Scattering of an *infraparticle*: the one particle sector in Nelson's massless model

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Abstract

In the one-particle sector of Nelson's massless model, we construct scattering states in the time dependent approach. On the so defined scattering subspaces, the convergence of the asymptotic Weyl operators related to the boson field as well as the asymptotic limit of the mean velocity of the infraparticle are established. The construction relies on some spectral results concerning the one-particle (improper) states of the system. Moreover, in the region of physical interest, we assume a positive bound from below for the second derivative of the ground state energy as a function of the total momentum, uniform in the limit of no infrared cut-off in the interaction term.

Contents

Introduction

In this paper we aim at describing the scattering behavior of a non-relativistic quantum particle interacting (only) with a quantized relativistic massless scalar field, when an ultraviolet cut-off is imposed on the interaction and no infrared regularization is adopted. This model is also known as the one-particle sector of the translation invariant Nelson's massless model [Ne.]. The interest in Nelson's massless model is related to the infrared features of Q.E.D., in spite of the various approximations here introduced: the charge is not described by a field (no pair production), an ultraviolet cut-off is imposed, the "photons" are scalar particles and the "electron" is a spinless non-relativistic particle. In particular, the analysis of the counterpart of Compton scattering in the given scalar model meets with problems (infrared divergences) analogous to the Q.E.D. case in some substantial respects.
The general features of the asymptotic states, as they arise from perturbative computations and from rigorous results in related solvable models (dipole approximation, see also [Bl.]), suggest the following intuitive picture: a "free" massive particle, that we call the electron or the charged particle, always surrounded by a cloud of asymptotic soft bosons, that we call photons even though they are scalar particles. The momenta distribution in the photon cloud, in the limit of zero energy, turns out to be linked to the electron asymptotic velocity according to a "Bloch and Nordsieck" [B.N.] type factor.
The infrared features of the model are at the origin of the following difficulties in the control of scattering:
Because of the massless dispersion of the bosons, the construction of the so called asymptotic *L.S.Z.* (Lehmann-Symanzik-Zimmermann) operators [L.S.Z.] associated with the boson field requires a careful application of Huyghens' principle for the decay estimates of the solutions of Klein Gordon equation (in this respect, see [R.S.]). In our context, due to the non-relativistic description of the charge, it implies a restriction of the physical Hilbert space to keep the asymptotic velocity of the charge smaller with respect to the speed of the light;
Due to the arbitrarily large number of photons emitted in the scattering at arbitrary long time, the explanation and the exact meaning of the asymptotic decoupling between the bosons and the non-relativistic particle require a different characterization of the "free" dynamics (which is only asymptotic) of the massive particle.

The structural issue, as far as spectral properties are concerned, consists in the disappearance of one-particle states from the joint spectrum of the operators Hamiltonian and total momentum, in other words the absence of a proper mass shell for the charged particle. In literature, particles sharing such feature are called infraparticles [Sc.]. Because of this missing ingredient, an asymptotic description based on concepts and techniques which stem from "Haag-Ruelle scattering theory" (see [Ha.]) is conceptually not adequate. Haag-Ruelle scattering theory provides a recipe to construct scattering states for quantum relativistic fields satisfying Wightman axioms and with mass gap. In our model, while the notion of relativistic locality can be easily replaced by a non-relativistic one (that is at fixed time), the absence of one-particle states is a genuine infrared feature which modifies the collision picture at a substantial level with the appearance of non-Fock representations for the asymptotic boson algebra. Moreover the rigorous definition of the asymptotic observables that describe the infraparticle cannot be accomplished without some further information on the (improper) mass shell structure. This is the main gap we want to fill in the present paper.

The first systematic analysis of scattering in the translationally invariant Nelson's model has been done by Fröhlich in two papers [Fr.1] and [Fr.2], where the second one provides spectral results exploited in [Fr.1]. In this study, indications coming from solvable models are mastered, two different approaches to collision theory are developed and many useful technical tools are provided. Starting from the intuitive picture, a recipe is attempted for the vector in the Hilbert space, $\psi^{out(in)}$, corresponding (in the Heisenberg picture) to an asymptotic electron of given wave function. According to the time-dependent approach to scattering, the vector $\psi^{out(in)}$ is singled out by the time convergence of a related approximating vector $\psi(t)$. The content of the present paper is strongly connected with Fröhlich's attempt to provide a definition for the generic vector $\psi(t)$, whose limit in time has to be consistent with an asymptotic description. Therefore a brief review of that work is carried on in a subsequent paragraph. Indeed it is surely crucial to outline first the physical and mathematical aspects involved, then we can better justify new conceptual and technical steps sufficient for a consistent collision theory both for the infraparticle and the bosons.

In the framework of general quantum field theory, analogous issues has been treated by Buchholz [Bu.1] [Bu.2], who established the asymptotic convergence of massless boson fields applying Huyghens' principle. Also the problem of the asymptotic description of an infraparticle has been addressed by Buchholz, Pormann and Stein [B.P.S.] in the context of a more general definition of the particle content of a theory. The infraparticles are described starting from *weights*; they are positive linear forms over the algebra of some operators that, in broad terms, represent detectors. The (pure) *weights* turn out to carry the properties of single one-particle improper states that means one-particle states with sharp energy-momentum p.

For Nelson's massless model with a confining potential, Gerard and Derezinski [D.G.] have recently faced the problem to define wave operators for non-Fock co-

herent sectors. In this respect, it is worthwhile to point out that the main physical feature in the model discussed in our paper is never expected in the confined case, namely the coexistence of inequivalent non-Fock representations of the asymptotic boson field labelled by the asymptotic (mean) velocity of the electron. The asymptotic completeness in the confined and infrared regularized case has been discussed by Gerard in [Ge.].
The asymptotic convergence of the radiation field in non-relativistic Q.E.D. has been established in [F.G.S.] for small energy configurations of the system. The approach is slightly different with respect to the point of view developed for the massless field in the present paper.

1 Preliminaries

1.1 Definition of the model

The physical system consists of a non-relativistic spin-less quantum particle of mass m, linearly coupled to a quantized relativistic scalar boson field, which is massless and real. The non-relativistic particle is described by position and momentum variables with usual *canonical commutation rules* (c.c.r.) $[x_l, p_j] = i\delta_{l,j}$ $(\hbar = 1)$ $l, j = 1, 2, 3$; the (scalar) boson field, which we will call also photon field, at time $t = 0$ is:

$$A(0,\mathbf{y}) = \frac{1}{\sqrt{2\pi}^3} \cdot \int \left(a^\dagger(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{y}} + a(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{y}} \right) \frac{d^3k}{\sqrt{2\,|\mathbf{k}|}}\,, \tag{1.1}$$

(having assumed $c = \hbar = 1$), where $a^\dagger(\mathbf{k}), a(\mathbf{k})$ are standard creation and annihilation operator valued tempered distributions obeying the c.c.r.

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{q})\right] = \delta^3(\mathbf{k} - \mathbf{q}) \quad , \quad [a(\mathbf{k}), a(\mathbf{q})] = \left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{q})\right] = 0.$$

The spatial translations are implemented by the total momentum

$$\mathbf{P} := \mathbf{p} + \int \mathbf{k} a^\dagger(\mathbf{k})\, a(\mathbf{k})\, d^3k\,. \tag{1.2}$$

The dynamics of the system is generated by the covariant Hamiltonian ($[H, \mathbf{P}] = 0$)

$$H := \frac{\mathbf{p}^2}{2m} + g \int_0^\kappa \left(a(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \frac{d^3k}{\sqrt{2}\,|\mathbf{k}|^{\frac{1}{2}}} + H^{ph} \tag{1.3}$$

where κ is an ultraviolet *cut-off*, g $(g > 0)$ is the coupling constant and H^{ph} is the free Hamiltonian of the photon field

$$H^{ph} := \int |\mathbf{k}|\, a^\dagger(\mathbf{k})\, a(\mathbf{k})\, d^3k\,. \tag{1.4}$$

The Hilbert space of the system is $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}$ where $\mathcal{F}$ is the Fock space with respect to the creation and annihilation operator valued distributions $\left\{a^\dagger(\mathbf{k}), a(\mathbf{k})\right\}$:

$$\mathcal{F} = \oplus_{j=0}^{\infty} S_j L^2\left[\left(\mathbb{R}^3\right)^j\right]\,. \tag{1.5}$$

An element of $\mathcal{H}$ is a sequence $\{\psi_n\}$ of functions on $\mathbb{R}^{3(n+1)}$ with $\|\psi\| < \infty$, where

$$\|\psi\|^2 = \sum_{n=0}^{\infty} \int \overline{\psi^n (\mathbf{x}, \mathbf{k}_1, ..\mathbf{k}_n)} \psi^n (\mathbf{x}, \mathbf{k}_1, ..\mathbf{k}_n) \, d^3k_1 ... d^3k_n d^3x$$

and each $\psi^n (\mathbf{x}, \mathbf{k}_1, ..\mathbf{k}_n)$ is symmetric in $\mathbf{k}_1, ..., \mathbf{k}_n$. The $n = 0$ component corresponds to the tensor product of the vacuum subspace of $\mathcal{F}$ with the non-relativistic particle space $L^2 (\mathbb{R}^3)$.

Standard results about H *and* P*:*

i) The operators

$$\mathbf{P} = \mathbf{p} \otimes I + I \otimes \int \mathbf{k} a^\dagger (\mathbf{k}) \, a (\mathbf{k}) \, d^3k \, ,$$

where I is the identity operator, are essentially self-adjoint (e.s.a.) in

$$D := \bigvee_{n \in \mathbb{N}} h \otimes \psi^n \, .$$

D is the set of the finite linear combinations of vectors of wave function $h (\mathbf{x}) \, \psi^n (\mathbf{k}_1, ...\mathbf{k}_n)$, where $h (\mathbf{x}) \in S \left(\mathbb{R}^3\right)$(the space of Schwartz test functions), $\psi^n (\mathbf{k}_1, ...\mathbf{k}_n) \in S^s \left(\mathbb{R}^{3n}\right)$ (symmetric Schwartz test functions) and $n \in \mathbb{N}$. Since $\mathbf{p}$ and $\int \mathbf{k} a^\dagger (\mathbf{k}) \, a (\mathbf{k}) \, d^3k$ are e.s.a. in $S \left(\mathbb{R}^3\right)$ and $\bigvee_{n \in \mathbb{N}} \psi^n$ respectively, the result easily follows for the $\mathbf{P}$ operators. The spectrum of each component of $\mathbf{P}$ is the real axis, the spectral measure is absolutely continuous with respect to the Lebesgue measure.

ii) The interaction term in the Hamiltonian is an infinitesimal small perturbation (in the sense of Kato) with respect to

$$H_0 := \frac{\mathbf{p}^2}{2m} + H^{ph} \, . \tag{1.6}$$

Hence H is bounded from below, it is e.s.a. in D and its self-adjointness domain (s.a.d.), $D (H)$, coincides with $D (H_0)$ (s.a.d. of H_0).

iii) The groups $e^{i\mathbf{a}\cdot\mathbf{P}}$and $e^{i\tau H}$, $\tau \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^3$, commute.

iv) The joint spectral decomposition of the Hilbert space with respect to the $\mathbf{P}$ operators is $\mathcal{H} = \int^{\oplus} \mathcal{H}_{\mathbf{P}} d^3P$ where $\mathcal{H}_{\mathbf{P}}$ is a copy of $\mathcal{F}$.
Indeed to the improper eigenvectors of the $\mathbf{P}$ operators, $\Phi^n_{\mathbf{P}}$, where

$$\Phi^n_{\mathbf{P}} (\mathbf{x}, \mathbf{k}_1, ..\mathbf{k}_n) := (2\pi)^{-\frac{3}{2}} \, e^{i(\mathbf{P} - \mathbf{k}_1 - ... - \mathbf{k}_n)\cdot\mathbf{x}} \varphi^n_{\mathbf{P}} (\mathbf{k}_1, ..\mathbf{k}_n) \qquad \varphi^n_{\mathbf{P}} (\mathbf{k}_1, ..\mathbf{k}_n) \in S^s \left(\mathbb{R}^{3n}\right) \, ,$$

we can relate a natural scalar product:

$$\left(\Phi'^n_{\mathbf{P}}, \Phi^m_{\mathbf{P}}\right) = \delta_{n,m} \int \overline{\varphi'^n_{\mathbf{P}} (\mathbf{k}_1, ..\mathbf{k}_n)} \varphi^m_{\mathbf{P}} (\mathbf{k}_1, ..\mathbf{k}_m) \, d^3k_1 ... d^3k_n \, . \tag{1.7}$$

The vector space $\overline{\bigvee_{n \in N} \Phi^n_{\mathbf{P}}}$ is defined as the closure of the finite linear combinations of the wave functions $\Phi^n_{\mathbf{P}} (\mathbf{x}, \mathbf{k}_1, ..\mathbf{k}_n)$ in the norm which arises from the scalar product (1.7). Starting from this space, we uniquely define the linear application

$$\mathbf{I}_{\mathbf{P}} : \overline{\bigvee_{n\in N} \Phi^{n}_{\mathbf{P}}} \rightarrow \mathcal{F}^{b} \tag{1.8}$$

by the prescription:

$$\mathbf{I}_{\mathbf{P}}\left(\Phi^{n}_{\mathbf{P}}\left(\mathbf{x},\mathbf{k}_{1},..\mathbf{k}_{n}\right)\right) = \frac{1}{\sqrt{n!}}\int b^{\dagger}\left(\mathbf{k}_{1}\right)....b^{\dagger}\left(\mathbf{k}_{n}\right)\varphi^{n}_{\mathbf{P}}\left(\mathbf{k}_{1},..\mathbf{k}_{n}\right)d^{3}k_{1}...d^{3}k_{n}\psi_{0}\,, \tag{1.9}$$

where $b\left(\mathbf{k}\right), b^{\dagger}\left(\mathbf{k}\right)$, which formally correspond to $a\left(\mathbf{k}\right)e^{i\mathbf{k}\cdot\mathbf{x}}, a^{\dagger}\left(\mathbf{k}\right)e^{-i\mathbf{k}\cdot\mathbf{x}}$, are annihilation and creation operator-valued tempered distributions in the Fock space $\mathcal{F}^{b}\cong\mathcal{F}$, and ψ_{0} is the related vacuum. The given norm for $\Phi^{n}_{\mathbf{P}}$ is equal to $\|\mathbf{I}_{\mathbf{P}}\left(\Phi^{n}_{\mathbf{P}}\right)\|_{\mathcal{F}}$ ($\|\cdot\|_{\mathcal{F}}$ is the Fock norm). The application $\mathbf{I}_{\mathbf{P}}$ is onto and isometric.

v) Since $[H,\mathbf{P}]=0$, we have that $H=\int H_{\mathbf{P}}d^{3}P$, where $H_{\mathbf{P}}:\mathcal{H}_{\mathbf{P}}\rightarrow\mathcal{H}_{\mathbf{P}}$ is e.s.a. in $D^{b}:=\bigvee_{n\in\mathbb{N}}\Phi^{n}_{\mathbf{P}}$. In terms of the variables $\mathbf{P}$, $b\left(\mathbf{k}\right), b^{\dagger}\left(\mathbf{k}\right)$, the operator $H_{\mathbf{P}}$ is written as follows:

$$H_{\mathbf{P}} = \frac{\left(\mathbf{P}^{ph}-\mathbf{P}\right)^{2}}{2m} + g\int_{0}^{\kappa}\left(b\left(\mathbf{k}\right)+b^{\dagger}\left(\mathbf{k}\right)\right)\frac{d^{3}k}{\sqrt{2\left|\mathbf{k}\right|}} + H^{ph}\,. \tag{1.10}$$

being $H^{ph}\equiv\int\left|\mathbf{k}\right|\,b^{\dagger}\left(\mathbf{k}\right)b\left(\mathbf{k}\right)d^{3}k$ and $\mathbf{P}^{ph}\equiv\int\mathbf{k}\,b^{\dagger}\left(\mathbf{k}\right)b\left(\mathbf{k}\right)d^{3}k$ when applied on the fiber spaces $\mathcal{H}_{\mathbf{P}}$.

Notations

We collect standard notations and some conventions which are used throughout the paper:

1) $C^{n}\left(\mathbb{R}^{3}\right)$ denotes the space of functions on $\mathbb{R}^{3}$ with continuous derivatives up to degree n, $C^{n}_{0}\left(\mathbb{R}^{3}\right)$ denotes the subset of compact support functions contained in $C^{n}\left(\mathbb{R}^{3}\right)$.
2) The symbol $|.|$ will denote the absolute vale for $\mathbb{C}$ numbers as well as the euclidean norm for vectors in $\mathbb{R}^{n}$, $n>1$. Multiplication between real numbers and scalar products between vectors in $\mathbb{R}^{n}$, $n>1$, are both denoted by the multiplication sign "$\cdot$".
3) Given a function $\chi\left(u\right)$, $supp\chi$ is the support of the function.
4) For 3-dimensional integrals we use only one integration symbol and the explicit integration bounds are referred to the radial part of the integration variable. If necessary the notations are less compressed.
5) The notation $s-lim$ means strong limit.
6) Given a self-adjoint operator A, $D\left(A\right)$ is the corresponding domain. The notation $h.c.$ means hermitian conjugate.
7) The operators, $\nabla E^{\sigma}\left(\mathbf{P}\right)$, $W_{\sigma}\left(\nabla E^{\sigma}\left(\mathbf{P}\right)\right)$, are functions of the total momentum operator $\mathbf{P}$. For brevity the dependence on $\mathbf{P}$ is some times differently indicated (e.g. $E^{\sigma}_{\mathbf{P}}$).

8) In the estimates that we produce throughout the paper, we generically call C all the multiplicative constants which are are time independent, uniform in the infrared cut-off and in the cell partition.

1.2 Fröhlich's construction

The issues and the results we are going to discuss concern the model with an ultraviolet cut-off and are connected to the infrared difficulties which affect the formulation of scattering theory. Focusing our attention on such aspect, we recall that in Fröhlich's paper [Fr.1] the following cases are investigated and compared:
1) The massive and the massless cases as far as the boson field is concerned;
2) Both the non-relativistic and the relativistic dispersion law for the charged particle kinetic energy in the Hamiltonian.

The scattering problem is studied in a time-dependent approach, by adapting the "Haag-Ruelle" framework [Ha.] to the mixed character of the model, in fact quantum mechanical non-relativistic matter coexists with a quantum relativistic field. The adopted procedure is successful as far as one particle states for the charge are available. It is always the case in presence of massive bosons; in the massless case only if an infrared regularization, for instance a cut-off, is imposed on the interaction. Starting from the one-particle states and the asymptotic limit of the *L.S.Z.* smeared field, the asymptotic picture is simply given by a free electron with a renormalized dispersion law and free bosons in the Fock representation. We recall that in the massless case the asymptotic convergence of the *L.S.Z.* smeared field requires, as additional condition, that the asymptotic velocity of the non-relativistic particle is strictly smaller with respect to the boson velocity (the speed of the light).
Such a physical description fails in the true (no infrared regularization) Nelson's massless model and two alternative scattering descriptions are therefore considered.
The first one is an attempt to generalize Haag-Ruelle theory by a limiting construction starting from the model with an infrared cut-off. This approach is reconsidered and developed in this paper, where it is proved to be consistent.
The second one assumes the existence of the asymptotic boson (free) algebra to define the time-space translation generators for the asymptotic charge as a difference: these are obtained by subtracting from the full generators the corresponding ones for the asymptotic bosons (for details see [Fr.1]). Similar concepts were later exploited in Q.E.D. (see [F.M.S.]) in the Wightman framework of quantum field theory. In that context a tentative recipe has been provided for the construction of the asymptotic charged fields.

The first approach (the only one we are interested in) requires a careful analysis of the one-particle improper states or, in other words, of the one-particle states corresponding to Hamiltonians with smaller and smaller infrared-cutoff σ in the interaction term. The underlying conjecture is that a sufficiently refined control on the one-particle states (which disappear from the Hilbert space $\mathcal{H}$ in the limit $\sigma \to 0$) should predict the low energy behavior of the boson cloud appearing in the scattering states. This aspect is clearly crucial in order to define an approximating vector $\psi(t)$ of ψ^{out}. To motivate why, in our opinion, the way followed in [Fr.1] is the correct one to understand the scattering behavior, we review that analysis before filling some conceptual steps towards a modified definition of $\psi(t)$ and the proof of its convergence in time.

The spectral results behind the definition of $\psi(t)$ in [Fr.1] are concerned with the ground states of the Hamiltonians $H_{\mathbf{P}}$, and they are achieved through a non constructive method already used by Glimm and Jaffe [G.J.]. The main results are:
The absence, in the not (infrared) regularized case, of a ground state for $H_{\mathbf{P}}$ in the Hilbert space $\mathcal{H}_{\mathbf{P}} \cong \mathcal{F}$ and its existence and uniqueness in the $\mathbf{P}-$dependent coherent representation of $\{b(\mathbf{k}), b^{\dagger}(\mathbf{k})\}$ with coherent factor $c(\mathbf{k})$ singled out by the infrared behavior:

$$c(\mathbf{k}) \rightarrow_{\mathbf{k}\rightarrow 0} -\frac{g}{\sqrt{2}\,|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E(\mathbf{P})\right)}\,; \tag{1.11}$$

The following properties which hold for the ground state energy $E(\mathbf{P})$:
The energy $E(\mathbf{P}) = E(|\mathbf{P}|)$ is absolutely continuous, therefore $\frac{\partial E(\mathbf{P})}{\partial|\mathbf{P}|}$ exists almost everywhere;
Concerning the derivative

$$\left|\frac{\partial E(\mathbf{P})}{\partial|\mathbf{P}|}\right| < 1 \qquad \text{for} \qquad \mathbf{P} : |\mathbf{P}| < M \quad \text{if} \qquad H_0 = \frac{\mathbf{p}^2}{2m} + H^{ph}$$

$$\left|\frac{\partial E(\mathbf{P})}{\partial|\mathbf{P}|}\right| < 1 \quad \text{for any} \qquad \mathbf{P} \in \mathbb{R}^{\mathbf{3}} \qquad \text{if} \quad H_0 = \sqrt{\mathbf{p}^2 + m^2} + H^{ph}\,.$$

A crucial technical tool is involved in the above results. It is an almost explicit expression for the action of $b(\mathbf{k})$ on the ground state $\psi^{\sigma}_{\mathbf{P}}$ of the Hamiltonian $H_{\mathbf{P},\sigma}$, i.e. with an infrared cut-off σ in the interaction term. The tool is

$$b(\mathbf{k})\,\psi^{\sigma}_{\mathbf{P}} = \frac{g}{\sqrt{2\,|\mathbf{k}|}}\left(\frac{1}{E^{\sigma}(\mathbf{P}) - |\mathbf{k}| - H_{\mathbf{P}-\mathbf{k},\sigma}}\right)\psi^{\sigma}_{\mathbf{P}} \qquad \sigma \leq |\mathbf{k}| \leq \kappa \tag{1.12}$$

where $E^{\sigma}(\mathbf{P})$ is the eigenvalue of $\psi^{\sigma}_{\mathbf{P}}$, $H_{\mathbf{P},\sigma}\psi^{\sigma}_{\mathbf{P}} = E^{\sigma}(\mathbf{P})\,\psi^{\sigma}_{\mathbf{P}}$. The resolvent formula (1.12) clearly plays an important role also in the present paper because it contains a structural information about the logarithmic divergence, in the infrared limit $\sigma \rightarrow 0$, of the boson number operator, $N := \int b^{\dagger}(\mathbf{k})\, b(\mathbf{k})\, d^3k$, evaluated on the ground state of $H_{\mathbf{P},\sigma}$.

Once the previous spectral information is known, the main issue is how to define a vector $\psi^{\sigma}_{h,\kappa_1}(t) \in \mathcal{H}$ with the property that $\lim_{t\rightarrow\infty,\sigma\rightarrow 0}\psi^{\sigma}_{h,\kappa_1}(t)$ represents, in Heisenberg picture, an asymptotic electron with wave function h in the asymptotic momentum of the charged particle, with the expected freely moving (soft) photon cloud surrounding it, where the boson frequency is up to the threshold κ_1. The wave function of the asymptotic (soft) photons which form the cloud is suggested by the spectral analysis of one-particle states. More precisely it is linked to the coherent representations (1.11) singled out in the limit $\sigma \rightarrow 0$ for different $\mathbf{P}$. This

interpretation of the limiting vectors requires an *a posteriori* justification from the action of the asymptotic observables (to be constructed) on them.
In order to construct the generic vector $\psi^{\sigma}_{h,\kappa_1}(t)$, Fröhlich starts from the wave function, in terms of the charged particle position operator and the bosons momenta variables, of a one-particle state corresponding to the model with (infrared) cut-off σ in the interaction term of the Hamiltonian (1.3). Let it be given by the sequence

$$\left\{\psi^{\sigma(n)}(\mathbf{x},\mathbf{k}_1,...,\mathbf{k}_n):\ n\in\mathbb{N}\quad \sum_n\int\left|\psi^{\sigma(n)}(\mathbf{x},\mathbf{k}_1,...,\mathbf{k}_n)\right|^2 d^3x d^3k_1...d^3k_n<\infty\right\}$$

where

$$\psi^{\sigma(n)}(\mathbf{x},\mathbf{k}_1,...,\mathbf{k}_n)=(2\pi)^{-\frac{3}{2}}\int e^{i\mathbf{p}\cdot\mathbf{x}}\widetilde{\psi}^{\sigma(n)}_{\mathbf{p}}(\mathbf{k}_1,...,\mathbf{k}_n)\,d^3p \tag{1.13}$$

In $\psi^{\sigma}_{h,\kappa_1}(t)$ the subscript h is referred to the wave function in $\mathbf{P}$ of the one-particle state. The support of h has to be restricted to a neighborhood of $\mathbf{P}=0$ such that $|\nabla E^{\sigma}(\mathbf{P})|<1$. In the expression (1.13) the $\mathbf{P}$-dependence is hidden in the (symmetric) function $\widetilde{\psi}^{\sigma(n)}_{\mathbf{p}}(\mathbf{k}_1,...,\mathbf{k}_n)$ where $\mathbf{p}$ is the charged particle momentum.

By the substitution $\mathbf{p}=\mathbf{P}-\mathbf{k}_1-...-\mathbf{k}_n$, we have

$$\begin{aligned}\psi^{\sigma(n)}(\mathbf{x},\mathbf{k}_1,...,\mathbf{k}_n) &= (2\pi)^{-\frac{3}{2}}\int e^{i\mathbf{P}\cdot\mathbf{x}}e^{-i(\mathbf{k}_1+..+\mathbf{k}_n)\cdot\mathbf{x}}\widetilde{\psi}^{\sigma(n)}_{\mathbf{p}}(\mathbf{k}_1,...,\mathbf{k}_n)\,d^3p\\ &= \int e^{i\mathbf{P}\cdot\mathbf{x}}e^{-i\mathbf{P}^{ph}\cdot\mathbf{x}}h(\mathbf{P})\,\psi^{\sigma(n)}_{\mathbf{P}}(\mathbf{k}_1,...,\mathbf{k}_n)\,d^3P\end{aligned} \tag{1.14}$$

where

$$(2\pi)^{-\frac{3}{2}}\,\widetilde{\psi}^{\sigma(n)}_{\mathbf{p}=\mathbf{P}-\mathbf{k}_1-..-\mathbf{k}_n}(\mathbf{k}_1,...,\mathbf{k}_n)=h(\mathbf{P})\,\psi^{\sigma(n)}_{\mathbf{P}}(\mathbf{k}_1,...,\mathbf{k}_n) \tag{1.15}$$

with the normalization

$$\sum_n\int\left|\psi^{\sigma(n)}_{\mathbf{P}}(\mathbf{k}_1,...,\mathbf{k}_n)\right|^2 d^3k_1....d^3k_n=1\,.$$

In order to properly control the action, on the one-particle state, of the Weyl operator "carrying" the boson cloud, one first defines the operator valued distributions $\{a(\mathbf{k}),a^{\dagger}(\mathbf{k})\}$ smeared out with functions $f(\mathbf{k},\mathbf{P})$, where $\mathbf{P}$ is the the total momentum, and then applied to the vector $\widetilde{\psi}^{\sigma(n)}_{\mathbf{p}}(\mathbf{k}_1,...,\mathbf{k}_n)$. Exploiting the decomposition of $\mathcal{H}$ on the spectrum of $\mathbf{P}$, the definition is:

$$\int a(\mathbf{k})\,f(\mathbf{k},\mathbf{P})\,d^3k\psi^{\sigma(n)}(\mathbf{x},\mathbf{k}_1,...,\mathbf{k}_n):= \tag{1.16}$$

$$= \int\int a(\mathbf{k})\,f(\mathbf{k},\mathbf{P})\,e^{i\mathbf{P}\cdot\mathbf{x}-i\mathbf{P}^{ph}\cdot\mathbf{x}}h(\mathbf{P})\,\psi^{\sigma(n)}_{\mathbf{P}}(\mathbf{k}_1,...,\mathbf{k}_n)\,d^3Pd^3k$$

$$= (2\pi)^{-\frac{3}{2}}\sqrt{n}\int e^{-i\mathbf{P}^{ph}\cdot\mathbf{x}+i\mathbf{P}\cdot\mathbf{x}}\int f(\mathbf{k},\mathbf{P})\,e^{-i\mathbf{k}\cdot\mathbf{x}}\widetilde{\psi}^{\sigma(n)}_{\mathbf{P}-\mathbf{k}-\mathbf{k}_2...-\mathbf{k}_n}(\mathbf{k},\mathbf{k}_2,..,\mathbf{k}_n)\,d^3kd^3P\,.$$

A similar procedure is used for the action of the operator $\int a^{\dagger}(\mathbf{k})\,f(\mathbf{k},\mathbf{P})\,d^3k$.

On the basis of the previous definitions, the final expression we are interested in can be handled after having expanded, in terms of the generator, the formal expression for the $L.S.Z.$ Weyl operator "carrying" the boson cloud. In other words, it means that the approximating vector $\psi^{\sigma}_{h,\kappa_1}(t)$ is defined starting from each projection on the $n-$particle subspace, namely:

$$\left(e^{-iH_\sigma t}\psi^{\sigma}_{h,k_1}(t)\right)^{(n)}(\mathbf{x},\mathbf{k}_1,..\mathbf{k}_n) := \tag{1.17}$$

$$= \left(\int e^{-iH^{ph}t} e^{-g\int_\sigma^{\kappa_1}\frac{a(\mathbf{k})-a^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)}d^3k} e^{iH^{ph}t}e^{-iE^\sigma(\mathbf{P})t}e^{i\mathbf{P}\cdot\mathbf{x}}e^{-i\mathbf{P}^{ph}\cdot\mathbf{x}}h(\mathbf{P})\,\psi^{\sigma}_{\mathbf{P}}d^3P\right)^{(n)}(\mathbf{k}_1,..\mathbf{k}_n)$$

$$= \left(\int e^{i\mathbf{P}\cdot\mathbf{x}}e^{-i\mathbf{P}^{ph}\cdot\mathbf{x}} e^{-g\int_\sigma^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|t-i\mathbf{k}\cdot\mathbf{x}}-h.c.}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)}d^3k} e^{-iE^\sigma(\mathbf{P})t}h(\mathbf{P})\,\psi^{\sigma}_{\mathbf{P}}d^3P\right)^{(n)}(\mathbf{k}_1,..\mathbf{k}_n)\,.$$

For a fixed infrared cut-off σ, the time limit of the vector $\psi^{\sigma}_{h,\kappa_1}(t)$ is obtained exploiting Hepp's method [He.]. The proof basically relies on the estimates of the expectation values of polynomials in $\left\{b(\mathbf{k}),b^\dagger(\mathbf{k})\right\}$ on the ground state $\psi^{\sigma}_{\mathbf{P}}$ (generalized resolvent formulas, see [Fr.1]) and on an implicit propagation estimate for the electron contained in the constraint $|\nabla E^\sigma(\mathbf{P})|<1$ which holds in a neighborhood of $\mathbf{P}=0$.

The ultimate motivation for the previous construction is however the limit in time of $\psi_{h,\kappa_1}(t)$ with no infrared cut-off σ. In the physical situation without infrared cut-off, it indeed represents a minimal (with respect to the photon cloud) description of an infraparticle in a scattering state. It means that a photon cloud of soft photons is unavoidable, i.e. κ_1 can be arbitrarily small but not zero. The subspace generated by such vectors can be seen as a one-particle subspace, up to an observability threshold in the energy of the asymptotic photons.

1.3 Minimal asymptotic electron

Let us inquire about the features and the problems of the previous construction. As already pointed out, no problem arises in the norm control and in the convergence in time of $\psi^{\sigma}_{h,\kappa_1}(t)$ as long as $\sigma\neq 0$, because the series expansion of the Weyl operators in terms of the generator can be controlled, basically, due to the regularity properties in $\mathbf{P}$ of $\psi^{\sigma}_{\mathbf{P}}$ in a neighborhood of $\mathbf{P}=0$. The situation changes drastically for $\sigma=0$. If we remove the cut-off σ in the expression (1.17), the definition of the vector at finite times becomes a delicate issue. The previous method fails because of divergences appearing in the series expansion of the Weyl operator

$$e^{-g\int_\sigma^{\kappa_1}\frac{a(\mathbf{k})-a^\dagger(\mathbf{k})}{\sqrt{2}|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E^\sigma(\mathbf{P})\right)}d^3k}\,. \tag{1.18}$$

The expansion is technically forced because the $\mathbf{P}-$fiber spaces $\mathcal{H}_{\mathbf{P}}$ are not preserved under the action of the operator (1.18). The definition at finite times of each $n-$component in (1.17) and summability in n is still well founded by assuming some regularity properties which can be eventually reconciled with the existence of the second derivative of the ground state energy $E(\mathbf{P})$. However, even in these assumptions, the time asymptotic behavior is practically out of control.

The difficulties coming from the expression without the cut-off σ are at the origin of an alternative and, for some aspects, conceptually different recipe for the approximating vector that we denote by $\psi_{h,\kappa_1}(t)$. The new main ingredients that we introduce are:
A convergence scheme based on a diagonal limiting procedure to better follow the slow asymptotic decoupling due to the interaction with infrared bosons. It means that the infrared cut-off in the approximating vectors is removed only asymptotically in time;
A constructive characterization of one particle states provided in [Pi.] which enable us to heavily use strong regularity properties (for details see Section 3 in [Pi.]);
The propagation estimate provided by the non-relativistic locality of the model, namely the decoupling mechanism used in Haag-Ruelle theory can be reproduced in terms of cluster properties at fixed time of the photon field and of the current density field associated to the electron.

Going to technical details, our proposal for the generic vector $\psi_{h,\kappa_1}(t)$ introduces:
A time dependent cut-off σ_t, which is removed at a rate faster than $\frac{1}{t}$;
The transformation of the integral in d^3P to a Riemann sum by a time dependent cell-partition of the $\mathbf{P}-$momentum space;
A phase factor, already somehow present in the tentative construction by Fröhlich for the case $\sigma = 0$, which is here exploited, in applying Cook's argument, as a function of the variable $\frac{\mathbf{x}(t)}{t} := e^{iHt}\frac{\mathbf{x}}{t}e^{-iHt}$, that is the electron mean velocity (at time t) up to a correction of order t^{-1}.

The two main differences with respect to Fröhlich's proposal, namely the diagonal limit and the cell partition of the $\mathbf{P}-$space, represent the building blocks of a strategy controlling, simultaneously, the logarithmic divergences arising in the two limits $\sigma \to 0$ and $t \to \infty$. To implement our strategy, the use of different time scales is crucial. They are basically:
The rate σ_t of the removal of the infrared cut-off, by which we approach the limit $\sigma = 0$ of no infrared cut-off cutting away the frozen degrees of freedom at the given time scale t;
The slower rate of the partition governed by an exponent ϵ, determined by the (estimated) time scale of the decoupling.

Let us anticipate the expected advantages of these constructive modifications in controlling the two quantities

$$\|\psi_{h,\kappa_1}(t)\|^2 \quad , \quad \|\psi_{h,\kappa_1}(t_2) - \psi_{h,\kappa_1}(t_1)\|^2 \tag{1.19}$$

that we will study in the paper.

1) By the transformation of the integral to a Riemann sum:

We can replace the series expansion of the Weyl operators by a "cell-expansion" in the $\mathbf{P}-$space which we can easily control exploiting the cluster property of the system. In this respect, we anticipate here that different values of $\mathbf{P}$ in the expansion correspond to different asymptotic velocities of the charged particle;
For all finite times, we deal with an expression in terms of bounded operators in the Hilbert space, that we can actually handle without considering, in general, any particular wave function representation but simply abstract calculus.

2) By introducing a time-dependent cut-off σ_t :
We can exploit the unitarity property of the Weyl operators as long as $\sigma_t > 0$. For each cell, it provides a-priori estimates without resumming contributions which are logarithmically divergent in the infrared limit. Moreover the a-priori estimates match easily with the power law decay of the vanishing quantities which neutralize the divergent terms;
We can extend or simply push to the limit of no infrared cut-off some properties which hold for the model with a fixed infrared cut-off. The properties are:
2i) The propagation estimate

$$e^{iHt} f\left(\frac{\mathbf{x}}{t}\right) e^{-iHt} \rightarrow_{t\rightarrow\infty} f\left(\nabla E\left(\mathbf{P}\right)\right) \qquad f \in C_0^{\infty}\left(\mathbb{R}^3\right)$$

which morally holds on one particle improper states, as it can be deduced from Theorem 4.2. This extrapolated property is nothing but the limiting case for $\sigma \rightarrow 0$ of the analogous convergence which can be easily proved in the case of a fixed $\sigma-$cutoff dynamics;
2ii) The fact that, for a fixed $\sigma-$cutoff dynamics, the one particle states are vacua for the annihilation part of the asymptotic boson field. It turns out to be extremely useful in treating the off-diagonal terms, with respect to the partition, of the quantities (1.19);

3) The phase factor is employed in Cook's argument, in analogy with Dollard's treatment of Coulomb scattering [Do.] (see also [K.F.]), though the present phase factor is only a technical tool in the following sense. In contrast to the Coulomb phase, it is in fact convergent for $t \rightarrow \infty$. It is seemingly avoidable, nevertheless it is helpful in our framework because provides a useful subtraction in the application of Cook's argument.

The explicit construction of the generic asymptotic vector will clarify the motivations for the strategy invoked so far. In the new recipe for the vector $\psi_{h,\kappa_1}(t)$, the key different points of view to be stressed are:
The construction of the vector is analyzed in terms of a "regular" block given by the one-particle states of transformed Hamiltonians $H^w_{\mathbf{P},\sigma}$ (see the expression (1.21) written later) and a "dressing" block which is different from the physical dressing photon cloud;
The infrared cut-off removal is an *a posteriori* result and a byproduct of the asymptotic decoupling.

Our construction should be simplified in order to treat generalizations, for instance more than one electron. Hopefully some constructive device is not necessary or can be made less cumbersome in a modified and improved recipe. However the present construction represents a starting point for simpler descriptions of the asymptotic decoupling and for a precise analysis of the involved time scales.
The entire construction is self-contained assuming the results in [Pi.] together with the resolvent formula (1.12) (for details, [Fr.1]). The only crucial constructive hypothesis not proven yet (but physically reasonable) concerns a positive bound from below for the second derivative of the ground state energy $E^{\sigma}(\mathbf{P})$ uniform in $\sigma > 0$ and in the region of $\mathbf{P}$ we are interested in.

1.3.1 Assumptions for the construction

Spectral properties

We recall some spectral results stated in [Pi.] which hold for $\mathbf{P} \in \Sigma$

$$\Sigma := \left\{ \mathbf{P} : |\mathbf{P}| < \frac{m}{20} \right\} ,$$

when the coupling constant g and the ratio $\frac{k}{m}$ are sufficiently small. The constraint on Σ reflects the mixed character of the model, which forces to restrict the physical region to the set $\{\mathbf{P} : |\mathbf{P}| < m\}$; the adopted more restrictive constraint is only due to technical reasons.

Given the Weyl operator

$$W_{\sigma}\left(\nabla E^{\sigma}_{\mathbf{P}}\right) = e^{-g\int_{\sigma}^{\kappa} \frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma}_{\mathbf{P}}\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}} \tag{1.20}$$

and the transformed Hamiltonian

$$H^{w}_{\mathbf{P},\sigma} := W_{\sigma}\left(\nabla E^{\sigma}_{\mathbf{P}}\right) H_{\mathbf{P},\sigma} W^{\dagger}_{\sigma}\left(\nabla E^{\sigma}_{\mathbf{P}}\right) , \tag{1.21}$$

the corresponding non degenerate ground eigenvector in $\mathcal{H}_{\mathbf{P}}$

$$\phi^{\sigma}_{\mathbf{P}} := \frac{-\frac{1}{2\pi i}\oint_{\gamma} \frac{1}{H^{w}_{\mathbf{P},\sigma}-E} dE\, \psi_0}{\left\| -\frac{1}{2\pi i}\oint_{\gamma} \frac{1}{H^{w}_{\mathbf{P},\sigma}-E} dE\, \psi_0 \right\|} \qquad \left\{ \gamma : \; E \in \mathbb{C} \,, \quad |E - E^{\sigma}_{\mathbf{P}}| = \frac{\sigma}{4} \right\} \tag{1.22}$$

is regular as function of σ and $\mathbf{P}$ in the space $\mathcal{F}^b$, according to the following results:

Theorem 3.2 [Pi.]

For $\mathbf{P} \in \Sigma$, the limit $s - \lim_{\sigma\to 0} \phi^{\sigma}_{\mathbf{P}} =: \phi_{\mathbf{P}}$ exists.
Moreover the convergence of $\phi^{\sigma}_{\mathbf{P}}$ to the non zero vector $\phi_{\mathbf{P}}$ in $\mathcal{H}_{\mathbf{P}} \cong \mathcal{F}^b$ and the convergence $\nabla E^{\sigma}_{\mathbf{P}} \to \nabla E_{\mathbf{P}}$ are estimated with errors at most of order $(\frac{\sigma}{\kappa})^{\frac{1}{4}-\delta}$ and $(\frac{\sigma}{\kappa})^{\frac{1}{4}}$ respectively, where $\delta > 0$ is arbitrarily small;

Lemma 3.3 [Pi.]

The following Hölder estimate holds:

$$|\nabla E^{\sigma}(\mathbf{P}) - \nabla E^{\sigma}(\mathbf{P}+\Delta\mathbf{P})| \leq C \cdot |\Delta\mathbf{P}|^{\frac{1}{16}}$$

where the constant C *is uniform in* $0 < \sigma < \kappa\epsilon$ *, in* $\mathbf{P}, \mathbf{P}+\Delta\mathbf{P} \in \Sigma$ *and* $\Delta\mathbf{P} \in \widehat{I}$*, where* $\widehat{I} := \left\{ \Delta\mathbf{P} : \frac{|\Delta\mathbf{P}|}{m} \leq \left(\frac{1}{3C_{\widehat{I}}}\right)^{\frac{8}{3}}, m^{\frac{3}{4}} |\Delta\mathbf{P}|^{\frac{1}{4}} \leq \kappa\epsilon \right\}$ *and* $C_{\widehat{I}}$ *is a constant sufficiently larger than* 1*.*

Theorem 3.4 [Pi.]

Under the constructive hypotheses, for $\frac{k}{m}$ *and* g *sufficiently small, the norm difference between* $\phi^{\sigma}_{\mathbf{P}}$ *and* $\phi^{\sigma}_{\mathbf{P}+\Delta\mathbf{P}}$ *is Hölder in* $|\Delta\mathbf{P}|$ *with coefficient* $\frac{1}{16} - \delta$*,* $\delta > 0$ *and arbitrarily small. The multiplicative constant,* C_{δ}*, is uniform in* $0 \leq \sigma < \kappa\epsilon$*, in* $\mathbf{P}, \mathbf{P}+\Delta\mathbf{P} \in \Sigma$ *and* $\Delta\mathbf{P} \in I$*,* I *a sufficiently small fixed ball around* $\Delta\mathbf{P} = 0$*.*

In our construction we will assume the results above (*Theorem 3.2* [Pi.], *Lemma 3.3* [Pi.] and *Theorem 3.4* [Pi.]) with coefficients $\frac{1}{4}$ and $\frac{1}{16}$ respectively, with no substantial difference for our procedure and the content of the final results.

From the analysis in [Fr.1] and [Pi.] we get that for $\mathbf{P} \in \Sigma$ we have small electron "velocities" that means $|\nabla E^{\sigma}(\mathbf{P})| < 1 \quad \forall\sigma$. In later constructions we assume that the upper frequency, κ_1, in the boson cloud of $\psi_{h,\kappa_1}(t)$ is small enough such that for $\mathbf{P} \in \Sigma$

$$|\nabla E^{\sigma}(\mathbf{P}+\mathbf{k})| < v^{max} < 1 \qquad \forall\sigma, \forall\mathbf{k} : 0 < |\mathbf{k}| \leq \kappa_1 \tag{1.23}$$

for a given and fixed $v^{max} > 0$.

Remark 1.1
We will treat the convergence in $\mathcal{H}$ of a vector given as a direct integral on the fiber spaces $\mathcal{H}_{\mathbf{P}}$. In order to avoid any confusion in dealing with vectors belonging to different fiber spaces, we will use explicitly the isomorphism $\mathbf{I}_{\mathbf{P}}$ in our notations differently from [Pi.]. Therefore, for instance, the property in *Theorem 3.4* [Pi.] is rewritten as follows:

$$\|\mathbf{I}_{\mathbf{P}+\Delta\mathbf{P}}(\phi^{\sigma}_{\mathbf{P}+\Delta\mathbf{P}}) - \mathbf{I}_{\mathbf{P}}(\phi^{\sigma}_{\mathbf{P}})\|_{\mathcal{F}} \leq C_{\delta} \cdot |\Delta\mathbf{P}|^{\frac{1}{16}-\delta} .$$

Spectral hypothesis

We also assume the following not proven hypothesis, which allows the construction of a (time-dependent) cell partition with the desired properties:

Hypothesis H0
For $\mathbf{P} \in \Sigma$ *, there exists a positive constant* $\frac{1}{m_r}$ *such that the following inequalities hold uniformly in* $\sigma > 0$*:*

$$\frac{\partial E^{\sigma}(\mathbf{P})}{\partial |\mathbf{P}|} \geq \frac{|\mathbf{P}|}{m_r} \qquad \textit{and} \qquad \frac{\partial^2 E^{\sigma}(\mathbf{P})}{\partial^2 |\mathbf{P}|} \geq \frac{1}{m_r} .$$

Assuming this hypothesis, the application

$$\mathbf{J}_\sigma : \mathbf{P} \rightarrow \nabla E^\sigma(\mathbf{P}) \qquad \mathbf{P} \in \Sigma,\ \sigma > 0 \tag{1.24}$$

is a bijection and the determinant of the Jacobian satisfies the inequality

$$det\, \mathbf{dJ}_\sigma = \frac{1}{|\mathbf{P}|^2}\left(\frac{\partial E^\sigma(\mathbf{P})}{\partial |\mathbf{P}|}\right)^2 \cdot \frac{\partial^2 E^\sigma(\mathbf{P})}{\partial^2 |\mathbf{P}|} \geq \frac{1}{m_r^3}\,;$$

concerning the calculation of the determinant, we recall that the function $E^\sigma(\mathbf{P})$ is invariant under rotations and belongs to $C^\infty(\mathbb{R}^3)$ (see [Fr.1]).
Under this assumption, given $O_\mathbf{P} \subset \Sigma$ and the corresponding region $O_{\nabla E^\sigma_\mathbf{P}}$ in the $\nabla E^\sigma_\mathbf{P}-$space, $O_\mathbf{P} = \mathbf{J}_\sigma^{-1}\left(O_{\nabla E^\sigma_\mathbf{P}}\right)$, the following relation holds between their volumes:

$$V_{O_\mathbf{P}} \leq m_r^3 \cdot V_{O_{\nabla E^\sigma_\mathbf{P}}}\,. \tag{1.25}$$

Cell partition

Let us consider a region contained in Σ, for convenience a cube of volume $V = L^3$.
We now construct a time-dependent, $t \gg 1$, cell-partition $\Gamma^{(t)}$ of the volume V, according to the following recipe:
At time $t \gg 1$, the linear dimension of each cell is $\frac{L}{2^n}$ where $n \in \mathbb{N},\ n \geq 1$, is such that

$$(2^n)^{\frac{1}{\epsilon}} \leq t < \left(2^{n+1}\right)^{\frac{1}{\epsilon}} \qquad \epsilon > 0$$

and the small exponent $\epsilon \ll 1$ is fixed only a posteriori.
This definition implies that the total number of cells at time t is $N(t) = (2^n)^3$, where $n = [\log_2 t^\epsilon]$, $[.]$ is the integer part. We call $\Gamma^{(t)}_j$ the j^{th} cell, centered in $\overline{\mathbf{P}_j}$, belonging to the partition, $\Gamma^{(t)}$, at time t.

1.3.2 Definition of the vector $\psi_{h,\kappa_1}(t)$

The generic vector $\psi_{h,\kappa_1}(t)$ is constructed starting from a one-particle state for the Hamiltonian H_{σ_t}, of wave function h in $\mathbf{P}-$variables.
A $\mathbf{P}-$dependent *L.S.Z.* Weyl operator, in properly evolved photon variables is applied, cell by cell, on the considered one-particle state. The smearing function in the generator of the Weyl operator has frequency support in the set $\sigma_t \leq |\mathbf{k}| \leq \kappa_1$ where $\sigma_t \rightarrow 0$ for $|t| \rightarrow +\infty$ and κ_1 is an arbitrarily small positive number which satisfies the constraint (1.23) . The behavior of the smearing function at $\mathbf{k} = 0$ is labelled by the spectral values of the operator $\nabla E^\sigma(\mathbf{P})$.

1) We start from the vector

$$\psi^{(t)}_{j,\sigma_t} := \int_{\Gamma^{(t)}_j} h(\mathbf{P})\, \psi^{\sigma_t}_\mathbf{P}\, d^3P \tag{1.26}$$

where:
$h(\mathbf{P}) \in C^1_0\left(\mathbb{R}^3 \setminus 0\right)$ has support inside the cube V ;
$\sigma_t = t^{-\beta}$, where $\beta \gg 1$ is fixed only a posteriori;

$\psi_{\mathbf{P}}^{\sigma_t} := W_{\sigma_t}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_t}\right)\phi_{\mathbf{P}}^{\sigma_t}$ is the unique ground state of $H_{\mathbf{P},\sigma_t}$.
Notice that

$$\left\|\psi_{j,\sigma_t}^{(t)}\right\| = \left(\int_{\Gamma_j^{(t)}} |h(\mathbf{P})|^2 \, d^3P\right)^{\frac{1}{2}}$$

is of order $(N(t))^{-\frac{1}{2}}$.

2) We consider for each $\psi_{j,\sigma_t}^{(t)}$ a corresponding dressing "cloud" carried by the *L.S.Z.* Weyl operator

$$e^{iHt} e^{-iH^{ph}t} \mathcal{W}_{\sigma_t}(\mathbf{v}_j) \, e^{iH^{ph}t} e^{-iH_{\sigma_t}t},$$

here

$$\mathcal{W}_{\sigma_t}(\mathbf{v}_j) := e^{-g\int_{\sigma_t}^{\kappa_1} \frac{a(\mathbf{k})-a^{\dagger}(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}} \tag{1.27}$$

where $\mathbf{v}_j \equiv \nabla E^{\sigma_t}\left(\overline{\mathbf{P}_j}\right)$ is the "velocity" at time t corresponding to the center $\overline{\mathbf{P}_j}$ of the cell $\Gamma_j^{(t)}$. In order not to overburden the notations, the time dependence of $\mathbf{v}_j$ is not explicit. However it can be easily recovered from the time which labels the corresponding cell $\Gamma_j^{(t)}$. This will be carefully taken into account in the study of the convergence of $\psi_{h,\kappa_1}(t)$, precisely in Subsection 3.1.1. The c-number $\mathbf{v}_j$ clearly commutes with the algebra generated by $\left\{a(\mathbf{k}), a^{\dagger}(\mathbf{k})\right\}$.

3) Finally we define:

$$\begin{aligned} \psi_{h,\kappa_1}(t) &:= e^{iHt} e^{-iH^{ph}t} \textstyle\sum_{j=1}^{N(t)} \mathcal{W}_{\sigma_t}(\mathbf{v}_j) \, e^{iH^{ph}t} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E_{\mathbf{P}}^{\sigma_t}, t\right)} e^{-iE_{\mathbf{P}}^{\sigma_t}t} \psi_{j,\sigma_t}^{(t)} \\ &= e^{iHt} \textstyle\sum_{j=1}^{N(t)} \mathcal{W}_{\sigma_t}(\mathbf{v}_j, t) \, e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E_{\mathbf{P}}^{\sigma_t}, t\right)} e^{-iE_{\mathbf{P}}^{\sigma_t}t} \psi_{j,\sigma_t}^{(t)} \end{aligned} \tag{1.28}$$

with the definitions

$$\mathcal{W}_{\sigma_t}(\mathbf{v}_j, t) := e^{-iH^{ph}t} \mathcal{W}_{\sigma_t}(\mathbf{v}_j) \, e^{iH^{ph}t} = e^{-g\int_{\sigma_t}^{\kappa_1} \frac{a(\mathbf{k})e^{i|\mathbf{k}|t} - a^{\dagger}(\mathbf{k})e^{-i|\mathbf{k}|t}}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}} \tag{1.29}$$

$$e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E_{\mathbf{P}}^{\sigma_t}, t\right)} := e^{-i\int_1^t \left\{ g^2 \int_{\sigma_t}^{\sigma_\tau^S} \frac{\cos\left(\mathbf{k}\cdot\nabla E_{\mathbf{P}}^{\sigma_t}\tau - |\mathbf{k}|\tau\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} d\Omega \, d|\mathbf{k}| \right\} d\tau} \tag{1.30}$$

where $\sigma_\tau^S = \tau^{-\alpha}$ with α, $0 < \alpha < 1$, fixed only a posteriori.
The definitions (1.29), (1.30) require some comments contained in the remarks below, which give us the opportunity to come back to the motivations of the recipe here presented.

Remark 1.2
As far as the boson cloud and the coherent factor (1.11)

$$-g\frac{1}{|\mathbf{k}|\sqrt{2|\mathbf{k}|}\left(1-\widehat{\mathbf{k}}\cdot\nabla E(\mathbf{P})\right)}$$

are concerned, by the introduction of the c-number $\mathbf{v}_j$ we implement in our formalism a crucial physical feature which is not exploited in all its consequences in Fröhlich's formalism. More precisely, the operator that actually labels the coherent factor is the asymptotic (mean) velocity of the electron, that, differently from $\nabla E(\mathbf{P})$, has to commute with the asymptotic boson algebra. The two operators would coincide on the one-particle states if the latter ones existed, as happens when a fixed infrared cut-off in the interaction is considered. We want to keep track of this concept in a limiting construction, involving the time dependent partition and the discretized velocities $\nabla E^{\sigma_t}\left(\overline{\mathbf{P}_j}\right)$. We will see that the chosen recipe for the dressing cloud is in the end a technically convenient way to approximate the operator asymptotic (mean) velocity of the electron inside the wave function of the photon cloud. The reason is that we can easily exploit the cluster property which implements the asymptotic orthogonality between off-diagonal terms in the partition of the vector. We stress that the operator asymptotic (mean) velocity of the electron is not constructed yet. We only use the values that it is expected to take on the minimal asymptotic electron states for k_1 smaller and smaller, in the region of momenta $\mathbf{P}$ which is physically meaningful. Moreover the expression given for the smearing function in the Weyl operators (1.29) encodes somehow, already at finite times, the commutation property we expect at asymptotic times between the asymptotic boson algebra and the asymptotic (mean) velocity of the electron, up to an error which becomes smaller and smaller as time increases and the partition gets finer and finer.

Remark 1.3
The introduction of the phase factor (1.30) is related to Cook's argument. The "fast" cut-off σ_τ is of order $\tau^{-\beta}$, where β is larger than 1. The integration bound σ_τ^S is a "slow" infrared cut-off, $\sigma_\tau^S = \tau^{-\alpha}$ where α is a positive number less than 1. The infinitesimal upper bound σ_τ^S for the integral in (1.30) enables us to replace the argument $\nabla E_{\mathbf{P}}^{\sigma_t}$ with $\frac{\mathbf{x}(t)}{t}$ (Corollary A3) for asymptotic times. Therefore we get that the time derivative of the phase factor kills an infrared tail term arising from the application of Cook's argument, which is not (absolutely) convergent as function of t. On the basis of partial estimates, α is eventually chosen sufficiently close to 1 and β large enough with respect to 1 in order to achieve the strong convergence of the vector $\psi_{h,\kappa_1}(t)$.

1.4 Survey of results and plan of the paper

After having constructed the generic vector $\psi_{h,\kappa_1}(t)$, we prove the existence of the strong limit

$$s-\lim_{t\to\pm\infty}\psi_{h,\kappa_1}(t) =: \psi_{h,\kappa_1}^{out(in)}\,.$$

The construction is explicitly performed in the case "out", the case "in" is completely analogous. In our notations we use the superscript $^{(in)}$ to mean either that an analogous structure holds for the ingoing case or to denote both the two ones,

for instance both the two asymptotic subspaces. However we do not claim anything about their relations.
By analogy with the regularized case, we define the invariant (under space-time translation) subspaces

$$\mathcal{H}^{1out(in)}_{\kappa_1} := \overline{\left\{ \bigvee \psi^{out(in)}_{h,\kappa_1}(\tau,\mathbf{a}) \;:\quad h(\mathbf{P}) \in C^1_0(\Sigma\setminus 0)\,,\, \tau\in\mathbb{R}\,,\mathbf{a}\in\mathbb{R}^3 \right\}}$$

where the subscript κ_1 denotes the upper frequency in the boson cloud and the vector $\psi^{out(in)}_{h,\kappa_1}(\tau,\mathbf{a})$ corresponds to the evolution τ in time and to a displacement $\mathbf{a}$ in space of the state associated to the vector $\psi^{out(in)}_{h,\kappa_1}$. Because of the presence of the boson cloud, the electronic wave functions $\{h\}$ cannot fully characterize the set of states we are interested in. The next step consists in adding "hard" asymptotic bosons as result of the limits

$$s-\lim_{t\to\pm\infty} e^{iHt}e^{-iH^{ph}t}e^{i\left(a(\mu)+a^\dagger(\mu)\right)}e^{iH^{ph}t}e^{-iHt}\psi^{out(in)}_{h,\kappa_1}(\tau,\mathbf{a}) =: \psi^{out(in)}_{h,\mu}$$

where $a^\dagger(\mu) := (a(\mu))^\dagger = \left(\int a(\mathbf{k})\,\widetilde{\mu}(\mathbf{k})\,d^3k\right)^\dagger$, $\mu(\mathbf{y}) = \int e^{-i\mathbf{k}\mathbf{y}}\widetilde{\mu}(\mathbf{k})\,d^3k$, $\widetilde{\mu}(\mathbf{k}) \in C^\infty_0\left(\mathbb{R}^3\setminus 0\right)$, and the dependence on κ_1,τ , $\mathbf{a}$ is omitted in the final expression.
Finally, the proposed scattering subspaces are

$$\mathcal{H}^{out(in)} := \overline{\left\{ \bigvee \psi^{out(in)}_{h,\mu} \;:\quad h(\mathbf{P}) \in C^1_0(\Sigma\setminus 0)\,, \widetilde{\mu}\in C^\infty_0(\mathbb{R}^3\setminus 0) \right\}}.$$

On these subspaces the C^∞_0 functions f of the variable $e^{iHt}\frac{\mathbf{x}}{t}e^{-iHt}$ converge. This means that the limits

$$s-\lim_{t\to\pm\infty} e^{iHt} f\left(\frac{\mathbf{x}}{t}\right) e^{-iHt}$$

exist and generate the commutative algebra $\mathcal{A}^{out(in)}_{\mathbf{v}_{el}}$. Analogously the canonical Weyl algebra $\mathcal{A}^{out(in)}_{ph}$associated to a free massless boson field is generated by the strong time limits of the *L.S.Z.* Weyl operators acting on the space $\mathcal{H}^{out(in)}$:

$$\mathcal{W}^{out(in)}(\zeta) := s-\lim_{t\to\pm\infty} e^{iHt}e^{-iH^{ph}t}e^{i\left(a(\zeta)+a^\dagger(\zeta)\right)}e^{iH^{ph}t}e^{-iHt} \tag{1.31}$$

with $\widetilde{\zeta}(\mathbf{k}) \in L^2\left(\mathbb{R}^3, \left(1+|\mathbf{k}|^{-1}\right)d^3k\right)$.
The two algebras $\mathcal{A}^{out(in)}_{\mathbf{v}_{el}}$, $\mathcal{A}^{out(in)}_{ph}$ are related to decoupled degrees of freedom and therefore commute each other.

Remark 1.4
A warning is necessary at this point. Previous definitions are arbitrary to some extent, due to the coherent function in the definition of the minimal asymptotic electron states, which is arbitrary except in the infrared limit. Nevertheless, through the (artificial) distinction between $\mathcal{H}^{1out(in)}_{\kappa_1}$ and $\mathcal{H}^{out(in)}$ we want to point out that:
From a technical point of view, our construction of the scattering subspaces is based on some, not unique, $\mathcal{H}^{1out(in)}_{\kappa_1}$ just to focus on the infrared dressing;
From a physical point of view, whether the "hard" photon cloud described by the smearing functions $\{\mu\}$ is totally removable, the photon cloud linked to the vectors

in $\mathcal{H}^{1out(in)}_{\kappa_1}$ is not completely removable. It means that all scattering states contain asymptotic photons.
The physical quantities must be independent of the choice of the "one-particle" subspace $\mathcal{H}^{1out(in)}_{\kappa_1}$, in particular of the choice of κ_1. Indeed, once Σ is fixed, if we considered a different $\mathcal{H}^{1out(in)}_{\kappa'_1}$, i.e. with a different upper frequency, $\kappa'_1 > \kappa_1$, in the photon cloud of the generic vector $\psi^{out(in)}_{h,\kappa'_1}$, it is not difficult to check that $\psi^{out(in)}_{h,\kappa'_1}$ can be expressed as a dressing of $\psi^{out(in)}_{h,\kappa_1}$ by means of asymptotic photons and, vice versa, that $\psi^{out(in)}_{h,\kappa_1}$ can be obtained from $\psi^{out(in)}_{h,\kappa'_1}$ subtracting asymptotic photons. Following the procedures developed in the next two sections, it is enough to properly choice the partition rates for combining estimates and to take into account the mechanism used in Subsection 3.1.1. This is important in order to identify limits obtained with different partition rates. In this analysis it is also necessary to assume the following result proved in [Fr.2] (formula 3.5)

$$\inf_{|\mathbf{k}|\geq\sigma_t}\left\{E^{\sigma_t}(\mathbf{P}-\mathbf{k})+|\mathbf{k}|-E^{\sigma_t}(\mathbf{P})\right\}=\Delta(\sigma_t,\mathbf{P})>0 \tag{1.32}$$

which holds for any $\mathbf{P}\in\mathbf{\Sigma}$ and provides the inequality (5.35) in Theorem A5 when κ'_1 is too large to fulfill the constraint (1.23).
Therefore the space $\mathcal{H}^{out(in)}$ and the algebras $\mathcal{A}^{out(in)}_{\mathbf{v}_{el}}$, $\mathcal{A}^{out(in)}_{ph}$ are independent of the construction of the "one-particle" space $\mathcal{H}^{1out(in)}_{\kappa_1}$ but depend only on Σ.

About the structure of the paper:
In Section 2 we study the time behavior of the norm of the approximating vector $\psi_{h,\kappa_1}(t)$;
In Section 3 we prove the strong convergence of $\psi_{h,\kappa_1}(t)$ for $t\to\infty$;
Section 4 contains the construction of the scattering subspace $\mathcal{H}^{out(in)}$, of $\mathcal{A}^{out(in)}_{\mathbf{v}_{el}}$ and $\mathcal{A}^{out(in)}_{ph}$;
Section 5 contains the Appendix where we collect some results employed in Sections 2,3,4. Lemmas and theorems in the Appendix are denoted by the letter A (i.e. Lemma A1).

2 Control of the norm of the approximating vector

The squared norm $(\psi_{h,\kappa_1}(t),\psi_{h,\kappa_1}(t))$ corresponds to:

$$\sum_{l,j=1}^{N(t)}\left(\mathcal{W}_{\sigma_t}(\mathbf{v}_l,t)\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},t\right)}e^{-iH_{\sigma_t}t}\psi^{(t)}_{l,\sigma_t},\mathcal{W}_{\sigma_t}(\mathbf{v}_j,t)\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},t\right)}e^{-iH_{\sigma_t}t}\psi^{(t)}_{j,\sigma_t}\right).$$

The diagonal terms are easily under control because their sum is constant in time:

$$\sum_{j=1}^{N(t)}\left(\psi^{(t)}_{j,\sigma_t},\psi^{(t)}_{j,\sigma_t}\right)=\sum_{j=1}^{N(t)}\int_{\Gamma^{(t)}_j}|h(\mathbf{P})|^2\,d^3P=\int|h(\mathbf{P})|^2\,d^3P\,. \tag{2.1}$$

The non trivial step consists in proving that it is indeed the limit of the squared norm provided the partition rate is properly chosen. For this purpose it is enough to show that each off-diagonal term in the sum $\sum_{l,j=1}^{N(t)}$ asymptotically vanishes with an order in t substantially not related to the dimension of the cell. In the end, we obtain that the sum of the off-diagonal terms, $\sum_{l,j=1,l\neq j}^{N(t)}$, vanishes for $t \to +\infty$, provided the exponent ϵ, which determines the growth rate of the total number of cells, $N(t) \leq t^{3\epsilon}$, is sufficiently small.

2.1 Control of the off-diagonal terms

The generic off-diagonal term is $(l \neq j)$

$$M_{l,j}(t) = \left(e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},t\right)}\psi^{(t)}_{l,\sigma_t}, e^{iH_{\sigma_t}t}\mathcal{W}_{\sigma_t,l,j}(t)\,e^{-iH_{\sigma_t}t}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},t\right)}\psi^{(t)}_{j,\sigma_t}\right) \quad (2.2)$$

where

$$\mathcal{W}_{\sigma_t,l,j}(t) := e^{-\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|t}-a^{\dagger}(\mathbf{k})e^{-i|\mathbf{k}|t}}{|\mathbf{k}|}\eta_{l,j}\left(\widehat{\mathbf{k}}\right)\frac{d^3k}{\sqrt{2|\mathbf{k}|}}} \quad (2.3)$$

and

$$\eta_{l,j}\left(\widehat{\mathbf{k}}\right) := \frac{g\widehat{\mathbf{k}}\cdot(\mathbf{v}_j-\mathbf{v}_l)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)\cdot\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)}\,. \quad (2.4)$$

Now, let us consider $M_{l,j}(t)$ as a two-variable function, by distinguishing the variable t, which parameterizes the partition $\Gamma^{(t)}$ and the infrared cut-off σ_t, from the variable, s, of the dynamical evolution. Then, for $s \geq t$ we define:

$$\widehat{M}_{l,j}(t,s) := \left(e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{l,\sigma_t}\,,\, e^{iH_{\sigma_t}s}\mathcal{W}_{\sigma_t,l,j}(s)\,e^{-iH_{\sigma_t}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{j,\sigma_t}\right) \quad (2.5)$$

where

$$\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right) :=$$

$$= \begin{cases} -\int_1^s \left\{g^2\int_{\tau\cdot\sigma_t}^{\tau\cdot\tau^{-\alpha}}\frac{\cos\left(\mathbf{q}\cdot\nabla E^{\sigma_t}_{\mathbf{P}}-|\mathbf{q}|\right)}{(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_l)}\frac{d\Omega d|\mathbf{q}|}{\tau}\right\}d\tau & \text{for } s: \quad s^{-\alpha}\geq\sigma_t \\ -\int_1^{\sigma_t^{-\frac{1}{\alpha}}} \left\{g^2\int_{\tau\cdot\sigma_t}^{\tau\cdot\tau^{-\alpha}}\frac{\cos\left(\mathbf{q}\cdot\nabla E^{\sigma_t}_{\mathbf{P}}-|\mathbf{q}|\right)}{(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_l)}\frac{d\Omega d|\mathbf{q}|}{\tau}\right\}d\tau & \text{for } s: \quad s^{-\alpha}<\sigma_t\,. \end{cases} \quad (2.6)$$

The property $\widehat{M}_{l,j}(t,t) \equiv M_{l,j}(t)$ follows by definition.

Theorem 2.1
Under the constructive assumptions and for $\alpha(<1)$ sufficiently close to 1, the following properties hold for the off-diagonal matrix elements $\widehat{M}_{l,j}(t,s)$:

I) $\widehat{M}_{l,j}(t,+\infty) := \lim_{s\to+\infty}\widehat{M}_{l,j}(t,s) = 0$

II) $|M_{l,j}(t)| = \left|\widehat{M}_{l,j}(t,t) - \widehat{M}_{l,j}(t,+\infty)\right| \leq C \cdot t^{-7\epsilon}$

provided $4\epsilon < \eta$ where η is a positive exponent α-dependent.

Proof
analysis of I)
For $s \geq t$, let us consider

$$\widehat{M}^{\lambda}_{l,j}(t,s) := \left(e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{l,\sigma_t}\,,\, e^{iH_{\sigma_t}s}\mathcal{W}^{\lambda}_{\sigma_t,l,j}(s)\,e^{-iH_{\sigma_t}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{j,\sigma_t}\right) \tag{2.7}$$

where

$$\mathcal{W}^{\lambda}_{\sigma_t,l,j}(s) := e^{-\lambda\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|s}-a^{\dagger}(\mathbf{k})e^{-i|\mathbf{k}|s}}{|\mathbf{k}|}\eta_{l,j}(\widehat{\mathbf{k}})\frac{d^3k}{\sqrt{2|\mathbf{k}|}}} \tag{2.8}$$

λ being a real parameter.
From the derivative[1] of $\widehat{M}^{\lambda}_{l,j}(t,s)$ with respect to the real parameter λ, the following differential equation is determined:

$$\frac{d\widehat{M}^{\lambda}_{l,j}(t,s)}{d\lambda} = -\lambda C_{l,j,\sigma_t}\cdot\widehat{M}^{\lambda}_{l,j}(t,s) + r^{\lambda}_{\sigma_t}(t,s) \tag{2.9}$$

where

$$C_{l,j,\sigma_t} = \int_{\sigma_t}^{\kappa_1}\left|\eta_{l,j}\left(\widehat{\mathbf{k}}\right)\right|^2\frac{d^3k}{2|\mathbf{k}|^3} \tag{2.10}$$

$$\begin{aligned} &r^{\lambda}_{\sigma_t}(t,s) = \\ &= \Bigg(-\mathcal{W}^{\lambda\dagger}_{\sigma_t,l,j}(s)\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{l,\sigma_t}\,, \\ &\qquad ,\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|s}\eta_{l,j}(\widehat{\mathbf{k}})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{j,\sigma_t}\Bigg) \\ &+\Bigg(\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})e^{i|\mathbf{k}|s}\eta_{l,j}(\widehat{\mathbf{k}})}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{l,\sigma_t}\,, \\ &\qquad ,\mathcal{W}^{\lambda}_{\sigma_t,l,j}(s)\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}e^{-iH_{\sigma_t}s}\psi^{(t)}_{j,\sigma_t}\Bigg) \end{aligned} \tag{2.11}$$

The solution of the differential equation (2.9) at $\lambda = 1$ is

$$\widehat{M}_{l,j}(t,s) = e^{-\frac{C_{l,j,\sigma_t}}{2}}\widehat{M}^{\lambda=0}_{l,j}(t,s) + \int_0^1 r^{\lambda'}_{\sigma_t}(t,s)\cdot e^{-\frac{C_{l,j,\sigma_t}}{2}\left(1-\lambda'^2\right)}d\lambda'\,. \tag{2.12}$$

Now, notice the following facts:

$\widehat{M}^{\lambda=0}_{l,j}(t,s) = 0\ \ \forall t,s$, because the $\mathbf{P}-$supports of $\psi^{(t)}_{j,\sigma_t}$ and $\psi^{(t)}_{l,\sigma_t}$, $l \neq j$, are disjoint;

[1] $\psi^{(t)}_{j,\sigma_t} \in D(H_{\sigma_t})$ implies that it belongs to the domains of the operators $a(f)$ and $a^{\dagger}(f)$, $f \in L^2\left(\mathbb{R}^3 \setminus B_{\sigma_t}\right)$ with $B_{\sigma_t} := \left\{\mathbf{k} \in \mathbb{R}^3 :\ |\mathbf{k}| \leq \sigma_t\right\}$; therefore the derivative with respect to λ is well defined.

Thanks to Theorem A5 (in Appendix), the vector

$$s-\lim_{s\to+\infty} e^{iH_{\sigma_t}s}\int_{\sigma_t}^{\kappa_1}\frac{a(\mathbf{k})\,e^{i|\mathbf{k}|s}\eta_{l,j}\left(\widehat{\mathbf{k}}\right)}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}e^{-iH_{\sigma_t}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_t},s\right)}\psi_{j,\sigma_t}^{(t)}$$

is well defined and can be written as

$$\int_{\sigma_t}^{\kappa_1}\frac{a_{\sigma_t}^{out}(\mathbf{k})\,\eta_{l,j}\left(\widehat{\mathbf{k}}\right)}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_t},\sigma_t^{-\frac{1}{\alpha}}\right)}\psi_{j,\sigma_t}^{(t)}\,,$$

where $a_{\sigma_t}^{out}(\mathbf{k})$ is the asymptotic annihilation operator-valued distribution corresponding to the dynamics governed by the Hamiltonian H_{σ_t};
Since the vector $\psi_{j,\sigma_t}^{(t)}$ is a vacuum vector for $\left\{a_{\sigma_t}^{out}(\mathbf{k})\right\}$ (see Theorem A5), we get

$$\lim_{s\to+\infty} r_{\sigma_t}^{\lambda}(t,s)=0\,. \tag{2.13}$$

Starting from the solution (2.12) and exploiting the dominated convergence theorem, we have

$$\widehat{M}_{l,j}(t,+\infty)=\lim_{s\to+\infty}\int_0^1 r_{\sigma_t}^{\lambda'}(t,s)\cdot e^{-\frac{C_{l,j,\sigma_t}}{2}\left(1-\lambda'^2\right)}d\lambda'=0\,. \tag{2.14}$$

analysis of II)
Let us consider:

$$\begin{aligned} &e^{-iH_{\sigma_t}s}\frac{d}{ds}\left(e^{iH_{\sigma_t}s}\mathcal{W}_{\sigma_t}(\mathbf{v}_l,s)\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E_{\mathbf{P}}^{\sigma_t},s\right)}e^{-iH_{\sigma_t}s}\right)\psi_{l,\sigma_t}^{(t)}= \\ &\qquad = i\mathcal{W}_{\sigma_t}(\mathbf{v}_l,s)\left(\varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)+\frac{d\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E_{\mathbf{P}}^{\sigma_t},s\right)}{ds}\right)e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E_{\mathbf{P}}^{\sigma_t},s\right)}e^{-iH_{\sigma_t}s}\psi_{l,\sigma_t}^{(t)} \end{aligned} \tag{2.15}$$

where

$$\varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s):=g^2\int_{\sigma_t}^{\kappa_1}\frac{\cos\left(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|\,s\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)}d\Omega d\,|\mathbf{k}|\ . \tag{2.16}$$

The formal derivative in (2.15) is operatorially well defined because $\psi_{j,\sigma_t}^{(t)}\in D\left(H_{\sigma_t}\right)\equiv D(H)$ [2].
First we discuss some preliminary quantities useful to estimate the norm of the

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[2] More precisely, the result follows because:
The operators H_{σ_t}, H^{ph}, $H_0=\frac{\mathbf{P}^2}{2m}+H^{ph}$ and $g\int_{\sigma_t}^{\kappa_1}\left(a(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}}+a^{\dagger}(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\frac{d^3k}{\sqrt{2|\mathbf{k}|}}$ have a common e.s.d. D;
The derivative

$$\frac{d\left(e^{iH^{ph}s}e^{-iH_{\sigma_t}s}\right)}{ds}$$

is an operator which has a closure;
Approximating the vectors in $D\left(H_{\sigma_t}\right)$ with vectors in D (in the norm $\|H_0\psi\|+\|\psi\|$) and applying the formal calculus we get convergent sequences.

expression (2.15).

i) From the definition (2.6) we have

$$\frac{d\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E_{\mathbf{P}}^{\sigma_t},s\right)}{ds}=\begin{cases}-g^2\int_{\sigma_t\cdot s}^{s^{1-\alpha}}\frac{\cos\left(\mathbf{q}\cdot\nabla E_{\mathbf{P}}^{\sigma_t}-|\mathbf{q}|\right)}{(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_l)}\frac{d\Omega d|\mathbf{q}|}{s} & \text{for } s<\sigma_t^{-\frac{1}{\alpha}}\\ 0 & \text{for } s\geq\sigma_t^{-\frac{1}{\alpha}}\,.\end{cases} \tag{2.17}$$

By analogy we define

$$\frac{d\gamma_{\sigma_t}\left(\mathbf{v}_l,\frac{\mathbf{x}}{s},s\right)}{ds}:=\begin{cases}-g^2\int_{\sigma_t\cdot s}^{s^{1-\alpha}}\frac{\cos\left(\mathbf{q}\cdot\frac{\mathbf{x}}{s}-|\mathbf{q}|\right)}{(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_l)}\frac{d\Omega d|\mathbf{q}|}{s} & \text{for } s<\sigma_t^{-\frac{1}{\alpha}}\\ 0 & \text{for } s\geq\sigma_t^{-\frac{1}{\alpha}}\,.\end{cases} \tag{2.18}$$

ii) The function $\varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)$ can be decomposed as

$$\varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)=\varphi^-_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)+\varphi^+_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s) \tag{2.19}$$

where the two terms on the right hand side of the equation (2.19) are defined as

$$\varphi^-_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s):=\begin{cases}g^2\int_{\sigma_t}^{s^{-\alpha}}\frac{\cos(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|s)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)}d\Omega d|\mathbf{k}| & \text{for } s<\sigma_t^{-\frac{1}{\alpha}}\\ 0 & \text{for } s\geq\sigma_t^{-\frac{1}{\alpha}}\end{cases} \tag{2.20}$$

$$\varphi^+_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s):=\begin{cases}g^2\int_{s^{-\alpha}}^{\kappa_1}\frac{\cos(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|s)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_l\right)}d\Omega d|\mathbf{k}| & \text{for } s<\sigma_t^{-\frac{1}{\alpha}}\\ \varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s) & \text{for } s\geq\sigma_t^{-\frac{1}{\alpha}}\end{cases} \tag{2.21}$$

iii) For implementing the propagation estimate concerning the position of the electron, taking into account Hypothesis H0 (***Spectral hypothesis***, Subsection 1.3.1) we can consider a $C_0^\infty\left(\mathbb{R}^3\setminus 0\right)$ function χ_h with the following property:

$$\chi_h\left(\nabla E_{\mathbf{P}}^{\sigma_t}\right)\equiv 1 \qquad \text{for} \qquad \mathbf{P}\in supph \tag{2.22}$$

uniformly in t.

What we want now to check is that the norm of the expression (2.15) goes to zero with an integrable rate substantially independent of the partition rate. For this purpose we exploit the decomposition (2.19) of $\varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)$ and the function $\chi_h\left(\nabla E_{\mathbf{P}}^{\sigma_t}\right)$. We break the expression (2.15) in separate contributions and estimate the norm for each of them to control the norm of the original vector:

$$\left\|\frac{d}{ds}\left(e^{iH_{\sigma_t}s}\mathcal{W}_{\sigma_t}(s)\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E_{\mathbf{P}}^{\sigma_t},s\right)}e^{-iH_{\sigma_t}s}\right)\psi^{(t)}_{l,\sigma_t}\right\|\leq \tag{2.23}$$

$$\leq \left\| \varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)\left(\chi_h\left(\nabla E^{\sigma_t}_{\mathbf{P}}\right)-\chi_h\left(\frac{\mathbf{x}}{s}\right)\right)e^{-iE^{\sigma_t}_{\mathbf{P}}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{l,\sigma_t}\right\|$$
$$+\left\|\varphi^{+}_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)\,\chi_h\left(\frac{\mathbf{x}}{s}\right)e^{-iE^{\sigma_t}_{\mathbf{P}}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{l,\sigma_t}\right\|$$
$$+\left\|\left(\varphi^{-}_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)+\frac{d\gamma_{\sigma_t}\left(\mathbf{v}_l,\frac{\mathbf{x}}{s},s\right)}{ds}\right)\chi_h\left(\frac{\mathbf{x}}{s}\right)e^{-iE^{\sigma_t}_{\mathbf{P}}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{l,\sigma_t}\right\|$$
$$+\left\|\frac{d\gamma_{\sigma_t}\left(\mathbf{v}_l,\frac{\mathbf{x}}{s},s\right)}{ds}\left(\chi_h\left(\nabla E^{\sigma_t}_{\mathbf{P}}\right)-\chi_h\left(\frac{\mathbf{x}}{s}\right)\right)e^{-iE^{\sigma_t}_{\mathbf{P}}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{l,\sigma_t}\right\|$$
$$+\left\|\left(-\frac{d\gamma_{\sigma_t}\left(\mathbf{v}_l,\frac{\mathbf{x}}{s},s\right)}{ds}+\frac{d\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}{ds}\right)\chi_h\left(\nabla E^{\sigma_t}_{\mathbf{P}}\right)e^{-iE^{\sigma_t}_{\mathbf{P}}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{l,\sigma_t}\right\|$$

Now we explain how in the expression above each term is controlled and why we can fix $\eta>0$ such that a leading order is $s^{-1}\cdot s^{-\eta}\cdot|\ln\sigma_t|^2\cdot t^{-\frac{3\epsilon}{2}}$.
From Lemma A4, Theorem A2 and Corollary A3 we deduce that:
After the subtraction of the infrared tail $\varphi^{-}_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)$ and exploiting the electron dispersion, the decoupling is estimated from above by

$$\sup_{\mathbf{x}}\left|\varphi^{+}_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)\,\chi_h\left(\frac{\mathbf{x}}{s}\right)\right|\leq C\cdot s^{-2}\cdot s^{\alpha}\,;$$

The remainder is controlled by combining the bounds

$$\begin{aligned}\sup_{\mathbf{x}}\left|\varphi_{\sigma_t,\mathbf{v}_l}(\mathbf{x},s)\right| &\leq C\cdot\left|\frac{\ln(\sigma_t)}{s}\right|\\ \sup_{\mathbf{x}}\left|\frac{d\gamma_{\sigma_t}\left(\mathbf{v}_l,\frac{\mathbf{x}}{s},s\right)}{ds}\right| &\leq C\cdot\left|\frac{\ln(\sigma_t)}{s}\right|\end{aligned}\tag{2.24}$$

with the propagation estimates

$$\begin{aligned}&\left\|\left(\chi_h\left(\nabla E^{\sigma_t}_{\mathbf{P}}\right)-\chi_h\left(\frac{\mathbf{x}}{s}\right)\right)e^{-iE^{\sigma_t}_{\mathbf{P}}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{l,\sigma_t}\right\|\leq\\ &\qquad\leq C\cdot s^{-\upsilon}\cdot|\ln\sigma_t|\cdot t^{-\frac{3\epsilon}{2}}\\ &\left\|\left(-\frac{d\gamma_{\sigma_t}\left(\mathbf{v}_l,\frac{\mathbf{x}}{s},s\right)}{ds}+\frac{d\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}{ds}\right)e^{-iE^{\sigma_t}_{\mathbf{P}}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{l,\sigma_t}\right\|\leq\\ &\qquad\leq C\cdot s^{-1}\cdot s^{-\upsilon}\cdot|\ln\sigma_t|\cdot t^{-\frac{3\epsilon}{2}}\end{aligned}\tag{2.25}$$

where $\upsilon>0$ for α sufficiently close to 1 and ϵ small enough (Theorem A2, Corollary A3). Moreover, assuming the following constraint

$$4\epsilon<\eta\,,\tag{2.26}$$

we easily obtain

$$|M_{l,j}(t)|\leq\tag{2.27}$$

$$\leq \left|\int_t^{+\infty} \frac{d}{ds}\left(e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E_{\mathbf{P}}^{\sigma_t},s\right)}\psi_{l,\sigma_t}^{(t)}\,,\, e^{iH_{\sigma_t}s}\mathcal{W}_{\sigma,l,j}(s)\,e^{-iH_{\sigma_t}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_t},s\right)}\psi_{j,\sigma_t}^{(t)}\right)ds\right|$$

$$\leq \int_t^{+\infty}\left\|\frac{d}{ds}\left\{e^{iH_{\sigma_t}s}\mathcal{W}_{\sigma_t}(\mathbf{v}_l,s)\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_l,\nabla E_{\mathbf{P}}^{\sigma_t},s\right)}e^{-iH_{\sigma_t}s}\psi_{l,\sigma_t}^{(t)}\right\}\right\|\cdot\left\|\psi_{j,\sigma_t}^{(t)}\right\|ds$$

$$+\int_t^{+\infty}\left\|\frac{d}{ds}\left\{e^{iH_{\sigma_t}s}\mathcal{W}_{\sigma_t}(\mathbf{v}_j,s)\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_t},s\right)}e^{-iH_{\sigma_t}s}\psi_{j,\sigma_t}^{(t)}\right\}\right\|\cdot\left\|\psi_{l,\sigma_t}^{(t)}\right\|ds$$

$$\leq C\cdot t^{-7\epsilon}$$

Hence the sum of the off-diagonal terms is bounded by $C\cdot t^{-\epsilon}$. □

3 Strong convergence of the approximating vector

In order to prove the strong convergence of $\psi_{h,\kappa_1}(t)$, we study the norm of the vector

$$\Delta_{t_2,t_1}\psi_{h,\kappa_1} := \psi_{h,\kappa_1}(t_2) - \psi_{h,\kappa_1}(t_1) \tag{3.1}$$

for arbitrary times $t_2 > t_1 \gg 1$.
For a time difference, $t_2 - t_1$, sufficiently large we have different partitions corresponding to t_2 and t_1 respectively and then $N(t_2) \neq N(t_1)$. The t_2-partition sum, $\sum_{j=1}^{N(t_2)}$, is therefore generally written as $\sum_{j=1}^{N(t_1)}\sum_{l(j)}$, where the index $l(j)$ counts the subcells, relative to the t_2-partition, which are contained in the j^{th} cell of $\Gamma^{(t_1)}$, with $1 \leq l(j) \leq \frac{N(t_2)}{N(t_1)}$. In accordance to these notations, the vector $\Delta_{t_2,t_1}\psi_{h,\kappa_1}$ corresponds to

$$\begin{aligned}\Delta_{t_2,t_1}\psi_{h,\kappa_1} = {} & e^{iHt_2}\sum_{j=1}^{N(t_1)}\sum_{l(j)}\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},t_2\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},\nabla E^{\sigma_{t_2}}(\mathbf{P}),t_2\right)}e^{-iH_{\sigma_{t_2}}t_2}\psi_{l(j),\sigma_{t_2}}^{(t_2)} \\ & -e^{iHt_1}\sum_{j=1}^{N(t_1)}\mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}(\mathbf{P}),t_1\right)}e^{-iH_{\sigma_{t_1}}t_1}\psi_{j,\sigma_{t_1}}^{(t_1)}\,.\end{aligned} \tag{3.2}$$

Our final goal is to obtain the following estimate

$$\|\Delta_{t_2,t_1}\psi_{h,\kappa_1}\| = \|\psi_{h,\kappa_1}(t_2) - \psi_{h,\kappa_1}(t_1)\| \leq C\cdot\frac{|\ln t_2|^2}{t_1^{2\rho}} \quad \rho > 0\,. \tag{3.3}$$

This estimate is sufficient to prove the strong Cauchy property of $\psi_{h,\kappa_1}(t)$ by a telescopic argument (Theorem 3.1).

3.1 Outline of the proof

Due to the constructive recipe, the time variation, $t_2 \to t_1$, yields many modifications in the vector $\psi_{h,k_1}(t)$. In addition to the time evolution, the partition $\Gamma^{(t)}$ and the infrared cut-off σ_t are time dependent. Before going into details, it

is worthwhile to explain the general mechanisms which prevent the rising of not convergent terms and then imply the Cauchy property.
The increase in the number of cells is a potential source of problems in the control of the difference (3.3). Concerning the norm of the piece of vector $\Delta_{t_2,t_1}\psi_{h,\kappa_1}$ corresponding to each cell in $\Gamma^{(t_2)}$, if we only used the bound coming from the restriction of the support in the $\mathbf{P}-$variable, the estimate for the norm of the entire vector would diverge like $N(t_2)^{\frac{1}{2}}$. On the other hand, the (non-relativistic) cluster property of the system implies that the components with different (electronic) "velocities" in the cell-partition are asymptotically orthogonal as vectors in the Hilbert space. In order to exploit this mechanism in Theorem 2.1, the rate of the partition is chosen slower than the decoupling rate (constraint (2.26)).
Besides the increase of number of cells in time, we must handle two delicate aspects concerning the convergence in each single cell, namely:
A correction to the asymptotic dynamics, by means of the phase factor, is required in the application of Cook's argument;
The regularity properties concerning the vector $\phi_{\mathbf{P}}^{\sigma_t}$ (Subsection 1.3.1) which come into game and must be exploited in order to check that the dressing cloud combined with the one particle state gives rise to a well defined vector in the limit $\sigma_t \to 0$.

Variation of the partition

As preliminary step in the analysis of $\Delta_{t_2,t_1}\psi_{h,\kappa_1}$, we control the variation of the approximating vector when the cell partition changes from $\Gamma^{(t_2)}$ to $\Gamma^{(t_1)}$, all the other variables remaining fixed at time t_2. This means that we perform the following replacements

$$\begin{array}{ccc} W_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},t_2\right) & \to & W_{\sigma_{t_2}}\left(\mathbf{v}_j,t_2\right) \qquad \mathbf{v}_{l(j)}\equiv\nabla E^{\sigma_{t_2}}_{\mathbf{P}_{l(j)}},\ \mathbf{v}_j\equiv\nabla E^{\sigma_{t_1}}_{\mathbf{P}_j} \\ e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},\nabla E^{\sigma_{t_2}}_{\mathbf{P}},t_2\right)} & \to & e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}_{\mathbf{P}},t_2\right)} \\ \psi^{(t_2)}_{l(j),\sigma_{t_2}} & \to & \psi^{(t_1)}_{j,\sigma_{t_2}} \end{array}$$

so that the corresponding modification of the approximating vector is
D0)

$$e^{iHt_2}\sum_{j=1}^{N(t_1)}\sum_{l(j)}\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},t_2\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},\nabla E^{\sigma_{t_2}}_{\mathbf{P}},t_2\right)}e^{-iH_{\sigma_{t_2}}t_2}\psi^{(t_2)}_{l(j),\sigma_{t_2}}\to$$

$$\to\quad e^{iHt_2}\sum_{j=1}^{N(t_1)}\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_j,t_2\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}_{\mathbf{P}},t_2\right)}e^{-iH_{\sigma_{t_2}}t_2}\psi^{(t_1)}_{j,\sigma_{t_2}}\ .$$

Once the partition is $\Gamma^{(t_1)}$, we can study the variation of the vector

$$\psi^{(t_1)}_{h,\kappa_1}(t):=\sum_{j}^{N(t_1)}e^{iHt}\mathcal{W}_{\sigma_t}\left(\mathbf{v}_j,t\right)e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},t\right)}e^{-iE^{\sigma_t}_{\mathbf{P}}\cdot t}\psi^{(t_1)}_{j,\sigma_t} \tag{3.4}$$

between $t=t_2$ and $t=t_1$. The initial constructive hypotheses are not sufficient to perform the time derivative:

$$\frac{d}{ds}\psi^{(t_1)}_{h,\kappa_1}(s)\mid_{s=t}\ .$$

In fact the (strong) convergence proved in [Pi.] for the vector

$$\phi^{\sigma_t}_{\mathbf{P}}=W_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}}\right)\psi^{\sigma_t}_{\mathbf{P}}$$

does not imply that $\phi^{\sigma_t}_{\mathbf{P}}$ is (strongly) Lipschitz in σ_t in a neighborhood of $\sigma_t=0$. Assuming the Hypothesis H0 and exploiting only the Hölder property of $\phi^{\sigma_t}_{\mathbf{P}}$ in neighborhoods of $\sigma_t=0$ and $\mathbf{P}=0$ (see ***Spectral properties***, Subsection 1.3.1) we perform some intermediate steps from $\psi^{(t_1)}_{h,\kappa_1}(t_2)$ to $\psi^{(t_1)}_{h,\kappa_1}(t_1)$ corresponding to finite differences and we study the norm of each contribution. In the next lines, we carefully single out the intermediate variation *D1)*, involving Cook's argument, and then the variations *D2)*, *D3.1)*, *D3.2)* and *D3.3*), related to the removal of infrared cut-off. Indeed the order in the subsequent modifications is important to get the desired estimate.

Cook's argument

The backwards time evolution, at fixed cut-off σ_{t_2}, corresponds to the modification
D1)

$$\sum_{j=1}^{N(t_1)}e^{iHt_2}\,\mathcal{W}_{\sigma_{t_2}}(\mathbf{v}_j,t_2)\,e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}_{\mathbf{P}},t_2\right)}e^{-iE^{\sigma_{t_2}}_{\mathbf{P}}t_2}\psi^{(t_1)}_{j,\sigma_{t_2}}\qquad\rightarrow$$

$$\rightarrow\qquad\sum_{j=1}^{N(t_1)}e^{iHt_1}\mathcal{W}_{\sigma_{t_2}}(\mathbf{v}_j,t_1)\,e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}_{\mathbf{P}},t_1\right)}e^{-iE^{\sigma_{t_2}}_{\mathbf{P}}t_1}\psi^{(t_1)}_{j,\sigma_{t_2}}\tag{3.5}$$

and the study of the difference consists in a standard Cook's argument with the subtraction of the infrared tail as in Theorem 2.1.

Variation of the infrared cut-off

Under the variation of the infrared cut-off, $\sigma_{t_2}\rightarrow\sigma_{t_1}$, the vector (3.5) changes as follows

$$\sum_{j=1}^{N(t_1)}e^{iHt_1}\mathcal{W}_{\sigma_{t_2}}(\mathbf{v}_j,t_1)\,e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}_{\mathbf{P}},t_1\right)}e^{-iE^{\sigma_{t_2}}_{\mathbf{P}}t_1}\psi^{(t_1)}_{j,\sigma_{t_2}}\qquad\rightarrow\tag{3.6}$$

$$\rightarrow\qquad\sum_{j=1}^{N(t_1)}e^{iHt_1}\mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}_{\mathbf{P}},t_1\right)}e^{-iE^{\sigma_{t_1}}_{\mathbf{P}}t_1}\psi^{(t_1)}_{j,\sigma_{t_1}}\,.$$

For convenience we consider each cell-vector in the sum at the line (3.6) as the composition of two blocks

$$e^{iHt_1}\mathcal{W}_{\sigma_{t_2}}(\mathbf{v}_j,t_1)\,W^{\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}_{\mathbf{P}}\right)\tag{3.7}$$

$$e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}_{\mathbf{P}},t_1\right)}e^{-iE^{\sigma_{t_2}}_{\mathbf{P}}t_1}W_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}_{\mathbf{P}}\right)\psi^{(t_1)}_{j,\sigma_{t_2}}\tag{3.8}$$

where we recall that $W_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)=e^{-g\int_{\sigma_{t_2}}^{\kappa}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$.
Let us call the operator in (3.7) *dressing block* and the vector in (3.8) *regular block.*

The contribution due to the infrared cut-off variation, $\sigma_{t_2}\to\sigma_{t_1}$, can therefore be split in:
The variation of the *regular block* (3.8)
D2)

$$\begin{array}{ll} & e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_1\right)}e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}t_1}W_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\psi_{j,\sigma_{t_2}}^{(t_1)}\qquad\to\\ \to & e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_1}},t_1\right)}e^{-iE_{\mathbf{P}}^{\sigma_{t_1}}t_1}W_{\sigma_{t_1}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\psi_{j,\sigma_{t_1}}^{(t_1)}\end{array}$$

which is substantially related to the convergence for $\sigma\to 0$ of the vectors $\phi_{\mathbf{P}}^{\sigma}$ in the fiber spaces $\mathcal{H}_{\mathbf{P}}$;

The variation of the *dressing block* (3.7) up to the term e^{iHt_1}
D3)

$$\begin{array}{ll} & \mathcal{W}_{\sigma_{t_2}}(\mathbf{v}_j,t_1)\,W_{\sigma_{t_2}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\qquad\to\\ \to & \mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W_{\sigma_{t_1}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\,.\end{array}$$

Let us analyze the variation *D3)* in further details. It can be written as

$$\begin{array}{ll} & \mathcal{W}_{\sigma_{t_2}}(\mathbf{v}_j,t_1)\,W_{\sigma_{t_2}}^{\dagger}(\mathbf{v}_j)\,W_{\sigma_{t_2}}(\mathbf{v}_j)\,W_{\sigma_{t_2}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\qquad\to\\ \to & \mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W_{\sigma_{t_1}}^{\dagger}(\mathbf{v}_j)\,W_{\sigma_{t_1}}(\mathbf{v}_j)\,W_{\sigma_{t_1}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\end{array}$$

where $W_{\sigma_{t_2}}(\mathbf{v}_j)=e^{-g\int_{\sigma_{t_2}}^{\kappa}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}$, so that we split the step *D3)* in three smaller ones:
D3.1)

$$\begin{array}{ll} & \mathcal{W}_{\sigma_{t_2}}(\mathbf{v}_j,t_1)\,W_{\sigma_{t_2}}^{\dagger}(\mathbf{v}_j)\,W_{\sigma_{t_2}}(\mathbf{v}_j)\,W_{\sigma_{t_2}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\qquad\to\\ \to & \mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W_{\sigma_{t_1}}^{\dagger}(\mathbf{v}_j)\,W_{\sigma_{t_2}}(\mathbf{v}_j)\,W_{\sigma_{t_2}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\end{array}$$

in this step the logarithmic divergence arising from the variation $\sigma_{t_2}\to\sigma_{t_1}$ in the two Weyl operator on the very left is neutralized by the strong Hölder property in $\mathbf{P}$ of the *regular block* (3.8), on which the full operator is applied;
D3.2)

$$\begin{array}{ll} & \mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W_{\sigma_{t_1}}^{\dagger}(\mathbf{v}_j)\,W_{\sigma_{t_2}}(\mathbf{v}_j)\,W_{\sigma_{t_2}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\qquad\to\\ \to & \mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W_{\sigma_{t_1}}^{\dagger}(\mathbf{v}_j)\,W_{\sigma_{t_2}}(\mathbf{v}_j)\,W_{\sigma_{t_2}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\end{array}$$

D3.3)

$$\begin{array}{ll} & \mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W_{\sigma_{t_1}}^{\dagger}(\mathbf{v}_j)\,W_{\sigma_{t_2}}(\mathbf{v}_j)\,W_{\sigma_{t_2}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\qquad\to\\ \to & \mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W_{\sigma_{t_1}}^{\dagger}(\mathbf{v}_j)\,W_{\sigma_{t_1}}(\mathbf{v}_j)\,W_{\sigma_{t_1}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\end{array}$$

the last two steps account respectively for the difference between the gradients $\nabla E^{\sigma_{t_2}}(\mathbf{P})$,$\nabla E^{\sigma_{t_1}}(\mathbf{P})$ and for the shift $\sigma_{t_2} \to \sigma_{t_1}$ in the two Weyl operators on the very right.

The analysis of each difference *D0)*, *D1)*, *D2)*, *D3.1)*, *D3.2)* and *D3.3)* is the content of the remaining part of the Section. The discussion is carried out in Subsections 3.1.1, 3.1.2, 3.1.3 and 3.1.4, where we describe the physical ingredients and the technical steps used to control them. It is rather detailed and complete with the use of some results proved in the Appendix. Though it is not always explicitly written, we assume that $\alpha(<1)$ is sufficiently close to 1 and that the constraint (2.26) is satisfied.
We here anticipate the result:
Let us assume $\beta > 1$ and large enough (which means that the removal of the infrared cut-off $\sigma_t = t^{-\beta}$ is sufficiently fast in time). Then the bounds (3.23), (3.32), (3.38), (3.47), (3.49) and (3.54) - which are obtained respectively for the norms of the vectors corresponding to the variations *D0)*, *D1)*, *D2)*, *D3.1)*, *D3.2)* and *D3.3)* - are such that a leading order term is

$$\left(\frac{\ln(t_2)}{t_1^{\rho}}\right)^2 .$$

with $\rho > 0$. We can now state the main theorem of the paper.

Theorem 3.1
For $\beta > 1$ large enough and α sufficiently close to 1, the vector $\psi_{h,\kappa_1}(t)$, with $\int |h(\mathbf{P})|^2 d^3P > 0$, converges strongly for $t \to +\infty$ to a non zero vector ψ^{out}_{h,κ_1}, with an error of order $\frac{1}{t^{\rho}}$ at most, where $\rho > 0$ is a proper small coefficient.

Proof
Starting from Theorem 2.1, the time scale related to the partition is tuned according to the constraint $4\epsilon < \eta$. Therefore, for $\beta > 1$ large enough and α sufficiently close to 1, we can estimate

$$\|\psi_{h,\kappa_1}(t_2) - \psi_{h,\kappa_1}(t_1)\| < C \cdot \left(\frac{\ln(t_2)}{t_1^{\rho}}\right)^2 \tag{3.9}$$

where $\rho > 0$ e $C > 0$ are independent of t_1 and t_2 $(t_2 \geq t_1 > \bar{t} \gg 1)$.

Now let us consider the sequence $\{t_1, t_1^2, ...t_1^n, ...\}$ and assume $t_1^n \leq t_2 < t_1^{n+1}$. Due to the norm properties, it follows that:

$$\|\psi_{h,\kappa_1}(t_2) - \psi_{h,\kappa_1}(t_1)\| \leq$$

$$\leq \left\|\psi_{h,\kappa_1}(t_1^2) - \psi_{h,\kappa_1}(t_1)\right\| + ... + \|\psi_{h,\kappa_1}(t_2) - \psi_{h,\kappa_1}(t_1^n)\| \tag{3.10}$$

$$\leq \frac{C}{t_1^{\rho}} \cdot \left\{\left(\frac{2}{t_1^{\frac{\rho}{2}}} \cdot \ln(t_1)\right)^2 + ... + \left(\frac{n+1}{t_1^{\frac{n\rho}{2}}} \cdot \ln(t_1)\right)^2\right\} \tag{3.11}$$

For t_1 sufficiently large, $t_1 \geq \widehat{t_1} > \overline{t} \gg 1$, the series inside the brackets in (3.11) is bounded by a constant less than 1.
We can conclude that $\forall t_1, t_2$, where $t_2 \geq t_1 \geq \widehat{t_1}$,

$$\left\| \psi_{h,\kappa_1}(t_2) - \psi_{h,\kappa_1}(t_1) \right\| \leq \frac{C}{t_1^{\rho}} . \tag{3.12}$$

Because of Theorem 2.1, the limiting vector is non zero if $\int |h(\mathbf{P})|^2 d^3P > 0$. $\square$

3.1.1 Variation of the partition

The squared norm of the difference *D0)* is

$$\begin{aligned} &\Bigg\| \sum_{j=1}^{N(t_1)} \sum_{l(j)} \mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}, t_2\right) e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}, \nabla E_{\mathbf{P}}^{\sigma_{t_2}}, t_2\right)} e^{-iH_{\sigma_{t_2}} t_2} \psi^{(t_2)}_{l(j),\sigma_{t_2}} - \\ &\qquad - \sum_{j=1}^{N(t_1)} \sum_{l(j)} \mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_{j}, t_2\right) e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{j}, \nabla E_{\mathbf{P}}^{\sigma_{t_2}}, t_2\right)} e^{-iH_{\sigma_{t_2}} t_2} \psi^{(t_2)}_{l(j),\sigma_{t_2}} \Bigg\|^2 \end{aligned} \tag{3.13}$$

where we have used that

$$\psi^{(t_1)}_{j,\sigma_{t_2}} = \int_{\Gamma^{(t_1)}_j} h(\mathbf{P})\, \psi_{\mathbf{P}}^{\sigma_{t_2}} d^3P = \sum_{l(j)} \int_{\Gamma^{(t_2)}_{l(j)}} h(\mathbf{P})\, \psi_{\mathbf{P}}^{\sigma_{t_2}} d^3P = \sum_{l(j)} \psi^{(t_2)}_{l(j),\sigma_{t_2}} .$$

For brevity, let us define

$$\begin{aligned} \widehat{\mathcal{W}}_{\sigma_{t_2}}\left(\mathbf{v}_j, t_2\right) : &= \mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_j, t_2\right) e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j, \nabla E_{\mathbf{P}}^{\sigma_{t_2}}, t_2\right)} \\ \widehat{\mathcal{W}}_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}, t_2\right) : &= \mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}, t_2\right) e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}, \nabla E_{\mathbf{P}}^{\sigma_{t_2}}, t_2\right)} \end{aligned} \tag{3.14}$$

so that the squared norm (3.13) can be written as follows

$$\begin{aligned} &\textstyle\sum_{j,j'=1}^{N(t_1)} \sum_{l(j),l'(j')} \Big(\left(\widehat{\mathcal{W}}_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}, t_2\right) - \widehat{\mathcal{W}}_{\sigma_{t_2}}\left(\mathbf{v}_{j}, t_2\right) \right) e^{-iH_{\sigma_{t_2}} t_2} \psi^{(t_2)}_{l(j),\sigma_{t_2}} , \\ &\qquad , \left(\widehat{\mathcal{W}}_{\sigma_{t_2}}\left(\mathbf{v}_{l'(j')}, t_2\right) - \widehat{\mathcal{W}}_{\sigma_{t_2}}\left(\mathbf{v}_{j'}, t_2\right) \right) e^{-iH_{\sigma_{t_2}} t_2} \psi^{(t_2)}_{l'(j'),\sigma_{t_2}} \Big) \end{aligned} \tag{3.15}$$

The sum of the terms where $j' \neq j$ and $l'(j) \neq l(j)$ vanishes for $t_2 \to +\infty$ and its rate is surely bounded (from above) by a quantity of order $t_2^{-\epsilon}$, as we can estimate by the same decoupling mechanism exploited in the norm control of $\psi_{h,k_1}(t)$ (Theorem 2.1). Keeping aside this estimate, we can focus on the following sum over the diagonal terms with respect to the partition $\Gamma^{(t_2)}$:

$$\sum_{j=1}^{N(t_1)} \sum_{l(j)} \left\langle \left(\begin{array}{c} 2 - \widehat{\mathcal{W}}^{\dagger}_{\sigma_{t_2}}\left(\mathbf{v}_j, t_2\right) \widehat{\mathcal{W}}_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}, t_2\right) \\ -\widehat{\mathcal{W}}^{\dagger}_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}, t_2\right) \widehat{\mathcal{W}}_{\sigma_{t_2}}\left(\mathbf{v}_j, t_2\right) \end{array} \right) \right\rangle_{e^{-iH_{\sigma_{t_2}} t_2} \psi^{(t_2)}_{l(j),\sigma_{t_2}}} \tag{3.16}$$

where, for a given operator O and a vector φ, $\langle O\rangle_\varphi$ denotes $(\varphi, O\varphi)$.

Now from the sum (3.16) we extract the leading contribution.
Let us start analyzing

$$\left\langle e^{-i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_2\right)}\mathcal{W}^\dagger_{\sigma_{t_2}}\left(\mathbf{v}_j,t_2\right)\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},t_2\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_2\right)}\right\rangle_{e^{-iH_{\sigma_{t_2}}t_2}\psi^{(t_2)}_{l(j),\sigma_{t_2}}} \tag{3.17}$$

and for $s\geq t_2$

$$\left\langle e^{iH_{\sigma_{t_2}}s}e^{-i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},s\right)}\mathcal{W}^\dagger_{\sigma_{t_2}}\left(\mathbf{v}_j,s\right)\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},s\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},\nabla E_{\mathbf{P}}^{\sigma_{t_2}},s\right)}e^{-iH_{\sigma_{t_2}}s}\right\rangle_{\psi^{(t_2)}_{l(j),\sigma_{t_2}}} \tag{3.18}$$

with the trivial property that (3.18) coincides with (3.17) for $s=t_2$.
The limit for $s\to+\infty$ of the expression (3.18) is:

$$e^{-\frac{C_{j,l(j),\sigma_{t_2}}}{2}}\cdot\left(e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},\sigma_{t_2}^{-\frac{1}{\alpha}}\right)}\psi^{(t_2)}_{l(j),\sigma_{t_2}},e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},\nabla E_{\mathbf{P}}^{\sigma_{t_2}},\sigma_{t_2}^{-\frac{1}{\alpha}}\right)}\psi^{(t_2)}_{l(j),\sigma_{t_2}}\right) \tag{3.19}$$

where $C_{j,l(j),\sigma_{t_2}}=\int_{\sigma_{t_2}}^{\kappa_1}\left|\eta_{j,l(j)}\left(\widehat{\mathbf{k}}\right)\right|^2\frac{d^3k}{2|\mathbf{k}|^3}$ and $\eta_{l(j),j}\left(\widehat{\mathbf{k}}\right):=\frac{g\widehat{\mathbf{k}}\cdot\left(\mathbf{v}_j-\mathbf{v}_{l(j)}\right)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)\cdot\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_{l(j)}\right)}$.

Then we rewrite the expectation value (3.17) as the limit for $s\to+\infty$ of the corresponding quantity (3.18) plus a remainder. The limit corresponds to the expression (3.19). The sum of the remainders over the cells in (3.16) amounts to a quantity which is at most of order $t_2^{-4\epsilon}$.

Hence the discussion is now restricted to the following sum

$$\sum_{j=1}^{N(t_1)}\sum_{l(j)}\left\langle 2-e^{i\Delta\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}-\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_2\right)}e^{-\frac{C_{l(j),j,\sigma_{t_2}}}{2}}-e^{i\Delta\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j-\mathbf{v}_{l(j)},\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_2\right)}e^{-\frac{C_{j,l(j),\sigma_{t_2}}}{2}}\right\rangle_{\psi^{(t_2)}_{l(j),\sigma_{t_2}}} \tag{3.20}$$

where

$$e^{i\Delta\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j-\mathbf{v}_{l(j)},\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_2\right)}:=e^{-i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},\sigma_{t_2}^{-\frac{1}{\alpha}}\right)}e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)},\nabla E_{\mathbf{P}}^{\sigma_{t_2}},\sigma_{t_2}^{-\frac{1}{\alpha}}\right)}. \tag{3.21}$$

An estimate from above of the sum (3.20) is given by

$$\sum_{j=1}^{N(t_1)}\sum_{l(j)}\left(\left\|\psi^{(t_2)}_{l(j),\sigma_{t_2}}\right\|^2\cdot C\cdot t_1^{-\frac{\epsilon}{16}}\cdot\left|\ln\left(\sigma_{t_2}\right)\right|\right)\leq C\cdot t_1^{-\frac{\epsilon}{16}}\cdot\left|\ln\left(\sigma_{t_2}\right)\right| \tag{3.22}$$

taking into account that

$$C_{l(j),j,\sigma_{t_2}}:=\int_{\sigma_{t_2}}^{\kappa_1}\left|\eta_{l(j),j}\left(\widehat{\mathbf{k}}\right)\right|^2\frac{d^3k}{2\left|\mathbf{k}\right|^3}$$

and
$$\left|e^{i\Delta\gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}-\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_2\right)}\right|$$
are both bounded by
$$C\cdot t_1^{-\frac{\epsilon}{16}}\cdot\left|\ln\left(\sigma_{t_2}\right)\right|$$
because of the difference $\mathbf{v}_j-\mathbf{v}_{l(j)}$ and the regularity properties of the gradient of the ground state energy (Subsection 1.3.1 and Lemma A1).
In the end, collecting all the partial estimates, we can conclude that the norm of the difference *D0)* is surely bounded from above by a quantity of order
$$t_1^{-\frac{\epsilon}{16}}\cdot\left|\ln\left(\sigma_{t_2}\right)\right|\ . \tag{3.23}$$

3.1.2 Cook's argument

The difference corresponding to the variation *D1)* is
$$\sum_{j=1}^{N(t_1)}\left\{e^{iHt_2}\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_j,t_2\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_2\right)}e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}t_2}\psi_{j,\sigma_{t_2}}^{(t_1)}-\right.$$
$$\left.-e^{iHt_1}\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_j,t_1\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_1\right)}e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}t_1}\psi_{j,\sigma_{t_2}}^{(t_1)}\right\}\ .$$
We estimate the contribution for each cell by expressing the difference of the two related vectors as the following integral from t_1 to t_2:
$$\int_{t_1}^{t_2}\frac{d}{ds}\left\{e^{iHs}\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_j,s\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},s\right)}e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}s}\psi_{j,\sigma_{t_2}}^{(t_1)}\right\}ds\ . \tag{3.24}$$
Moreover we use the inequality
$$\begin{aligned}&\left\|\int_{t_1}^{t_2}\tfrac{d}{ds}\left\{e^{iHs}\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_j,s\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},s\right)}e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}s}\psi_{j,\sigma_{t_2}}^{(t_1)}\right\}ds\right\|\leq\\&\leq\int_{t_1}^{t_2}\left\|\tfrac{d}{ds}\left\{e^{iHs}\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_j,s\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},s\right)}e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}s}\psi_{j,\sigma_{t_2}}^{(t_1)}\right\}\right\|ds\ .\end{aligned} \tag{3.25}$$

The derivative in the expression (3.24) can be split as follows:
$$ie^{iHs}\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_j,s\right)\left(\varphi_{\sigma_{t_2},\mathbf{v}_j}\left(\mathbf{x},s\right)+\frac{d\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},s\right)}{ds}\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},s\right)}e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}s}\psi_{j,\sigma_{t_2}}^{(t_1)} \tag{3.26}$$
$$+\quad ie^{iHs}\mathcal{W}_{\sigma_{t_2}}\left(\mathbf{v}_j,s\right)\left(H-H_{\sigma_{t_2}}\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},s\right)}e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}s}\psi_{j,\sigma_{t_2}}^{(t_1)} \tag{3.27}$$
where the term (3.26) is analogous to the expression (2.15) in Theorem 2.1 a part from the evolution operator on the very left.

Term (3.26)
As in Theorem 2.1, Section 2, we decompose $\varphi_{\sigma_{t_2},\mathbf{v}_j}\left(\mathbf{x},s\right)$ as

$$\varphi^{-}_{\sigma_{t_2},\mathbf{v}_j}(\mathbf{x},s)+\varphi^{+}_{\sigma_{t_2},\mathbf{v}_j}(\mathbf{x},s) \tag{3.28}$$

where:

$$\begin{aligned}\varphi^{-}_{\sigma_{t_2},\mathbf{v}_j}(\mathbf{x},s): &= g^2\int_{\sigma_{t_2}}^{s^{-\alpha}}\frac{\cos(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|\,s)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}d\Omega d|\mathbf{k}| \\ \varphi^{+}_{\sigma_{t_2},\mathbf{v}_j}(\mathbf{x},s): &= g^2\int_{s^{-\alpha}}^{\kappa_1}\frac{\cos(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|\,s)}{\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}d\Omega d|\mathbf{k}|\ .\end{aligned} \tag{3.29}$$

By the same procedure, exploiting the subtraction of the infrared tail $\varphi^{-}_{\sigma_{t_2},\mathbf{v}_j}(\mathbf{x},s)$ by means of the derivative of $e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}_{\mathbf{P}},s\right)}$ and assuming the constraint (2.26), we obtain the following estimate from above for the norm of the expression (3.26):

$$C\cdot s^{-1}\cdot s^{-4\epsilon}\cdot t_1^{-\frac{3\epsilon}{2}}\cdot(\ln\sigma_{t_2})^2\ .$$

Term (3.27)
As far as the norm is concerned, the vector (3.27) is equivalent to the vector

$$\begin{aligned}&\left(H-H_{\sigma_{t_2}}\right)e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}_{\mathbf{P}},s\right)}\psi^{(t_1)}_{j,\sigma_{t_2}}= \\ &=e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}_{\mathbf{P}},s\right)}g\int_0^{\sigma_{t_2}}\left(b(\mathbf{k})+b^{\dagger}(\mathbf{k})\right)\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\psi^{(t_1)}_{j,\sigma_{t_2}}\end{aligned} \tag{3.30}$$

whose norm can be estimated starting from

$$\begin{aligned}&\left\|g\int_0^{\sigma_{t_2}}\left(b(\mathbf{k})+b^{\dagger}(\mathbf{k})\right)\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\psi^{(t_1)}_{j,\sigma_{t_2}}\right\|\le \\ &\le\left\|g\int_0^{\sigma_{t_2}}b^{\dagger}(\mathbf{k})\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\psi^{(t_1)}_{j,\sigma_{t_2}}\right\| \\ &\le C\cdot\sigma_{t_2}\cdot t_1^{-\frac{3\epsilon}{2}}\end{aligned} \tag{3.31}$$

because $b(\mathbf{k})\,\psi^{(t_1)}_{j,\sigma_{t_2}}=0$ for $\mathbf{k}\in\{\mathbf{k}:\ |\mathbf{k}|\le\sigma_{t_2}\}$.
In conclusion the norm of the vector corresponding to the difference *D1)* is bounded by a quantity of order

$$t_1^{-\frac{5\epsilon}{2}}\cdot(\ln\sigma_{t_2})^2+t_2\cdot\sigma_{t_2}\cdot t_1^{\frac{3\epsilon}{2}}\ . \tag{3.32}$$

3.1.3 Variation of the infrared cut-off: *regular block*

Let us study the difference

$$\sum_{j=1}^{N(t_1)}e^{iHt_1}\mathcal{W}_{\sigma_{t_2}}(\mathbf{v}_j,t_1)\,W^{\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}_{\mathbf{P}}\right)\left[\begin{array}{l}e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_2}}_{\mathbf{P}},t_1\right)}W_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}_{\mathbf{P}}\right)e^{-iE^{\sigma_{t_2}}_{\mathbf{P}}t_1}\psi^{(t_1)}_{j,\sigma_{t_2}}- \\ -e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}_{\mathbf{P}},t_1\right)}W_{\sigma_{t_1}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)e^{-iE^{\sigma_{t_1}}_{\mathbf{P}}t_1}\psi^{(t_1)}_{j,\sigma_{t_1}}\end{array}\right]$$

for each single cell in $\Gamma^{(t_1)}$. The norm of the cell-vector is controlled as follows

$$\left\| e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_1\right)} e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}t_1} W_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\psi_{j,\sigma_{t_2}}^{(t_1)} - \right.$$

$$\left. - e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_1}},t_1\right)} e^{-iE_{\mathbf{P}}^{\sigma_{t_1}}t_1} W_{\sigma_{t_1}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\psi_{j,\sigma_{t_1}}^{(t_1)}\right\| \leq$$

$$\leq \quad \left\| \left[e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_1\right)} - e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_1}},t_1\right)} \right] e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}t_1} W_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\psi_{j,\sigma_{t_2}}^{(t_1)} \right\| \tag{3.33}$$

$$+ \left\| e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_1}},t_1\right)} \left[e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}t_1} - e^{-iE_{\mathbf{P}}^{\sigma_{t_1}}t_1} \right] W_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\psi_{j,\sigma_{t_2}}^{(t_1)} \right\| \tag{3.34}$$

$$+ \left\| e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_1}},t_1\right)} e^{-iE_{\mathbf{P}}^{\sigma_{t_1}}t_1} \left[W_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\psi_{j,\sigma_{t_2}}^{(t_1)} - W_{\sigma_{t_1}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\psi_{j,\sigma_{t_1}}^{(t_1)} \right] \right\| \tag{3.35}$$

and each term (3.33), (3.34), (3.35) is infinitesimal in the limit in which the infrared cut-off is removed. The arguments for this are the following ones.

Term (3.33)
We can easily estimate

$$\left\| \left[e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_1\right)} - e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_1}},t_1\right)} \right] e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}t_1} W_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\psi_{j,\sigma_{t_2}}^{(t_1)} \right\| \quad \leq$$

$$\leq \sup_{\mathbf{P}\in\Sigma} \left| e^{i\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_1\right)} - e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_1}},t_1\right)} \right| \cdot \left\| \psi_{j,\sigma_{t_2}}^{(t_1)} \right\|$$

and the sup is bounded in terms of (see Lemma A1):

$$\left|\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_2}},t_1\right) - \gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E_{\mathbf{P}}^{\sigma_{t_1}},t_1\right)\right| \leq C\cdot(\sigma_{t_1})^{\frac{1}{4}}\cdot t_1^{2(1-\alpha)} + C\cdot t_1\cdot\sigma_{t_1} \tag{3.36}$$

so that for α sufficiently close to 1 we can surely provide a bound of order $t_1^{-\frac{3\epsilon}{2}+1}\cdot(\sigma_{t_1})^{\frac{1}{4}}$ for the expression (3.33);
Term (3.34)
Because of the regularity properties of the energy $E_{\mathbf{P}}^{\sigma}, \mathbf{P}\in\Sigma$, we easily get

$$\left| e^{-iE_{\mathbf{P}}^{\sigma_{t_2}}t_1} - e^{-iE_{\mathbf{P}}^{\sigma_{t_1}}t_1} \right| \leq C\cdot\sigma_{t_1}^{\frac{1}{4}}\cdot t_1\,. \tag{3.37}$$

which implies a bound from above of order $\sigma_{t_1}^{\frac{1}{4}}\cdot t_1^{-\frac{3\epsilon}{2}+1}$ for the norm (3.34).
Term (3.35)
Exploiting the ***Spectral properties***, Subsection 1.3.1, we can estimate:

$$\left\| W_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\psi_{j,\sigma_{t_2}}^{(t_1)} - W_{\sigma_{t_1}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\psi_{j,\sigma_{t_1}}^{(t_1)} \right\| =$$

$$= \left\| \int_{\Gamma_j^{(t_1)}} h(\mathbf{P})\left(W_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\psi_{\mathbf{P}}^{\sigma_{t_2}} - W_{\sigma_{t_1}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\psi_{\mathbf{P}}^{\sigma_{t_1}}\right) d^3P \right\|$$

$$\leq \left\{ \int_{\Gamma_j^{(t_1)}} |h_{\mathbf{P}}|^2 \left\| \mathbf{I}_{\mathbf{P}}\left(\phi_{\mathbf{P}}^{\sigma_{t_2}}\right) - \mathbf{I}_{\mathbf{P}}\left(\phi_{\mathbf{P}}^{\sigma_{t_1}}\right) \right\|_{\mathcal{F}}^2 d^3P \right\}^{\frac{1}{2}}$$

$$\leq C \cdot (\sigma_{t_1})^{\frac{1}{4}} \cdot t_1^{-\frac{3\epsilon}{2}} .$$

In conclusion the norm of the difference *D2)* is surely bounded by

$$C \cdot t_1^{\frac{3\epsilon}{2}+1} \cdot (\sigma_{t_1})^{\frac{1}{4}} . \tag{3.38}$$

3.1.4 Variation of the infrared cut-off: *dressing block*

analysis of D3.1)

We first define $\varphi_{j,\sigma_{t_1}}^{(t_1)}$ the *regular block* corresponding to the time t_1

$$\varphi_{j,\sigma_{t_1}}^{(t_1)} := e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j, \nabla E_{\mathbf{P}}^{\sigma_{t_1}}, t_1\right)} e^{-iE_{\mathbf{P}}^{\sigma_{t_1}} t_1} W_{\sigma_{t_1}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right) \psi_{j,\sigma_{t_1}}^{(t_1)} . \tag{3.39}$$

Then the difference involved in the step *D3.1)* can be written as:

$$\sum_{j=1}^{N(t_1)} e^{iHt_1} \left(\mathcal{W}_{\sigma_{t_2}}(\mathbf{v}_j, t_1) W_{\sigma_{t_2}}^{\dagger}(\mathbf{v}_j) - \mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j, t_1) W_{\sigma_{t_1}}^{\dagger}(\mathbf{v}_j) \right) W_{\sigma_{t_2}}(\mathbf{v}_j) W_{\sigma_{t_2}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right) \varphi_{j,\sigma_{t_1}}^{(t_1)} . \tag{3.40}$$

We can restrict the analysis to each single cell. By standard algebraic steps we have

$$\mathcal{W}_{\sigma_{t_2}}(\mathbf{v}_j, t_1) W_{\sigma_{t_2}}^{\dagger}(\mathbf{v}_j) - \mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j, t_1) W_{\sigma_{t_1}}^{\dagger}(\mathbf{v}_j) =$$

$$= \mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j, t_1) W_{\sigma_{t_1}}^{\dagger}(\mathbf{v}_j) \left(e^{ig^2 \int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{\sin(-|\mathbf{k}|t_1 + \mathbf{k}\cdot\mathbf{x})}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)^2} d^3k} \right) \left(e^{-g\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1} - e^{i\mathbf{k}\cdot\mathbf{x}}\right) - h.c.}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}} - I \right) \tag{3.41}$$

$$+ \mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j, t_1) W_{\sigma_{t_1}}^{\dagger}(\mathbf{v}_j) \left(e^{ig^2 \int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{\sin(-|\mathbf{k}|t_1 + \mathbf{k}\cdot\mathbf{x})}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)^2} d^3k} - I \right) \tag{3.42}$$

Since the vector

$$W_{\sigma_{t_2}}(\mathbf{v}_j) W_{\sigma_{t_2}}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right) \varphi_{j,\sigma_{t_1}}^{(t_1)}$$

belongs to the domain of the operator

$$-g \int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1} - e^{i\mathbf{k}\cdot\mathbf{x}}\right) - h.c.}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}$$

we can use the identity

$$\left(e^{-g\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1} - e^{i\mathbf{k}\cdot\mathbf{x}}\right) - h.c.}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} \frac{d^3k}{\sqrt{2|\mathbf{k}|}}} - I \right) =$$

$$= -g\int_0^1 e^{-g\lambda\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right)-h.c.}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}} d\lambda \int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right)-h.c.}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}\,. \tag{3.43}$$

Moreover we can estimate

$$\left\|\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right)-h.c.}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}} W_{\sigma_{t_2}}(\mathbf{v}_j)\, W^\dagger_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}}\right\| \leq$$

$$\begin{aligned}
\leq\quad & \left\|\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a(\mathbf{k})\left(e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}} W_{\sigma_{t_2}}(\mathbf{v}_j)\, W^\dagger_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}}\right\| \\
& + \left\|\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{a^\dagger(\mathbf{k})\left(e^{-i|\mathbf{k}|t_1}-e^{-i\mathbf{k}\cdot\mathbf{x}}\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}} W_{\sigma_{t_2}}(\mathbf{v}_j)\, W^\dagger_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}}\right\|
\end{aligned}$$

$$\leq\quad 2\left\|\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{b(\mathbf{k})\left(e^{i|\mathbf{k}|t_1-i\mathbf{k}\cdot\mathbf{x}}-1\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}} W_{\sigma_{t_2}}(\mathbf{v}_j)\, W^\dagger_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}}\right\| \tag{3.44}$$

$$+\left\langle \int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{\left|e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right|^2}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)^2} d^3k \right\rangle^{\frac{1}{2}}_{W_{\sigma_{t_2}}(\mathbf{v}_j)W^\dagger_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}}}\,. \tag{3.45}$$

As checked in Lemma A6, the two expressions (3.44) and (3.45) are logarithmically divergent in t_2 but vanishing with a power law in t_1 due to the smoothness of the *regular block* in its $\mathbf{P}$-dependence and because of the upper integration bound σ_{t_1}. Concerning the term (3.42) the corresponding norm

$$\left\|\left(e^{ig^2\int_{\sigma_{t_2}}^{\sigma_{t_1}} \frac{\sin(-|\mathbf{k}|t_1+\mathbf{k}\cdot\mathbf{x})}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)^2}d^3k} - I\right) W_{\sigma_{t_2}}(\mathbf{v}_j)\, W^\dagger_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}}\right\| \tag{3.46}$$

can be treated like the expression (3.45).

In the end we obtain that the norm of the term *D3.1)* is surely bounded by a quantity of order:

$$t_1^{3\epsilon}\cdot|\ln\sigma_{t_2}|\cdot(\sigma_{t_1})^{\frac{1}{16}}\,. \tag{3.47}$$

analysis of D3.2)

The difference to be analyzed is

$$\begin{aligned}
&\sum_{j=1}^{N(t_1)} e^{iHt_1}\mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\, W^\dagger_{\sigma_{t_1}}(\mathbf{v}_j)\, W_{\sigma_{t_2}}(\mathbf{v}_j)\, W^\dagger_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_2}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}} \\
-&\sum_{j=1}^{N(t_1)} e^{iHt_1}\mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\, W^\dagger_{\sigma_{t_1}}(\mathbf{v}_j)\, W_{\sigma_{t_2}}(\mathbf{v}_j)\, W^\dagger_{\sigma_{t_2}}\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}}\,.
\end{aligned}$$

For each single cell, by an argument similar to that one described in Theorem A2 for the expression (5.18), we have

$$\left\|\left(W^{\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}_{\mathbf{P}}\right)-W^{\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)\right)W_{\sigma_{t_1}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)\psi^{(t_1)}_{j,\sigma_{t_1}}\right\| \leq \quad C\cdot(\sigma_{t_1})^{\frac{1}{4}}\cdot|\ln\sigma_{t_2}|\cdot t_1^{-\frac{3\epsilon}{2}} \tag{3.48}$$

so that the norm of the vector corresponding to the variation *D3.2)* is bounded by

$$C\cdot t_1^{\frac{3\epsilon}{2}}\cdot(\sigma_{t_1})^{\frac{1}{4}}\cdot|\ln\sigma_{t_2}|\,. \tag{3.49}$$

analysis of D3.3)

The difference involved in this step is

$$\sum_{j=1}^{N(t_1)} e^{iHt_1}\mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,W^{\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\,W_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}}-$$

$$-\sum_{j=1}^{N(t_1)} e^{iHt_1}\mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\,e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}_{\mathbf{P}},t_1\right)}e^{-iE^{\sigma_{t_1}}_{\mathbf{P}}t_1}\psi^{(t_1)}_{j,\sigma_{t_1}}\,.$$

This variation can be written as

$$\sum_{j=1}^{N(t_1)} e^{iHt_1}\mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)\left(W\,|^{\sigma_{t_1}}_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{\dagger}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)-I\right)e^{i\gamma_{\sigma_{t_1}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t_1}}_{\mathbf{P}},t_1\right)}e^{-iE^{\sigma_{t_1}}_{\mathbf{P}}t_1}\psi^{(t_1)}_{j,\sigma_{t_1}} \tag{3.50}$$

with the definitions

$$W\,|^{\sigma_{t_1}}_{\sigma_{t_2}}(\mathbf{v}_j) \quad := \quad W^{\dagger}_{\sigma_{t_1}}(\mathbf{v}_j)\,W_{\sigma_{t_2}}(\mathbf{v}_j)=e^{-g\int^{\sigma_{t_1}}_{\sigma_{t_2}}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}} \tag{3.51}$$

$$W^{\dagger}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right) \quad := \quad W^{\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)W_{\sigma_{t_1}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)=e^{g\int^{\sigma_{t_1}}_{\sigma_{t_2}}\frac{b(\mathbf{k})-b^{\dagger}(\mathbf{k})}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}} \tag{3.52}$$

The discussion of this contribution requires the study of the squared norm of the vector (3.50) and the control of the off-diagonal terms, with respect to the cell-partition $\Gamma^{(t_1)}$, in the corresponding scalar product. We first check that the sum of the off-diagonal terms vanishes for $t_1\to+\infty$ with a rate fast enough. Then we turn to consider the diagonal contribution.

off-diagonal terms

Let us consider the generic $l-j$ term and try to reply the same procedure as in Theorem 2.1, by the insertion of the real parameter λ in the dressing operator $\mathcal{W}_{\sigma_{t_1}}(\mathbf{v}_j,t_1)$ and the subsequent derivative with respect to λ. The only obstacles in repeating the usual steps come from the lack of commutativity between

$$\int^{\kappa_1}_{\sigma_{t_1}}\frac{a(\mathbf{k})e^{i|\mathbf{k}|s}\eta_{l,j}\left(\widehat{\mathbf{k}}\right)}{|\mathbf{k}|}\frac{d^3k}{\sqrt{2|\mathbf{k}|}} \quad \text{and} \quad W^{\dagger}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)$$

$$\chi_h\left(\frac{\mathbf{x}}{s}\right) \quad \text{and} \quad W^{\dagger}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)$$

$$\frac{d\gamma_{\sigma_{t_1}}\left(\mathbf{v}_l,\frac{\mathbf{x}}{s},s\right)}{ds} \quad \text{and} \quad W^{\dagger}\,|^{\sigma_{t_1}}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_1}}_{\mathbf{P}}\right)$$

in the study of the limit $s\to\infty$. However these problems can be easily circumvented by means of an "$\frac{\epsilon}{3}$ argument", by exploiting the resolvent equation (1.12) in addition to the usual regularity properties. Therefore an analogous estimate is obtained: the sum of the absolute values of the off-diagonal terms is bounded by $C\cdot t_1^{-\epsilon}\cdot|\ln\sigma_{t_2}|$.

diagonal terms

Considering that the norm

$$\left\|\left(W\,|_{\sigma_{t_2}}^{\sigma_{t_1}}\,(\mathbf{v}_j)\,W\,|_{\sigma_{t_2}}^{\sigma_{t_1}}\,\left(\nabla E_{\mathbf{P}}^{\sigma_{t_1}}\right)-I\right)\psi_{j,\sigma_{t_1}}^{(t_1)}\right\| \tag{3.53}$$

can be estimated from above in terms of a quantity of order

$$\sup_{\mathbf{P}\in\Gamma_j^{(t_1)}}|\mathbf{v}_j-\nabla E^{\sigma_{t_1}}(\mathbf{P})|\cdot|\ln\sigma_{t_2}|\cdot t_1^{-\frac{3\epsilon}{2}}$$

with

$$\sup_{\mathbf{P}\in\Gamma_j^{(t_1)}}\left|\nabla E^{\sigma_{t_1}}(\mathbf{P})-\nabla E^{\sigma_{t_1}}\left(\overline{\mathbf{P}_j}\right)\right|\leq C\cdot t_1^{-\frac{\epsilon}{16}}\,,$$

the sum of the diagonal terms amounts to a contribution of order $t_1^{-\frac{\epsilon}{16}}\cdot|\ln\sigma_{t_2}|$ at most.

We can conclude that the norm of the vector corresponding to the difference *D3.3)* is bounded by

$$C\cdot t_1^{-\frac{\epsilon}{16}}\cdot|\ln\sigma_{t_2}|\,. \tag{3.54}$$

4 Scattering subspaces and asymptotic observables

In this Section, at first we consider the family of vectors $\psi_{h,\kappa_1}^{out(in)}(\tau,\mathbf{a})$ corresponding to the evolution τ in time and to a displacement $\mathbf{a}$ in space of the state associated to the vector $\psi_{h,\kappa_1}^{out(in)}$ previously constructed. Then we construct the covariant, under space-time translations, subspace $\mathcal{H}_{\kappa_1}^{1out(in)}$ as the norm closure of the finite linear combinations of vectors in the set $\left\{\psi_{h,\kappa_1}^{out(in)}(\tau,\mathbf{a})\right\}$:

$$\mathcal{H}_{\kappa_1}^{1out(in)}:=\overline{\left\{\bigvee\psi_{h,\kappa_1}^{out(in)}(\tau,\mathbf{a})\;:\quad h(\mathbf{P})\in C_0^1(\Sigma\setminus 0)\,,\,\tau\in\mathbb{R}\,,\mathbf{a}\in\mathbb{R}^3\right\}}$$

Later, in Theorem 4.1, we define the vectors $\left\{\psi_{h,\mu}^{out(in)}\right\}$ obtained from the strong (time) limit of the *L.S.Z.* Weyl operators, with smearing functions$\left\{\mu\,:\,\widetilde{\mu}(\mathbf{k})\in C_0^\infty(\mathbb{R}^3\setminus 0)\right\}$, applied to the total set $\left\{\psi_{h,\kappa_1}^{out(in)}(\tau,\mathbf{a})\right\}$ of the Hilbert space $\mathcal{H}_{\kappa_1}^{1out(in)}$. The norm closure of the finite linear combinations of the vectors in the set $\left\{\psi_{h,\mu}^{out(in)}\right\}$ is a reasonable candidate for the scattering subspace $\mathcal{H}^{out(in)}$. The physical meaning of this definition stems from the characterization of the states belonging to $\mathcal{H}^{out(in)}$

in terms of quantum numbers associated with the asymptotic variables which are well defined on them: the asymptotic photon Weyl operators and the asymptotic electron mean velocity. In Theorem 4.2 the asymptotic convergence of the C_0^∞ functions of the the variable $e^{iHt}\frac{\mathbf{x}}{t}e^{-iHt}$ is established on the vectors of $\mathcal{H}^{out(in)}$. These functions generate the commutative algebra $\mathcal{A}^{out(in)}_{\mathbf{v}_{el}}$. In Theorem 4.4, we construct the canonical Weyl algebra $\mathcal{A}^{out(in)}_{ph}$, generated by the strong limits of the *L.S.Z.* Weyl operators smeared with functions $\left\{\zeta : \widetilde{\zeta}(\mathbf{k}) \in L^2\left(\mathbb{R}^3, \left(1+|\mathbf{k}|^{-1}\right)d^3k\right)\right\}$ and acting on the space $\mathcal{H}^{out(in)}$. The algebra $\mathcal{A}^{out(in)}_{ph}$ is associated with a free massless boson field and commutes with the algebra $\mathcal{A}^{out(in)}_{\mathbf{v}_{el}}$ as a consequence of the asymptotic decoupling.
The spectral restriction on the electron (mean) velocity (strictly less than 1) implies a restriction of $\mathcal{H}^{out(in)}$, as subspace of $\mathcal{H}$, explained with the partial non-relativistic character of the model. However no issue regarding completeness is addressed in our discussion, even under the restriction on the energy configurations of the system.

Definition of the vector $\psi^{out}_{h,\kappa_1}(\tau,\mathbf{a})$.

Applying the operator $e^{-i\mathbf{a}\cdot\mathbf{P}}e^{-iH\tau}$ to the generic vector ψ^{out}_{h,κ_1}, we obtain:

$$\begin{aligned}
&e^{-i\mathbf{a}\cdot\mathbf{P}}e^{-iH\tau}\psi^{out}_{h,\kappa_1} = \\
&= s-\lim_{t\to+\infty} e^{-i\mathbf{a}\cdot\mathbf{P}}e^{-iH\tau}e^{iHt}\sum_{j=1}^{N(t)}\mathcal{W}_{\sigma_t}(\mathbf{v}_j,t)\,e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},t\right)}e^{-iE^{\sigma_t}_{\mathbf{P}}(t-\tau)}e^{-iE^{\sigma_t}_{\mathbf{P}}\tau}\psi^{(t)}_{j,\sigma_t} \\
&= s-\lim_{t\to+\infty} e^{iHt}\sum_{j=1}^{N(t+\tau)}\left[e^{-g\int_{\sigma_{t+\tau}}^{\kappa_1}\frac{a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{a}}e^{i|\mathbf{k}|(\tau+t)}-h.c.}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}}e^{i\gamma_{\sigma_{t+\tau}}\left(\mathbf{v}_j,\nabla E^{\sigma_{t+\tau}}_{\mathbf{P}},t+\tau\right)}e^{-iE^{\sigma_{t+\tau}}_{\mathbf{P}}t}\right]\psi^{(t+\tau)}_{j,\sigma_{t+\tau}}(\tau,\mathbf{a}) \\
&= s-\lim_{t\to+\infty}\psi^{\tau,\mathbf{a}}_{h,k_1}(t)
\end{aligned} \tag{4.1}$$

where

$$\psi^{(t+\tau)}_{j,\sigma_{t+\tau}}(\tau,\mathbf{a}) := \int_{\Gamma^{(t+\tau)}_j} e^{-i\mathbf{a}\cdot\mathbf{P}}e^{-iE^{\sigma_{t+\tau}}_{\mathbf{P}}\tau}h(\mathbf{P})\,\psi^{\sigma_{t+\tau}}_{\mathbf{P}}d^3P$$

and $\psi^{\tau,\mathbf{a}}_{h,\kappa_1}(t)$ corresponds to the approximating vector $\psi_{h,\kappa_1}(t)$ with translated wave function both in the electron and the photon variables. The final equality (4.1) can be easily derived exploiting the estimates involved in the construction of ψ^{out}_{h,κ_1}. Therefore the definition

$$\psi^{out}_{h,\kappa_1}(\tau,\mathbf{a}) := e^{-i\mathbf{a}\cdot\mathbf{P}}e^{-iH\tau}\psi^{out}_{h,\kappa_1} \tag{4.2}$$

is consistent with the expected asymptotic interpretation. □

The definition of the subspace of the minimal asymptotic electron states is

$$\mathcal{H}^{1out(in)}_{\kappa_1} := \overline{\left\{\bigvee \psi^{out}_{h,\kappa_1}(\tau,\mathbf{a}) \quad h(\mathbf{P})\in C^1_0\left(\mathbb{R}^3\setminus 0\right),\, supph\subset\Sigma\,,\,\tau\in\mathbb{R}\,,\mathbf{a}\in\mathbb{R}^3\right\}} \tag{4.3}$$

Definition of the scattering spaces

In next theorem we construct the vector $\psi^{out(in)}_{h,\mu}$ starting from a vector $\psi^{out(in)}_{h,\kappa_1}(\tau,\mathbf{a})$ in $\mathcal{H}^{1out(in)}_{\kappa_1}$ and a cloud of photons represented by an *L.S.Z.* Weyl operator with smearing function μ. Concerning notations we omit the dependence on $\kappa_1,\tau,\mathbf{a}$ due to $\psi^{out(in)}_{h,\kappa_1}(\tau,\mathbf{a})$.

Theorem 4.1
The strong limit

$$s-lim_{t\to+\infty}\psi_{h,\mu}(t):=\psi^{out}_{h,\mu} \tag{4.4}$$

exists, where:

$$\begin{aligned}
&\psi_{h,\mu}(t):=e^{iHt}e^{i\left(a(\mu_t)+a^{\dagger}(\mu_t)\right)}e^{-iHt}\psi_{h,\kappa_1}(t)\ ;\\
&\qquad\widetilde{\mu}(\mathbf{k})\in C_0^{\infty}\left(\mathbb{R}^3\setminus 0\right)\ ,\ \widetilde{\mu}_t(\mathbf{k}):=e^{i|\mathbf{k}|t}\widetilde{\mu}(\mathbf{k})\ ;\\
&a^{\dagger}(\mu):=(a(\mu))^{\dagger}=\left(\int a(\mathbf{k})\,\widetilde{\mu}(\mathbf{k})\,d^3k\right)^{\dagger}\ ;\\
&\psi^{out}_{h,k_1}\in\mathcal{H}^{1out}_{k_1}\,.
\end{aligned}$$

Proof
Taking into account Theorem 3.1, the function $\chi_h(\nabla E^{\sigma_t}_{\mathbf{P}})$ introduced in Theorem 2.1 and Huyghens' principle, the result follows from Cook's argument once the constraint (2.26) is assumed. □

The scattering subspaces

$$\mathcal{H}^{out(in)}:=\overline{\left\{\bigvee\psi^{out(in)}_{h,\mu}\ :\quad h(\mathbf{P})\in C^1_0(\Sigma\setminus 0)\,,\widetilde{\mu}\in C^{\infty}_0(\mathbb{R}^3\setminus 0)\right\}} \tag{4.5}$$

are invariant under space-time translations because the subspaces $\mathcal{H}^{1out(in)}_{\kappa_1}$ are invariant by construction.

Asymptotic algebras

Theorem 4.2
The C_0^{∞} functions f of the variable $e^{iHt}\frac{\mathbf{x}}{t}e^{-iHt}$, that is the electron mean velocity (at time t) up to a correction of order t^{-1}, have strong limits in $\mathcal{H}^{out}$ for $t\to+\infty$, namely

$$s-lim_{t\to+\infty}e^{iHt}f\left(\frac{\mathbf{x}}{t}\right)e^{-iHt}\psi^{out}_{h,\mu}=:\psi^{out}_{h\cdot f_{\nabla E},\mu} \tag{4.6}$$

where $f_{\nabla E}(\mathbf{P}):=lim_{\sigma\to 0}f(\nabla E^{\sigma}(\mathbf{P}))$.

Proof
Exploiting Theorem 3.1 and using the fact that the operators

$$f\left(\frac{\mathbf{x}}{t}\right)\ ,\qquad e^{iHt}e^{i\left(a(\mu_t)+a^{\dagger}(\mu_t)\right)}e^{-iHt}$$

are uniformly bounded in t, we obtain

$$s-lim_{t\to+\infty}e^{iHt}f\left(\frac{\mathbf{x}}{t}\right)e^{-iHt}\psi^{out}_{h,\mu}=$$

$$= \quad s-lim_{t\to+\infty}e^{iHt}f\left(\frac{\mathbf{x}}{t}\right)e^{i\left(a(\mu_t)+a^\dagger(\mu_t)\right)}e^{-iHt}\psi^{out}_{h,\kappa_1}$$

$$= \quad s-lim_{t\to+\infty}e^{iHt}e^{i\left(a(\mu_t)+a^\dagger(\mu_t)\right)}f\left(\frac{\mathbf{x}}{t}\right)e^{-iHt}\psi^{out}_{h,\kappa_1} \tag{4.7}$$

$$= \quad s-lim_{t\to+\infty}e^{iHt}e^{i\left(a(\mu_t)+a^\dagger(\mu_t)\right)}e^{-iHt}\psi^{out}_{h\cdot f_{\nabla E},\kappa_1} \tag{4.8}$$

where the last step, from (4.7) to (4.8), is proved by means of the same technique used in Theorem A2 and the notation $\psi^{out}_{h\cdot f_{\nabla E},\kappa_1}$ is justified starting from the regularity properties of $\nabla E^\sigma(\mathbf{P})$.
The extension to all of $\mathcal{H}^{out}$ is straightforward because $f\left(\frac{\mathbf{x}}{t}\right)$ is uniformly bounded in t and the set $\bigvee\psi^{out}_{h,\mu}$ is dense in $\mathcal{H}^{out}$, by construction. $\square$

We call $\mathcal{A}^{out(in)}_{\mathbf{v}_{el}}$ the norm closure of the *algebra generated by the C_0^∞ functions of the asymptotic electron mean velocity defined in $\mathcal{H}^{out(in)}$ by the strong limits (4.6) (for the *out* and the *in* case respectively) in Theorem 4.2.

Corollary 4.3
In the space $\mathcal{H}^{out}$, the unitary operators

$$\left\{\mathcal{W}^{out}(\zeta):\widetilde{\zeta}(\mathbf{k})\in L^2\left(\mathbb{R}^3,\left(1+|\mathbf{k}|^{-1}\right)d^3k\right)\right\}$$

are well defined starting from the strong limit:

$$\mathcal{W}^{out}(\zeta):=s-\lim_{t\to+\infty}e^{iHt}e^{i\left(a(\zeta_t)+a^\dagger(\zeta_t)\right)}e^{-iHt}\,. \tag{4.9}$$

The following properties hold:

i) The operators $\left\{\mathcal{W}^{out}(\zeta):\widetilde{\zeta}\in L^2\left(\mathbb{R}^3,\left(1+|\mathbf{k}|^{-1}\right)d^3k\right)\right\}$ satisfy the Weyl commutation rules

$$\mathcal{W}^{out}(\zeta)\,\mathcal{W}^{out}(\zeta')=\mathcal{W}^{out}(\zeta'+\zeta)\,e^{-\frac{\rho(\zeta,\zeta')}{2}} \tag{4.10}$$

where $\rho(\zeta,\zeta')=2iIm\left(\int\overline{\widetilde{\zeta}}(\mathbf{k})\,\widetilde{\zeta'}(\mathbf{k})\,d^3k\right)$;

ii) The mapping $\mathbb{R}\ni s\to\mathcal{W}^{out}(s\mu)$ defines a strongly continuous, one parametric group of unitary operators.

Proof
The existence of

$$s-\lim_{t\to+\infty}e^{iHt}\mathcal{W}(\zeta_t)\,e^{-iHt}\psi^{out}_{h,\mu} \tag{4.11}$$

with

$$\mathcal{W}(\zeta_t):=e^{i\left(a(\zeta_t)+a^\dagger(\zeta_t)\right)}$$

on a generic $\psi_{h,\mu}^{out}$ implies that the bounded operators $W^{out}(\zeta)$ can be extended from the dense set $\bigvee \psi_{h,\mu}^{out}$ to all of $\mathcal{H}^{out}$, by continuity.
In order to prove the existence of the limit (4.11), let us consider for $t_2 > t_1$ the difference

$$e^{iHt_1}\mathcal{W}(\zeta_{t_1})\, e^{-iHt_1}\psi_{h,\mu}^{out} - e^{iHt_2}\mathcal{W}(\zeta_{t_2})\, e^{-iHt_2}\psi_{h,\mu}^{out} \tag{4.12}$$

and a sequence of functions $\left\{\mu^n(\mathbf{k}) \in C_0^\infty\left(\mathbb{R}^3 \setminus 0\right)\; n \in \mathbb{N}\right\}$ such that

$$\|\zeta - \mu^n\|_{L^2\left(\mathbb{R}^3,\left(1+|\mathbf{k}|^{-1}\right)d^3k\right)} \to_{n\to+\infty} 0\,.$$

Then we exploit the following identity

$$e^{iHt_1}\mathcal{W}(\zeta_{t_1})\, e^{-iHt_1}\psi_{h,\mu}^{out} - e^{iHt_2}\mathcal{W}(\zeta_{t_2})\, e^{-iHt_2}\psi_{h,\mu}^{out} =$$

$$= \quad e^{iHt_1}\mathcal{W}(\zeta_{t_1})\, e^{-iHt_1}\psi_{h,\mu}^{out} - e^{iHt_1}\mathcal{W}(\zeta_{t_1})\, e^{-iHt_1}\psi_{h,\mu}(t_1) \tag{4.13}$$

$$+ \quad e^{iHt_1}\mathcal{W}(\zeta_{t_1})\, e^{-iHt_1}\psi_{h,\mu}(t_1) - e^{iHt_1}\mathcal{W}(\mu_{t_1}^n)\, e^{-iHt_1}\psi_{h,\mu}(t_1) \tag{4.14}$$

$$+ \quad e^{iHt_1}\mathcal{W}(\mu_{t_1}^n)\, e^{-iHt_1}\psi_{h,\mu}(t_1) - e^{iHt_2}\mathcal{W}(\mu_{t_2}^n)\, e^{-iHt_2}\psi_{h,\mu}(t_2) \tag{4.15}$$

$$+ \quad e^{iHt_2}\mathcal{W}(\mu_{t_2}^n)\, e^{-iHt_2}\psi_{h,\mu}(t_2) - e^{iHt_2}\mathcal{W}(\zeta_{t_2})\, e^{-iHt_2}\psi_{h,\mu}(t_2) \tag{4.16}$$

$$+ \quad e^{iHt_2}\mathcal{W}(\zeta_{t_2})\, e^{-iHt_2}\psi_{h,\mu}(t_2) - e^{iHt_2}\mathcal{W}(\zeta_{t_2})\, e^{-iHt_2}\psi_{h,\mu}^{out} \tag{4.17}$$

and observe that:
For (4.13) and (4.17), the corresponding norms are at most of order $t_1^{-\rho}$ and $t_2^{-\rho}$ respectively, for some positive ρ, because of Theorem 4.1;
For (4.14) (and equivalently (4.16)), we can estimate

$$\left\| e^{iHt_1}\mathcal{W}(\zeta_{t_1})\, e^{-iHt_1}\psi_{h,\mu}(t_1) - e^{iHt_1}\mathcal{W}(\mu_{t_1}^n)\, e^{-iHt_1}\psi_{h,\mu}(t_1)\right\| \le$$

$$\le C\cdot\|\zeta-\mu^n\|_{L^2\left(\mathbb{R}^3,\left(1+|\mathbf{k}|^{-1}\right)d^3k\right)}\cdot\left\|H^{ph\frac{1}{2}}\left(\frac{1}{H_{\sigma_{t_1}}+a}\right)^{\frac{1}{2}}\right\|\cdot\left\|\left(H_{\sigma_{t_1}}+a\right)^{\frac{1}{2}} e^{-iHt_1}\psi_{h,\mu}(t_1)\right\|$$

where a is a sufficiently large positive number. Both the positive constant C and the two norms

$$\left\|H^{ph\frac{1}{2}}\left(\frac{1}{H_{\sigma_{t_1}}+a}\right)^{\frac{1}{2}}\right\|\;,\;\left\|\left(H_{\sigma_{t_1}}+a\right)^{\frac{1}{2}} e^{-iHt_1}\psi_{h,\mu}(t_1)\right\|$$

are uniformly bounded in t_1 (in t_2 for the analogous expression in the case (4.16));

For (4.15), at fixed n, the norm is infinitesimal for $t_1 \to +\infty$ because of Theorem 4.1.

Hence the convergence (4.9) trivially follows. Moreover the so defined operators

$$\left\{\mathcal{W}^{out}(\zeta) : \widetilde{\zeta}(\mathbf{k}) \in L^2\left(\mathbb{R}^3, \left(1+|\mathbf{k}|^{-1}\right)d^3k\right)\right\}$$

are unitary in $\mathcal{H}^{out}$.

Concerning the properties i) and ii):
i) The operator

$$\mathcal{W}^{out}(\zeta)\,\mathcal{W}^{out}(\zeta') : \mathcal{H}^{out} \to \mathcal{H}^{out}$$

is the time limit of the equal time product of the corresponding approximating operators (4.9). The approximating operators obey the Weyl rules by construction. Hence the property is satisfied in the limit.
ii) This property follows basically by means of the same approximation arguments used to justify the limit (4.9). □

Theorem 4.4
For the asymptotic boson algebra $\mathcal{A}_{ph}^{out(in)}$ defined as the norm closure of the *algebra generated by the set of unitary operators (4.9) (in the *out* and in the *in* case respectively) acting on $\mathcal{H}^{out(in)}$, the following properties hold:

i) Starting from the τ-evolved generators

$$e^{iH\tau}\mathcal{W}^{out(in)}(\zeta)\,e^{-iH\tau} = \mathcal{W}^{out(in)}(\zeta_{-\tau}) \tag{4.18}$$

where $\zeta_{-\tau}$ ($\widetilde{\zeta}_{-\tau}(\mathbf{k}) := e^{-i|\mathbf{k}|\tau}\widetilde{\zeta}(\mathbf{k})$) is the freely evolved test function ζ in the time τ, the automorphism α_τ of $\mathcal{A}_{ph}^{out(in)}$ is uniquely defined:

$$\alpha_\tau\left(\mathcal{W}^{out(in)}(\zeta)\right) := \mathcal{W}^{out(in)}(\zeta_{-\tau}). \tag{4.19}$$

Therefore $\mathcal{A}_{ph}^{out(in)}$ is the Weyl algebra associated with the free massless scalar field;

ii) The algebra $\mathcal{A}_{ph}^{out(in)}$ commutes with the algebra $\mathcal{A}_{\mathbf{v}_{el}}^{out(in)}$.

Proof
i) The τ-evolved generators $e^{iH\tau}\mathcal{W}^{out(in)}(\zeta)\,e^{-iH\tau}$ are well defined on $\mathcal{H}^{out(in)}$ because $e^{-iH\tau}$:$\mathcal{H}^{out(in)} \to \mathcal{H}^{out(in)}$. By inserting the expression (4.9) for $\mathcal{W}^{out(in)}(\zeta)$, we easily get the equality $e^{iH\tau}\mathcal{W}^{out(in)}(\zeta)\,e^{-iH\tau} = \mathcal{W}^{out(in)}(\zeta_{-\tau})$. The Weyl commutation rules are trivially conserved by α_τ because

$$\rho\left(\zeta_{-\tau}, \zeta'_{-\tau}\right) = 2iIm\left(\int \overline{\widetilde{\zeta}(\mathbf{k})}\widetilde{\zeta}'(\mathbf{k})\,d^3k\right) = \rho\left(\zeta, \zeta'\right)\ . \tag{4.20}$$

Hence α_τ can be uniquely extended to all the algebra $\mathcal{A}_{ph}^{out(in)}$.
ii) By an approximation argument, the considered property follows from the definition of the generators of $\mathcal{A}_{\mathbf{v}_{el}}^{out(in)}$ and $\mathcal{A}_{ph}^{out(in)}$ in Theorem 4.2 and Corollary 4.3 respectively. □

5 APPENDIX

In the following lemmas and theorems we assume the properties discussed in Subsection 1.3.1. As in the previous Sections, we use the convention to generically call C the constants which are time independent, uniform in the infrared cut-off and in the cell partition. The bounds are intended from above, unless otherwise indicated.

We now provide some results about the phase factor " $e^{i\gamma}$ " which enters in the definition of the approximating vector $\psi_{h,k_1}(t)$.

Lemma A1
Under the assumptions for the construction (Subsection 1.3.1) and because of the definition (2.6), the following estimates hold:

$$\left|\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j, \nabla E_{\mathbf{P}}^{\sigma_{t_2}}, (\sigma_{t_2})^{-\frac{1}{\alpha}}\right) - \gamma_{\sigma_{t_2}}\left(\mathbf{v}_{l(j)}, \nabla E_{\mathbf{P}}^{\sigma_{t_2}}, (\sigma_{t_2})^{-\frac{1}{\alpha}}\right)\right| \quad \leq C \cdot \left|\mathbf{v}_j - \mathbf{v}_{l(j)}\right| \tag{5.1}$$

where $\mathbf{v}_j \equiv \nabla E_{\mathbf{P}_j}^{\sigma_{t_1}}$ and $\mathbf{v}_{l(j)} \equiv \nabla E_{\mathbf{P}_{l(j)}}^{\sigma_{t_2}}$ are related to the partitions $\Gamma^{(t_1)}$ and $\Gamma^{(t_2)}$ respectively;
for $t_2 > t_1 \gg 1$

$$\left|\gamma_{\sigma_{t_2}}\left(\mathbf{v}_j, \nabla E_{\mathbf{P}}^{\sigma_{t_2}}, t_1\right) - \gamma_{\sigma_{t_1}}\left(\mathbf{v}_j, \nabla E_{\mathbf{P}}^{\sigma_{t_1}}, t_1\right)\right| \leq C \cdot (\sigma_{t_1})^{\frac{1}{4}} \cdot t_1^{2(1-\alpha)} + C \cdot t_1 \cdot \sigma_{t_1}; \tag{5.2}$$

for $\mathbf{q} \in \left\{\mathbf{q} : \ |\mathbf{q}| < s^{(1-\alpha)}\ ,\ 1 > \alpha > 0\right\}$

$$\left|\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E_{\mathbf{P}}^{\sigma_t}, s\right) - \gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E_{\mathbf{P}+\frac{\mathbf{q}}{s}}^{\sigma_t}, s\right)\right| \leq C \cdot s^{-\cdot\frac{\alpha}{16}} \cdot s^{2(1-\alpha)}\,. \tag{5.3}$$

Proof
The bounds can be obtained by standard computations taking into account the assumptions and the results in the paragraph ***Spectral properties***, Subsection 1.3.1. Moreover the bounds are intended not to be optimal but sufficient for our purposes. □

The following theorem and the related corollary are concerned with the convergence "$e^{iHt}\frac{\mathbf{x}}{t}e^{-iHt} \to_{t\to\infty} \nabla E(\mathbf{P})$".

Theorem A2
Under the assumptions for the construction (Subsection 1.3.1), for $0 < \alpha\,(< 1)$ sufficiently close to 1 and $\epsilon > 0$ sufficiently small, the following propagation estimate holds true:

$$\left\|\int \widetilde{\chi}_h(\mathbf{q})\left(e^{-i\mathbf{q}\cdot\nabla E_{\mathbf{P}}^{\sigma_t}} - e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}\right) d^3q e^{-iE_{\mathbf{P}}^{\sigma_t}s} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E_{\mathbf{P}}^{\sigma_t}, s\right)} \psi_{j,\sigma_t}^{(t)}\right\| \leq$$

$$\leq C \cdot s^{-\upsilon} \cdot t^{-\frac{3\epsilon}{2}} \cdot |\ln \sigma_t| \qquad \upsilon > 0$$

where $\widetilde{\chi}_h$ is the Fourier transformed of χ_h, $s \geq t \gg 1$ and $\mathbf{v}_j \equiv \nabla E_{\mathbf{P}_j}^{\sigma_t}$ is referred to the partition $\Gamma^{(t)}$.

Proof
Let us start from the following Hilbert inequality:

$$\left\|\int \widetilde{\chi}_h(\mathbf{q})\left(e^{-i\mathbf{q}\cdot\nabla E^{\sigma_t}_{\mathbf{P}}}-e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}\right)d^3q e^{-iE^{\sigma_t}_{\mathbf{P}}s}e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{j,\sigma_t}\right\|\leq$$

$$\leq\left\|\int \widetilde{\chi}_h(\mathbf{q})\left(e^{-i\mathbf{q}\cdot\nabla E^{\sigma_t}_{\mathbf{P}}}-e^{i\left(E^{\sigma_t}_{\mathbf{P}}-E^{\sigma_t}_{\mathbf{P}+\frac{\mathbf{q}}{s}}\right)s}\right)d^3q e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{j,\sigma_t}\right\| \tag{5.4}$$

$$+\left\|\int \widetilde{\chi}_h(\mathbf{q})e^{i\left(E^{\sigma_t}_{\mathbf{P}}-E^{\sigma_t}_{\mathbf{P}+\frac{\mathbf{q}}{s}}\right)s}\left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}-1\right)d^3q e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{j,\sigma_t}\right\| \tag{5.5}$$

In order to estimate the integrals in (5.4) and (5.5), we distinguish "large" and "small" $\mathbf{q}$:
For large $\mathbf{q}$, that is $\left\{\mathbf{q}:\ |\mathbf{q}|\geq s^{(1-\alpha)}\right\}$, we exploit that $\widetilde{\chi}_h(\mathbf{q})\in S\left(\mathbb{R}^3\right)$, therefore

$$\forall n\in\mathbb{N}\quad \exists\, C_n>0\quad \text{s.t. } |\widetilde{\chi}_h(\mathbf{q})|<C_n\cdot\frac{1}{|\mathbf{q}|^n}\quad \text{for}\quad |\mathbf{q}|>1; \tag{5.6}$$

For small $\mathbf{q}$, that is $\left\{\mathbf{q}:\ |\mathbf{q}|<s^{(1-\alpha)}\right\}$, the Hölder properties in $\mathbf{P}$ of $\nabla E^{\sigma_t}_{\mathbf{P}}$ and of $\phi^{\sigma_t}_{\mathbf{P}}$ provide the desired result.

Term (5.4)
The inequality

$$\left\|\int \widetilde{\chi}_h(\mathbf{q})\left(e^{-i\mathbf{q}\cdot\nabla E^{\sigma_t}_{\mathbf{P}}}-e^{i\left(E^{\sigma_t}_{\mathbf{P}}-E^{\sigma_t}_{\mathbf{P}+\frac{\mathbf{q}}{s}}\right)s}\right)d^3q e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{j,\sigma_t}\right\|\leq$$

$$\leq C\cdot\left\|\psi^{(t)}_{j,\sigma}\right\|\cdot\int_{s^{(1-\alpha)}}^{+\infty}|\widetilde{\chi}_h(\mathbf{q})|\,d^3q+C\cdot\left\|\psi^{(t)}_{j,\sigma}\right\|\cdot\frac{\int_0^{s^{(1-\alpha)}}|\mathbf{q}|\cdot|\widetilde{\chi}_h(\mathbf{q})|\,d^3q}{s^{\frac{\alpha}{16}}}$$

holds because of the ***Spectral properties***, Subsection 1.3.1, which imply:

$$\begin{array}{rcl} sE^{\sigma_t}_{\mathbf{P}}-sE^{\sigma_t}_{\mathbf{P}+\frac{\mathbf{q}}{s}} & = & -\mathbf{q}\cdot\nabla E^{\sigma_t}_{\mathbf{P}'} \qquad \text{with} \qquad |\mathbf{P}-\mathbf{P}'|\leq\left|\frac{\mathbf{q}}{s}\right| \\ \left|\nabla E^{\sigma_t}_{\mathbf{P}}-\nabla E^{\sigma_t}_{\mathbf{P}'}\right| & \leq & C\cdot|\mathbf{P}-\mathbf{P}'|^{\frac{1}{16}} \end{array}$$

Therefore the term (5.4) is surely bounded by a quantity of order

$$s^{-\frac{\alpha}{16}}\cdot t^{-\frac{3\epsilon}{2}}\,.$$

Term (5.5)

Let us start from the trivial inequality

$$\left\|\int \widetilde{\chi}_h(\mathbf{q})e^{i\left(E^{\sigma_t}_{\mathbf{P}}(\mathbf{P})-E^{\sigma_t}_{\mathbf{P}+\frac{\mathbf{q}}{s}}\right)s}\left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}-1\right)d^3q e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)}\psi^{(t)}_{j,\sigma_t}\right\|\leq$$

$$\leq \left\| \int_{s^{(1-\alpha)}}^{+\infty} \widetilde{\chi}_h(\mathbf{q})\, e^{i\left(E^{\sigma_t}_{\mathbf{P}} - E^{\sigma_t}_{\mathbf{P}+\frac{\mathbf{q}}{s}}\right)s} \left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}} - 1\right) d^3q e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E^{\sigma_t}_{\mathbf{P}}, s\right)} \psi^{(t)}_{j,\sigma_t} \right\| \tag{5.7}$$

$$+ \left\| \int_{0}^{s^{(1-\alpha)}} \widetilde{\chi}_h(\mathbf{q})\, e^{i\left(E^{\sigma_t}_{\mathbf{P}} - E^{\sigma_t}_{\mathbf{P}+\frac{\mathbf{q}}{s}}\right)s} \left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}} - 1\right) d^3q e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E^{\sigma_t}_{\mathbf{P}}, s\right)} \psi^{(t)}_{j,\sigma_t} \right\| . \tag{5.8}$$

The integral in (5.7) involves large $\mathbf{q}$, therefore it is easily under control thanks to (5.6). For the second term (5.8) we add and subtract the same quantities to eventually obtain three expressions, (5.10), (5.11) and (5.12), which can be controlled due to:
The convergence rate of the vector $\phi^{\sigma}_{\mathbf{P}}$ for $\sigma \to 0$;
The regularity properties in $\mathbf{P}$ of $h(\mathbf{P})$, $e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E^{\sigma_t}_{\mathbf{P}}, s\right)}$ and $\phi^{\sigma}_{\mathbf{P}}$ as a vector in $\mathcal{F}^b$;
The vanishing (for $s \to \infty$) volume $O_{\frac{\mathbf{q}}{s}}$ which is the difference between the cell $\Gamma^{(t)}_j$ and the same cell under a displacement $\frac{\mathbf{q}}{s}$.
In the derivation of (5.10), (5.11) and (5.12) we warn the reader about the following crucial facts:
i) Both the vectors $\psi^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}$, $\widehat{\psi}^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}} := e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}\psi^{\sigma_t}_{\mathbf{P}}$ belong to the same fiber space $\mathcal{H}_{\mathbf{P}-\frac{\mathbf{q}}{s}}$;
ii) As vectors in Fock space, $\widehat{\psi}^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}$ and $\psi^{\sigma_t}_{\mathbf{P}}$ coincide

$$\mathbf{I}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}}\psi^{\sigma_t}_{\mathbf{P}}\right) \equiv \mathbf{I}_{\mathbf{P}}\left(\psi^{\sigma_t}_{\mathbf{P}}\right) \tag{5.9}$$

where the isomorphism $\mathbf{I}_{\mathbf{P}}$ is defined by (1.9) in Subsection 1.1.
iii) Inside the integral

$$\int_{\Gamma^{(t)}_j} e^{i\left(E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}} - E^{\sigma_t}_{\mathbf{P}}\right)s} h_{\mathbf{P}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E^{\sigma_t}_{\mathbf{P}}, s\right)} \widehat{\psi}^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}} d^3P$$

the integration variable $\mathbf{P}$ is the spectral value of the corresponding (vectorial) operator.
vi) In an expression like

$$\mathbf{I}_{\mathbf{P}-\frac{\mathbf{q}}{\mathbf{s}}}\left(W^{\dagger}_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{\mathbf{s}}}\right) W_{\sigma_t}\left(\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{\mathbf{s}}}\right) \psi^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{\mathbf{s}}}\right)$$

$\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{\mathbf{s}}}$ is the vectorial operator in $\mathcal{H}_{\mathbf{P}-\frac{\mathbf{q}}{s}}$ obtained by the multiplication of the identity operator with the gradient of the ground state energy evaluated in $\mathbf{P} - \frac{\mathbf{q}}{s}$.
The treatment of (5.8) proceeds as follows

$$\left\| \int_{0}^{s^{(1-\alpha)}} \widetilde{\chi}_h(\mathbf{q})\, e^{i\left(E^{\sigma_t}_{\mathbf{P}} - E^{\sigma_t}_{\mathbf{P}+\frac{\mathbf{q}}{s}}\right)s} \left(e^{-i\mathbf{q}\cdot\frac{\mathbf{x}}{s}} - 1\right) d^3q e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E^{\sigma_t}_{\mathbf{P}}, s\right)} \psi^{(t)}_{j,\sigma_t} \right\| =$$

$$= \left\| \int_{0}^{s^{(1-\alpha)}} \widetilde{\chi}_h(\mathbf{q}) \int_{\Gamma^{(t)}_j} e^{i\left(E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}} - E^{\sigma_t}_{\mathbf{P}}\right)s} h_{\mathbf{P}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j, \nabla E^{\sigma_t}_{\mathbf{P}}, s\right)} \widehat{\psi}^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}} d^3P d^3q \right.$$

$$-\int_0^{s^{(1-\alpha)}} \widetilde{\chi}_h(\mathbf{q}) \int_{\Gamma_j^{(t)}} e^{i\left(E^{\sigma_t}_{\mathbf{P}}-E^{\sigma_t}_{\mathbf{P}+\frac{\mathbf{q}}{s}}\right)s} h_{\mathbf{P}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)} \psi^{\sigma_t}_{\mathbf{P}} d^3P d^3q \Bigg\|$$

$$= \Bigg\| \int_0^{s^{(1-\alpha)}} \widetilde{\chi}_h(\mathbf{q}) \int_{\Gamma_j^{(t)}} e^{i\left(E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}-E^{\sigma_t}_{\mathbf{P}}\right)s} h_{\mathbf{P}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)} \widehat{\psi}^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}} d^3P d^3q \tag{5.10}$$

$$-\int_0^{s^{(1-\alpha)}} \widetilde{\chi}_h(\mathbf{q}) \int_{\Gamma_j^{(t)}} e^{i\left(E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}-E^{\sigma_t}_{\mathbf{P}}\right)s} h_{\mathbf{P}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)} \psi^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}} d^3P d^3q$$

$$+\int_0^{s^{(1-\alpha)}} \widetilde{\chi}_h(\mathbf{q}) \int_{\Gamma_j^{(t)}} e^{i\left(E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}-E^{\sigma_t}_{\mathbf{P}}\right)s} h_{\mathbf{P}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)} \psi^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}} d^3P d^3q \tag{5.11}$$

$$-\int_0^{s^{(1-\alpha)}} \widetilde{\chi}_h(\mathbf{q}) \int_{\Gamma_j^{(t)}} e^{i\left(E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}-E^{\sigma_t}_{\mathbf{P}}\right)s} h_{\mathbf{P}-\frac{\mathbf{q}}{s}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)} \psi^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}} d^3P d^3q$$

$$+\int_0^{s^{(1-\alpha)}} \widetilde{\chi}_h(\mathbf{q}) \int_{\Gamma_j^{(t)}} e^{i\left(E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}-E^{\sigma_t}_{\mathbf{P}}\right)s} h_{\mathbf{P}-\frac{\mathbf{q}}{s}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)} \psi^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}} d^3P d^3q \tag{5.12}$$

$$-\int_0^{s^{(1-\alpha)}} \widetilde{\chi}_h(\mathbf{q}) \int_{\Gamma_j^{(t)}} e^{i\left(E^{\sigma_t}_{\mathbf{P}}-E^{\sigma_t}_{\mathbf{P}+\frac{\mathbf{q}}{s}}\right)s} h_{\mathbf{P}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)} \psi^{\sigma_t}_{\mathbf{P}} d^3P d^3q \Bigg\|$$

We now study the three differences (5.10), (5.11), (5.12) which are controlled respectively by

$$\int_0^{s^{(1-\alpha)}} |\widetilde{\chi}_h(\mathbf{q})| \left\{ \int_{\Gamma_j^{(t)}} |h_{\mathbf{P}}|^2 \left\| \mathbf{I}_{\mathbf{P}}\left(\psi^{\sigma_t}_{\mathbf{P}}\right) - \mathbf{I}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\psi^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\right) \right\|^2_{\mathcal{F}} d^3P \right\}^{\frac{1}{2}} d^3q \tag{5.13}$$

$$\int_0^{s^{(1-\alpha)}} |\widetilde{\chi}_h(\mathbf{q})| \left\{ \int_{\Gamma_j^{(t)}} \left| \Delta^{\mathbf{P}}_{\mathbf{P}-\frac{\mathbf{q}}{s}} [\, h_{\mathbf{P}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)} \,] \right|^2 \left\| \mathbf{I}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\psi^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\right) \right\|^2_{\mathcal{F}} d^3P \right\}^{\frac{1}{2}} d^3q \tag{5.14}$$

$$\int_0^{s^{(1-\alpha)}} |\widetilde{\chi}_h(\mathbf{q})| \left\{ \int_{O_{\frac{\mathbf{q}}{s}}} |h_{\mathbf{P}}|^2 \left\| \mathbf{I}_{\mathbf{P}}\left(\psi^{\sigma_t}_{\mathbf{P}}\right) \right\|^2_{\mathcal{F}} d^3P \right\}^{\frac{1}{2}} d^3q \tag{5.15}$$

where

$$\Delta^{\mathbf{P}}_{\mathbf{P}-\frac{\mathbf{q}}{s}} [\, h_{\mathbf{P}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)} \,] := \tag{5.16}$$

$$= h_{\mathbf{P}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s\right)} - h_{\mathbf{P}-\frac{\mathbf{q}}{s}} e^{i\gamma_{\sigma_t}\left(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}-\frac{\mathbf{q}}{s}},s\right)} .$$

and $O_{\frac{\mathbf{q}}{s}}$ is the difference between the cell $\Gamma_j^{(t)}$ and the same cell under a displacement $\frac{\mathbf{q}}{s}$.

Difference (5.10)

Using the fact that $\mathbf{P} \in \Gamma_j^{(t)} \subset \Sigma$, we estimate:

$$\left\| \mathbf{I_P}\left(\psi_{\mathbf{P}}^{\sigma_t}\right) - \mathbf{I}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(\psi_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\right\|_{\mathcal{F}} =$$

$$= \left\| \mathbf{I_P}\left(W_{\sigma_t}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_t}\right) W_{\sigma_t}\left(\nabla E_{\mathbf{P}}^{\sigma_t}\right)\psi_{\mathbf{P}}^{\sigma_t}\right) - \right.$$

$$\left. -\mathbf{I}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(W_{\sigma_t}^{\dagger}\left(\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right) W_{\sigma_t}\left(\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\right\|_{\mathcal{F}}$$

$$\leq \left\| \mathbf{I_P}\left(W_{\sigma_t}\left(\nabla E_{\mathbf{P}}^{\sigma_t}\right)\psi_{\mathbf{P}}^{\sigma_t}\right) - \mathbf{I}_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(W_{\sigma_t}\left(\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\right\|_{\mathcal{F}} \tag{5.17}$$

$$+\left\| \mathbf{I}_{\mathbf{P}-\frac{\mathbf{q}}{\mathbf{s}}}\left(\left(W_{\sigma_t}^{\dagger}\left(\nabla E_{\mathbf{P}}^{\sigma_t}\right) - W_{\sigma_t}^{\dagger}\left(\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\right) W_{\sigma_t}\left(\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\right\|_{\mathcal{F}} \tag{5.18}$$

Now notice that:
The norm (5.17) is bounded by a quantity of order $\left(\frac{|\mathbf{q}|}{s}\right)^{\frac{1}{16}}$ as consequence of the Hölder regularity in $\mathbf{P}$ of the vector $\phi_{\mathbf{P}}^{\sigma_t}$ (***Spectral properties***, Subsection 1.3.1);
The norm (5.18) can be estimated starting from the norm of the vector

$$I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(g\int_{\sigma_t}^{\kappa}\frac{\widehat{\mathbf{k}}\cdot\left(\nabla E_{\mathbf{P}}^{\sigma_t} - \nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\left(b\left(\mathbf{k}\right) - b^{\dagger}\left(\mathbf{k}\right)\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_t}\right)\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)}\frac{d^3k}{\sqrt{2\,|\mathbf{k}|}} W_{\sigma_t}\left(\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right) \tag{5.19}$$

therefore in terms of the following quantities:

$$\left(\int_{\sigma_t}^{\kappa}\left(\frac{g\widehat{\mathbf{k}}\cdot\left(\nabla E_{\mathbf{P}}^{\sigma_t} - \nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)}{\sqrt{2}\,|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}}^{\sigma_t}\right)\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)}\right)^2 d^3k\right)^{\frac{1}{2}} \tag{5.20}$$

which is bounded by

$$C\cdot\left|\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t} - \nabla E_{\mathbf{P}}^{\sigma_t}\right|\cdot|\ln\sigma_t|^{\frac{1}{2}} \leq C\cdot\left|s^{-\alpha}\right|^{\frac{1}{16}}\cdot|\ln\sigma_t|^{\frac{1}{2}}\ ;$$

$$\left(\int_{\sigma_t}^{\kappa}\left\| I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(b\left(\mathbf{k}\right) W_{\sigma_t}\left(\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\right\|_{\mathcal{F}}^2 d^3k\right)^{\frac{1}{2}} = \tag{5.21}$$

$$= \left(\int_{\sigma_t}^{\kappa}\left\| I_{\mathbf{P}-\frac{\mathbf{q}}{s}}\left(W_{\sigma_t}\left(\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\left(b\left(\mathbf{k}\right) + \frac{g\chi_{\sigma_t}^{\kappa}\left(\mathbf{k}\right)}{\sqrt{2}\,|\mathbf{k}|^{\frac{3}{2}}\left(1-\widehat{\mathbf{k}}\cdot\nabla E_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)}\right)\psi_{\mathbf{P}-\frac{\mathbf{q}}{s}}^{\sigma_t}\right)\right\|_{\mathcal{F}}^2 d^3k\right)^{\frac{1}{2}}$$

for which, using the resolvent equation (1.12) in Subsection 1.2

$$b(\mathbf{k})\,\psi_{\mathbf{P},\sigma_t} = \frac{g}{\sqrt{2|\mathbf{k}|}}\left(\frac{1}{E^{\sigma_t}_{\mathbf{P}} - |\mathbf{k}| - H_{\mathbf{P}-\mathbf{k},\sigma_t}}\right)\psi^{\sigma_t}_{\mathbf{P}} \qquad \sigma_t \le |\mathbf{k}| \le \kappa\,, \tag{5.22}$$

it is easy to provide a bound by a quantity $\mathcal{O}\left(|\ln\sigma_t|^{\frac{1}{2}}\cdot t^{-\frac{3\epsilon}{2}}\right)$ (uniform in s) which is enough for our purposes.
The difference (5.10), through the term (5.13), can be estimated in terms of

$$C\cdot\left(s^{-\alpha}\right)^{\frac{1}{16}}\cdot t^{-\frac{3\epsilon}{2}} + C\cdot\left(s^{-\alpha}\right)^{\frac{1}{16}}\cdot t^{-\frac{3\epsilon}{2}}\cdot|\ln\sigma_t|$$

so that it is surely bounded by a quantity of order

$$s^{-\cdot\frac{\alpha}{16}}\cdot t^{-\frac{3\epsilon}{2}}\cdot|\ln\sigma_t|\,. \tag{5.23}$$

Differences (5.11) and (5.12)

They are easily under control because:
$h\in C^1_0\left(\mathbb{R}^3\setminus 0\right)$ and the estimate (5.3) in Lemma A2 holds; this implies that for (5.14) and then for (5.11) we can surely provide a bound with the quantity

$$C\cdot s^{-\cdot\frac{\alpha}{16}}\cdot s^{2(1-\alpha)}\cdot t^{-\frac{3\epsilon}{2}}\,; \tag{5.24}$$

Starting from a difference between volumes, the expression (5.15) can be bounded by a quantity of order

$$\sup_{|\mathbf{q}|\le s^{(1-\alpha)}}\left(\frac{|\mathbf{q}|}{s}\right)^{\frac{1}{2}}\cdot t^{-\epsilon} \le s^{-\frac{\alpha}{2}}\cdot t^{-\epsilon}\,. \tag{5.25}$$

Conclusion.

For α, $0<\alpha\,(<1)$, sufficiently close to 1 and $\epsilon>0$ sufficiently small, there exists $\upsilon>0$ such that the sum of the terms (5.4) and (5.5) is bounded by

$$C\cdot s^{-\upsilon}\cdot|\ln\sigma_t|\cdot t^{-\frac{3\epsilon}{2}}\,.$$

□

Corollary A3
Under the same assumptions as in Theorem A2, for $s\gg 1$ and such that $s^{-\alpha}\ge\sigma_t$, the norm of the vector

$$\left[\int_{\sigma_t\cdot s}^{s\cdot s^{-\alpha}}\left(\frac{\cos\left(\mathbf{q}\cdot\frac{\mathbf{x}}{s}-|\mathbf{q}|\right)}{(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_j)} - \frac{\cos\left(\mathbf{q}\cdot\nabla E^{\sigma_t}_{\mathbf{P}}-|\mathbf{q}|\right)}{(1-\widehat{\mathbf{q}}\cdot\mathbf{v}_j)}\right)\frac{d\Omega d|\mathbf{q}|}{s}\right]e^{-iE^{\sigma_t}_{\mathbf{P}}s}e^{i\gamma_{\sigma_t}(\mathbf{v}_j,\nabla E^{\sigma_t}_{\mathbf{P}},s)}\psi^{(t)}_{j,\sigma_t}$$

(with $\mathbf{v}_j\equiv\nabla E^{\sigma_t}_{\overline{\mathbf{P}}_j}$) is surely bounded by a quantity of order

$$s^{-1}\cdot s^{-\upsilon}\cdot|\ln\sigma_t|\cdot t^{-\frac{3\epsilon}{2}} \tag{5.26}$$

for some $\upsilon > 0$.
Proof
The proof proceeds along the same lines as for the terms (5.4) and (5.8) in Theorem A2. $\square$

In the next lemma we provide some upper estimates for the absolute value of the function

$$\varphi_{t^{-\alpha},\mathbf{v}_j}(\mathbf{x},s) := g^2 \int_{t^{-\alpha}}^{\kappa_1} \int\int \frac{\cos(\mathbf{k}\cdot\mathbf{x} - |\mathbf{k}|\, s)}{\left(1 - \widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)} d\Omega d\,|\mathbf{k}|$$

where $1 > \alpha > 0$, $t \gg 1$ and $s > t$, $\mathbf{v}_j \equiv \nabla E^{\sigma_t}_{\mathbf{P}_j}$. The proof only requires some integrations by parts; therefore it is left to the reader.

Lemma A4
The following two bounds hold for $t \gg 1$ and $s > t$:

Uniformly in $\mathbf{x} \in \mathbb{R}^3$

$$\left|\varphi_{t^{-\alpha},\mathbf{v}_j}(\mathbf{x},s)\right| \le C \cdot \frac{\ln t}{s}\,; \tag{5.27}$$

In the region $\left\{\mathbf{x} \in \mathbb{R}^3 : \ (1-\rho')\, s < |\mathbf{x}| < (1-\rho)\, s \quad , \quad 0 < \rho < \rho' < 1\right\}$

$$\left|\varphi_{t^{-\alpha},\mathbf{v}_j}(\mathbf{x},s)\right| \le C_{\rho,\rho'} \cdot \frac{t^\alpha}{s^2} \tag{5.28}$$

where the positive constant $C_{\rho,\rho'}$ depends on ρ, ρ'.

We now discuss some properties for the annihilation operator associated to the asymptotic boson field when the Hamiltonian is H_{σ_t}, $\sigma_t > 0$.

Theorem A5
The limit

$$s - \lim_{s\to+\infty} e^{iH_{\sigma_t}s} \int_{\sigma_t}^{\kappa_1} \frac{a(\mathbf{k})\, e^{i|\mathbf{k}|s}\eta_{l,j}\left(\widehat{\mathbf{k}}\right)}{|\mathbf{k}|} \frac{d^3k}{\sqrt{2\,|\mathbf{k}|}} e^{-iH_{\sigma_t}s}\psi^{(t)}_{j,\sigma_t} =: a^{out}_{\sigma_t}(\check{\eta}_{l,j})\,\psi^{(t)}_{j,\sigma_t}$$

is well defined, with

$$\check{\eta}_{l,j}(\mathbf{k}) := \frac{\eta_{l,j}\left(\widehat{\mathbf{k}}\right)}{|\mathbf{k}|\cdot\sqrt{2\,|\mathbf{k}|}}\chi^{\kappa_1}_{\sigma_t}(\mathbf{k}) \tag{5.29}$$

where $\chi^{\kappa_1}_{\sigma_t}(\mathbf{k})$ is the characteristic function of the set $\{\mathbf{k} : \sigma_t < |\mathbf{k}| \le \kappa_1\}$. The vector $a^{out(in)}_{\sigma_t}(\check{\eta}_{l,j})\,\psi^{(t)}_{j,\sigma_t}$ belongs to $D(H_{\sigma_t})$.
Under the assumption that $\mathbf{P}+\mathbf{k} \in \Sigma$ for $\mathbf{P}$ and $\mathbf{k}$ belonging respectively to $\Gamma^{(t)}_j$ and $\{\mathbf{k} : 0 < |\mathbf{k}| \le \kappa_1\}$, the following identity holds:

$$a^{out(in)}_{\sigma_t}(\check{\eta}_{l,j})\,\psi^{(t)}_{j,\sigma_t} = 0\,.$$

Proof
The existence of the limit is a simple application of the propagation estimate in Theorem A2 and of Cook's argument by using the function $\chi_h\left(\nabla E_{\mathbf{P}}^{\sigma_t}\right)$ as in Theorem 2.1 and exploiting the estimates (5.27) and (5.28) in Lemma A4.
The vector $a_{\sigma_t}^{out(in)}\left(\check{\eta}_{l,j}\right)\psi_{j,\sigma_t}^{(t)}$ belongs to $D\left(H_{\sigma_t}\right)$. Indeed, for each s, the vector

$$H_{\sigma_t}e^{iH_{\sigma_t}s}e^{-iH^{ph}s}a\left(\check{\eta}_{l,j}\right)e^{iH^{ph}s}e^{-iH_{\sigma_t}s}\psi_{j,\sigma_t}^{(t)} \tag{5.30}$$

is well defined because

$$a\left(\check{\eta}_{l,j}\right)\psi_{j,\sigma_t}^{(t)}\subset D\left(H_{\sigma_t}\right)\ . \tag{5.31}$$

The inclusion (5.31) is proved by an approximation argument, exploiting the fact that $|\mathbf{k}|^{\frac{1}{2}}\cdot\check{\eta}_{l,j}\left(\mathbf{k}\right)\in L^2\left(\mathbb{R}^3,d^3k\right)$ and $\psi_{j,\sigma_t}^{(t)}\subset D\left(H_{\sigma_t}^2\right)$. Now, since H_{σ_t} is a closed operator, it is enough to prove the convergence for $s\to+\infty$. We rewrite the vector (5.30) as

$$\begin{aligned} H_{\sigma_t}e^{iH_{\sigma_t}s}e^{-iH^{ph}s}a\left(\check{\eta}_{l,j}\right)e^{iH^{ph}s}e^{-iH_{\sigma_t}s}\psi_{j,\sigma_t}^{(t)} &= \\ = \quad e^{iH_{\sigma_t}s}e^{-iH^{ph}s}a\left(\check{\eta}_{l,j}\right)e^{iH^{ph}s}e^{-iH_{\sigma_t}s}E^{\sigma_t}\left(\mathbf{P}\right)\psi_{j,\sigma_t}^{(t)} & \\ +e^{iH_{\sigma_t}s}\left[H_{\sigma_t}-H^{ph},e^{-iH^{ph}s}a\left(\check{\eta}_{l,j}\right)e^{iH^{ph}s}\right]e^{-iH_{\sigma_t}s}\psi_{j,\sigma_t}^{(t)} & \\ +e^{iH_{\sigma_t}s}\left[H^{ph}\ ,\ e^{-iH^{ph}s}a\left(\check{\eta}_{l,j}\right)e^{iH^{ph}s}\right]e^{-iH_{\sigma_t}s}\psi_{j,\sigma_t}^{(t)}\ . & \end{aligned} \tag{5.32}$$

Because of the first part of the theorem and the following equality which holds on $D\left(H_{\sigma_t}\right)$

$$\left[H^{ph}\ ,\ e^{-iH^{ph}s}a\left(\check{\eta}_{l,j}\right)e^{iH^{ph}s}\right]=-\int_{\sigma_t}^{\kappa_1}a\left(\mathbf{k}\right)e^{i|\mathbf{k}|s}\frac{\eta_{l,j}\left(\widehat{\mathbf{k}}\right)}{\sqrt{2\left|\mathbf{k}\right|}}d^3k\ ,$$

each term on the right hand side of the expression (5.32) has a well defined limit for $s\to+\infty$.

We denote as $\left(\psi_{j,\sigma_t}^{(t)}\right)_{\mathbf{P}+\mathbf{k}}$ the projection of $\psi_{j,\sigma_t}^{(t)}$ on the fiber space $\mathcal{H}_{\mathbf{P}+\mathbf{k}}$. Starting from the spectral decomposition of $\mathcal{H}$ with respect to the $\mathbf{P}$ operators and because of the equation (5.32), we deduce that

$$\int_{\sigma_t}^{\kappa_1}a_{\sigma_t}^{out}\left(\mathbf{k}\right)\frac{\eta_{l,j}\left(\widehat{\mathbf{k}}\right)}{|\mathbf{k}|\sqrt{2\left|\mathbf{k}\right|}}\left(\psi_{j,\sigma_t}^{(t)}\right)_{\mathbf{P}+\mathbf{k}}d^3k \tag{5.33}$$

is a vector in $\mathcal{H}_{\mathbf{P}}$ and it belongs to the domain of $H_{\mathbf{P},\sigma_t}$. Then the procedure consists in studying the mean value of the positive operator

$$H_{\mathbf{P},\sigma_t}-E^{\sigma_t}\left(\mathbf{P}\right)+\Delta$$

on the given vector in $\mathcal{H}_{\mathbf{P}}$:

$$\left\langle H_{\mathbf{P},\sigma_t}-E^{\sigma_t}\left(\mathbf{P}\right)+\Delta\right\rangle_{\int_{\sigma_t}^{\kappa_1}a_{\sigma_t}^{out}(\mathbf{k})\frac{\eta_{l,j}\left(\widehat{\mathbf{k}}\right)}{|\mathbf{k}|\sqrt{2|\mathbf{k}|}}\left(\psi_{j,\sigma_t}^{(t)}\right)_{\mathbf{P}+\mathbf{k}}d^3k} \tag{5.34}$$

where Δ is a properly small positive number.
The condition (1.23) implies the inequality

$$E^{\sigma_t}(\mathbf{P}+\mathbf{k})-|\mathbf{k}|-E^{\sigma_t}(\mathbf{P})<0 \tag{5.35}$$

for $\mathbf{k}\in\{\mathbf{k}\,:\,0<|\mathbf{k}|\leq\kappa_1\}$, so that we can conclude that the original vector (5.33) is zero. $\square$

Next Lemma provides the estimates of the expressions (3.44) and (3.45) involved in the control of the difference *D3.1)* in Subsection 3.1.4.

Lemma A6
Under the assumptions for the construction (Subsection 1.3.1) and for α, $0<\alpha\,(<1)$, sufficiently close to 1, the estimates below are valid:

$$\left\| g\int_{\sigma_{t_2}}^{\sigma_{t_1}}\frac{b(\mathbf{k})\left(e^{i|\mathbf{k}|t_1-i\mathbf{k}\cdot\mathbf{x}}-1\right)}{|\mathbf{k}|\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)}\frac{d^3k}{\sqrt{2|\mathbf{k}|}}W_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}_{\mathbf{P}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}}\right\|\leq C\cdot t_1^{1-\epsilon}\cdot|\ln\sigma_{t_2}|\cdot(\sigma_{t_1})\,;$$

$$\left\langle\int_{\sigma_{t_2}}^{\sigma_{t_1}}\frac{\left|e^{i|\mathbf{k}|t_1}-e^{i\mathbf{k}\cdot\mathbf{x}}\right|^2}{2|\mathbf{k}|^3\left(1-\widehat{\mathbf{k}}\cdot\mathbf{v}_j\right)^2}d^3k\right\rangle^{\frac{1}{2}}_{W_{\sigma_{t_2}}(\mathbf{v}_j)W^{\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}_{\mathbf{P}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}}}\leq C\cdot t_1^{\frac{1}{2}-\frac{5\epsilon}{4}}\cdot(|\ln\sigma_{t_2}|)^{\frac{1}{2}}\cdot(\sigma_{t_1})^{\frac{1}{16}}$$

where $\mathbf{v}_j\equiv\nabla E^{\sigma_{t_1}}_{\mathbf{P}_j}$.
Proof
The proof is only outlined because the estimates involve similar procedures as in Theorem A2 on the basis of the known spectral properties. The key ingredients to be exploited are:
The pull-through formula for the action of $b(\mathbf{k})$;
The $\mathbf{k}$-regularity of

$$\left(e^{i|\mathbf{k}|t_1}e^{-i\mathbf{k}\cdot\mathbf{x}}-1\right)W_{\sigma_{t_2}}(\mathbf{v}_j)\,W^{\dagger}_{\sigma_{t_2}}\left(\nabla E^{\sigma_{t_2}}_{\mathbf{P}}\right)\varphi^{(t_1)}_{j,\sigma_{t_1}} \tag{5.36}$$

which is related to the ***Spectral properties***, Subsection 1.3.1;
The fact that the considered momenta $\mathbf{k}$ belong to the set $\{\mathbf{k}:\ |\mathbf{k}|\leq\sigma_{t_1}\}$. $\square$

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