^2\theta \tag{5.7}$$

that g is a monotone increasing function with $g(-\pi/2) = -\pi/2$ and $g(\pi/2) = \pi/2$. Hence $g : [-\pi/2, \pi/2] \to [-\pi/2, \pi/2]$. As θ goes to $\pm\pi/2$, $\sec^2\theta/\zeta^2 \to 1$ and $g'(\theta) \to p^2$. Since $p < 1$, $\theta = \pm\pi/2$ are stable fixed point of g. To find the unstable fixed point we observe that

$$\tan g(\theta) + \cot\varphi = \frac{1}{p^2}(\tan\theta + \cot\varphi) \ ,$$

together with the fixed point equation $g(\theta) = \theta$ and $p < 1$, implies

$$\tan\theta^u + \cot\varphi = 0 \tag{5.8}$$

which is equivalent to $\theta^u = \varphi - \pi/2$. At $\theta = \theta^u$ equation (5.6) reads $\zeta(\theta^u) = p^2\cot\varphi$ which together with $\sec^2\theta^u = 1 + \tan^2\theta^u = 1 + \cot^2\varphi$ and (5.7) gives

$$g'(\theta^u) = \frac{1}{p^2} \, .$$

A direct computation yields

$$g''(\theta) = \frac{2p^2 \sec^2\theta}{(p^4+\zeta^2)^2}\left[(p^4+\zeta^2)\tan\theta - \zeta\sec^2\theta\right] \tag{5.9}$$

from where we get the two inflection points by solving $(p^4+\zeta^2)\tan\theta - \zeta\sec^2\theta = 0$ for $\tan\theta$:

$$\theta^i_{\pm} = \tan^{-1}\left(\mp(\cot\varphi)^{\pm 1}\left(1 \pm \frac{p^2}{\sec^2\varphi} + O\left(p^4\right)\right)\right) \tag{5.10}$$

gives $\theta^i_+ = \theta^u + O(p^2)$ and $\theta^i_- = \varphi + O(p^2)$, if $\varphi \in [0, \pi/2)$ and $\theta^i_- = \varphi - \pi + O(p^2)$ if $\varphi \in [\pi/2, \pi)$. To show that $g'(\theta) \leq O(1)$ if we are out of a p –neighborhood of θ^i_+, it is enough to observe that, using

$$\tan(\theta^u + p) = \tan\theta^u + p\sec^2\theta^u - p^2\tan\theta^u\,\sec^2\theta^u + O\left(p^3\right) ,$$

(5.8) and (5.7), we have

$$\begin{aligned} g'(\theta^u + p) &= \frac{p^2\csc^2\varphi\,(1 - 2p\cot\varphi + O(p^2))}{p^2\csc^4\varphi\left(1 - 2p\cot\varphi\dfrac{2+\cot^2\varphi}{1+\cot^2\varphi} + O(p^2)\right)} \\ &= \frac{1}{\csc^2\varphi}\,(1 + O(p)) \end{aligned}$$

and the last assertion follows from (5.10).

Since equation (5.5) follows from a similar calculation, this concludes the proof of Proposition 5.4.

□

The function $g : \mathbb{R} \longrightarrow [-\pi/2, \pi/2]$ is periodic and monotone in each interval $[-\pi/2, \pi/2] + m\pi$, $m \in \mathbb{Z}$. So $\theta_k(\varphi)$, given by (4.8), is a piecewise monotone function of φ. Note that the derivative at right coincides with the derivative at left of every discontinuity point of g and this property is inherited by $\theta_k(\varphi)$. Our aim is to use Theorem 5.2 to all segment of monotonicity of $\theta'_k(\varphi)$ but this can be accomplished only if (4.8) is replaced by

$$\psi_k = G(\psi_{k-1}) + \beta^k\varphi\,. \tag{5.11}$$

where $G : \mathbb{R} \longrightarrow \mathbb{R}$ is a smooth function defined in the whole line as

$$G(\theta + m\pi) = g(\theta) + m\pi\ , \qquad \frac{-\pi}{2} \leq \theta < \frac{\pi}{2}\ ,\ m \in \mathbb{Z}\,. \tag{5.12}$$

The following result shows that, as far as (4.9) is concerned, there is no loss in replacing $\{\theta_k(\varphi)\}_{k\geq 0}$ by the sequence of smooth functions $\{\psi_k(\varphi)\}_{k\geq 0}$. Note that, in view of (4.10), f given by (3.11) is a periodic function with period π.

Proposition 5.5 *$f(\psi_k) = f(\theta_k)$ holds for all $k \in \mathbb{N}$ and f periodic of period π.*

Proof. Proposition 5.5 follows if, for every k, $\psi_k = \theta_k + l\pi$ holds for some $l \in \mathbb{N}$. We prove it by induction. For $k = 1$, we have

$$\psi_1 = G(\psi_0) + \beta\varphi .$$

If $\psi_0 = \theta_0$ or $\psi_0 = \theta_0 + m\pi$, then $\psi_1 = \theta_1 + l\pi$ by (5.12). Let us assume that $\psi_k = \theta_k + m\pi$ holds for $k = n$. Then

$$\begin{aligned} \psi_{n+1} &= G(\psi_n) + \beta^n\varphi \\ &= G(\theta_n + m\pi) + \beta^n\varphi \\ &= g(\theta_n) + l\pi + \beta^n\varphi = \theta_{n+1} + l\pi \end{aligned}$$

In the third line we have used the decomposition $\theta_n = \theta + r\pi$, $\theta \in [-\pi/2, \pi/2)$ and $r = l - m \in \mathbb{N}$, and the fact $g(\theta) = g(\theta + r\pi) = g(\theta_n)$.

□

We are now ready to state and prove our result.

Theorem 5.6 *The sequence $\{\psi_k(\varphi)\}_{k\geq 0}$ given by (5.11), (5.12) and (3.13) is u.d. mod π for almost all $\varphi \in (0, \pi)$ and for all $\psi_0 \in [0, \pi]$, $p \in (0, 1)$ and β such that $p^2(\beta - 1) > c_2$ (c_2 as in (5.5)) holds.*

Proof. We begin by determining the condition under which $\psi_k'(\varphi) - \psi_m'(\varphi)$ is bounded from below uniformly in $\varphi \in I \subset (0, \pi)$.

If $p^2\beta > c_2 = 1 + O(p^2)$, then from (5.11), (5.12) and (5.5)

$$\begin{aligned} \psi_k'(\varphi) &= g'(\psi_{k-1})\psi_{k-1}'(\varphi) + \beta^k \\ &\leq \frac{c_2}{p^2}\psi_{k-1}' + \beta^k \\ &\leq \frac{1}{1 - c_2/(p^2\beta)}\beta^k , \end{aligned} \tag{5.13}$$

by induction. Analogously,

$$\begin{aligned} \psi_k'(\varphi) &\geq c_1 p^2 \psi_{k-1}'(\varphi) + \beta^k \\ &\geq \frac{1}{1 - c_1 p^2/\beta}\beta^k . \end{aligned} \tag{5.14}$$

which will be used later. Equations (5.13) and (5.5) imply

$$\begin{aligned}
\psi_k'(\varphi) - \psi_{k-1}'(\varphi) &= \left(g'(\psi_{k-1}) - 1\right)\, \psi_{k-1}'(\varphi) + \beta^k \\
&\geq -\left(1-p^2\right)\psi_{k-1}' + \beta^k \\
&\geq \frac{\beta p^2 - c_2 - p^2 + p^4}{p^2\beta - c_2}\beta^k
\end{aligned}$$

and hence

$$\psi_k'(\beta) - \psi_m'(\beta) \geq K\beta^k \tag{5.15}$$

where

$$K = K(p,\beta) = \frac{\beta\left(\beta p^2 - c_2 - p^2 + p^4\right)}{\left(p^2\beta - c_2\right)(\beta - 1)}$$

is a strictly positive constant by hypothesis.

To complete the proof of Theorem 5.6, we need an estimate on the cardinality of the partition $\mathcal{P}_{k,m} = \{I_i\}$ of $I \subset (0,\pi)$ into pairwise disjoint open intervals $I_i = (a_i, a_{i+1})$ such that $\psi_k'(\varphi) - \psi_m'(\varphi)$ is a monotone function in each I_i and $\bigcup_i \overline{I}_i = I$ ($\overline{A}$ means the closure of A).

Lemma 5.7 *There exist a finite constant C such that card$(\mathcal{P}_{k,m}) \leq C\,\beta^k$ holds for any $m < k$.*

We defer the proof of Lemma 5.7 to the end of this section. Returning to estimates (5.4) of the proof of Theorem 5.2 together with Lemma 5.7 and (5.15), we have

$$\begin{aligned}
\sum_{m=1}^{k-1}\left|\int_I e^{2i(\psi_k(\varphi)-\psi_m(\varphi))}d\varphi\right| &\leq \sum_{m=1}^{k-1}\sum_{I_i\in\mathcal{P}_{k,m}}\left|\int_{I_i} e^{2i(\psi_k(\varphi)-\psi_m(\varphi))}d\varphi\right| \\
&\leq \sum_{m=1}^{k-1}\sum_i \frac{2}{|\psi_k'(a_i) - \psi_m'(a_i)|} \\
&\leq \frac{4C}{K}\sum_{m=1}^{n}\frac{1}{m} \leq \frac{4C}{K}\ln 3n
\end{aligned}$$

and this implies that $(1/n)\left\|S_n\right\|_{L_2(I)}^2$ is summable and, in view of Theorem 5.1, concludes the proof of Theorem 5.6.

$\square$

Proof of Lemma 5.7. Define

$$\zeta_k = \tan\psi_k \tag{5.16}$$

and

$$\omega_k = \tan \beta^k \varphi \tag{5.17}$$

for any $k \in \mathbb{N}$, with $\omega = \omega_0 = \tan\varphi$, and note that $(\zeta_k)_{k\geq 0}$ satisfies a rational linear (Möbius) transformation

$$\zeta_k = \frac{1 - p^2 + p^2\omega\omega_{k-1} + \omega\zeta_{k-1}}{p^2\omega - (1-p^2)\,\omega_{k-1} - \omega\omega_{k-1}\zeta_{k-1}} \tag{5.18}$$

with $\zeta_0 = \tan\theta_0$, in view of (5.11), (3.13) and

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\ \tan\beta}\ . \tag{5.19}$$

In the following, we are going to show that $\zeta_k = C_k/D_k$ is a rational function with polynomials C_k and D_k in ω of degree not exceeding $C\beta^k$ for some constant C depending on β.

Taking the derivative of (5.16) with respect to φ, we have $\zeta_k' = \sec^2\psi_k\ \psi_k'$, which leads to the conclusion that

$$\psi_k' = \frac{\zeta_k'}{1+\zeta_k^2} \tag{5.20}$$

and

$$\psi_k'' = \frac{(1+\zeta_k^2)\,\zeta_k'' - 2\zeta_k\,(\zeta_k')^2}{(1+\zeta_k^2)^2} \tag{5.21}$$

are rational function of ω. Since the number of zeros of $\psi_k''(\varphi)$ counts the monotone intervals of $\psi_{j+1}'(\varphi)$ in $I \subset (0,\pi)$, Lemma 5.7 will be established by the fundamental theorem of Algebra and the fact that, as ω varies over $\mathbb{R}$,

$$\omega = \tan\varphi \tag{5.22}$$

has exactly two solutions in $(0,\pi)$, $\varphi_1(\omega)$ and $\varphi_2(\omega)$.

We begin by showing that $\omega_k = A_{\beta^k}/B_{\beta^k}$ is a rational function with A_n and B_n polynomials in ω of order n and $n-1$, if n is odd, or $n-1$ and n, if n is even, respectively. Using (5.19),

$$\frac{A_n}{B_n} = \tan((n-1)\varphi + \varphi) = \frac{\tan(n-1)\varphi + \omega}{1 - \omega\tan(n-1)\varphi}$$

implies the recursion

$$\begin{aligned} A_n &= A_{n-1} + \omega B_{n-1} \\ B_n &= B_{n-1} - \omega A_{n-1} \end{aligned}$$

with initial data $A_0 = 0$ and $B_0 = 1$, which by iteration yields

$$\begin{aligned} A_{2n}(\omega) &= (-1)^{n-1} 2n\omega^{2n-1} + \text{ lower order terms} \\ B_{2n}(\omega) &= (-1)^{n} \omega^{2n} + \text{ lower order terms} \end{aligned} \tag{5.23}$$

and

$$\begin{aligned} A_{2n+1}(\omega) &= (-1)^{n} \omega^{2n+1} + \text{ lower order terms} \\ B_{2n+1}(\omega) &= (-1)^{n} (2n+1)\omega^{2n} + \text{ lower order terms} . \end{aligned} \tag{5.24}$$

Now, let us assume that $C_k(\omega)$ and $D_k(\omega)$ are polynomials of degree

$$\deg (C_k) , \ \ \deg (D_k) \leq C\beta^k . \tag{5.25}$$

Clearly, since $\zeta_0 = \tan\theta_0$ is a constant in ω, C_0 and D_0 satisfy (5.25) for any $C \geq 0$. From (5.18),

$$\begin{aligned} C_{k+1} &= \left(1-p^2\right) B_{\beta^k} D_k + p^2\omega A_{\beta^k} D_k + \omega B_{\beta^k} C_k \\ D_{k+1} &= p^2\omega B_{\beta^k} D_k - \left(1-p^2\right) A_{\beta^k} D_k - \omega A_{\beta^k} C_k \end{aligned}$$

are polynomials in ω whose degree, in view of (5.23), (5.24) and (5.25), is bounded by

$$C\beta^k + \beta^k + 1 \leq C\beta^{k+1}$$

provided

$$C \geq \frac{\beta+1}{\beta(\beta-1)} .$$

This concludes the induction and the proof of Lemma 5.7. Note that (5.21) is a rational function, $\psi''_k = \dfrac{P_k}{Q_k}$, of ω with polynomials P_k and Q_k of degree not exceeding $10C\beta^k$ and, from the fundamental theorem of Algebra and observation (5.22), the number of zeros of $\psi''_k(\varphi) - \psi''_m(\varphi)$, $m < k$, in $[a, b] \subset (0, \pi)$ does not exceed $20C\beta^k$.

□

Remark 5.8

1. *Details on various properties of $\psi'_k(\varphi)$, including the distribution of zeros of $\psi''_k(\varphi)$, can be obtained using the informations of Proposition 5.4. In particular, one can show that the set $M_k = \bigcap_{j\leq k} \left\{\varphi \in I : \left(\psi''_k/\psi'_k\right)' (\varphi) \geq 0\right\}$ has full Lebesgue measure, i.e. $|I \backslash M_k| = O\left(p^2\right)|I|$, as $p \to 0$; each connected segment of M_k has at most one zero of $\psi''_k(\varphi)$ and there are at most $C\beta^k$ of them. The complementary set $I\backslash M_k$ is more difficult to deal with.*

2. *Poles of $\psi_k'(\varphi)$ (and of all other derivatives of ψ_k) are located outside the real line since, from (5.18) and (5.20), there is no real solution of $1+\zeta_k^2 = 0$. Note that the discriminant Δ of the second degree equation for ζ_{k-1} is always negative and goes to zero as $p \to 0$. The poles of $\psi_k''(\varphi)$ can be shown to agree with the set Ω_k, defined as the solutions of (5.2), in the limit $p \to 0$.*

3. *For simplicity, we have restricted ourselves to the parameter region $p^2(\beta-1) > c_2 = 1 + O\left(p^2\right)$ in which $\psi_k'(\varphi)$ is a monotone increasing function. From the proof of Lemma 5.7, $\psi_k'(\varphi)$ is a rational function of ω whose degree does not exceed $C\beta^k$. This allows us to apply a more refined version of Koksma's theorem suited for trigonometric sequences (see Theorem 4.4 of [KN]) in order to conclude that $(\psi_k)_{k\geq 0}$ is u.d. mod π for almost every φ in the complementary region $p^2(\beta-1) < c_2$. Detailed analysis of $(\psi_k)_{k\geq 0}$ in the latter region together with a numerical investigation will be presented separately [AMW].*

6 Conclusion: Open Problems

Theorem 1.3 shows a spectral transition from a continuous spectrum for κ sufficiently small, to dense pure point, for κ sufficiently close to one. As a consequence of the one–dimensional nature of the models considered, the continuous part is not a.c. but s.c., a fact which is explained by Theorem 1.7 of [KLS].

A simple extension of the present model to "higher dimensions" is possible: the "separable case". For instance, consider the "two dimensions" Hamiltonian

$$H = J \otimes I + I \otimes J \tag{6.1}$$

with I the identity matrix. The characteristic function

$$f_H(\lambda) = \int e^{i\lambda x} d\rho_H(x)$$

of the spectral measure $d\rho_H$ of H is equal to

$$f_H(\lambda) = f^2(\lambda) \tag{6.2}$$

where

$$f(\lambda) = \int e^{i\lambda x} d\rho(x)$$

is the characteristic function of the measure $d\rho$, given by (2.1). So, $\mu_H = d\rho_H = d\rho * d\rho$ is the convolution of $\mu = d\rho$ with itself [C]. We are not able to conclude, however, that (1.9) holds for μ_H, because the convolution of two s.c. measures may be absolutely continuous. An example is provided by the identity

$$\frac{\sin\lambda}{\lambda} = \left(\prod_{k=1}^{\infty} \cos\frac{\lambda}{2^{2k-1}}\right)\left(\prod_{k=1}^{\infty} \cos\frac{\lambda}{2^{2k}}\right)$$

(see [C], Exercise 9, p. 165), where the l.h.s. is a characteristic function of an a.c. distribution (because it belongs to $L^2(\mathbb{R}, dx)$ – see Exercise 11, p. 159 of [C]), and each infinite product on the r.h.s. is the characteristic function of the s.c. distribution associated to the standard Cantor set of zero Lebesgue measure, obtained by removing the "middle thirds" starting from $[0,1]$. This is an example of (6.2), with μ_H a.c. and μ s.c. Thus, already in the case (6.1), it might happen that a transition from a.c. to (dense) p.p. spectrum takes place.

A Equivalence of Matrix Norms

Let $\|A\|_U := \|UAU^{-1}\|$ be a matrix norm defined by the invertible matrix U given by (2.7) with $\varphi \in (0,\pi)$, where

$$\|B\| = \sup_{\mathbf{v}\in\mathbb{C}^2} \frac{\|B\mathbf{v}\|}{\|\mathbf{v}\|}$$

is the spectral matrix norm induced by the Euclidean vector norm on $\mathbb{C}^2$. We have

Proposition A.1 (Norm Equivalence)

$$C^{-1}\|A\| \le \|A\|_U \le C\|A\|$$

holds with

$$C = \sqrt{\frac{1+|\cos\varphi|}{1-|\cos\varphi|}}\,.$$

Proof. Given $\mathbf{v} = (v_1, v_2) \in \mathbb{C}^2$, we have

$$\|U\mathbf{v}\|^2 = |v_1|^2 + |v_2|^2 - 2\Re e\,(v_1\overline{v_2})\,\cos\varphi$$

which together with

$$0 \le |v_1 \pm v_2|^2 = |v_1|^2 + |v_2|^2 \pm 2\Re e\,(v_1\overline{v_2})\,,$$

yields

$$(1-|\cos\varphi|)\,\|\mathbf{v}\|^2 \le \|U\mathbf{v}\|^2 \le (1+|\cos\varphi|)\,\|\mathbf{v}\|^2 \tag{A.1}$$

and the vector norm $\|\mathbf{v}\|_U := \|U\mathbf{v}\|$ is equivalent to the norm $\|\mathbf{v}\|$. Since U define an isomorphism in $\mathbb{C}^2$,

$$\|A\|_U = \sup_{\mathbf{v}\in\mathbb{C}^2} \frac{\|UAU^{-1}\mathbf{v}\|}{\|\mathbf{v}\|} = \sup_{\mathbf{v}\in\mathbb{C}^2} \frac{\|UA\mathbf{v}\|}{\|U\mathbf{v}\|} = \sup_{\mathbf{v}\in\mathbb{C}^2} \frac{\|A\mathbf{v}\|_U}{\|\mathbf{v}\|_U}$$

and the result follows from (A.1).

□

Proposition A.1 and (3.9) imply

$$\|T(n;\lambda)\| \leq C \sup_{\theta_0} \left\|UT(n;\lambda)U^{-1}\mathbf{v}_0\right\| \leq C \sup_{\theta_0} R_{N+1} \tag{A.2}$$

and

$$\|T(n;\lambda)\| \geq C^{-1} \left\|UT(n;\lambda)U^{-1}\mathbf{v}_0\right\| = C^{-1} R_{N+1} \tag{A.3}$$

for n so that $a_{N+1} < n \leq a_N$ holds for some N. The analog Theorem 2.3 of Kiselev, Last and Simon [KLS] uses explicitly the fact that T, and consequently $T_U = UTU^{-1}$, is a unimodal matrix. In this case,

$$\|T_U\mathbf{v}_0\|^2 = \cos^2(\theta_0 - \theta^*)\,\|T\|_U^2 + \sin^2(\theta_0 - \theta^*)\,\|T\|_U^{-2} \,,$$

where θ^* is the angle in which the supremum is attained, can be used to recover their result. This will not be necessary in this paper.

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References

[A] N. I. Akhiezer. "The classical moment problem and some related questions in analysis", Oliver & Boyd (1965)

[AMW] R. M. Angelo, D. H. U. Marchetti and W. F. Wreszinski. "Composition of nonlinear map possessing an attractive and a repulsive fixed points with an exponential translation", in Preparation.

[C] Kai Lai Chung. "A course in Probability theory", Academic Press - 2nd edition (1974)

[H] J. S. Howland. "Quantum stability" in *Schrödinger operators*, Ed. E. Balslev. Lecture Notes in Physics **403**, Springer Verlag (1992)

[K] Abel Klein. "Extended States in the Anderson Model on the Bethe Lattice", Adv. Math. **133**, 163-184 (1998)

[KLS] Alexander Kiselev, Yoram Last and Barry Simon. "Modified Prüfer and EFGP transforms and the spectral analysis of one–dimensional Schrödinger operators", Commun. Math. Phys. **194**, 1-45 (1997)

[KN] L. Kuipers and H. Niederrreiter. "Uniform Distribution of Sequences" John Wiley & Sons (1974)

[LS] Yoram Last and Barry Simon. "Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one–dimensional Schrödinger operators", Invent. Math. **135**, 329-367 (1999)

[PBM] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev. "Integrals and Series. Vol. 1. Elementary functions", Gordon & Breach (1986)

[S] Barry Simon. "The classical moment problem as a self-adjoint finite difference operator", Adv. Math. **137**, 82-203 (1998)

[S1] Michael Reed and Barry Simon. "Methods of Modern Mathematical Physics IV: Analysis of Operators", Academic Press 1978.